Motivic knot theory

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Algebraic geometry in a nutshell

Let F be a (perfect) field. Geometrical objects of interest are subsets of F^n which are zeroes of polynomials or complements of such subsets, for instance:

- F^n (no polynomial);
- the unit circle $\{(x, y) \in F^2, x^2 + y^2 - 1 = 0\}$ (1 p.);
- the diagonal line $\{(x, y) \in F^2, x - y = 0\}$ (1 p.);
- their intersection $\{(x, y) \in F^2, x^2 + y^2 - 1 = 0, x - y = 0\}$ (2 p.);
- the origin $\{0\} \subset F^2$: $\{(x, y) \in F^2, x = 0, y = 0\}$ (2 p.);
- $F^2 \setminus \{0\}$...

In practice, we replace these with schemes, for instance F^n is replaced with the affine n-space $\mathbb{A}^n_F$ and $F^n \setminus \{0\}$ is replaced with the scheme $\mathbb{A}^n_F \setminus \{0\}$.

Knot theory in a nutshell

Topological objects of interest are knots and links.

- A **knot** is a (closed) topological subspace of the 3-sphere $\mathbb{S}^3$ which is homeomorphic to the circle $\mathbb{S}^1$.
- An **oriented knot** is a knot with a "continuous" local trivialization of its tangent bundle, or equivalently of its normal bundle (the ambient space being oriented). There are two orientation classes.
- A **link** is a finite union of disjoint knots. A link is **oriented** if all its components (i.e. its knots) are oriented.
- The **linking number** of an (oriented) link with two components is the number of times one of the components turns around the other component.

Oriented knots and links in algebraic geometry

Recall that for all $n \geq 1$, $\mathbb{S}^n$ has the same homotopy type as $\mathbb{R}^{n+1} \setminus \{0\}$.

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Link with two components

A link with two components is a couple of knots $\varphi_i : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ with disjoint images Z_i (where $i \in \{1, 2\}$).

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Intuition

Think of the normal sheaf of Z_i in $\mathbb{A}^4_F \setminus \{0\}$ as a two-dimensional vector space and think of a trivialization of it as a basis of this vector space.

Orientation classes

Fact

The orientation classes are parametrized by the elements of $F^*/(F^*)^2$ (where $(F^*)^2 = \{a \in F^*, \exists b \in F^*, a = b^2\}$).

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If $F = \mathbb{C}$ then $F^*/(F^*)^2$ has one element.

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If $F = \mathbb{C}$ then $F^*/(F^*)^2$ has one element.
If $F = \mathbb{Q}$ then $F^*/(F^*)^2$ has infinitely many elements (the classes of the integers without square factors).

The Hopf link

We fix coordinates x, y, z, t for $\mathbb{A}^4_F$ and u, v for $\mathbb{A}^2_F$ once and for all.

- The image of the Hopf link:

$$\{x = 0, y = 0\} \sqcup \{z = 0, t = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$$

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$$\varphi_1 : (x, y, z, t) \leftrightarrow (0, 0, u, v), \varphi_2 : (x, y, z, t) \leftrightarrow (u, v, 0, 0)$$

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- The orientation of the Hopf link:

$$o_1 : \overline{x}^* \wedge \overline{y}^* \mapsto 1, o_2 : \overline{z}^* \wedge \overline{t}^* \mapsto 1$$

A variant of the Hopf link

- The image is the same as the Hopf link's image:

$$\{x = y, y = 0\} \sqcup \{z = 0, a \times t = 0\} \subset \mathbb{A}^4_F \setminus \{0\} \text{ with } a \in F^*$$

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Motivic homotopy theory

Overview

Motivic homotopy theory (a.k.a. $\mathbb{A}^1$-homotopy theory) is a homotopy theory on smooth schemes of finite type over a “nice” base scheme (in our case the perfect field F).

The idea is to replace the unit interval $[0, 1]$ with the affine line $\mathbb{A}^1_F$.

References on motivic homotopy theory

- The foundations were laid out in Morel and Voevodsky's article $\mathbb{A}^1$*-homotopy theory of schemes* (1999)
- Its specificities when the base scheme is a perfect field were laid out in Morel's book $\mathbb{A}^1$*-algebraic topology over a field* (2012)
- The nLab page *Motivic homotopy theory* is nicely done and has plenty of references

Motivic spheres

There are two analogues of the circle $[0,1]/\{0,1\}$ in motivic homotopy theory: $S^1 := \mathbb{A}^1_F/\{0,1\}$ and the multiplicative group $\mathbb{G}_m := \mathbb{G}_{m,F}$.

Motivic spheres

For all $i, j \in \mathbb{Z}$, we denote by S^i the i-th smash-product of S^1 and we call the smash-product $S^i \wedge \mathbb{G}_m^{\wedge j}$ (in the stable homotopy category) a motivic sphere.

