FORMAL VECTOR SPACES IN MIXED CHARACTERISTIC

DISSERTATION

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Abstract

Let k be a perfect field of positive characteristic p. We study formal $\mathbb{Q}_p$-vector spaces over k, i.e. $\mathbb{Q}_p$-vector space objects in the category of formal k-schemes. This is inspired by Weinstein's work [19] in equal characteristic and by questions on Banach-Colmez spaces asked by Le Bras (cf. [8, Question 7.16]). Our main result is that, if $\mathcal{F}$ is a formal $\mathbb{Q}_p$-vector space represented by a ring $R = k[\![X_1^{1/p^\infty}, \ldots, X_d^{1/p^\infty}]\!]$ of fractional formal power series, then $\mathcal{F}$ is isomorphic to the universal formal cover $\widetilde{G}$ of a p-divisible group G over k in case $d = 1$. For higher dimensions, we extend Drinfeld's equivalence between p-divisible groups and Tate k-groups. We then prove that, if $d \geq 2$, then, under a natural continuity condition, $\mathcal{F}$ is isomorphic to the universal formal cover $\widetilde{G}$ of what we call a generalized p-divisible group G. We also briefly discuss the étale case and how to deal with infinite dimensional $\mathbb{Q}_p$-Banach space representations of $\mathrm{Gal}(\overline{k}/k)$.

Zusammenfassung

Sei k ein perfekter Körper der positiven Charakteristik p. Wir untersuchen formale $\mathbb{Q}_p$-Vektorräume über k, d.h. $\mathbb{Q}_p$-Vektorraumobjekte in der Kategorie der formalen k-Schemata. Inspiriert wird das durch die Arbeit [19] von Weinstein in gleicher Charakteristik und durch Fragen von Le Bras über Banach-Colmez-Räume (vgl. [8, Question 7.16]). Wird ein formaler $\mathbb{Q}_p$-Vektorraum $\mathcal{F}$ durch einen Ring $k[\![X_1^{1/p^\infty}, \ldots, X_d^{1/p^\infty}]\!]$ gebrochener formaler Potenzreihen repräsentiert, so besagt unser Hauptresultat, dass $\mathcal{F}$ im Fall $d = 1$ isomorph ist zur universellen formalen Überlagerung $\widetilde{G}$ einer p-divisible Gruppe G über k. Für höhere Dimensionen verallgemeinern wir eine Äquivalenz von Drinfeld zwischen p-divisiblen Gruppen und k-Tategruppen. Im Fall $d \geq 2$ zeigen wir dann, dass $\mathcal{F}$ unter einer natürlichen Stetigkeitsbedingung isomorph ist zur universellen formalen Überlagerung $\widetilde{G}$ einer sogenannten verallgemeinerten p-divisible Gruppe G. Wir diskutieren auch kurz den étalen Fall und wie man unendlichdimensionale $\mathbb{Q}_p$-Banachraumdarstellungen von $\mathrm{Gal}(\overline{k}/k)$ behandeln kann.

Contents

Introduction

Let p be a prime number, and let k be a perfect field of characteristic p. If K is a non-Archimedean local field of characteristic p whose residue field is contained in k, then Weinstein introduced and studied formal K-vector spaces over k in [19]. These objects are used in studying the Lubin-Tate tower and its group actions in equal characteristic.

Passing to mixed characteristic, formal $\mathbb{Q}_p$-vector spaces over k arise naturally as universal formal covers $\widetilde{G} = \varprojlim_p G$ of p-divisible groups G over k. Implicitly, this functor already appears in the work of Tate, [16]. It was studied systematically by Fontaine (cf. [5, Chapitre V, §1]), and more recently by Scholze and Weinstein, who also introduced its name (cf. [14, §3.1]). Universal formal covers of p-divisible groups over valuation rings of perfectoid fields of mixed characteristic play a prominent role in p-adic Hodge theory and in the theory of Banach-Colmez spaces. In [8], for example, Le Bras asks if any representable Banach-Colmez space is isomorphic to the universal formal cover of a p-divisible group (cf. [8, Question 7.16]). Fargues partly answers this question by establishing an equivalence of categories between certain classes of p-divisible rigid analytic groups and Banach-Colmez spaces via the universal formal cover functor (cf. [4, Théorème 3.3]). In a much more down-to-earth situation, a similar question was asked by Kedlaya in the language of perfect formal group laws over k (cf. [7, Question 2]).

With these questions in mind, we set up a general framework to study formal $\mathbb{Q}_p$-vector spaces over k. We always assume our base field k to be discrete as a topological space. We define a formal $\mathbb{Q}_p$-vector space $\mathcal{F}$ over k to be a representable functor from the category of prodiscrete k-algebras into that of $\mathbb{Q}_p$-vector spaces (cf. Definition 4.1). By abuse of notation, we write $\mathcal{F} = \mathrm{Spf}(R)$ if R is the representing algebra of $\mathcal{F}$. It is known that, if G is a connected p-divisible group of dimension d over k, then $\mathcal{F} = \widetilde{G}$ is represented by $R = k[\![X_1^{1/p^\infty}, \dots, X_d^{1/p^\infty}]\!]$ (cf. [14, Proposition 3.1.3.(iii)]). On the other hand, if G is an étale p-divisible group over k, then $\mathcal{F} = \widetilde{G}$ is represented by $\mathcal{C}\big(T[1/p], \overline{k}\big)^\Gamma$, where $\Gamma = \mathrm{Gal}(\overline{k}/k)$, and $T[1/p]$ denotes the group $\underline{T}(G)[1/p](\overline{k})$ of $\overline{k}$-points of the rational Tate module $\underline{T}(G)[1/p]$ of G (cf. Example 3.4).

In order to recognize a formal $\mathbb{Q}_p$-vector space over k as a universal formal cover, we rely on a second description of $\widetilde{G} = \mathrm{Spf}(R)$. Namely, there is an isomorphism $\widetilde{G} \simeq \underline{T}(G)[1/p]$ of formal k-schemes (cf. Proposition 3.1.(ii)), where $\underline{T}(G) = \varprojlim_n G[p^n]$ is the Tate module of G viewed as a profinite affine group scheme over k. In the connected case, the representing algebra of $\underline{T}(G)$ is $R/(X_1, \dots, X_d)$ (cf. Proposition 3.5).

Coming back to our motivating question, let us first consider a formal $\mathbb{Q}_p$-vector space $\mathcal{F}$ represented by a k-algebra of the form $R = k[\![X^{1/p^\infty}]\!]$, where $X = (X_1, \dots, X_d)$. Just as in the classical situation, we find that $\mathcal{F}$ is given by a family $G(X, Y) \in k[\![X^{1/p^\infty}, Y^{1/p^\infty}]\!]^d$ of d fractional formal power series $G = (G_1, \dots, G_d)$ in $2d$ variables X and Y satisfying the usual axioms of a d-dimensional commutative formal group law (cf. Definition 1.20 and Proposition 1.21). Objects of this kind were coined perfect formal group laws by

Kedlaya in [7]. The above description of the Tate module in the connected case suggests to analyze whether or not the ideal $(X) \subseteq R$ is a topological Hopf ideal for the Hopf structure induced by $\mathcal{F}$. In contrast to classical formal group laws, this does not directly follow from the axioms because a perfect formal group law can in principle be of the form $G(X,Y) = X + Y +$ mixed *lower* terms.

In case $d = 1$, we show, by elementary methods, that this does not happen. In fact, the law of associativity implies that $G(X,Y) = X + Y +$ mixed higher terms (cf. Proposition 4.12). Consequently, the ideal $(X) \subset R$ is a topological Hopf ideal, and $H = \operatorname{Spec}\big(R/(X)\big)$ is an affine group scheme over k (cf. Corollary 4.15). Using only power series methods, we can show that H is a Tate k-group in the sense of Drinfeld (cf. [2, Definition 3.1.1], as well as our Corollary 4.17). If G denotes the corresponding p-divisible group under Drinfeld's equivalence (cf. [2, §3.1.3], as well as our Corollary 2.11), we obtain $\mathcal{F} \simeq \widetilde{G}$, giving a positive answer to our motivating question in the case $d = 1$ (cf. Corollary 4.18):

Theorem. *For any formal $\mathbb{Q}_p$-vector space $\mathcal{F} = \operatorname{Spf}\left(k[\![X^{1/p^\infty}]\!]\right)$ of dimension $d = 1$, there is a p-divisible group $\mathcal{G}$ over k whose universal formal cover is isomorphic to $\mathcal{F}$.*

If $d \geq 2$, then we prove that the ideal $(X) = (X_1, \ldots, X_d) \subseteq R$ need not be a topological Hopf ideal even for formal $\mathbb{Q}_p$-vector spaces isomorphic to universal formal covers of p-divisible groups, simply by considering the p-divisible formal group law of $\mu_{p^\infty} \times \mu_{p^\infty}$, and changing it by a suitable fractional coboundary (cf. Corollary 4.19):

Theorem. *There are formal $\mathbb{Q}_p$-vector spaces represented by $k[\![X^{1/p^\infty}]\!]$ with $d \geq 2$ such that the ideal $(X) = (X_1, \ldots, X_d)$ generated by the variables is not a Hopf ideal of $k[\![X^{1/p^\infty}]\!]$.*

Surprisingly, it is not hard to construct other open ideals of definition $(X) \subsetneq \mathcal{I} \subseteq R$ that are topological Hopf ideals (cf. Corollary 4.21). Consequently, we obtain an affine group scheme $H = \operatorname{Spec}(R/\mathcal{I})$ over k, and it follows, from general results on commutative affine group schemes over k, that H is p-adically separated and complete, i.e. that the canonical map $H \to \varprojlim_n \operatorname{coker}(H \xrightarrow{p^n} H)$ is an isomorphism (cf. Proposition 4.22). Moreover, multiplication with p is injective on H because this is true on $\mathcal{F} = \operatorname{Spf}(R)$. Thus, H satisfies the axioms of a Tate k-group, except for the finiteness of $H/pH = \operatorname{coker}(H \xrightarrow{p} H)$, which we are unable to settle.

Fortunately, it turns out that Drinfeld's equivalence extends under these relaxed conditions (cf. Proposition 2.10) to give an equivalence with a category of what we call generalized p-divisible groups (cf. Definition 2.1). These are defined almost in the same manner as their classical counterparts, but without the usual finiteness conditions on the torsion points. The universal formal cover can also be defined for the generalized p-divisible group G corresponding to H, and gives, under a natural continuity assumption on the multiplication by p, an isomorphism $\mathcal{F} \simeq \widetilde{G}$ as formal $\mathbb{Q}_p$-vector spaces over k, pointing towards a positive answer to our motivating question also in the case $d \geq 2$ (cf. Corollary 4.23):

Theorem. *Let $d \geq 2$, and let $\mathcal{F} = \operatorname{Spf}\left(k[\![X_1^{1/p^\infty}, \dots, X_d^{1/p^\infty}]\!]\right)$ be a formal $\mathbb{Q}_p$-vector space over k. Assume that, for any $f \in \mathfrak{m}_R$, we have $\lim_{n\to\infty}[p^n](f) = 0$. Then, there is a generalized p-divisible group G over k with $\widetilde{G} \simeq \mathcal{F}$.*

We also give a tentative definition of an étale formal $\mathbb{Q}_p$-vector space $\mathcal{F}$ over k (cf. Definition 4.2). In Proposition 4.7, we give a criterion that ensures that $\mathcal{F} \simeq \widetilde{G}$ for some étale p-divisible group G over k. We know that the $\overline{k}$-points $\widetilde{G}(\overline{k})$ of the universal formal cover of an étale p-divisible group G over k is a finite dimensional $\mathbb{Q}_p$-vector space with a continuous action of $\Gamma = \operatorname{Gal}(\overline{k}/k)$ (cf. Corollary 3.2). In order to construct examples of such beyond p-divisible groups, we show that the functor $\widetilde{G}$ admits a straightforward generalization to a functor $\mathcal{F} = \underline{V}$, where V is any continuous $\mathbb{Q}_p$-Banach space representation of Γ. In the infinite dimensional case, the functor $\mathcal{F}$ is not representable in the strict sense; however, we show that introducing strict k-linear inductive limits on a certain space of uniformly continuous maps allows us to compute it on discrete k-algebras (cf. Lemma 4.9). The use of these topologies seems a novel aspect of our work that might deserve further consideration.

We provide a brief overview of each of the chapters:

In Chapter 1, our article starts with a background material on commutative affine group schemes over a field k, with a particular focus on profinite group schemes. It is in this chapter where we study prodiscrete k-algebras, and fix our definition of formal schemes accordingly. Next, we introduce and discuss the notion of strict k-linear inductive limits on linearly topologized k-vector spaces, which runs quite parallel to Schneider's treatment of the notion for locally convex topological spaces in [12, §5, E2]. We finish the chapter by studying fractional formal power series rings and perfect formal group laws over k.

