Linking of motivic spheres

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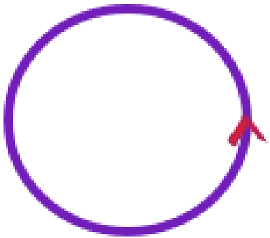
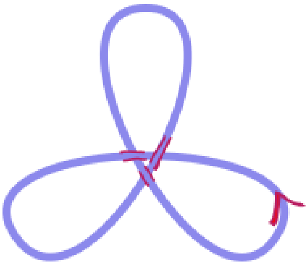
Contents

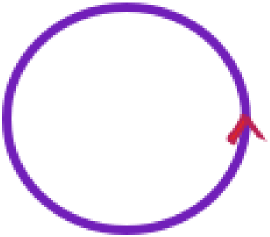


The unknot



The trefoil knot

A **knot** is a (closed) topological subspace of the 3-sphere $\mathbb{S}^3$ which is homeomorphic to the circle $\mathbb{S}^1$ (+ a tameness condition e.g. smoothness).



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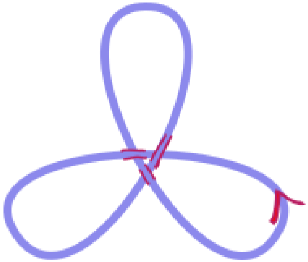
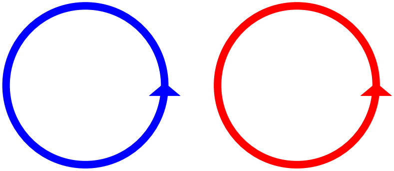
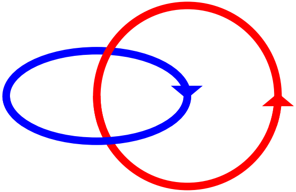


The trefoil knot

A **knot** is a (closed) topological subspace of the 3-sphere $\mathbb{S}^3$ which is homeomorphic to the circle $\mathbb{S}^1$ (+ a tameness condition e.g. smoothness).
An **oriented knot** is a knot with a "continuous" local trivialization of its tangent bundle, or equivalently of its normal bundle.

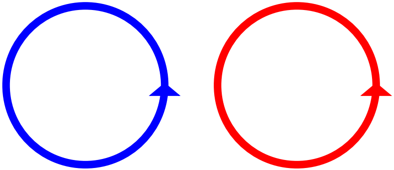


The unlink with two components

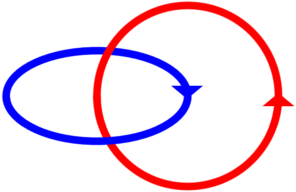


The Hopf link

A **link** is a finite union of disjoint knots (called components).

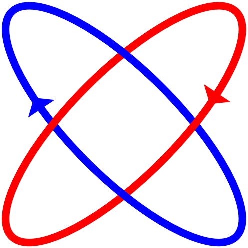


The unlink with two components
(linking number $= 0$)

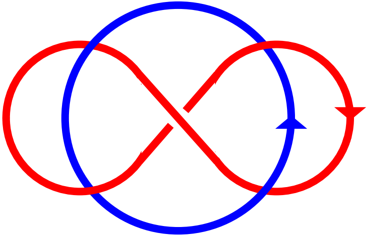


The Hopf link
(linking number $= 1$)

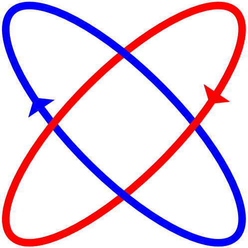
A **link** is a finite union of disjoint knots (called components).
The **linking number** of an oriented link with two components is the number of times one of the components turns around the other component (the sign indicating the direction).



The Solomon link
(linking number $= 2$)



The Whitehead link
(linking number $= 0$)



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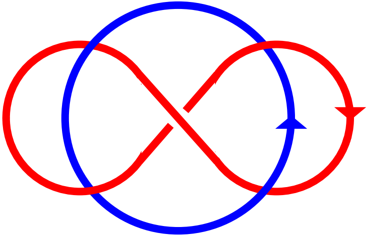
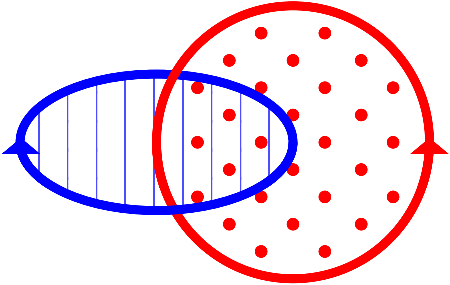