Note that the projective line $\mathbb{P}^1 := \mathbb{P}^1_F$ is equal to $S^1 \wedge \mathbb{G}_m$ in the stable homotopy category.

Intuition

Think of $\mathbb{P}^1$ as the set of lines in F^2, i.e. $\{[x:y], (x,y) \in F^2 \setminus \{0\}\}$ with $[\lambda x : \lambda y] = [x : y]$ for all $\lambda \in F^*$.

Morel's Theorem

Objects of interest

The groups of morphisms $[S^i \wedge \mathbb{G}_m^{\wedge j}, S^k \wedge \mathbb{G}_m^{\wedge l}] = [S^{i-k}, \mathbb{G}_m^{\wedge(l-j)}]$ in the stable homotopy category.

Similarly to the fact that the stable homotopy group $\pi_i^s(S_0) = 0$ if $i < 0$, the group $[S^i, \mathbb{G}_m^{\wedge j}]$ is equal to 0 if $i < 0$ (with $j \in \mathbb{Z}$).

Morel's theorem

Morel gave a presentation by generators and relations of the graded ring with unit $\bigoplus_{n\in\mathbb{Z}}[S^0, \mathbb{G}_m^{\wedge n}]$ (where the product is given by the smash-product).

The generators are denoted $[a] \in [S^0, \mathbb{G}_m]$ for every $a \in F^*$ and $\eta \in [S^0, \mathbb{G}_m^{\wedge(-1)}] = [\mathbb{A}^2 \setminus \{0\}, \mathbb{P}^1]$ which sends (x, y) to $[x : y]$.

Milnor-Witt K-theory

Definition

The graded ring with unit $K_*^{MW}(F) := \bigoplus_{n\in\mathbb{Z}} [S^0, \mathbb{G}_m^{\wedge n}]$ is called the Milnor-Witt K-theory ring of F. We denote $K_n^{MW}(F) := [S^0, \mathbb{G}_m^{\wedge n}]$.

We denote $\langle a \rangle = \eta[a] + 1 \in K_0^{MW}(F)$ for every $a \in F^*$.

Fact

If $n \leq 0$ then every element of $K_n^{MW}(F)$ is a $\mathbb{Z}$-linear combination of $\langle a \rangle \eta^{-n}$ with $a \in F^*$.

The Rost-Schmid ring: An analogue of the singular cohomology ring

To a smooth F-scheme Y, an integer $j \in \mathbb{Z}$ and an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ we associate the corresponding Rost-Schmid complex

$$\bigoplus_{i \geq 0} \bigoplus_{p \text{ point of codim } i \text{ in } Y} K^{\mathrm{MW}}_{j-i}(\kappa(p)) \otimes \text{ a twist which depends on } p \text{ and } \mathcal{L}.$$

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For every $i \in \mathbb{N}_0$, we denote the i-th cohomological group of this complex (called a Rost-Schmid group) by $H^i(Y, \underline{K}^{\mathrm{MW}}_j\{\mathcal{L}\})$. We denote $H^i(Y, \underline{K}^{\mathrm{MW}}_j) := H^i(Y, \underline{K}^{\mathrm{MW}}_j\{\mathcal{O}_Y\})$.

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To a smooth F-scheme Y, an integer $j \in \mathbb{Z}$ and an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ we associate the corresponding Rost-Schmid complex

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We have an intersection product

$$\cdot : H^i(Y, \underline{K}_j^{\mathrm{MW}}) \times H^{i'}(Y, \underline{K}_{j'}^{\mathrm{MW}}) \to H^{i+i'}(Y, \underline{K}_{j+j'}^{\mathrm{MW}})$$

which makes $\bigoplus_{i \in \mathbb{N}_0, j \in \mathbb{Z}} H^i(Y, \underline{K}_j^{\mathrm{MW}})$ into a graded $K_0^{\mathrm{MW}}(F)$-algebra.

Boundary maps

Definition

A boundary triple is a 5-tuple (Z, i, X, j, U), or abusively a triple (Z, X, U), with $i : Z \to X$ a closed immersion and $j : U \to X$ an open immersion such that the image of U by j is the complement in X of the image of Z by i, where Z, X, U are smooth F-schemes of pure dimensions. The boundary map associated to this boundary triple is the morphism

$$\partial : \mathcal{C}^\bullet(U, \underline{K}_*^{\mathrm{MW}}) \to \mathcal{C}^{\bullet+1+d_Z-d_X}(Z, \underline{K}_{*+d_Z-d_X}^{\mathrm{MW}}\{\nu_Z\})$$

induced by the differential d of the Rost-Schmid complex $\mathcal{C}(X, \underline{K}_*^{\mathrm{MW}})$, i.e.:

$$\partial = i^* \circ d \circ j_*$$

The localization long exact sequence: An analogue of the cohomology long exact sequ. of a pair