In Chapter 2, by loosening some finiteness condition in the definition of a p-divisible group, we start by defining generalized p-divisible groups over a field k. Assuming k is perfect of characteristic $p > 0$, we generalize Drinfeld's equivalence (cf. [2, 3.1.3 in §3]) of categories between the category of p-divisible groups over k and the category of Tate k-groups accordingly. As we have discussed above, both the classical equivalence and its generalization are vital in establishing the main results Corollary 4.18 and Corollary 4.23, respectively.

In Chapter 3, we study the universal formal cover functor on generalized, as well as on classical, p-divisible groups over k, mostly making use of its relation with the Tate modules (cf. Proposition 3.1). We also prove that the universal formal cover functor on the category of p-divisible groups up to isogeny over a perfect (discrete) field k of characteristic $p > 0$ is fully faithful.

In Chapter 4, we obtain most of our new results we have been describing above.

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1 Foundations

In this chapter, we collect necessary foundations, and set down notational conventions. Let k be a field (endowed with the discrete topology throughout the article).

1.1 Recalls on affine group schemes

Denote by $\mathrm{GSch}_k^{\mathrm{aff}}$ the category of commutative affine group schemes over k, and by Hopf_k the category of cocommutative k-Hopf algebras. The category $\mathrm{GSch}_k^{\mathrm{aff}}$ is Abelian (cf. [18, Ex 12, §16]), and there is an equivalence

$$\mathrm{Spec} : \mathrm{Hopf}_k \to \mathrm{GSch}_k^{\mathrm{aff}}$$

of categories (cf. [18, Theorem in §1.4]). Since objects of interest will always be (co)commutative, we prefer to drop these adjectives from the writing although some of the results we will be giving are more generally true for also non-(co)commutative setting.

Given a map $f : G \to H$ of affine group schemes over k, the kernel of f is defined as the group functor taking a k-algebra R to the group $\ker\big(G(R) \to H(R)\big)$, and is represented by $G \times_H 1$. Assuming $G = \mathrm{Spec}(A)$ and $H = \mathrm{Spec}(B)$ with $A, B \in \mathrm{Hopf}_k$, the cokernel $\mathrm{coker}(f)$ of f is represented by the G-invariant subalgebra B^G of B, which is given by

$$B^G = \{b \in B \mid (\varphi \circ \mathrm{id}_B) \circ \Delta = 1 \otimes b \in A \otimes_k B\},$$

where $\varphi : B \to A$ is the algebra map in Hopf_k corresponding to f, and $\Delta : B \to B \otimes_k B$ is the comultiplication of B (cf. [1, Theorem 1 in §6, Chapter II]).

Observe that, if $(G_i)_{i\in I}$ is an inverse system of affine group schemes $G_i = \mathrm{Spec}(A_i)$ with $A_i \in \mathrm{Hopf}_k$, then it formally follows that

$$\varprojlim_i G_i = \varprojlim_i \mathrm{Spec}(A_i) \simeq \varprojlim_i \mathrm{Hom}_k(A_i, \bullet) \simeq \mathrm{Hom}_k(\varinjlim_i A_i, \bullet) \simeq \mathrm{Spec}(\varinjlim_i A_i),$$

where Hom_k denotes (here and throughout the whole text) the Hom space in the category of k-algebras. This implies that the category $\mathrm{GSch}_k^{\mathrm{aff}}$ has projective limits. Moreover, the anti-equivalence between $\mathrm{GSch}_k^{\mathrm{aff}}$ and Hopf_k restricts to an anti-equivalence between the full subcategory $\mathrm{GSch}_k^{\mathrm{fin}}$ of finite group schemes over k and the full subcategory $\mathrm{Hopf}_k^{\mathrm{fin}}$ of finite dimensional k-Hopf algebras. We then call a projective limit $G = \varprojlim_{i\in I} G_i \in \mathrm{GSch}_k^{\mathrm{aff}}$ of finite group schemes $G_i \in \mathrm{GSch}_k^{\mathrm{fin}}$ a profinite group scheme over k, and denote by $\mathrm{Pro}(\mathrm{GSch}_k^{\mathrm{fin}})$ the category of profinite group schemes over k. The representing k-algebra $A = \varinjlim_{i\in I} A_i \in \mathrm{Hopf}_k$ of G is then an inductive limit of finite dimensional k-Hopf algebras A_i, which we dually call an indfinite k-Hopf algebra. We denote the category of indfinite k-Hopf algebras by $\mathrm{Ind}(\mathrm{Hopf}_k^{\mathrm{fin}})$.

In the rest of the text, we often refrain from indicating the indexing sets of the limits, and suggest instead that they should become clear from the corresponding lowercase letters in place. For instance, a projective limit $\varprojlim_i A_i$ should be simply understood to be taken over an indexing set I.

Now, we want to describe the morphisms in the category $\mathrm{Ind}(\mathrm{Hopf}_k^{\mathrm{fin}})$. Let $B = \varinjlim_j B_j$ and $A = \varinjlim_i A_i \in \mathrm{Ind}(\mathrm{Hopf}_k^{\mathrm{fin}})$, and consider a map $B \to A$. The universal property of the inductive limit gives maps $B_j \to A$ for each $j \in J$, and upon identifying the image of the canonical map $A_i \to A$ with a finite dimensional k-Hopf subalgebra of A, we view A as a filtered union of finite dimensional k-Hopf subalgebras A_i. This implies that each map $B_j \to A$ factors through an injection $A_k \hookrightarrow A$ for some $k \in I$. Moreover, the limit over I can be taken over the indices i such that $i \geq k$ (cf. [9, Theorem 1, §3, Ch IX]) to give us

$$\mathrm{Hom}_k(B, A) \simeq \varprojlim_j \varinjlim_i \mathrm{Hom}_k(B_j, A_i).$$

This implies an analogous relation for the Hom set in $\mathrm{GSch}_k^{\mathrm{aff}}$. Consequently, assuming $G = \varprojlim_i G_i$ and $H = \varprojlim_j H_j$ in $\mathrm{Pro}(\mathrm{GSch}_k^{\mathrm{fin}})$ are represented respectively by A and B in $\mathrm{Ind}(\mathrm{Hopf}_k^{\mathrm{fin}})$, it follows that

$$\mathrm{Hom}(G, H) \simeq \varprojlim_j \varinjlim_i \mathrm{Hom}(G_i, H_j),$$

where the plain Hom symbol (without sub- or superscript) is always meant to denote the set of natural transformations between k-group functors.

The category $\mathrm{Pro}(\mathrm{GSch}_k^{\mathrm{fin}})$ of profinite group schemes over k as a full subcategory of $\mathrm{GSch}_k^{\mathrm{aff}}$ is completely characterized by its underlying topological space in the following sense:

Proposition 1.1. *An affine group scheme over k is profinite if and only if its underlying topological space is profinite.*

Proof. Let $G = \mathrm{Spec}(A) \in \mathrm{Pro}(\mathrm{GSch}_k^{\mathrm{fin}})$, so that $A = \varinjlim_i A_i \in \mathrm{Ind}(\mathrm{Hopf}_k^{\mathrm{fin}})$ is a filtered union of finite dimensional Hopf k-subalgebras A_i of A. We want to show that $\mathrm{Spec}(A)$ is profinite. By [15, Tag 0905], it suffices to show that every prime ideal of A is maximal. Let $\mathfrak{p}$ be a prime ideal of A, and let $\mathfrak{m}$ be a maximal ideal of A containing $\mathfrak{p}$. Since A_i is finite dimensional over k, every prime ideal of A_i is maximal. So the prime ideals $\mathfrak{p} \cap A_i$ and $\mathfrak{m} \cap A_i$ of A_i are both maximal, and must coincide by maximality. But A is a union of A_i, so we must have $\mathfrak{p} = \mathfrak{m}$, and $\mathfrak{p}$ is maximal.

Conversely, assume that $G = \mathrm{Spec}(A)$ is an affine group scheme such that $\mathrm{Spec}(A)$ is profinite. Use [18, 2nd Theorem in §3.3] to write $A = \varinjlim_i A_i$ as a filtered union of finitely generated Hopf subalgebras A_i of A, so that each inclusion $A_i \to A$ is faithfully flat by [18, Theorem in §14.1]. We would like to see that each A_i is indeed finite dimensional over k. According to [15, Tag 06LH], it is enough to show that each prime ideal of A_i

is maximal (for then, A_i would have Krull dimension 0, and hence be Artinian of finite k-dimension). To this end, let $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A_i)$ with $\mathfrak{p}'$ maximal such that $\mathfrak{p} \subseteq \mathfrak{p}'$. Using that $A_i \to A$ is faithfully flat, let $\mathfrak{q}' \in \mathrm{Spec}(A)$ be a prime ideal mapping to $\mathfrak{p}'$ under the (surjective) map $\mathrm{Spec}(A) \to \mathrm{Spec}(A_i)$ of spectra, so that $\mathfrak{q}' \cap A_i = \mathfrak{p}'$. Using [15, Tag 00HS], we have that there is a prime ideal $\mathfrak{q} \subseteq \mathfrak{q}'$ with $\mathfrak{q} \cap A_i = \mathfrak{p}$. But then, it follows that $\mathfrak{q} = \mathfrak{q}'$ as, by [15, Tag 0905], $\mathrm{Spec}(A)$ is Hausdorff. Thus, $\mathfrak{p} = \mathfrak{p}'$, and $\mathfrak{p}$ is maximal, as desired. $\square$

Furthermore, we have:

Proposition 1.2. *The category Pro*$(\mathrm{GSch}_k^{\mathrm{fin}})$ *is Abelian. Moreover, it is closed under extensions: if*

$$1 \to G' \xrightarrow{f} G \xrightarrow{g} G'' \to 1$$

is an exact sequence in $\mathrm{GSch}_k^{\mathrm{aff}}$*, then we have* $G \in$ *Pro*$(\mathrm{GSch}_k^{\mathrm{fin}})$ *if and only if* $G', G'' \in$ *Pro*$(\mathrm{GSch}_k^{\mathrm{fin}})$*.*

Proof. It is clear that the trivial group scheme $\mathrm{Spec}(k)$ is profinite. Observe also that a product of two profinite group schemes $\varprojlim_i G_i$ and $\varprojlim_j H_j$ is again a profinite group scheme $\varprojlim_{(i,j)}(G_i \times H_j)$. Since the category $\mathrm{GSch}_k^{\mathrm{aff}}$ is Abelian, by [11, Proposition 5.92], to prove that $\mathrm{Pro}(\mathrm{GSch}_k^{\mathrm{fin}})$ is Abelian, it is enough to see that it has kernels and cokernels. So we let $f : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ be a map of profinite group schemes over k, where, as usual, A and B are filtered unions of their finite dimensional k-Hopf subalgebras. But then, the representing algebra $A \otimes_B k$ of $\ker(f)$, being a quotient of A by a Hopf ideal, is a filtered union of finite dimensional k-Hopf subalgebras itself. Similarly, the representing algebra B^G of $\mathrm{coker}(f)$ is a subalgebra of B, and hence is as well a filtered union of finite dimensional k-Hopf subalgebras. This proves that $\ker(f)$ and $\mathrm{coker}(f)$ are profinite group schemes over k, and thus, $\mathrm{Pro}(\mathrm{GSch}_k^{\mathrm{fin}})$ is an Abelian category.

For the second assertion, assume that G' and G'' are profinite. Write $G = \mathrm{Spec}(A)$ and $G' = \mathrm{Spec}(B)$. Let A_i be a Hopf subalgebra of A that is finitely generated over k (cf. [18, 2nd Theorem in §3.3]). Let B_i be the image of A_i under the homomorphism $A \to B$ of Hopf algebras corresponding to $f : G' \to G$, so it is itself a Hopf algebra. Let $G'_i := \mathrm{Spec}(B_i)$ and $G_i := \mathrm{Spec}(A_i)$. Let G''_i be the cokernel of the closed immersion $G'_i \to G_i$ (corresponding to the surjection $A_i \to B_i$). We have a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & G' \arrow[r, "f"] \arrow[d] & G \arrow[r, "g"] \arrow[d] & G'' \arrow[r] \arrow[d, dotted, "\pi"] & 1 \\
1 \arrow[r] & G'_i \arrow[r] & G_i \arrow[r] & G''_i \arrow[r] & 1
\end{tikzcd}$$

with exact rows, where the homomorphism $\pi : G'' \to G''_i$ uniquely exists by the universal property of $\mathrm{coker}(f)$. Write $G'' = \mathrm{Spec}(C)$ and $G''_i = \mathrm{Spec}(C_i)$.