The Whitehead link
(linking number $= 0$)

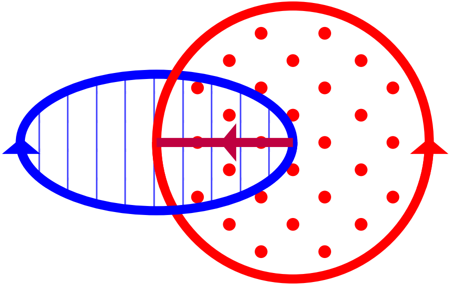
The linking number is a complete invariant of oriented links with two components for link homotopy (i.e. $L = K_1 \sqcup K_2$ and $L' = K_1' \sqcup K_2'$ are link homotopic if and only if they have the same linking number).

Defining the linking number: Seifert surfaces



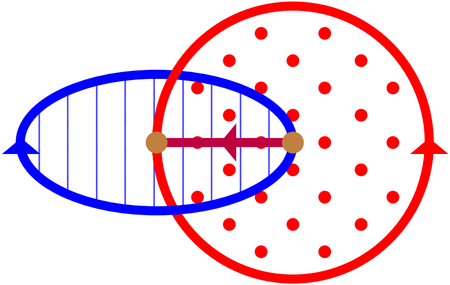
The class S_1 in $H^1(\mathbb{S}^3 \setminus L) \simeq H_2^{\mathrm{BM}}(\mathbb{S}^3, L)$ of Seifert surfaces of the oriented knot K_1 is the **unique** class that is sent by the **boundary map** to the (oriented) fundamental class of K_1 in $H^0(K_1) \subset H^0(L)$.

Defining the linking number: intersection of S. surfaces



This intersection corresponds to the **cup-product** $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$.

Defining the linking number: boundary of int. of S. surf.



This corresponds to $\partial(S_1 \cup S_2) \in H^1(L) \simeq H^1(K_1) \oplus H^1(K_2)$, which we call the **linking class**. Writing $\partial(S_1 \cup S_2) = (\sigma_1, \sigma_2)$, the **linking number** is $r((i_1)_*(\sigma_1)) \in \mathbb{Z}$ with $(i_1)_* : H^1(K_1) \to H^3(\mathbb{S}^3)$ induced by the inclusion.

Definition

The **linking couple** is the couple of integers $(h_1(\sigma_1), h_2(\sigma_2))$ with $h_i : H^1(K_i) \simeq \mathbb{Z}$ induced by the volume form ω_{K_i} of K_i.

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More generally, the linking class, the linking number and the linking couple can be defined in a similar manner to what we have done for two disjoint m-spheres $\mathbb{S}^m$ in the $(2m+1)$-sphere $\mathbb{S}^{2m+1}$ (with $m \in \mathbb{N}$).

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Important fact

The linking couple is equal to $(\pm n, \pm n)$ with n the linking number.

Linking numbers in general

The linking number can actually be defined in a much more general case:

- if M^n is an oriented n-dimensional manifold (as defined in [Seifert and Threlfall, *Lehrbuch der Topologie / A textbook of top.* Chapter X], e.g. $\mathbb{S}^n$, $\mathbb{RP}^n$ (if n is odd, for orientability) or $\mathbb{CP}^{\frac{n}{2}}$ (if n is even))

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- and if A^{k-1} and B^{n-k} are disjoint oriented homologically trivial submanifolds of M^n of respective dimensions $k-1$ and $n-k$
- then the **linking number** of A^{k-1} and B^{n-k} is the intersection number of C^k with B^{n-k}, where C^k is a k-dimensional singular chain of boundary A^{k-1} (e.g. C^k is a k-dimensional oriented submanifold of M^n whose oriented boundary is A^{k-1}).

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- Examples: $M^n = \mathbb{S}^n$, $A^{k-1} \simeq \mathbb{S}^{k-1}$, $B^{n-k} \simeq \mathbb{S}^{n-k}$. If in addition $k-1 = n-k$, then the definition of the linking number we have presented before agrees with this definition.

Contents

Smooth models of motivic spheres

From now on, F denotes a perfect field.

Definition

Let $i, j \in \mathbb{N}_0$. A smooth model of the motivic sphere $S^i \wedge \mathbb{G}_m^{\wedge j}$ is a smooth finite-type F-scheme which is $\mathbb{A}^1$-homotopic to $S^i \wedge \mathbb{G}_m^{\wedge j}$, where S^i is the i-th smash-product of the simplicial circle S^1 and $\mathbb{G}_m^{\wedge j}$ is the j-th smash-product of the multiplicative group.