Theorem

Let (Z, i, X, j, U) be a boundary triple. The boundary map induces a morphism $\partial : H^{n+d_X-d_Z}(U, \underline{K}^{\mathrm{MW}}_{m+d_X-d_Z}) \to H^{n+1}(Z, \underline{K}^{\mathrm{MW}}_m\{\nu_Z\})$ and we have the following long exact sequence, called the localization long exact sequence:

$$\dots \longrightarrow H^n(Z, \underline{K}^{\mathrm{MW}}_m\{\nu_Z\}) \xrightarrow{i_*} H^{n+d_X-d_Z}(X, \underline{K}^{\mathrm{MW}}_{m+d_X-d_Z}) \xrightarrow{j^*}$$

$$\xrightarrow{j^*} H^{n+d_X-d_Z}(U, \underline{K}^{\mathrm{MW}}_{m+d_X-d_Z}) \xrightarrow{\partial} H^{n+1}(Z, \underline{K}^{\mathrm{MW}}_m\{\nu_Z\}) \longrightarrow \dots$$

Punctured affine spaces are analogues of spheres

Let $n \geq 2$ be an integer, $i \in \mathbb{N}_0, j \in \mathbb{Z}$. The Rost-Schmid group

$$H^i(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) \text{ is isomorphic to } \begin{cases} K^{\mathrm{MW}}_j(F) & \text{if } i = 0 \\ K^{\mathrm{MW}}_{j-n}(F) & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases}.$$

This is similar to the fact in classical homotopy theory that $H^i(\mathbb{S}^{n-1})$ is

$$\text{isomorphic to } \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases}.$$

Note that $\mathbb{A}^n_F \setminus \{0\} = S^{n-1} \wedge \mathbb{G}_m^{\wedge n}$ in the stable homotopy category.

The linking number and its analogue

Let $L = K_1 \sqcup K_2$ be an oriented link (in knot theory) and $\mathscr{L}$ be an oriented link with two components (in motivic knot theory), i.e. a couple of closed immersions $\varphi_i : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ with disjoint images Z_i and orientation classes $\overline{o_i}$. We denote $Z := Z_1 \sqcup Z_2$.

The linking number and its analogue: step 1

Knot theory

The class S_i in $H^1(\mathbb{S}^3 \setminus L) \simeq H_2^{\mathrm{BM}}(\mathbb{S}^3, L)$ of Seifert surfaces of the oriented knot K_i is the unique class that is sent by the boundary map to the (oriented) fundamental class of K_i in $H^0(K_i) \subset H^0(L)$.

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Motivic knot theory

We define an analogue $[o_i] \in H^0(Z_i, \underline{K}_{-1}^{\mathrm{MW}}\{\nu_{Z_i}\})$ of the oriented fundamental class of each oriented component of $\mathscr{L}$ then we define the Seifert class $\mathcal{S}_i$ as the unique class in $H^1(X \setminus Z, \underline{K}_1^{\mathrm{MW}})$ that is sent by the boundary map to the oriented fundamental class $[o_i] \in H^0(Z, \underline{K}_{-1}^{\mathrm{MW}}\{\nu_Z\})$.

The linking number and its analogue: step 2

Knot theory

The linking class of L is the image of the cup-product $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$ by the boundary map $\partial : H^2(\mathbb{S}^3 \setminus L) \to H^3(\mathbb{S}^3, \mathbb{S}^3 \setminus L) \simeq H^1(L)$.

The linking number and its analogue: step 2

Knot theory

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Motivic knot theory

We define the quadratic linking class of $\mathscr{L}$ as the image of the intersection product $\mathcal{S}_1 \cdot \mathcal{S}_2 \in H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}})$ by the boundary map $\partial : H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}}) \to H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\})$.

The linking number and its analogue: step 3

Knot theory

The linking number of $L = K_1 \sqcup K_2$ is the integer $n \in \mathbb{Z}$ such that the linking class in $H^1(L) = \mathbb{Z}[\omega_{K_1}] \oplus \mathbb{Z}[\omega_{K_2}]$ is equal to $(n[\omega_{K_1}], -n[\omega_{K_2}])$ (where ω_{K_i} is the volume form of the oriented knot K_i).

The linking number and its analogue: step 3

Knot theory

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Motivic knot theory

We define the quadratic linking degree of $\mathscr{L}$ as the image of the quadratic linking class of $\mathscr{L}$ by the isomorphism
$H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\}) \to H^1(Z, \underline{K}_0^{\mathrm{MW}}) \to$
$H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) \oplus H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) \to K_{-2}^{\mathrm{MW}}(F) \oplus K_{-2}^{\mathrm{MW}}(F)$.

We fixed an isomorphism $H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) \to K_{-2}^{\mathrm{MW}}(F)$ once and for all. Recall that $K_{-2}^{\mathrm{MW}}(F)$ is generated by the $\langle a \rangle \eta^2$ with $a \in F^*$.