Suppose we have shown that B_i and C_i are finitely generated over k. The commutativity of the right square implies that the homomorphism $C_i \to C$ is injective because $A_i \to A$ and $C_i \to A_i$ are. But then, the homomorphisms $B_i \to B$ and $C_i \to C$ of Hopf algebras are faithfully flat with $\mathrm{Spec}(B)$ and $\mathrm{Spec}(C)$ profinite, and the second half of the above proof implies that B_i and C_i are finite dimensional over k. Moreover, the exactness of the second row gives us an isomorphism $B_i \otimes_k A_i \simeq A_i \otimes_{C_i} A_i$ of A_i-modules. As B_i is finite dimensional over k, the left hand side is a finitely generated A_i-module. Thus, so must be the right hand side $A_i \otimes_{C_i} A_i$. Since the map $C_i \to A_i$ corresponding to the cokernel $G_i \to G_i''$ is faithfully flat, this implies that A_i is a finitely generated module over C_i. But then, A_i is finite dimensional over k because C_i is. Thus, $A \in \mathrm{Ind}(\mathrm{Hopf}_k^{\mathrm{fin}})$, and $G \in \mathrm{Pro}(\mathrm{GSch}_k^{\mathrm{fin}})$.

Now, to see that B_i is indeed finitely generated, simply recall that A_i is finitely generated, and $G_i' \to G_i$ is a closed immersion. Hence, being a quotient of A_i, B_i is also finitely generated over k. As for the finite dimensionality of C_i, note that we have a faithfully flat inclusion $C_i \hookrightarrow A_i$ of Hopf algebras with A_i finitely generated. Let $\varepsilon : C_i \to k$ be the augmentation of C_i, and $I := \ker(\varepsilon) \subseteq C_i$ the augmentation ideal. Then, the ideal IA_i in A_i generated by I is finitely generated. Moreover, faithful flatness implies that $IA_i \simeq I \otimes_{C_i} A_i$, and hence, I is finitely generated (as a C_i-module). Assume $c_1, \ldots, c_n \in I$ are generators of I, and let D be a finitely generated Hopf subalgebra of C_i containing the generators $c_1, \ldots, c_n$, with augmentation ideal J. But then, the kernel of the quotient map $\mathrm{Spec}(C_i) \to \mathrm{Spec}(D)$ (induced by the inclusion $D \hookrightarrow C_i$), which is represented by C_i/JC_i, is trivial as $JC_i = I$. This implies $\mathrm{Spec}(C_i) \to \mathrm{Spec}(D)$ is an isomorphism, and $C_i = D$ is a finitely generated k-algebra.

Finally, assume $G = \mathrm{Spec}(A)$ is profinite, where A is a filtered union of its finite dimensional k-Hopf subalgebras. We need to see that G' and G'' are profinite. But the exactness of the sequence $1 \to G' \to G \to G'' \to 1$ implies that the representing algebra of G' is a quotient of A, and that the represening algebra of G'' is a subalgebra of A. This proves that G and G'' are profinite as in the first paragraph of the proof. $\square$

Let Γ denote the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ of k throughout the text. We will quite often refer to the standard equivalence between the category $\mathrm{GSch}_k^{\text{ét}}$ of étale group schemes over k and the category of finite (discrete) groups on which Γ acts continuously as group automorphisms, as well as its formal generalizations to related ind- and pro-objects (recall that Γ acting on a group X as group automorphisms means that, for any $\gamma \in \Gamma$ and $x_1, x_2 \in X$, the action satisfies $\gamma \cdot (x_1 x_2) = (\gamma \cdot x_1)(\gamma \cdot x_2)$). Therefore, we would like to record it here:

Proposition 1.3. *There is an equivalence of categories between the category* $\mathrm{GSch}_k^{\text{ét}}$ *and the category of finite groups on which* Γ *acts continuously as group automorphisms. Given* $G = \mathrm{Spec}(A) \in \mathrm{GSch}_k^{\text{ét}}$*, this equivalence is given by*

$$G \mapsto G(\overline{k}) = \mathrm{Hom}_k(A, \overline{k}).$$

A quasi-inverse is given by mapping a finite group X with a continuous Γ-action by group automorphisms to the étale group scheme over k whose representing algebra is the k-algebra $\mathcal{C}(X, \overline{k})^\Gamma$ of continuous maps from A to $\overline{k}$ that commute with the action of Γ.

Proof. See [18, Theorem in §6.4]. □

Note that the above equivalence could also be given in terms of the anti-equivalence of categories between the category of étale k-algebras and the category of finite Γ-sets, given by $A \mapsto \mathrm{Hom}_k(A, \overline{k})$ (cf. [18, Theorem in §6.3]). Therefore, we will hardly make a distinction between the implied equivalences of categories.

We say that a group scheme G over k is proétale if it can be written as a projective limit $G \simeq \varprojlim_i G_i$ of étale group schemes G_i over k. Denote by $\mathrm{GSch}_k^{\text{ét}}$ the full subcategory of $\mathrm{GSch}_k^{\text{fin}}$ consisting of étale group schemes over k, and by $\mathrm{Pro}(\mathrm{GSch}_k^{\text{ét}})$ the category of proétale groups schemes over k. Observe that the equivalence in the above proposition formally extends to an equivalence between the category $\mathrm{Pro}(\mathrm{GSch}_k^{\text{ét}})$ and the category of profinite Γ-sets. Since proétale group schemes are affine, morphisms in the category $\mathrm{Pro}(\mathrm{GSch}_k^{\text{ét}})$ are maps of affine group schemes over k.

Recall that an algebraic group scheme over k is an affine group scheme that is represented by a finitely generated k-algebra. Letting G be an affine group scheme over k, use [18, Corollary in §3.3] to write $G = \varprojlim_i G_i$, where each G_i is an algebraic group scheme over k . By [18, Theorem in §6.7], each G_i sits in an exact sequence

$$1 \to G_i^\circ \to G_i \to G_i^{\text{ét}} \to 1,$$

where G_i° is the connected component of the identity of G_i (cf. [15, Tag 0B7R]), and $G_i^{\text{ét}}$ is represented by the largest separable subalgebra A_i^{sep} of A_i (so that $G_i^{\text{ét}} \in \mathrm{GSch}_k^{\text{fin}}$, by definition). Taking the projective limit in $\mathrm{GSch}_k^{\text{aff}}$ over I, we see that any affine group scheme $G \in \mathrm{GSch}_k^{\text{aff}}$ sits in an exact sequence

$$1 \to G^\circ \to G \to G^{\text{ét}} \to 1,$$

where $G^{\text{ét}} = \varprojlim_i G_i^{\text{ét}}$ is a proétale group scheme, and $G^\circ = \varprojlim_i G_i^\circ$ is the connected component of the identity of G. This follows because projective limits are left exact in $\mathrm{GSch}_k^{\text{aff}}$ as in any Abelian category, and are also right exact as inductive limits are exact in Hopf_k (cf. [3, Proposition A6.4]). It then follows from [18, Theorem in §6.8] that, if G is profinite, and k is perfect, then the sequence splits canonically, and G is a direct sum of a proétale group scheme and a connected group scheme over k. To see this, consider the splitting exact sequence

$$1 \to G_i^\circ \to G_i \to G_i^{\text{ét}} \to 1,$$

where G_i is finite for each $i \in I$, and let $f_i : G_i \to G_i^\circ$ be the canonical section for $G_i^\circ \to G_i$. Then, the f_i are compatible, and

$$f : \varprojlim G_i \to \varprojlim G_i^\circ \simeq (\varprojlim G_i)^\circ$$

defined by $f := \varprojlim_i f_i$, is a section for $G^\circ \to G$, and hence, the sequence $1 \to G^\circ \to G \to G^{\text{ét}} \to 1$ splits.

1.2 Prodiscrete k-algebras

Recall that a prodiscrete k-algebra is a projective limit $\varprojlim_i A_i$ of discrete k-algebras A_i endowed with the projective limit topology. A topological k-algebra admitting a basis of open neighborhoods of zero consisting of ideals is called linearly topologized. Recall also that a linearly topologized k-algebra A is called separated and complete if the canonical map $A \to \varprojlim_I A/I$, where I runs over the open ideals of A, is bijective.

We start with the following elementary lemma:

Lemma 1.4. *We have*

(i) Let A be a linearly topologized, separated and complete k-algebra. If the projective limit $\varprojlim_I A/I$, where I runs over the open ideals of A, is endowed with the prodiscrete topology, then the canonical map $A \to \varprojlim_I A/I$ is a topological isomorphism.

(ii) Any prodiscrete k-algebra is a separated and complete linearly topologized k-algebra.

Proof. For part (i), first note that, for any open ideal I of A, the quotient A/I is a discrete space. This makes the natural surjection $A \to A/I$ continuous, and hence, $A \to \varprojlim_I A/I$ is continuous. Conversely, given an open ideal J of A, we have that the image of J under the bijection $A \to \varprojlim_I A/I$ is precisely the kernel of the projection map $\varprojlim_I A/I \to A/J$, so that it is open.

For part (ii), letting $A = \varprojlim_i A_i$ be a prodiscrete k-algebra, the projection $p_i : A \to A_i$ is continuous for each i, and $\ker(p_i)$ for varying i forms a basis of open neighborhoods of zero consisting of ideals. Thus, we have a topological isomorphism $A = \varprojlim_I A/I \simeq \varprojlim_i A/\ker(p_i)$ as, for every open I, we have $\ker(p_i) \subseteq I$ for some i. $\square$

Remark 1.5. Suppose $A = \varprojlim_i A_i$ is a prodiscrete k-algebra. Letting $A_i' := p_i(A)$ be the image of the projection $p_i : A \to A_i$, the inclusion $A_i' \hookrightarrow A_i$ for each $i \in I$ induces a topological isomorphism $\varprojlim_i A_i' \to \varprojlim_i A_i$. Thus, we may, and will, assume that, in $A = \varprojlim_i A_i$, all the transition maps $f_{ij} : A_j \to A_i$ for $i \leq j$ are surjective.

Given a prodiscrete k-algebra $A = \varprojlim_i A_i$, we will often be interested in the question under which conditions the natural map

$$\varinjlim_i \mathrm{Hom}_k(A_i, B) \to \mathrm{Hom}_k^{\mathrm{cont}}(A, B)$$

is a bijection for any discrete k-algebra B. The following lemma directly addresses this question:

Lemma 1.6. *Let $A = \varprojlim_i A_i$ be a prodiscrete k-algebra such that the transition maps $f_{ij} : A_j \to A_i$ for $i \leq j$ are all surjective. Let B be a discrete k-algebra. Then, a k-algebra map $\varphi : A \to B$ is continuous if and only if φ factors through a projection $pr_j : A \to A_j$ for some j.*

Proof. If $\varphi : A \to B$ factors through a projection pr_j, then, being a composition of two continuous maps pr_j and $A_j \to B$, it is continuous.

For the other direction, assume $\varphi : A \to B$ is continuous. Then, the kernel $\ker(\varphi) = \varphi^{-1}(0)$ is open in A as 0 is open in the discrete space B. By definition of the prodiscrete topology, it follows that

$$A \cap \prod_i U_i \subseteq \ker(\varphi),$$

where, for a finite set $J \subseteq I$, $U_i = \{0\}$ for all $i \in J$, and $U_i = A_i$ for all $i \in I \setminus J$. Assume now that $a = (a_i)_i \in A$ with $\mathrm{pr}_j(a) = a_j = 0$. Then, if $k \in J$ with $k \leq j$, we get, by the surjectivity assumption, that

$$a_k = f_{kj}(a_j) = f_{kj}(0) = 0,$$

so that $a_k = 0$ whenever $k \leq j$. Thus, we get that $a \in A$ is an element of the above intersection, and hence, $\ker(\mathrm{pr}_j) \subseteq \ker \varphi$. Since $A/\ker(p_j) \simeq A_j$ by our surjectivity assumption, the map φ factors as desired. $\square$

As the dual category of $\mathrm{Pro}(\mathrm{GSch}_k^{\text{ét}})$ under the Spec functor (cf. §1.1), consider the category of indétale k-algebras, whose objects are inductive limits $\varinjlim_i A_i$ of étale k-algebras A_i. Then, Proposition 1.3 formally extends to an anti-equivalence between the category of indétale k-algebras and profinite Γ-sets, taking an indétale k-algebra $\varinjlim_i A_i$ to the profinite Γ-set

$$\mathrm{Hom}_k(\varinjlim_i A_i, \overline{k})^\Gamma \simeq \left(\varprojlim_i \mathrm{Hom}_k(A_i, \overline{k})\right)^\Gamma \simeq \varprojlim_i \mathrm{Hom}_k(A_i, k)^\Gamma.$$

Conversely, if X is a profinite Γ-set, the corresponding indétale k-algebra is $\mathcal{C}(X, \overline{k})^\Gamma$.

Remark 1.7. This anti-equivalence transforms direct products of k-algebras into topological disjoint unions, and vice versa. In particular, given k-algebras A and B, the map

$$\mathrm{Hom}_k(A, \overline{k})^\Gamma \amalg \mathrm{Hom}_k(B, \overline{k})^\Gamma \to \mathrm{Hom}_k(A \times B, \overline{k})^\Gamma$$

given by composing with the projection from $A \times B$, is a well-defined bijection.