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- $\mathbb{P}^1_F$ is a smooth model of the motivic sphere $S^1 \wedge \mathbb{G}_m$.
- $\mathbb{A}^n_F \setminus \{0\}$ is a smooth model of the motivic sphere $S^{n-1} \wedge \mathbb{G}_m^{\wedge n}$.

Results in Asok, Doran and Fasel's 2016 article

- Beware: not every motivic sphere has a smooth model! In fact, if $i > j$, the motivic sphere $S^i \wedge \mathbb{G}_m^{\wedge j}$ does not have a smooth model.

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- For each $n \in \mathbb{N}_0$, Q_{2n} is a smooth model of $S^n \wedge \mathbb{G}_m^{\wedge n}$, where:

$$Q_{2n} := \mathrm{Spec}(F[x_1, \ldots, x_n, y_1, \ldots, y_n, z]/(\sum_{i=1}^{n} x_i y_i - z(1+z)))$$

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- For each $n \in \mathbb{N}$, Q_{2n-1} is a smooth model of $S^{n-1} \wedge \mathbb{G}_m^{\wedge n}$, where:

$$Q_{2n-1} := \mathrm{Spec}(F[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\sum_{i=1}^{n} x_i y_i - 1))$$

Links in algebraic geometry

Let F be a perfect field and X be a smooth finite-type irred. F-scheme.

Link with two components

A link with two components in X is a couple of disjoint smooth finite-type irreducible closed F-subschemes Z_1 and Z_2 of X such that:

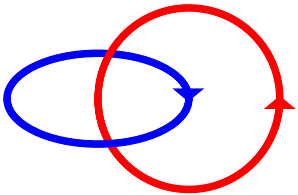
- Z_1 and Z_2 have the same codimension c in X;
- $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$ and $H^c(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$ for some $j_1 \leq 0$;
- $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$ and $H^c(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$ for some $j_2 \leq 0$.

Example: $Z_1 \simeq \mathbb{A}^2_F \setminus \{0\}$ and $Z_2 \simeq \mathbb{A}^2_F \setminus \{0\}$ disjoint closed F-subschemes of $X = \mathbb{A}^4_F \setminus \{0\}$.

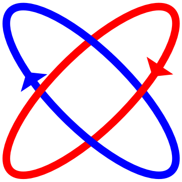
The Hopf link and the Solomon link

Here F is a perfect field of characteristic different from 2. We fix coordinates x, y, z, t for $\mathbb{A}^4_F$ once and for all.

- The Hopf link: $Z_1 = \{z = x, t = y\}$ and $Z_2 = \{z = -x, t = -y\}$ in $X = \mathbb{A}^4_F \setminus \{0\}$
- The Solomon link: $Z_1 = \{z = x^2 - y^2, t = 2xy\}$ and $Z_2 = \{z = -x^2 + y^2, t = -2xy\}$ in $X = \mathbb{A}^4_F \setminus \{0\}$



The Hopf link (linking number $= 1$)



The Solomon link (linking number $= 2$)

Oriented links in algebraic geometry

An orientation o_i of Z_i is an isomorphism from the determinant (i.e. the maximal exterior power) of the normal sheaf $\mathcal{N}_{Z_i/X}$ of Z_i in X to the tensor product of an invertible $\mathcal{O}_{Z_i}$-module $\mathcal{L}_i$ with itself:

$$o_i : \nu_{Z_i} := \det(\mathcal{N}_{Z_i/X}) \simeq \mathcal{L}_i \otimes \mathcal{L}_i$$

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Orientation classes

Two orientations $o_i : \nu_{Z_i} \to \mathcal{L}_i \otimes \mathcal{L}_i$ and $o_i' : \nu_{Z_i} \to \mathcal{L}_i' \otimes \mathcal{L}_i'$ of Z_i represent the same orientation class of Z_i if there exists an isomorphism $\psi : \mathcal{L}_i \simeq \mathcal{L}_i'$ such that $(\psi \otimes \psi) \circ o_i = o_i'$.

The link (Z_1, Z_2) together with an orientation class $\overline{o_1}$ of Z_1 and an orientation class $\overline{o_2}$ of Z_2 is an oriented link with two components.

Oriented fundamental classes and Seifert classes

Let $i \in \{1, 2\}$.

Definition

- We define the **oriented fundamental class** $[o_i]_{j_i}$ with respect to $j_i \leq 0$ as the unique class in $H^0(Z_i, \underline{K}_{j_i}^{\mathrm{MW}}\{\nu_{Z_i}\})$ that is sent by $\widetilde{o}_i$ to the class of η^{-j_i} in $H^0(Z_i, \underline{K}_{j_i}^{\mathrm{MW}})$.