The Hopf link

Recall that we fixed coordinates x, y, z, t for $\mathbb{A}^4_F$ and u, v for $\mathbb{A}^2_F$.

- The image of the Hopf link:

$$\{x = 0, y = 0\} \sqcup \{z = 0, t = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$$

- The parametrization of the Hopf link:

$$\varphi_1 : (x, y, z, t) \leftrightarrow (0, 0, u, v), \varphi_2 : (x, y, z, t) \leftrightarrow (u, v, 0, 0)$$

- The orientation of the Hopf link:

$$o_1 : \overline{x}^* \wedge \overline{y}^* \mapsto 1, o_2 : \overline{z}^* \wedge \overline{t}^* \mapsto 1$$

The quadratic linking degree of the Hopf link

Or. fund. classes	$\eta \otimes (\overline{x}^* \wedge \overline{y}^*)$	|	$\eta \otimes (\overline{z}^* \wedge \overline{t}^*)$
Seifert classes	$\langle x \rangle \otimes \overline{y}^*$	|	$\langle z \rangle \otimes \overline{t}^*$
Apply int. prod.	$\langle xz \rangle \otimes (\overline{t}^* \wedge \overline{y}^*)$		
Quad. link. class	$-\langle z \rangle \eta \otimes (\overline{t}^* \wedge \overline{x}^* \wedge \overline{y}^*)$	$\oplus$	$\langle x \rangle \eta \otimes (\overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$
Apply $\widetilde{o_1} \oplus \widetilde{o_2}$	$-\langle z \rangle \eta \otimes \overline{t}^*$	$\oplus$	$\langle x \rangle \eta \otimes \overline{y}^*$
Apply $\varphi_1^* \oplus \varphi_2^*$	$-\langle u \rangle \eta \otimes \overline{v}^*$	$\oplus$	$\langle u \rangle \eta \otimes \overline{v}^*$
Apply $\partial \oplus \partial$	$-\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$	$\oplus$	$\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$
Quad. link. degree	$-\eta^2$	$\oplus$	η^2

A variant of the Hopf link

- The image is the same as the Hopf link's image:

$$\{x = y, y = 0\} \sqcup \{z = 0, a \times t = 0\} \subset \mathbb{A}^4_F \setminus \{0\} \text{ with } a \in F^*$$

- The parametrization is the same:

$$\varphi_1 : (x, y, z, t) \leftrightarrow (0, 0, u, v), \varphi_2 : (x, y, z, t) \leftrightarrow (u, v, 0, 0)$$

- The orientation is different:

$$o_1 : \overline{x - y}^* \wedge \overline{y}^* \mapsto 1, o_2 : \overline{z}^* \wedge \overline{at}^* \mapsto 1$$

The quadratic linking degree of a variant of the Hopf link

$[o_1^{var}] = \eta \otimes \overline{x-y}^* \wedge \overline{y}^* = [o_1^{Hopf}]$ $[o_2^{var}] = \eta \otimes \overline{z}^* \wedge \overline{at}^* = \langle a \rangle [o_2^{Hopf}]$

since $\begin{pmatrix} x-y \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ since $\begin{pmatrix} z \\ at \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}$

$\mathcal{S}_1^{var} = \mathcal{S}_1^{Hopf}$ $\mathcal{S}_2^{var} = \langle a \rangle \mathcal{S}_2^{Hopf}$

$$\mathcal{S}_1^{var} \cdot \mathcal{S}_2^{var} = \langle a \rangle \mathcal{S}_1^{Hopf} \cdot \mathcal{S}_2^{Hopf}$$
$$\partial(\mathcal{S}_1^{var} \cdot \mathcal{S}_2^{var}) = \langle a \rangle \partial(\mathcal{S}_1^{Hopf} \cdot \mathcal{S}_2^{Hopf})$$

The quadratic linking degree is $(-\langle a \rangle \eta^2, \eta^2)$.

Fact

Let $\mathscr{L}$ be an oriented link with two components of quadratic linking degree $(d_1, d_2) \in K_{-2}^{\mathrm{MW}}(F) \oplus K_{-2}^{\mathrm{MW}}(F)$. Let $a = (a_1, a_2)$ be a couple of elements of F^* and $\mathscr{L}_a$ be the link obtained from $\mathscr{L}$ by changing the orientation o_1 into $o_1 \circ (\times a_1)$ and the orientation o_2 into $o_2 \circ (\times a_2)$. Then $\mathrm{Qlc}_{\mathscr{L}_a} = \langle a_1 a_2 \rangle \, \mathrm{Qlc}_{\mathscr{L}}$ and $\mathrm{Qld}_{\mathscr{L}_a} = (\langle a_2 \rangle d_1, \langle a_1 \rangle d_2)$.