Lemma 1.8. *Let $\varphi : A \to B$ be a homomorphism of indétale k-algebras with associated map $f : X \to Y$ of profinite Γ-sets. Then, the following are equivalent:*

(i) f is an open immersion of topological spaces, i.e. $f(X)$ is open, and f is a homeomorphism onto its image.

(ii) The map φ is surjective, and the ideal $\ker(\varphi) \subseteq A$ is finitely generated.

(iii) We have $A \simeq B \times C$ for some k-algebra C such that φ is the projection to B.

Proof. We start by proving $(i) \Rightarrow (iii)$. Let us first assume that k is algebraically closed. Observe that, if $A = \varinjlim_i A_i$ for étale k-algebras A_i, then $Y = \mathrm{Hom}_k(A, k)$, and

$$\mathcal{C}(Y,k) \simeq \mathcal{C}\big(\varprojlim_i \mathrm{Hom}_k(A_i,k),k\big) \simeq \varinjlim_i \mathcal{C}\big(\mathrm{Hom}_k(A_i,k),k\big) \simeq \varinjlim_i A_i = A.$$

Similarly, we have $B \simeq \mathcal{C}(X,k)$, and the map φ is given by

$$\mathcal{C}(Y,k) \to \mathcal{C}(X,k) \ g \mapsto g \circ f.$$

Since f is open, and Y is profinite, it follows that $f(X) \subseteq Y$ is both open and closed. So we can write $Y = f(X) \amalg (Y \setminus f(X))$, and hence,

$$\mathcal{C}(Y,k) \simeq \mathcal{C}(f(X),k) \times \mathcal{C}(Y \setminus f(X),k), \ g \mapsto \big(g\mid_{f(X)}, g\mid_{Y\setminus f(X)}\big).$$

But $f : X \to f(X)$ is a homeomorphism by assumption, so that $\mathcal{C}(f(X),k) \simeq \mathcal{C}(X,k)$, as needed. If k is not algebraically closed, one replaces k by $\overline{k}$, and passes to Γ-invariants.

The implication $(iii) \Rightarrow (ii)$ being clear, we are left with proving $(ii) \Rightarrow (i)$. Assume, without loss of generality, that k is algebraically closed, so that we have

$$I := \ker(\varphi) = \{g \in \mathcal{C}(Y,k) \mid g\mid_{f(X)} = 0\}.$$

Since $\mathcal{C}(Y,k)$ is an inductive limit of finite k-algebras, every prime ideal of $\mathcal{C}(Y,k)$ is maximal, and these are in a bijective correspondance with the points $y \in Y$ via

$$y \mapsto \mathfrak{m}_y := \{g \in \mathcal{C}(Y,k) \mid g(y) = 0\}.$$

We first claim that, for every maximal ideal ideal $\mathfrak{m}_y$ of $\mathcal{C}(Y,k)$, either $I\mathcal{C}(Y,k)_{\mathfrak{m}_y} = 0$ or $I\mathcal{C}(Y,k)_{\mathfrak{m}_y} = \mathcal{C}(Y,k)_{\mathfrak{m}_y}$. To this end, write

$$\mathcal{C}(Y,k)_{\mathfrak{m}_y} = \varinjlim_{y \in U} \mathcal{C}(U,k),$$

where U ranges over the open neighborhoods of y in Y. Assume first that $y \in f(X)$, implying $g(y) = 0$ for every $g \in I$. But as k is discrete, any element of $\mathcal{C}(Y,k)$ is locally constant, and we get that g is 0 in an open neighborhood U of y for every $g \in I$. Consequently, $I\mathcal{C}(Y,k)_{\mathfrak{m}_y} = 0$ if $y \in f(X)$. Next, assume $y \notin f(X)$. Since $f(X)$ is compact, and Y is Hausdorff, there is an open neighborhood U of y with $U \cap f(X) = \emptyset$. But then, the characteristic function g of U lies in I, and satisfies $g\mid_U = 1$. This gives $I\mathcal{C}(Y,k)_{\mathfrak{m}_y} = \mathcal{C}(Y,k)_{\mathfrak{m}_y}$ if $y \notin f(X)$.

Now, by [15, Tag 04PS], we see that the ideal I is pure (that is, the A-module A/I is flat), and hence, by [15, Tag 05KK], we conclude that the ideal I is generated by an idempotent. Since φ is surjective, this yields a decomposition $A \simeq B \times I$ as in part *(iii)*. We need to see that $f : X \to Y$ is an open immersion of topological spaces. By the previous remark, we have a bijection

$$\mathrm{Hom}_k(B, k) \amalg \mathrm{Hom}_k(I, k) \to \mathrm{Hom}_k(A, k)$$

given by composing with the projections $\mathrm{pr}_B : B \times I \to B$ and $\mathrm{pr}_I : B \times I \to I$. This implies that the map $f : X \to Y$ is given by inclusion, and hence is a homeomorphism onto its image. Furthermore, since A and B are indétale, so is I. This gives that Y is a disjoint union of profinite sets $f(X)$ and $\mathrm{Hom}_k(I, k)$. Thus, the complement $\mathrm{Hom}_k(I, k)$ of $f(X)$ is closed. This completes the proof. □

Corollary 1.9. *Let $A = \varprojlim_i A_i$ be a prodiscrete k-algebra such that each A_i is an indétale k-algebra, and such that the transition maps $A_j \to A_i$ for $i \leq j$ satisfy the equivalent statements of the above lemma. Then, there is a locally profinite topological space X with a continuous action of Γ such that $A = \mathcal{C}(X, \overline{k})^\Gamma$ as prodiscrete k-algebras. As sets, we have $X = \mathrm{Hom}_k^{cont}(A, \overline{k})$.*

Proof. Write $A_i = \mathcal{C}(X_i, \overline{k})^\Gamma$, where X_i is a profinite Γ-set for each i. By assumption, every transition map $A_j \to A_i$ is induced by an open immersion $X_i \to X_j$ of profinite Γ-sets. Set $X := \varinjlim_i X_i$, and endow it with the inductive limit topology (that is, the finest topology making the inclusions $X_i \to X$ continuous). Then, X is a locally profinite space with a continuous Γ-action. Moreover, we have

$$A = \varprojlim_i A_i \simeq \varprojlim_i \mathcal{C}(X_i, \overline{k})^\Gamma \simeq \mathcal{C}\big(\varinjlim_i X_i, \overline{k}\big)^\Gamma \simeq \mathcal{C}\big(X, \overline{k}\big)^\Gamma$$

as prodiscrete k-algebras. Also, since $X_i = \mathrm{Hom}_k(A_i, \overline{k})$, we get, using Lemma 1.6, that

$$X = \varinjlim_i X_i = \varinjlim_i \mathrm{Hom}_k(A_i, \overline{k}) \simeq \mathrm{Hom}_k^{\mathrm{cont}}(\varprojlim_i A_i, \overline{k}) = \mathrm{Hom}_k^{\mathrm{cont}}(A, \overline{k}).$$

□

We close this section by fixing a definition for formal schemes over k. Although there are different conventions in the literature, we stick to the following fairly general and slightly abusive terminology:

Definition 1.10. *(i) A formal scheme over k is a representable set-valued functor on the category of prodiscrete k-algebras. If the representing prodiscrete k-algebra of a formal scheme $\mathcal{F}$ over k is A, we write $\mathcal{F} = \mathrm{Spf}(A)$. So a formal scheme $\mathcal{F}$ over k is, in other words, a set-valued functor on the category of prodiscrete k-algebras such that, for any prodiscrete k-algebra B, we have*

$$\mathcal{F}(B) = \mathrm{Spf}(A)(B) \simeq \mathrm{Hom}_k^{cont}(A, B).$$

(ii) A formal group over k is a representable functor from the category of prodiscrete k-algebras to the category of Abelian groups.

Remark 1.11. We have:

(i) Let $\mathcal{F}$ be a formal scheme over k with representing algebra $A = \varprojlim_i A_i$. As usual, we assume that the transition maps $A_j \to A_i$, $i \leq j$, are surjective (cf. Remark 1.5). By Lemma 1.6, we then have

$$\mathcal{F}(\bullet) = \mathrm{Spf}(A)(\bullet) \simeq \mathrm{Hom}_k^{\mathrm{cont}}(\varprojlim_i A_i, \bullet) \simeq \varinjlim_i \mathrm{Hom}_k(A_i, \bullet) \simeq \varinjlim_i \mathrm{Spec}(A_i)(\bullet),$$

i.e. a formal scheme over k is, in particular, a proprepresentable functor on the category of discrete k-algebras. This implies that our definition of formal schemes is more general than Fontaine's (cf. [5, §4.1 in Chapter 1]), where a formal scheme is defined as a prorepresentable functor on the category of finite discrete k-algebras. Note that the surjectivity condition in Lemma 1.6 is automatic in that case (cf. [5, §4.1 in Chapter I]).

(ii) Suppose we are given an inductive system $(\mathcal{F}_i)_i$ of affine schemes over k with $\mathcal{F}_i = \mathrm{Spec}(A_i)$, where A_i is a k-algebra for each i. Assume that, for any $i \leq j$, the transition map $F_i \to F_j$ is a closed immersion. Consider the inductive limit $\mathcal{F} := \varinjlim_i \mathcal{F}_i$, defined pointwise on the category of k-algebras. Endowing the k-algebra $A := \varprojlim A_i$ with the prodiscrete topology, Lemma 1.6 implies that $\mathcal{F}$ is prorepresented by A. Therefore, we can formally extend it to the category of prodiscrete k-algebras:

$$\mathcal{F}(\varprojlim_j B_j) \simeq \varprojlim_j \mathcal{F}(B_j)$$

for any prodiscrete k-algebra $\varprojlim_j B_j$. Thus, $\mathcal{F}$ is a formal scheme over k with representing algebra A.

(iii) Similarly, given an inductive system $(\mathcal{G}_i)_i$ of closed immersions $G_i \to G_j$, $i < j$, between affine group schemes over k, the inductive limit $\mathcal{G} := \varinjlim_i \mathcal{G}_i$ can be viewed as a formal group over k (cf. item *(ii)*) simply because the category of Abelian groups has inductive limits. Note, however, that given a formal group with representing algebra satisfying the condition of Lemma 1.6, it is not clear whether it can be written as an inductive limit of affine group schemes over k (cf. item *(i)*). In fact, the discrete quotients A_i do not have any reason to be Hopf algebras anymore.

1.3 Strict k-linear inductive limits

Recall that a topological k-vector space is called linearly topologized if it has a basis of open neighborhoods of zero consisting of k-subspaces.

Inspired by Schneider's treatment of strict inductive limit of locally convex topological vector spaces over non-Archimedean fields (cf. [12, §5 E.2]), we define strict k-linear inductive limits for linearly topologized k-vector spaces:

Definition 1.12. *Let $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$ be an increasing sequence of linearly topologized k-vector spaces X_n. Let $\mathcal{T}_n$ denote the topology of X_n for each $n \geq 1$. Assume*

$$\mathcal{T}_{n+1} \mid_{X_n} = \mathcal{T}_n$$

for each $n \geq 1$, and equip the union

$$X := \bigcup_{n \geq 1} X_n$$

with the k-linear inductive limit topology $\mathcal{T}$ with respect to the inclusions $f_n : X_n \to X$ (that is, $\mathcal{T}$ is the finest k-linear topology on X making the inclusion f_n continuous for all $n \geq 1$). Then, X, with its topology $\mathcal{T}$, is called the strict k-linear inductive limit of the sequence $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$.

Assume the setup of the above definition. Let $\mathcal{B}_n$ denote the basis of open neighborhoods of zero consisting of k-subspaces of the linearly topologized k-vector space X_n for each $n \geq 1$. The following lemma shows that the strict k-linear inductive limit $\mathcal{T}$ on X could equivalently be defined by an analogous condition on the $\mathcal{B}_n$ for $n \geq 1$:

Lemma 1.13. *The condition that $\mathcal{T}_{n+1} \mid_{X_n} = \mathcal{T}_n$ for each $n \geq 1$ is equivalent to the condition that $\mathcal{B}_{n+1} \mid_{X_n} = \mathcal{B}_n$ for each $n \geq 1$.*

Proof. Assume that $\mathcal{T}_{n+1} \mid_{X_n} = \mathcal{T}_n$ for each $n \geq 1$. Clearly, $\mathcal{B}_{n+1}|_{X_n} \subseteq \mathcal{B}_n$. Conversely, let $B_n \in \mathcal{B}_n$. Since $B_n \in \mathcal{T}_n$, the assumption implies that there is $T_{n+1} \in \mathcal{T}_{n+1}$ with $T_{n+1} \cap X_n = B_n$. Since the topology $\mathcal{T}_{n+1}$ is k-linear, there is $B'_{n+1} \in \mathcal{B}_{n+1}$ such that $B'_{n+1} \subseteq T_{n+1}$. Consider now $B_{n+1} := B'_{n+1} + B_n \in \mathcal{B}_{n+1}$. Then, $B_{n+1} \cap X_n = (B'_{n+1} + B_n) \cap X_n \subseteq T_{n+1} \cap X_n = B_n$. The reverse inclusion is obvious.