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- We define the **Seifert class** $\mathcal{S}_{o_i, j_i}$ with respect to j_i as the unique class in $H^{c-1}(X \setminus Z, \underline{K}_{j_i+c}^{\mathrm{MW}})$ that is sent by the boundary map ∂ to the oriented fundamental class $[o_i]_{j_i} \in H^0(Z, \underline{K}_{j_i}^{\mathrm{MW}}\{\nu_Z\})$.

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The assumptions $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_i+c}) = 0$ and $H^c(X, \underline{K}^{\mathrm{MW}}_{j_i+c}) = 0$ made earlier are there to ensure the unicity and the existence resp. of the Seifert class.

The (ambient) quadratic linking class / degree

The quadratic linking class

We define the **quadratic linking class** with respect to (j_1, j_2) as the image of the intersection product $\mathcal{S}_{o_1,j_1} \cdot \mathcal{S}_{o_2,j_2}$ by the boundary map $\partial : H^{2c-2}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) \to H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\})$.

The **quadratic linking degree** (couple) is the image of the quadratic linking class by an isomorphism.

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The **quadratic linking degree** (couple) is the image of the quadratic linking class by an isomorphism.

The ambient quadratic linking class

We define the **ambient quadratic linking class** with respect to (j_1, j_2) as the image of the part of the quadratic linking class which is in $H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_1}\})$ by the morphism $(i_1)_* : H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_1}\}) \to H^{2c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c})$.

The **ambient quadratic linking degree** is the image of the ambient quadratic linking class by an isomorphism.

Computations for the oriented Hopf link

- The oriented Hopf link: $Z_1 = \{z = x, t = y\}$ with $o_1 : \overline{z-x}^* \wedge \overline{t-y}^* \mapsto 1 \otimes 1$ and $Z_2 = \{z = -x, t = -y\}$ with $o_2 : \overline{z+x}^* \wedge \overline{t+y}^* \mapsto 1 \otimes 1$ in $X = \mathbb{A}^4_F \setminus \{0\}$

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- Its quadratic linking class is $-\langle z+x\rangle\eta \otimes (\overline{t+y}^* \wedge \overline{z-x}^* \wedge \overline{t-y}^*) \oplus \langle z-x\rangle\eta \otimes (\overline{t-y}^* \wedge \overline{z+x}^* \wedge \overline{t+y}^*)$ in $H^1(Z_1, \underline{K}_0^{\mathrm{MW}}\{\nu_{Z_1}\}) \oplus H^1(Z_2, \underline{K}_0^{\mathrm{MW}}\{\nu_{Z_2}\})$

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In the cases $Q_n \sqcup Q_n \to Q_{n+\lfloor \frac{n}{2} \rfloor+1} = X$ with $n \in \{2, 3, 4\}$, the only conditions which are not verified are the ones which are there to ensure the existence of Seifert classes ($H^c(X, \underline{K}^{MW}_{j_1+c}) = 0$ and $H^c(X, \underline{K}^{MW}_{j_2+c}) = 0$).

Examples of $Q_2 \sqcup Q_2 \to \mathbb{A}^4_F \setminus \{0\}$ $(j_1 = -1 = j_2)$

Assume $F \neq \mathbb{Z}/2\mathbb{Z}$. Let $a \neq b \in F^*$. $Z_1 = \{xy = z(z+1), t = a\}$ and $Z_2 = \{xy = z(z+1), t = b\}$ are of ambient quadratic linking degree 0 and of quadratic linking degree $(0, 0)$.

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Assume the characteristic of F to be different from 2 and 3.
$Z_1 = \{xy = z(z+1), t = 1\}$ and $Z_2 = \{xy = t(t+1), z = 2\}$ (with the orientation classes and parametrisations which you can guess) are of ambient quadratic linking degree 0 and of quadratic linking degree $(-1, -1) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$.

Examples of $Q_2 \sqcup Q_2 \to Q_4$ ($j_1 = -1 = j_2$)

For both examples, assume F of characteristic different from 2. Recall that $Q_4 = \mathrm{Spec}(F[x_1, y_1, x_2, y_2, z]/(x_1y_1 + x_2y_2 - z(z+1)))$.

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$Z_1 = \{x_1y_1 = (z-1)z, y_2 = 1\}$ and $Z_2 = \{x_1y_1 = (z+1)(z+2), x_2 = 1\}$ (with the orientation classes and parametrisations which you can guess) are of quadratic linking degree $(\langle 2\rangle, \langle 2\rangle) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$.

Thanks for your attention!