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Similarly, changes of parametrizations of the link can only multiply each component of the quadratic linking degree by elements of the form $\langle a \rangle$ with $a \in F^*$ (and do not change the quadratic linking class).

Fact

Let $\mathscr{L}$ be an oriented link with two components of quadratic linking degree $(d_1, d_2) \in K^{\mathrm{MW}}_{-2}(F) \oplus K^{\mathrm{MW}}_{-2}(F)$. Let $a = (a_1, a_2)$ be a couple of elements of F^* and $\mathscr{L}_a$ be the link obtained from $\mathscr{L}$ by changing the orientation o_1 into $o_1 \circ (\times a_1)$ and the orientation o_2 into $o_2 \circ (\times a_2)$. Then $\mathrm{Qlc}_{\mathscr{L}_a} = \langle a_1 a_2 \rangle \, \mathrm{Qlc}_{\mathscr{L}}$ and $\mathrm{Qld}_{\mathscr{L}_a} = (\langle a_2 \rangle d_1, \langle a_1 \rangle d_2)$.

Similarly, changes of parametrizations of the link can only multiply each component of the quadratic linking degree by elements of the form $\langle a \rangle$ with $a \in F^*$ (and do not change the quadratic linking class).

We want invariants of the quadratic linking degree. (Similarly to the absolute value of the linking number in knot theory)

Why a "quadratic" linking degree?

- The (commutative) ring with unit $K_0^{\mathrm{MW}}(F)$ is isomorphic to the Grothendieck-Witt ring $\mathrm{GW}(F)$ of F via $\langle a \rangle \in K_0^{\mathrm{MW}}(F) \leftrightarrow \langle a \rangle \in \mathrm{GW}(F)$.
- For all $n < 0$, the abelian group $K_n^{\mathrm{MW}}(F)$ is isomorphic to the Witt group $\mathrm{W}(F)$ of F via $\langle a \rangle \eta^{-n} \in K_n^{\mathrm{MW}}(F) \leftrightarrow \langle a \rangle \in \mathrm{W}(F)$.

The real definition of the quadratic linking degree

We define the quadratic linking degree of $\mathscr{L}$ as the image of the quadratic linking class of $\mathscr{L}$ by the isomorphism $H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\}) \to K_{-2}^{\mathrm{MW}}(F) \oplus K_{-2}^{\mathrm{MW}}(F) \to \mathrm{W}(F) \oplus \mathrm{W}(F)$.

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The Grothendieck-Witt ring of F and the Witt ring of F (and underlying Witt group of F) are constructed from symmetric bilinear forms on F. If F is of characteristic different from 2 (i.e. $2 \neq 0$ in F) then they are also constructed from quadratic forms.

Interlude: symmetric bilinear forms and quadratic forms

Definition

- A bilinear form on an F-vector space V of finite dimension is a bilinear map $b : V \times V \to F$. It is symmetric if for all $v, w \in V$, $b(v, w) = b(w, v)$.
- If F is of characterisitic different from 2, a quadratic form on V is a map $q : V \to F$ such that the map $b : \begin{cases} V \times V & \to & F \\ (x, y) & \mapsto & \frac{1}{2}(q(x+y) - q(x) - q(y)) \end{cases}$ is a symmetric bilinear form such that for all $x \in V$, $b(x, x) = q(x)$. We call b the polar form of q.

Note that if $b : V \times V \to F$ is a symmetric bilinear form and F is of characterisitic different from 2 then $q : \begin{cases} V & \to & F \\ x & \mapsto & b(x, x) \end{cases}$ is a quadratic form (of polar form b).

Definition

Let $b : V \times V \to F$ and $b' : V' \times V' \to F$ be symmetric bilinear forms.

- The (orthogonal) sum of b and b' is the symmetric bilinear form $b \perp b' : (V \oplus V') \times (V \oplus V') \to F$ which sends $((x, x'), (y, y'))$ to $b(x, y) + b'(x', y')$.
- The (tensor) product of b and b' is the symmetric bilinear form $b \otimes b' : (V \otimes V') \times (V \otimes V') \to F$ which sends $(\sum_{i \in I} x_i \otimes x_i', \sum_{j \in J} y_j \otimes y_j')$ to $\sum_{(i,j) \in I \times J} b(x_i, y_j) \times b'(x_i', y_j')$.

Definition

- The symmetric bilinear form $b : V \times V \to F$ is non-degenerate if 0 is the only element x of V which verifies that for all $y \in V$, $b(x,y) = 0$.
- Two non-degenerate symmetric bilinear forms $b : V \times V \to F$ and $b' : V' \times V' \to F$ are isometric if there exists a linear isomorphism $u : V \to V'$ such that for all $x, y \in V$, $b(x,y) = b'(u(x), u(y))$.