For the other direction, assume that $\mathcal{B}_{n+1} \mid_{X_n} = \mathcal{B}_n$ for each $n \geq 1$. The inclusion $\mathcal{T}_{n+1} \mid_{X_n} \subseteq \mathcal{T}_n$ being clear, suppose $T_n \in \mathcal{T}_n$. Let $x \in T_n$. Then, there is $B_n \in \mathcal{B}_n$ with $x + B_n \subseteq T_n$. By assumption, there is $B_{n+1} \in \mathcal{B}_{n+1}$ such that $B_{n+1} \cap X_n = B_n$. This gives $(x + B_{n+1}) \cap X_n = x + (B_{n+1} \cap X_n) = x + B_n \subseteq T_n$. Consequently, T_n is open for the induced topology $\mathcal{T}_{n+1} \mid_{X_n}$, as needed. $\square$

Keep the notation from the above discussion. The following lemma provides a clearer understanding of the topology $\mathcal{T}$:

Lemma 1.14. *Consider the collection $\mathcal{B}$ of subspaces of X of the form $\sum_n B_n$, where $B_n \in \mathcal{B}_n$ for all $n \geq 1$. Then, $\mathcal{B}$ forms a basis of open neighborhoods of zero for the topology $\mathcal{T}$.*

Proof. Observe that, for any element $\sum_n B_n$ of $\mathcal{B}$, the intersection $\sum_n B_n \cap X_m$ is a subspace of X_m containing B_m, hence is open in X_m, showing that $\mathcal{B} \subseteq \mathcal{T}$. Also, suppose $T \in \mathcal{T}$ with $0 \in T$. By definition, we have $X_n \cap T \in \mathcal{T}_n$ for all $n \geq 1$. By the k-linearity of $\mathcal{T}_n$, we get that there exists $B_n \in \mathcal{B}_n$ with $B_n \subseteq X_n \cap T$ for each $n \geq 1$. Then, $B := \sum_n B_n \subseteq T$, as required. □

The following lemma summarizes some basic properties of the strict k-linear inductive limit:

Lemma 1.15. *With the notation as above, we have:*

(i) For every $n \geq 1$, the inclusion $X_n \to X$ is a topological embedding, i.e. $\mathcal{T}|_{X_n} = \mathcal{T}_n$.

(ii) If X_n is Hausdorff for all $n \geq 1$, then so is X.

(iii) If X_n is closed in X_{n+1} for all $n \geq 1$, then X_n is closed in X for all $n \geq 1$.

(iv) If X_n is complete for all $n \geq 1$, then so is X.

Proof. For the first three items, we follow the arguments in [12, Proposition 5.5], and make use of Lemma 1.13 and Lemma 1.14.

For item (i), we prove that $\mathcal{B}|_{X_n} = \mathcal{B}_n$ for each $n \geq 1$. So fix $n \geq 1$, and assume $B_n \in \mathcal{B}_n$ is given. We need to find an element B in $\mathcal{B}$ such that $B \cap X_n = B_n$. Note that, by Lemma 1.13, we can inductively find $B_{n+m} \in \mathcal{B}_{n+m}$ such that $B_n = X_n \cap B_{n+m}$ for all $m \in \mathbb{N}$. Then, $B := \bigcup_{m \geq n} B_m \in \mathcal{B}$ with $B \cap X_n = B_n$ as required.

For item (ii), let $a, b \in X$ with $a \neq b$. Then, for a large enough $n \geq 1$, we have $a, b \in X_n$, so that there exists $B_n \in \mathcal{B}_n$ with $a - b \notin B_n$ by the Hausdorff assumption. As above, construct $B \in \mathcal{B}$ with $B \cap X_n = B_n$. Then, $a + B$ and $b + B$ are open disjoint neighboorhoods of a and b in X.

For item (iii), fix $n \geq 1$, and assume $x \in X \setminus X_n$. We need to find $B \in \mathcal{B}$ with $(x + B) \cap X_n = \emptyset$. By assumption, $x \in X_m$ for some $m \geq n$. Since X_n is closed in X_m, there is $B_m \in \mathcal{B}_m$ with $(x + B_m) \cap X_n = \emptyset$. As in the above items, there is $B \in \mathcal{B}$ with $B \cap X_m = B_m$. Then, $(x + B) \cap X_n = (x + B_m) \cap X_n = \emptyset$, as required.

Finally, the last part follows as in [12, Lemma 7.9]. □

1.4 Perfect formal group laws

Let p be a prime number. Assume that k is perfect of characteristic $p > 0$ through the discussion that follows.

Given any $d \in \mathbb{N}$, the ring

$$R := k[\![X_1^{1/p^\infty}, \ldots, X_d^{1/p^\infty}]\!]$$

of fractional formal power series over k in d variables is defined as the completion of the inductive limit

$$\varinjlim_{X_i \mapsto X_i^p} k[\![X_1, \ldots, X_d]\!]$$

with respect to the ideal $(X_1, \ldots, X_d)$. To ease the notation, we write (X) for any row vector $(X_1, \ldots, X_d)$ as long as it causes no confusion.

Remark 1.16. We refer to the number d of variables in R as the dimension of the fractional formal power series ring R although, strictly speaking, it is not clear that this is a well-defined notion.

We have the following description for the ring R:

Lemma 1.17. *An element of R is given by a fractional formal power series*

$$\sum_{\alpha=(\alpha_i)_i \in \mathbb{N}[1/p]^d} c_\alpha X^\alpha = \sum_{\alpha=(\alpha_i)_i \in \mathbb{N}[1/p]^d} c_\alpha X_1^{\alpha_1} \cdots X_d^{\alpha_d}$$

satisfying the condition that, for every $\beta \in \mathbb{N}$, there are only finitely many nonzero coefficients $c_\alpha \in k$ with

$$|\alpha| = \alpha_1 + \cdots + \alpha_d < \beta.$$

Moreover, this condition is equivalent to the condition that, for every $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^d$, there are only finitely many nonzero coefficients $c_\alpha \in k$ with $\alpha_i < \beta_i$ for every $i = 1, \ldots, d$.

Proof. Let R' be the set of fractional power series as described in the statement of the lemma. We first want to check that R' is a k-algebra. Other axioms being obvious to check, we need to see that the product

$$\sum_\alpha c_\alpha X^\alpha \cdot \sum_\beta d_\beta X^\beta = \sum_\gamma \Big(\sum_{\alpha+\beta=\gamma} c_\alpha d_\beta \Big) X^\gamma$$

of two elements of R' lies in R'. But, for any γ, choosing n with $\gamma < n$, we know, by assumption, that there are only finitely many nonzero c_α and d_β as $\alpha, \beta < n$, so that the sum

$$\sum_{\alpha+\beta=\gamma} c_\alpha d_\beta$$

is finite for any γ. Moreover, since for any $\theta \in \mathbb{N}$, there are only finitely many nonzero c_α and d_β with $|\alpha|, |\beta| < \theta/2$, we see that there are only finitely many nonzero $\sum_{\alpha+\beta=\gamma} c_\alpha d_\beta$ with $|\gamma| = |\alpha| + |\beta| < \theta$. This shows that the product lies in R'. Next, observe that the algebra

$$\varinjlim_{X \mapsto X^p} k[\![X]\!]$$

is a dense subalgebra of R': any element $\sum_\alpha c_\alpha X^\alpha$ of R' can be approximated by the elements

$$\sum_{|\alpha|<n} c_\alpha X^\alpha \in \varinjlim_{X \mapsto X^p} k[\![X]\!] = \bigcup_{n \in \mathbb{N}} k[\![X^{1/p^n}]\!].$$

Furthermore, we have that R' is (X)-adically separated and complete. Indeed, an inverse of the natural map

$$R' \to \varprojlim_n R'/(X)^n,\ f \mapsto \big(f \bmod (X^n)\big)_n,$$

is given by

$$(\overline{f_0}, \overline{f_1}, \overline{f_2}, \dots) \mapsto f_0 + \sum_{n=0} (f_{n+1} - f_n).$$

Observe that, if $(g_0, g_1, g_2, \dots)$ is another representative in $(\overline{f_0}, \overline{f_1}, \overline{f_2}, \dots)$, then

$$\Big(f_0 + \sum_{n=0} (f_{n+1} - f_n)\Big) - \Big(g_0 + \sum_{n=0} (g_{n+1} - g_n)\Big) = (f_0 - g_0) + \sum_{n=0} \big((f_{n+1} - g_{n+1}) + (g_n - f_n)\big)$$

converges to 0 as n tends to infinity because $f_n - g_n \in (X^n)$ for all $n \in \mathbb{N}$. So the map is well-defined. Thus, we have proved that $R' = R$.

Finally, to prove the last statement, note that, given $\beta \in \mathbb{N}$, if $\alpha_i < \beta/d$ for all $i = 1, \dots, d$, then $|\alpha| < d\beta/d = \beta$. Conversely, given $\beta_1, \dots, \beta_d \in \mathbb{N}$, if $\alpha_1 + \dots \alpha_d < \beta_i$ for all $i = 1, \dots, d$, then $\alpha_i < \beta_i$ for all $i = 1, \dots, d$. $\square$

Recall that, for a perfect field k of positive characteristic, a topological k-algebra A is called perfect if the p-power map on A is bijective.

We have:

Lemma 1.18. *Let A a perfect topological k-algebra. Let $\overset{\circ\circ}{A}$ denote its ideal of topologically nilpotent elements. Then, the map*

$$\mathrm{Hom}_k^{cont}(R, A) \to \big(\overset{\circ\circ}{A}\big)^d,\ \varphi \mapsto (\varphi(X_1), \dots, \varphi(X_d)),$$

is a bijection.

Proof. Note that, for any $i = 1, \dots, d$, the continuity for the (X)-adic topology on R implies that $\varphi(X_i)^n = \varphi(X_i^n)$ converges to 0 as n tends to infinity, so that the map is well-defined.

To see that it is surjective, given $a = (a_1, \dots, a_d) \in \big(\overset{\circ\circ}{A}\big)^d$, consider the substitution map

$$\sum_\alpha c_\alpha X^\alpha \mapsto \sum_\alpha c_\alpha a^\alpha.$$

It is continuous and well-defined because the p-power map on A is bijective by assumption, so that the elements of the form a_i^{1/p^n}, for any $i = 1, \ldots, d$ and $n \in \mathbb{N}$, lie in A. Moreover, it maps to $a \in (\overset{\infty}{A})^d$.

Finally, assume $\varphi(X_i) = 0$ for all $i = 1, \ldots, d$. We need to see that φ is then the 0-map. But again, by assumption, for each $i = 1, \ldots, d$, we have that $\varphi(X_i^{1/p^n}) = \varphi(X_i)^{1/p^n}$ has a unique value in A for any $n \in \mathbb{N}$. This implies that φ is completely determined by the images of X_i for $i = 1, \ldots, d$, and hence is 0. □

Now, assume that A is a prodiscrete k-algebra. Recall that the inverse perfection $A^\flat$ of A is defined as the projective limit

$$A^\flat := \varprojlim_{(\cdot)^p} A$$

of A over $\mathbb{N}$ with respect to the p-power map $a \mapsto a^p$ on the elements $a \in A$. Note that $A^\flat$ has naturally the structure of a k-algebra as $k^\flat \simeq k$ that is explicitly given by $x \cdot (a_n)_n = (x^{1/p^n} a_n)_n$ for all $x \in k$ and $(a_n)_n \in A^\flat$. Moreover:

Lemma 1.19. *Let A be a prodiscrete k-algebra. Endowing $A^\flat$ with the projective limit topology, the canonical map*

$$\mathrm{Hom}_k^{cont}(R, A^\flat) \to \mathrm{Hom}_k^{cont}(R, A),$$

given by composing with the projection $A^\flat \to A$ to the first coordinate, is a bijection.

Proof. One checks that an inverse to the map is given by $\phi \mapsto \big(r \mapsto (\phi(r^{1/p^n}))_n\big)$ for any $\phi \in \mathrm{Hom}_k^{\mathrm{cont}}(R, A)$. □

In the rest of this section, we would like to extend the definition of a formal group law to the fractional setting, and prove some elementary results about them. Fix $d \geq 1$, and let

$$S := R \hat{\otimes}_k R \simeq k[\![X^{1/p^\infty}, Y^{1/p^\infty}]\!] = k[\![X_1^{1/p^\infty}, \ldots, X_d^{1/p^\infty}, Y_1^{1/p^\infty}, \ldots, Y_d^{1/p^\infty}]\!].$$

Note that S is then a perfect k-algebra as k is perfect.

Definition 1.20. *Let $d \geq 1$. A d-dimensional commutative perfect formal group law G over k is a set of d fractional formal power series $G_i(X, Y) \in S$, $i = 1, \ldots, d$, such that*

$$G(X, Y) := \big(G_1(X, Y), \ldots, G_d(X, Y)\big) \in S^d$$

satisfies the following conditions:

(i) $G(X, 0) = X$,

(ii) $G(X, Y) = G(Y, X)$,

(iii) G(G(X, Y), Z) = G(X, G(Y,Z)).

We obtain:

Proposition 1.21. *Consider a formal group $\mathcal{F}$ over k represented by R. Then, there is a perfect formal group law $G(X,Y) \in S^d$ of dimension d over k such that, for any perfect prodiscrete k-algebra A and for all $x, y \in \mathcal{F}(A)$, the addition in the Abelian group $\mathcal{F}(A)$ is given by the rule*

$$x +_G y = G(x, y).$$

Proof. Let A be a perfect prodiscrete k-algebra. By Lemma 1.18, there is then a bijection $\mathcal{F}(A) \simeq (\overset{\infty}{A})^d$. We let now $X = (X_1, \ldots, X_d), Y = (Y_1, \ldots, Y_d) \in (\overset{\infty}{A})^d$, and define, in this group, $G(X,Y) := X + Y \in (\overset{\infty}{A})^d$. Then, for each $i = 1, \ldots, d$, we get that $G_i(X,Y) \in S$ with $G_i(0,0) = 0$, and the rest of the proof carries over from the classical situation.

For instance, to prove that $G(X,Y)$ satisfies the condition (i) of Definition 1.20 above, let $\varphi : S \to S$ be the k-algebra map defined by $X_i \mapsto X_i$ and $Y_i \mapsto 0$ for all $i = 1, \ldots, d$. The representability of G then implies

$$\begin{aligned} G(X,0) = G(\varphi(X), \varphi(Y)) = G(\varphi)(G(X,Y)) = G(\varphi)(X) + G(\varphi)(Y) &= \varphi(X) + \varphi(Y) \\ &= X. \end{aligned}$$

For the proof of the last asserion, let $\varphi : S \to A$ be the k-algebra map defined by $X_i \mapsto x_i$ and $Y_i \mapsto y_i$ for each $i = 1, \ldots, d$. We obtain

$$\begin{aligned} x +_G y = \varphi(X) + \varphi(Y) = G(\varphi)(X) + G(\varphi)(Y) = G(\varphi)(G(X,Y)) &= G(\varphi(X), \varphi(Y)) \\ &= G(x,y), \end{aligned}$$

as required. □

Remark 1.22. The perfect formal group law G rules the group structure of $\mathcal{F}(A)$ also if A is not necessarily perfect. This is because, by Lemma 1.19, there is an isomorphism $\mathcal{F}(A) \simeq \mathcal{F}(A^\flat)$ of groups, where $A^\flat$ is a perfect k-algebra by definition.

Proposition 1.23. *We keep the assumptions of the above proposition. There exists a d-tuple*

$$\iota(X) = (\iota_1(X), \ldots, \iota_d(X)) \in R^d$$

of fractional formal power series in $X_1, \ldots, X_d$, called the inverse of $X = (X_1, \ldots, X_d)$, satisfying $\iota(0) = 0$, $G(X, \iota(X)) = 0$, and $\iota(\iota(X)) = X$. Moreover, for any perfect prodiscrete k-algebra A, the inverse of an element $x = (x_1, \ldots, x_d)$ in the group $\mathcal{F}(A)$ is given by $\iota(x)$.

Proof. The first part follows as in the classical case (cf. [20, Corollary 1.5]). For the second assertion, let A be a perfect prodiscrete k-algebra, and $x = (x_1, \ldots, x_d) \in \mathcal{F}(A)$. Let $y = (y_1, \ldots, y_d)$ be the inverse of x in the Abelian group $\mathcal{F}(A)$. If $\varphi : R \to A$ is the k-algebra map defined by $X_i \mapsto y_i$, we then get

$$y = \varphi(X) = \varphi(-\iota(X)) = -\iota(\varphi(X)) = -\iota(y) = \iota(x).$$

$\square$

Remark 1.24. Suppose that we are conversely given $G(X, Y) \in S^d$, a perfect formal group law of dimension d over the perfect field k of characteristic $p > 0$. Let A be a prodiscrete k-algebra. Then, for any $x = (x_1, \ldots, x_d) \in \big(\overset{\infty}{A^\flat}\big)^d$ and $f(X) \in R$, the expression $f(x)$ makes sense, and we define a group operation in $\big(\overset{\infty}{A^\flat}\big)^d$ via

$$x +_G y := G(x, y)$$

for any $x, y \in \big(\overset{\infty}{A^\flat}\big)^d$. That this really defines a group follows from the axioms on G (cf. Definition 1.20) and Proposition 1.23 above (cf. [20, Remark 1.6]). Therefore, by Lemma 1.18, we obtain an induced group structure on $\mathrm{Hom}_k^{\mathrm{cont}}(R, A^\flat) \simeq \mathrm{Hom}_k^{\mathrm{cont}}(R, A)$ for any prodiscrete k-algebra A, giving a formal group $\mathcal{F}$ represented by R. Therefore, given a perfect formal group law over a perfect field k of characteristic $p > 0$, we will not distinguish it from its corresponding formal group over k, and vice versa.

In order to construct certain counterexamples later, we need the following result, which is a slight generalization of the classical case in [6, Proposition 6.5]:

Proposition 1.25. *Assume given $F = (F_1, \ldots, F_m) \in k[\![X^{1/p^\infty}, Y^{1/p^\infty}]\!]^m$ and $G = (G_1, \ldots, G_n) \in k[\![U^{1/p^\infty}, V^{1/p^\infty}]\!]^n$, perfect formal group laws over k of dimensions m and n, respectively. Let*

$$f(X) = (f_1(X), \ldots, f_n(X)) \in k[\![X^{1/p^\infty}]\!]^n$$

be such that $f(0) = 0$. Put

$$\Delta f(X, Y) := f(X) +_G f(Y) -_G f(X +_F Y).$$

Then,

$$E(U, X, V, Y) := (U +_G V -_G \Delta f(X, Y), X +_F Y) \in k[\![U^{1/p^\infty}, X^{1/p^\infty}, V^{1/p^\infty}, Y^{1/p^\infty}]\!]$$

is a perfect formal group law, and the map

$$i(U, X) := \big(U +_G f(X), X\big) \in k[\![U^{1/p^\infty}, X^{1/p^\infty}]\!]$$

gives an isomorphism $i : G \times F \to E$ of perfect formal group laws over k.

Proof. It is a routine to check that E is a perfect formal group law. We only compute to see that i is a homomorphism from $G \times F$ to E:

$$\begin{aligned}
E\big(i(U,X), i(V,Y)\big) &= E\big(U +_G f(X), X, V +_G f(Y), Y\big) \\
&= \big(U +_G f(X) +_G V +_G f(Y) -_G \Delta f(X,Y), X +_F Y\big) \\
&= \big(G(U,V) +_G f(F(X,Y)), F(X,Y)\big) \\
&= i\big(G(U,V), F(X,Y)\big) \\
&= i\big((G \times F)(U,X,V,Y)\big).
\end{aligned}$$

Finally, to see that i is indeed an isomorphism, we consider the map

$$j(U,X) := \big(U -_G f(X), X\big) \in k[\![U^{1/p^\infty}, X^{1/p^\infty}]\!].$$

It is then a straightforward computation to see that j is a homomorphism $E \to G \times F$ of perfect formal group laws. It is also clear that

$$i(j(U,X)) = j(i(U,X)) = (U,X).$$

$\square$

2 Generalized Drinfeld equivalence

Recall that we always assume our affine group schemes over the field k to be commutative.

Consider an inductive system $(G_n)_{n\in\mathbb{N}}$ of affine group schemes over k such that, for each $n \in \mathbb{N}$, the sequence

$$1 \to G_n \to G_{n+1} \xrightarrow{p^n} G_{n+1}$$

is exact, i.e. there are closed immersions $i_n : G_n \to G_{n+1}$ such that $G_n \simeq G_{n+1}[p^n]$ for each $n \in \mathbb{N}$. By iterating i_n, we define

$$i_{n,m} := i_{n+m-1} \circ \cdots \circ i_n : G_n \to G_{n+m}$$

for every $m \in \mathbb{N}$. We then have $G_{n+m}[p^n] = G_n$ by an induction argument on $m \geq 1$. Indeed, the case $m = 1$ being part of definition, assuming $G_{n+m}[p^n] = G_n$, we have, for any k-algebra A, that

$$\begin{aligned} G_{n+m+1}[p^n](A) = G_{n+m+1}[p^{n+m}](A) \cap G_{n+m+1}[p^n](A) &= G_{n+m}(A) \cap G_{n+m+1}[p^n](A) \\ &= G_{n+m}[p^n](A) \\ &= G_n(A), \end{aligned}$$

where the first equality follows from the fact that

$$G_{n+m+1}[p^n](A) \subseteq G_{n+m+1}[p^{n+m}](A),$$

the second one by assumption, the third one by viewing $G_{n+m}(A)$ as a subgroup of $G_{n+m+1}(A)$ via i_{n+m}, and the last one by the induction hypothesis.

Observe that, since $i_n : G_n \to G_{n+1}$ is a closed immersion, it follows that the map $A_{n+1} \to A_n$ is surjective, where $\mathrm{Spec}(A_n) = G_n$ for each $n \in \mathbb{N}$. Hence, by Remark 1.11.(iii), given such an inductive system $(G_n)_n$ of affine group schemes over k as above, we identify it with the formal group

$$G := \varinjlim_n G_n = \varinjlim_n (G_n, i_n),$$

whose representing algebra is the prodiscrete k-algebra $\varprojlim_n A_n$. Then, the equality $G_{n+m}[p^n] = G_n$ implies, for any $m \in \mathbb{N}$, that

$$G[p^m] = \varinjlim_n G_n[p^m] = G_m.$$

Consider the multiplication by p^m on G_{n+m}. Clearly, its image is annihilated by p^n as $p^n p^m G_{n+m} = p^{n+m} G[p^{n+m}] = 0$, and hence is contained in $G_{n+m}[p^n] = G_n$. It follows that the map p^m factors through a map

$$j_{m,n} : G_{n+m} \to G_n$$

such that $p^m = i_{n,m} \circ j_{m,n}$ for all $n, m \geq 0$.

We are ready to define generalized p-divisible groups:

Definition 2.1. *A generalized p-divisible group over k is a formal group $G = \varinjlim_n G_n$, where each G_n is an affine group scheme over k, satisfying*

(i) for each $n \in \mathbb{N}$, the sequence

$$1 \to G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{p^n} G_{n+1}$$

is exact, i.e. there are closed immersions $i_n : G_n \to G_{n+1}$ such that $G_n \simeq G_{n+1}[p^n]$, and

(ii) for each $n, m \in \mathbb{N}$, the above-constructed map $j_{m,n}$ is surjective.

We define morphisms of generalized p-divisible groups as in the case of classical p-divisible groups (cf. [16, §2]).

Remark 2.2. A p-divisible group over k is a generalized p-divisible group since, given a p-divisible group $G = \varinjlim_n G_n$, the sequence

$$1 \to G_m \xrightarrow{i_{m,n}} G_{n+m} \xrightarrow{j_{m,n}} G_n \to 1$$

is exact (cf. [16, §2]). On the other hand, if $G = \varinjlim_n G_n$ is a generalized p-divisible group over k, then the above sequence is still exact. Indeed, exactness at the left and right follows from the definition of a generalized p-divisible group, while the exactness in the middle by

$$\ker(j_{m,n}) = \ker(p^m : G_{n+m} \to G_{n+m}) = G_{n+m}[p^m] = G_m.$$

Example 2.3. For a commutative affine group scheme G over k, the kernel $G[p^n]$ of multiplication by p^n for any $n \in \mathbb{N}$ is an affine group scheme over k. Then, the inductive limit $\varinjlim_n G[p^n]$ is a generalized p-divisible group if the map $j_{m,n} : G[n+m] \to G[n]$ constructed above is surjective for all $n, m \in \mathbb{N}$.

Definition 2.4. *Given a generalized p-divisible group $G = \varinjlim_n G_n$ over k, let $j_n := j_{1,n}$ for any $n \in \mathbb{N}$, and consider the inverse system $(G_n, j_n : G_{n+1} \to G_n)_{n \in \mathbb{N}}$. We define the Tate module of G as the projective limit*

$$\underline{T}(G) := \varprojlim_n (G_{n+1}, j_n).$$

Remark 2.5. (i) By a common abuse of notation, we often write

$$\underline{T}(G) = \varprojlim_p G_n = \varprojlim_p G[p^n],$$

where the transition maps $G[p^{n+1}] \to G[p^n]$ are multiplication by p for all $n \in \mathbb{N}$. Since, as a map $G[p^{n+1}] \xrightarrow{p} G[p^{n+1}]$, multiplication by p satisfies $pG[p^{n+1}] \subseteq G[p^n]$, and $p = i_n \circ j_n$, this abuse indeed makes sense.