This gives a structure of commutative semiring (commutative monoid + commutative product) on the isometry classes. Grothendieck's construction gives a commutative ring: the Grothendieck-Witt ring of F. Its elements are $\mathbb{Z}$-linear combinations of the classes $\langle a \rangle : \begin{cases} F \times F & \to & F \\ (x,y) & \mapsto & axy \end{cases}$ of symmetric bilinear forms (with $a \in F^*$).

If F is of characteristic $\neq 2$, as a quadratic form $\langle a \rangle : \begin{cases} F & \to & F \\ x & \mapsto & ax^2 \end{cases}$.

Definition

- The hyperbolic plane $b_h : F^2 \times F^2 \to F$ is the symmetric bilinear form which sends $((x_1, y_1), (x_2, y_2))$ to $x_1 y_2 + x_2 y_1$.
- Two non-degenerate symmetric bilinear forms $b : V \times V \to F$ and $b' : V' \times V' \to F$ are Witt-equivalent if there exist $m, n \geq 0$ integers such that $b \perp mb_h$ is isometric to $b' \perp nb_h$.

This gives a structure of commutative ring on the Witt-equivalence classes: the Witt ring of F. Its elements are sums of classes
$\langle a \rangle : \begin{cases} F \times F & \to & F \\ (x, y) & \mapsto & axy \end{cases}$ of symmetric bilinear forms (with $a \in F^*$).

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Presentations of GW(F) and W(F)

- As a commutative ring (resp. abelian group), the Grothendieck-Witt ring (resp. group) GW(F) is generated by the $\langle a \rangle$ for $a \in F^*$ subject to the relations :
    - $\langle ab^2 \rangle = \langle a \rangle$ for all $a, b \in F^*$;
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- This last relation corresponds to the vanishing of the hyperbolic plane.

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- $W(\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ via the rank modulo 2
- For all $p \equiv 3 \mod 4$, $W(\mathbb{Z}/p\mathbb{Z}) \simeq \mathbb{Z}/4\mathbb{Z}$ via the signature modulo 4

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- $\mathrm{W}(\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ via the rank modulo 2
- For all $p \equiv 3 \mod 4$, $\mathrm{W}(\mathbb{Z}/p\mathbb{Z}) \simeq \mathbb{Z}/4\mathbb{Z}$ via the signature modulo 4
- For all $p \equiv 1 \mod 4$, $\mathrm{W}(\mathbb{Z}/p\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ via the "signature couple" modulo 2 (if $a \in \mathbb{Z}/p\mathbb{Z}$ is not a square, $\displaystyle\sum_{i=1}^{p} \langle 1 \rangle + \sum_{j=1}^{q} \langle a \rangle$ is sent to $(p \mod 2, q \mod 2)$)

It is difficult in general to know if two elements of the Witt group $\mathrm{W}(F)$ are equal. For instance, let $a, b, c, d \in F^*$ such that d is not a square in F^* and such that (1) and (2) below are well-defined. Can you tell which of the two following elements of $\mathrm{W}(F)$ is equal to $\langle a \rangle + \langle b \rangle$? (There is exactly one which is equal to $\langle a \rangle + \langle b \rangle$)

1. $\langle (a+b)c^2 + (a+b)abd \rangle + \langle (a+b)(c^2 + abd)abd \rangle$
2. $\langle (a+b)c^2 + (a+b)abd^2 \rangle + \langle (a+b)(c^2 + abd^2)ab \rangle$

Recall that the relations in $\mathrm{W}(F)$ are:

- $\langle ab^2 \rangle = \langle a \rangle$ for all $a, b \in F^*$;
- $\langle a \rangle + \langle b \rangle = \langle a+b \rangle + \langle (a+b)ab \rangle$ for all $a, b \in F^*$ such that $a + b \in F^*$;
- $\langle 1 \rangle + \langle -1 \rangle = 0$.

Solution

The second one is equal to $\langle a\rangle + \langle b\rangle$. Indeed:

$$\begin{aligned}
&\langle a\rangle + \langle b\rangle = \langle (a+b)c^2\rangle + \langle (a+b)abd^2\rangle \\
&= \langle (a+b)c^2 + (a+b)abd^2\rangle + \langle (a+b)(c^2+abd^2)ab(a+b)^2c^2d^2\rangle \\
&= \langle (a+b)c^2 + (a+b)abd^2\rangle + \langle (a+b)(c^2+abd^2)ab\rangle
\end{aligned}$$

To see that the first one is different from $\langle a\rangle + \langle b\rangle$, we will use one of the invariants presented later in this talk.

Invariants by multiplication by $\langle a \rangle$ for all $a \in F^*$

Case $F = \mathbb{R}$

If $F = \mathbb{R}$, the absolute value of an element of $\mathrm{W}(\mathbb{R}) \simeq \mathbb{Z}$ is invariant by multiplication by $\langle a \rangle$ for all $a \in F^*$.

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General case

The rank modulo 2 is invariant by multiplication by $\langle a \rangle$ for all $a \in F^*$.