(ii) Since the category of affine group schemes over k has projective limits (cf. §1.1), it follows, from part (i), that $\underline{T}(G)$ is an affine group scheme over k, whose representing algebra is $\varinjlim_p A_n$, where $G_n = \operatorname{Spec}(A_n)$ for each $n \in \mathbb{N}$.

Example 2.6. For an étale p-divisible group G over k, it follows that the Tate module $\underline{T}(G)$ of G is a proétale group scheme that is represented by

$$\mathcal{C}(T, \overline{k})^\Gamma \simeq \varinjlim_n \mathcal{C}(T/p^nT, \overline{k})^\Gamma,$$

where we used that $T = \underline{T}(G)(\overline{k})$ is p-adically separated and complete as an Abelian group, and that the natural maps $T/p^{n+1}T \to T/p^nT$ are surjective for all $n \in \mathbb{N}$ (recall that $G[p^n](\overline{k}) \simeq T/p^nT$ for all $n \geq 0$).

Let G be an affine group scheme over k. For each $n \in \mathbb{N}$, consider the map $p^n : G \to G$, the multiplication by p^n, as an element of the Abelian group $\operatorname{End}(G)$. By the universal property of cokernel, we have a map $G \to \operatorname{coker}(G \xrightarrow{p^n} G)$ for each $n \in \mathbb{N}$. Denoting $\operatorname{coker}(G \xrightarrow{p^n} G)$ simply by G/p^nG, we obtain a canonical map

$$\varphi : G \to \varprojlim_n G/p^nG = \varprojlim_n \operatorname{coker}(G \xrightarrow{p^n} G)$$

of affine group schemes over k.

Definition 2.7. *An affine group scheme G over a field k is said to be p-adically separated and complete if the map φ is an isomorphism.*

Remark 2.8. Given a p-adically separated and complete affine group scheme G over k, it is rarely true that, for any k-algebra A, the Abelian group $G(A)$ is p-adically separated and complete as well. Indeed, as the functor $\operatorname{Hom}_k(-, A)$ is left exact on the (Abelian) category Hopf_k, the exact sequence

$$1 \to G \xrightarrow{p^n} G \to G/p^nG \to 1$$

gives an injection

$$G(A)/p^nG(A) \to (G/p^nG)(A)$$

for any $n \in \mathbb{N}$. In particular, we see that $G(A)$ is p-adically separated and complete as an Abelian group if $\operatorname{Hom}_k(-, A)$ is an exact functor on Hopf_k. This holds, for example, when $A = \overline{k}$.

From here on until the end of this chapter, assume that k is perfect of characteristic $p > 0$. The first part of the following definition was given in [2, Definition 3.1.1]:

Definition 2.9. *(i) A Tate k-group is a p-adically separated and complete affine group scheme G over k such that* $\ker(G \xrightarrow{p} G) = 0$, *and such that* $\operatorname{coker}(G \xrightarrow{p} G)$ *is finite.*

(ii) A generalized Tate k-group is a p-adically separated and complete affine group scheme G over k such that $\ker(G \xrightarrow{p} G) = 0$.

We have the following generalization of the result in [2, 3.1.3 in §3]:

Proposition 2.10. *The functor*

$$G = \varinjlim_{n} G_n \mapsto \underline{T}(G) = \varprojlim_{p} G_n$$

gives an equivalence of categories between the category of generalized p-divisible groups over k and the category of generalized Tate k-groups.

Proof. Let G be a generalized p-divisible group over k. Note first that, as the transition maps j_n in the affine group scheme $\underline{T}(G)$ are surjective, the projection $\underline{T}(G) \to G[p^n]$ is surjective for any $n \in \mathbb{N}$. We claim that the kernel of this surjection is $p^n\underline{T}(G)$. We begin by showing that, for any k-algebra A, we have

$$\ker\big(\underline{T}(G)(A) \to G[p^n](A)\big) = p^n\big(\underline{T}(G)(A)\big).$$

Indeed, any element of $\underline{T}(G)(A)$ is a sequence of the form $(0, a_1, a_2, \dots)$ with $a_i \in G[p^i](A)$, and $p \cdot a_{i+1} = a_i$ for each $i \geq 1$, so that, for any n, $p^n \cdot (0, a_1, a_2, \dots)$ is a shift of the sequence $(0, a_1, a_2, \dots)$ by n entries to the right. Conversely, if $(0, a_1, a_2, \dots) \in \underline{T}(G)(A)$ such that $a_n = 0$, the relations $p \cdot a_{i+1} = a_i$ for $i \geq 1$ clearly force $a_1 = a_2 = \cdots = a_n = 0$. This proves the claimed equality above.

Observe that the above argument also implies that, being a shift to the right on its points, multiplication by p^n is injective on $\underline{T}(G)$ for any $n \in \mathbb{N}$, giving in particular that

$$\ker(\underline{T}(G) \xrightarrow{p} \underline{T}(G)) = 0.$$

Moreover, it implies that $p^n\big(\underline{T}(G)(A)\big) \simeq \underline{T}(G)(A) \simeq p^n\underline{T}(G)(A)$ as Abelian groups for any k-algebra A, and consequently, that

$$\underline{T}(G)/p^n\underline{T}(G) \simeq G[p^n]$$

as affine group schemes. Considering the projective limit of both sides over $\mathbb{N}$, we see that $\underline{T}(G)$ is a generalized Tate k-group, and we have defined a functor $\underline{T}$ from the category of generalized p-divisible groups over k to the category of generalized Tate k-groups.

To construct a quasi-inverse, assume given a generalized Tate k-group T. Then, the cokernel $\operatorname{coker}(T \xrightarrow{p^n} T) = T/p^nT$ is an affine group scheme over k, so that

$$\underline{G}(T) = \varinjlim_{n} (T/p^nT)$$

with the transition maps being multiplication by p is an inductive limit of affine group schemes over k. Since $\ker(T \xrightarrow{p} T) = 0$, we have $p^nT \simeq p^{n+1}T$ via multiplication by p, so that the map $T/p^nT \xrightarrow{p} T/p^{n+1}T$ is injective, and thus, there are exact sequences

$$1 \to T/p^nT \xrightarrow{p} T/p^{n+1}T \xrightarrow{p^n} T/p^{n+1}T$$

for each $n \in \mathbb{N}$. Moreover, the kernel of the map $T/p^{n+m}T \xrightarrow{p^m} T/p^{n+m}T$ is contained in $p^nT/p^{n+m}T$, so that it factors over the reduction $j_{m,n} : T/p^{n+m}T \to T/p^nT$ with $p^m = i_{n,m} \circ j_{m,n}$, where $i_{n,m} : T/p^nT \to T/p^{n+m}T$ is the multiplication by p^m. In particular, $j_{m,n}$ is surjective for all $n, m \in \mathbb{N}$, and $\underline{G}(T)$ is a generalized p-divisible group over k.

Finally, we need to see that the constructions $\underline{T}$ and $\underline{G}$ are quasi-inverses of each other. Indeed, if G is a generalized p-divisible group over k, then

$$\underline{G}(\underline{T}(G)) = \varinjlim_n \big(\underline{T}(G)/p^n\underline{T}(G)\big) \simeq \varinjlim_n G[p^n] = G.$$

Conversely, if T is a generalized Tate k-group, then $\underline{G}(T)[p^n] = T/p^nT$, and hence,

$$\underline{T}(\underline{G}(T)) = \varprojlim_n T/p^nT = T.$$

$\square$

We easily obtain [2, 3.1.3 in §3] as a corollary of this proposition:

Corollary 2.11. *The above equivalence restricts to an equivalence of categories between the category of p-divisible groups over k and the category of Tate k-groups.*

Proof. Given a p-divisible group G, it follows by definition that

$$\operatorname{coker}\big(\underline{T}(G) \xrightarrow{p} \underline{T}(G)\big) \simeq G[p]$$

is finite.

Conversely, given a Tate k-group T, the exact sequence $1 \to T \xrightarrow{p} T \to T/pT \to 1$ induces an exact sequence

$$1 \to T/p^nT \to T/p^{n+1}T \to T/pT \to 1$$

of finite group schemes over k for each $n \in \mathbb{N}$. The multiplicativity of orders of such group schemes over exact sequences (cf. [16, §1.3]) imply that the order $\operatorname{ord}(T/p^nT)$ of T/p^nT is $\big(\operatorname{ord}(T/pT)\big)^n$ by induction on n. Now, we want to see that $\operatorname{ord}(T/p^nT) = p^{nh}$ for some $h \in \mathbb{N}$, for which it is enough to prove that T/pT has order p^h for some $h \in \mathbb{N}$.

To this end, consider the short exact sequence (cf. §1.1)

$$1 \to (T/pT)^\circ \to T/pT \to (T/pT)^{\text{ét}} \to 1.$$

We have, from [18, 1st Corollary in §14.4], that the order of $(T/pT)^\circ$ is a power of p. Also, as $(T/pT)^{\text{ét}}$ is annihilated by p, so is the finite Abelian group $(T/pT)^{\text{ét}}(\overline{k})$, implying that the order of $(T/pT)^{\text{ét}}$ is a power of p. Thus, the multiplicativity of the order implies that the order of T/pT is a power of p, say p^h for some $h \in \mathbb{N}$, as desired. $\square$

Remark 2.12. One can define Tate k-groups over any field k by adding the assumption that the order of T/pT is a power of p (note that we have just proved that this is automatic when k is perfect of characteristic $p > 0$). The above corollary then remains true.

In combination with Corollary 2.11, the following standard fact shows that Tate k-groups are nothing but finite rank $\mathbb{Z}_p$-modules with a continuous Γ-action:

Corollary 2.13. *There is an equivalence between the category of étale p-divisible groups of height h over k and the category of free $\mathbb{Z}_p$-modules of rank h with a continuous Γ-action.*

Proof. Let G be an étale p-divisible group of height h over k. Consider the $\overline{k}$-points

$$T := \underline{T}(G)(\overline{k}) = \varprojlim_{p} G[p^n](\overline{k})$$

of the Tate module $\underline{T}(G)$ of G. It follows that $G[p^n](\overline{k})$ is a finite group of order p^{nh} for each $n \in \mathbb{N}$, for some $h \in \mathbb{N}$, so that $h \in \mathbb{N}$ is the height of G (cf. [16, p. 12]), and thus, $T \simeq \varprojlim_n \mathbb{Z}/p^{nh}\mathbb{Z} \simeq \mathbb{Z}_p^h$. Furthermore, T endowed with the projective limit topology naturally carries a continuous action of Γ, induced from the continuous action of Γ on the finite discrete spaces $G_n(\overline{k}) = G[p^n](\overline{k})$ for each n.

Conversely, assume that T is a free $\mathbb{Z}_p$-module of rank $h \in \mathbb{N}$ with a continuous Γ-action. The quotients T/p^nT are then finite Γ-sets for each $n \in \mathbb{N}$. Hence, by Proposition 1.3, they give rise to étale group schemes G_n over k such that $G_n(\overline{k}) = T/p^nT$, and the exact sequence

$$1 \to T/p^nT \xrightarrow{p} T/p^{n+1}T \xrightarrow{p^n} T/p^{n+1}T$$

of Γ-sets uniquely determines an exact sequence

$$1 \to G_n \xrightarrow{p} G_{n+1} \xrightarrow{p^n} G_{n+1}$$

on the corresponding étale group schemes over k such that the order of G_n is p^{nh} for each $n \in \mathbb{N}$, for some $h \in \mathbb{N}$, i.e. an étale p-divisible group $\varinjlim_n G_n$ of height h.

Finally, it is clear from the proof of the above proposition that these two constructions are inverses of each other. $\square$

Corollary 2.14. *Let k be a perfect field of characteristic different from p. Then the category of p-divisible groups over k is equivalent to the category of free $\mathbb{Z}_p$-modules of finite rank with a continuous action of the Galois group* $\mathrm{Gal}(\overline{k}/k)$*.*

Proof. Let $G = \varinjlim_n G_n$ be a p-divisible group over k. Then, each finite group scheme G_n is étale over k (cf. [10, Proposition 13.7]), and the result follows by the previous proposition. $\square$

Example 2.15. (i) Consider the constant p-divisible group $G = (\mathbb{Q}_p/\mathbb{Z}_p)^h$ of height $h \in \mathbb{N}$. By Corollary 2.13, the $\overline{k}$-points $\underline{T}(G)(\overline{k})$ of its Tate module is then $\mathbb{Z}_p^h$ with the trivial action of the Galois group Γ.