Definition

Let $d = \sum_{i=1}^{n} \langle a_i \rangle \in \mathrm{W}(F)$. There exists a unique sequence of abelian groups $Q_{d,k}$ and of elements $\Sigma_k(d) \in Q_{d,k}$, where k ranges over the nonnegative even integers, such that:

- $Q_{d,0} = \mathrm{W}(F)$ and $\Sigma_0(d) = 1 \in Q_{d,0}$;
- for each positive even integer k, $Q_{d,k}$ is the quotient group $Q_{d,k-2}/(\Sigma_{k-2}(d))$;
- for each positive even integer k, $\Sigma_k(d) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \langle \prod_{1 \leq j \leq k} a_{i_j} \rangle \in Q_{d,k}$.

General case

The Σ_k are invariant by multiplication by $\langle a \rangle$ for all $a \in F^*$.

- $\Sigma_2 : \begin{cases} \mathrm{W}(F) & \to & \mathrm{W}(F)/(1) \\ \displaystyle\sum_{i=1}^{n} \langle a_i \rangle & \mapsto & \displaystyle\sum_{1 \leq i < j \leq n} \langle a_i a_j \rangle \end{cases}$ (if $n < 2$, it sends $\displaystyle\sum_{i=1}^{n} \langle a_i \rangle$ to 0)

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- $\Sigma_4 : \begin{cases} \mathrm{W}(F) & \to & \bigcup_{d \in \mathrm{W}(F)} (\mathrm{W}(F)/(1))/(\Sigma_2(d)) \\ \sum_{i=1}^{n} \langle a_i \rangle & \mapsto & \sum_{1 \leq i < j < k < l \leq n} \langle a_i a_j a_k a_l \rangle \end{cases}$

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- $\Sigma_4 : \begin{cases} \mathrm{W}(F) & \to & \bigcup_{d \in \mathrm{W}(F)} (\mathrm{W}(F)/(1))/(\Sigma_2(d)) \\ \sum_{i=1}^{n} \langle a_i \rangle & \mapsto & \sum_{1 \leq i < j < k < l \leq n} \langle a_i a_j a_k a_l \rangle \end{cases}$
- We only want to compare $\Sigma_4(d)$ and $\Sigma_4(d')$ if $\Sigma_2(d) = \Sigma_2(d')$.

$\Sigma_2(\langle (a+b)c^2 + (a+b)abd\rangle + \langle (a+b)(c^2+abd)abd\rangle) =$
$\langle ((a+b)c^2+(a+b)abd)(a+b)(c^2+abd)abd\rangle = \langle abd\rangle \neq \langle ab\rangle \in \mathrm{W}(F)/(1)$ since d is not a square in F^*. Since $\Sigma_2(\langle a\rangle + \langle b\rangle) = \langle ab\rangle$,

$$\langle a\rangle + \langle b\rangle \neq \langle (a+b)c^2 + (a+b)abd\rangle + \langle (a+b)(c^2+abd)abd\rangle \in \mathrm{W}(F)$$

Application: invariants of the quadratic linking degree

Let $\mathscr{L}$ be an oriented link with two components (in motivic knot theory). We denote by $(d_1, d_2) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$ its quadratic linking degree.

- If $F = \mathbb{R}$ then the absolute value of d_1 and the absolute value of d_2 are invariant under changes of orientations o_1, o_2 and of parametrizations of $\varphi_1, \varphi_2 : \mathbb{A}^2_{\mathbb{R}} \setminus \{0\} \to \mathbb{A}^4_{\mathbb{R}} \setminus \{0\}$.

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- The rank modulo 2 of d_1 and the rank modulo 2 of d_2 are invariant under changes of orientations o_1, o_2 and of parametrizations of $\varphi_1, \varphi_2 : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$.

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- For every positive even integer k, $\Sigma_k(d_1)$ and $\Sigma_k(d_2)$ are invariant under changes of orientations o_1, o_2 and of parametrizations of $\varphi_1, \varphi_2 : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$.

Another Hopf link

From now on, F is a perfect field of characteristic different from 2. Recall that we fixed coordinates x, y, z, t for $\mathbb{A}^4_F$ and u, v for $\mathbb{A}^2_F$.

- The image is different from the Hopf link we saw before:

  $$\{z = x, t = y\} \sqcup \{z = -x, t = -y\} \subset \mathbb{A}^4_F \setminus \{0\}$$

  But the change of coordinates $x' = z - x$, $y' = t - y$, $z' = z + x$, $t' = t + y$ would give $\{x' = 0, y' = 0\} \sqcup \{z' = 0, t' = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$.

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- The parametrization is $\varphi_1 : (x, y, z, t) \leftrightarrow (u, v, u, v)$ and $\varphi_2 : (x, y, z, t) \leftrightarrow (u, v, -u, -v)$.
- The orientation is the following:

$$o_1 : \overline{z - x}^* \wedge \overline{t - y}^* \mapsto 1, o_2 : \overline{z + x}^* \wedge \overline{t + y}^* \mapsto 1$$

- This Hopf link is an analogue of the Hopf link in knot theory! In knot theory, the Hopf link is given by $\{z = x, t = y\} \sqcup \{z = -x, t = -y\}$ in $\mathbb{S}^3_\varepsilon = \{(x, y, z, t) \in \mathbb{R}^4, x^2 + y^2 + z^2 + t^2 = \varepsilon^2\}$ for ε small enough and has linking number 1.

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- For every positive even integer k, the image by Σ_k of each component is 0.

The Solomon link

- In knot theory, the Solomon link is given by $\{z = x^2 - y^2, t = 2xy\} \sqcup \{z = -x^2 + y^2, t = -2xy\}$ in $\mathbb{S}^3_\varepsilon$ for ε small enough and has linking number 2.
- In motivic knot theory, the image of the Solomon link is:

$$\{z = x^2 - y^2, t = 2xy\} \sqcup \{z = -x^2 + y^2, t = -2xy\} \subset \mathbb{A}^4_F \setminus \{0\}$$

- The parametrization is $\varphi_1 : (x, y, z, t) \leftrightarrow (u, v, u^2 - v^2, 2uv)$ and $\varphi_2 : (x, y, z, t) \leftrightarrow (u, v, -u^2 + v^2, -2uv)$.
- The orientation is the following:

$$o_1 : \overline{z - x^2 + y^2}^* \wedge \overline{t - 2xy}^* \mapsto 1, o_2 : \overline{z + x^2 - y^2}^* \wedge \overline{t + 2xy}^* \mapsto 1$$

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- If $F = \mathbb{R}$, the absolute value of each component is 2.
- The rank modulo 2 of each component is 0.
- For every positive even integer k, the image by Σ_k of each component is 0.
- More generally, we have analogues of the torus links $T(2, 2n)$ (of linking number n); the quadratic linking degree of $T(2, 2n)$ is $(n, -n) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$, which gives n as absolute value if $F = \mathbb{R}$, n modulo 2 as rank modulo 2, and 0 for the Σ_k.

Binary links

- The image of the binary link B_a with $a \in F^* \setminus \{-1\}$:

$$\{f_1 = 0, g_1 = 0\} \sqcup \{f_2 = 0, g_2 = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$$

with $f_1 = t - ((1+a)x - y)y$, $g_1 = z - x(x-y)$,
$f_2 = t + ((1+a)x - y)y$, $g_2 = z + x(x-y)$.

- The parametrization of the binary link B_a:

$$\varphi_1 : (x,y,z,t) \leftrightarrow (u, v, \quad ((1+a)u - v)v, \quad u(u-v))$$

$$\varphi_2 : (x,y,z,t) \leftrightarrow (u, v, -((1+a)u - v)v, -u(u-v))$$

- The orientation of the binary link B_a:

$$o_1 : \overline{f_1}^* \wedge \overline{g_1}^* \mapsto 1, o_2 : \overline{f_2}^* \wedge \overline{g_2}^* \mapsto 1$$

Or. fund. cyc.	$\eta \otimes (\overline{f_1}^* \wedge \overline{g_1}^*)$	$\vert$	$\eta \otimes (\overline{f_2}^* \wedge \overline{g_2}^*)$
Seifert divisors	$\langle f_1 \rangle \otimes \overline{g_1}^*$	$\vert$	$\langle f_2 \rangle \otimes \overline{g_2}^*$
Apply inter. prod.	$\langle f_1 f_2 \rangle \otimes (\overline{g_2}^* \wedge \overline{g_1}^*) \cdot (z, x - y)$ $+\langle f_1 f_2 \rangle \otimes (\overline{g_2}^* \wedge \overline{g_1}^*) \cdot (z, x)$		
...	...		
Apply $\partial \oplus \partial$	$(1 + \langle a \rangle)\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$	$\oplus$	$-(1 + \langle a \rangle)\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$
Quad. lk. deg.	$1 + \langle a \rangle$	$\oplus$	$-(1 + \langle a \rangle)$

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- If $F = \mathbb{R}$, the absolute value of each component is $\begin{cases} 2 \text{ if } a > 0 \\ 0 \text{ if } a < 0 \end{cases}$.
- Σ_2 of each component is $\langle a \rangle \in \mathrm{W}(F)/(1)$. For instance, if $F = \mathbb{Q}$, Σ_2 distinguishes between all the B_p with p prime numbers. $\Sigma_4 = 0$ etc.

Everything new I presented can be found in my preprint “The quadratic linking degree”:

- HAL: Clémentine Lemarié--Rieusset. THE QUADRATIC LINKING DEGREE. 2022. ⟨hal-03821736⟩
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Thanks for your attention!