(ii) Let k be a field of characteristic different from p. Then, the p-divisible group $\mu_{p^\infty} := \varinjlim_n \mu_{p^n}$, where

$$\mu_{p^n} := \operatorname{Spec}\left(k[t]/(t^{p^n} - 1)\right),$$

is étale over k of height 1. So, by the above corollary, we get that $\underline{T}(\mu_{p^\infty})(\overline{k})$ is a free $\mathbb{Z}_p$-module of rank one with the action of the Galois group $\operatorname{Gal}(\overline{k}/k)$ given by the p-adic cyclotomic character $\chi : \mathcal{G} \to \mathbb{Z}_p^\times$. Clearly, if the characteristic of k is p, then μ_{p^∞} is connected.

Remark 2.16. It is a natural question how to define good notions of étaleness and connectedness for generalized p-divisible groups. Note that there is an obvious way to extend connectedness to the generalized setting. Namely, one could call a generalized p-divisible group $G = \varinjlim_n G_n$ connected if each affine group scheme $G_n = \operatorname{Spec}(A_n)$ is connected as the prime spectrum of A_n in the Zariski topology. We will also briefly touch on the notion of étaleness later.

3 Universal formal cover functor

Let k be a field, and G a generalized p-divisible group over k. We define the universal formal cover $\widetilde{G}$ of G as the group functor

$$\widetilde{G} := \varprojlim_p G = G \xleftarrow{p} G \xleftarrow{p} \cdots$$

on the category of prodiscrete k-algebras, where the projective limit is taken over $\mathbb{N}$ with respect to the multiplication by p on G.

Note that, writing $G = \varinjlim_n G_n$, we see that the functor $\widetilde{G}$ takes a discrete k-algebra A to the group

$$\widetilde{G}(A) = \varprojlim_p G(A) \simeq \varprojlim_p \varinjlim_n G_n(A),$$

so that, for a prodiscrete k-algebra $A = \varprojlim_i A_i$, we obtain

$$\widetilde{G}(A) = \varprojlim_p G(\varprojlim_i A_i) \simeq \varprojlim_p \varprojlim_i G(A_i) \simeq \varprojlim_i \varprojlim_p G(A_i) \simeq \varprojlim_i \varprojlim_p \varinjlim_n G_n(A_i).$$

Moreover, by definition, $G(A_i)$ has a $\mathbb{Z}_p$-module structure, and, as the multiplication by p on the projective limit $\widetilde{G}(A_i)$ is bijective, $\widetilde{G}(A_i)$ has the structure of a module over the localization $\mathbb{Z}_p[1/p] = \mathbb{Q}_p$. It thus follows that

$$\widetilde{G}(A) \simeq \varprojlim_i \widetilde{G}(A_i)$$

is a $\mathbb{Q}_p$-vector space for any prodiscrete k-algebra A.

From here on, we assume that the field k is perfect of characteristic $p > 0$. Consider the inductive limit

$$\underline{T}(G)\left[\frac{1}{p}\right] := \varinjlim_p \underline{T}(G)$$

of affine group schemes over k. By definition, $\underline{T}(G)[p] = 0$, so that, assuming $\underline{T}(G)$ is represented by the k-algebra A, the prodiscrete k-algebra $\varprojlim_p A$ satisfies the condition of Lemma 1.6. Thus, $\underline{T}(G)[1/p]$ can be viewed as a formal group represented by $\varprojlim_p A$ (cf. Remark 1.11.(iii)).

The next proposition provides a few useful relations between the group functors $\underline{T}(G)$, $\underline{T}(G)[1/p]$ and $\widetilde{G}$:

Proposition 3.1. *Let the notation be as above. We have:*

(i) Endowing the representing algebra of $\underline{T}(G)$ with the discrete topology, there is an exact sequence

$$0 \to \underline{T}(G) \to \widetilde{G} \to G$$

of group functors on the category of prodiscrete k-algebras.

(ii) There is an isomorphism

$$\widetilde{G} \simeq \underline{T}(G)\left[\frac{1}{p}\right]$$

as group functors on the category of prodiscrete k-algebras. In particular, the universal formal cover $\widetilde{G}$ is a formal group over k, and hence is representable on the category of prodiscrete k-algebras.

(iii) For any k-algebra A, we have $\underline{T}(G)(A) \simeq \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, G(A))$. For any prodiscrete k-algebra A, we have $\widetilde{G}(A) \simeq \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, G(A))$.

Proof. Let A be a discrete k-algebra. For part (i) (which is already observed in [5, §1, Chapter V] for the case of formal groups in Fontaine's sense), we need to show that

$$\underline{T}(G)(A) = \varprojlim_p G[p^n](A) = \ker\left(\varprojlim_p G(A) \xrightarrow{\mathrm{pr}_0} G(A)\right) = \ker(\widetilde{G}(A) \xrightarrow{\mathrm{pr}_0} G(A)).$$

But the elements of $\underline{T}(G)(A)$ are of the form $(0, a_1, a_2, \dots)$, which maps to 0 under the projection $\widetilde{G}(A) \to G(A)$. Conversely, any element of the kernel of pr_0 is of the form $(0, a_1, a_2, \dots)$, which satisfies, by induction, that $p^n a_n = 0$ for all $n \in \mathbb{N}$. So the sequence is exact on the category of discrete k-algebras. The left-exactness of the projective limit ensures that the sequence remains exact when we pass to the category of prodiscrete k-algebras.

For part (ii), consider the map

$$\varphi : \underline{T}(G)(A)[1/p] \to \widetilde{G}(A),\ (0, a_1, a_2, \dots) \otimes p^{-r} \mapsto (a_r, a_{r+1}, a_{r+2}, \dots).$$

It is clear that φ is injective. For surjectivity, assume that $(a_0, a_1, a_2, \dots) \in \widetilde{G}(A)$ is given. Let $s \geq 1$ be such that $a_0 \in G[p^s]$. Then,

$$(a_0, a_1, a_2, \dots) \in \varprojlim_p G[p^{s+n}](A),$$

and we see that

$$(0, p^{s-1}a_0, p^{s-1}a_1, \dots) \otimes p^{-s} \in \underline{T}(G)(A)[1/p]$$

maps to

$$(p^{s-1}a_{s-1}, p^{s-1}a_s, p^{s-1}a_{s+1}, \dots) = (a_0, a_1, a_2, \dots).$$

Note finally that we have observed, earlier in this section, that the functors $\underline{T}(G)[1/p]$ and $\widetilde{G}$ commute with projective limits. Thus, the result extends to the category of prodiscrete k-algebras.

Finally, for the last part of the proposition (which gives the definitions used by Fontaine in [5, Chapter V]), one easily checks that the maps

$$\underline{T}(G)(A) \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, G(A)),\ (0, a_1, a_2, \dots) \mapsto (\overline{p^{-n}\mathbb{Z}_p} \mapsto a_n),$$

and

$$\widetilde{G}(A) \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, G(A)),\ (a_0, a_1, a_2, \dots) \mapsto (p^{-n}x \mapsto xa_n),$$

for which we use the isomorphisms $\mathbb{Q}_p/\mathbb{Z}_p \simeq \varinjlim_n p^{-n}\mathbb{Z}_p/\mathbb{Z}_p$ and $\mathbb{Q}_p \simeq \varinjlim_n p^{-n}\mathbb{Z}_p$, are bijective. Also, the isomorphism $\widetilde{G}(A) \simeq \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, G(A))$ we proved for discrete k-algebras formally extends to an isomorphism on prodiscrete k-algebras by commuting the limit. $\square$

Recall that the category of p-divisible groups over k up to isogeny is obtained by localizing the category of p-divisible groups with respect to the multiplication by p. For two p-divisible groups G and H over k, the group of morphisms in this category, denoted by $\mathrm{Hom}(G, H)[1/p]$, is then given by the formula

$$\mathrm{Hom}(G, H)[1/p] := \mathrm{Hom}(G, H) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

The next corollary gives a Galois-theoretic description of universal formal covers of étale p-divisible groups:

Corollary 3.2. *Let G be an étale p-divisible group over k. Then, the $\overline{k}$-points $\widetilde{G}(\overline{k})$ of the universal formal cover $\widetilde{G}$ is a finite dimensional $\mathbb{Q}_p$-vector space with a continuous action of Γ. Conversely, given such a $\mathbb{Q}_p$-vector space V, there is a unique étale p-divisible group G over k up to isogeny such that $\widetilde{G}(\overline{k}) \simeq V$.*

Proof. By Corollary 2.13, we know that $\underline{T}(G)(\overline{k}) \simeq \mathbb{Z}_p^h$ for some $h \in \mathbb{N}$ with a continuous Γ-action. By part (ii) of the above proposition, we have $\widetilde{G}(\overline{k}) \simeq \underline{T}(G)(\overline{k}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, giving us $\widetilde{G}(\overline{k}) \simeq \mathbb{Z}_p^h \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathbb{Q}_p^h$ with a continuous Γ-action. This justifies the first claim.

For the second one, let the finite dimensional $\mathbb{Q}_p$-vector space $V \simeq \mathbb{Q}_p^h$, $h \in \mathbb{N}$, have a continuous Γ-action. We need to show that, up to isogeny, there is a unique étale p-divisible group G over k with $\widetilde{G}(\overline{k}) = V$. To this end, let T be a Γ-stable lattice of V, existence of which is ensured as follows: choose any lattice T_0, and consider the stabilizer $H \subseteq \Gamma$ of T_0. Then, Γ/H is a finite set, due to H being open, and Γ being compact; and the sum

$$T := \sum_{g \in \Gamma/H} g(T_0)$$

is a Γ-stable lattice of V by definition. Hence, T is of the form $T \simeq \mathbb{Z}_p^h$ with a continuous Γ-action. Using Corollary 2.13 again, we get that it uniquely corresponds to an étale p-divisible group G with $\underline{T}(G)(\overline{k}) \simeq T$, as desired.

For the assertion that G is determined up to isogeny, assume there is another étale p-divisible group H over k such that $\widetilde{H}(\overline{k}) \simeq V$. Note that the kernel U' of the projection

$$V \simeq \widetilde{H}(\overline{k}) \simeq \varprojlim_p H(\overline{k}) \to H(\overline{k}),\ (x_n)_{n \in \mathbb{N}} \mapsto x_0,$$

is open and $\mathbb{Z}_p$-stable. Since U' is an open bounded $\mathbb{Z}_p$-lattice, we obtain an isomorphism $V \simeq \varprojlim_n V/p^n U'$, and it follows that the kernel of any projection

$$V \simeq \varprojlim_p H(\overline{k}) \to H(\overline{k})$$

is of the form $p^m U'$ for some $m \in \mathbb{N}$. Hence, we get that $p^m U' \subseteq U$ for some $m \in \mathbb{N}$. Conversely, the openness of U' implies that $p^n U \subseteq U'$ for some $n \in \mathbb{N}$. Therefore, the Γ-invariant map

$$G(\overline{k}) \simeq V/U \xrightarrow{p^n} V/U' \simeq H(\overline{k}) \xrightarrow{p^m} V/U = G(\overline{k})$$

shows that G and H are isogeneous, and the proof is complete. □

Remark 3.3. Observe that a description of the universal formal cover of a connected p-divisible group G over k in terms of the $\overline{k}$-points $\underline{T}(G)(\overline{k})$ of its Tate module $T(G)$ as in the étale case above is not possible as $\underline{T}(G)(\overline{k})$ vanishes. Indeed, by definition, G_i is represented by a finite local algebra A_i over k for any i, so that any k-algebra map $A_i \to \overline{k}$ factors through the inclusion $A_i/\mathfrak{m} \to \overline{k}$, yielding $\underline{T}(G)(\overline{k}) = 0$.

Example 3.4. If G is an étale p-divisible group, it follows, from Example 2.6 and Proposition 3.1.(ii), that the representing algebra of the universal formal cover $\widetilde{G}$ of G is

$$\varinjlim_p \mathcal{C}(\underline{T}(G)(\overline{k}), \overline{k})^\Gamma \simeq \mathcal{C}(\varprojlim_p \underline{T}(G)(\overline{k}), \overline{k})^\Gamma \simeq \mathcal{C}(\widetilde{G}(\overline{k}), \overline{k})^\Gamma.$$

Recall that our base field k is assumed to be perfect of characteristic $p > 0$. As in the étale case above, we can describe the representing algebra of the universal formal cover of a connected p-divisible group over k once we know the representing algebra of its Tate module $\underline{T}(G)$. This is what we do next:

Proposition 3.5. *Let G be a connected p-divisible group over k. Then the Tate module $\underline{T}(G)$ of G is represented by the k-algebra*

$$k[\![X^{1/p^\infty}]\!]/(X) = k[\![X_1^{1/p^\infty}, \dots, X_d^{1/p^\infty}]\!]/(X_1, \dots, X_d),$$

where $d \in \mathbb{N}$ is the dimension of G.

Proof. This is the result [14, Proposition 3.3.1]. Here, we give an explanatory proof by mimicking the argument for the proof of [14, Proposition 3.1.3.(iii)].

Write $G = \varinjlim_i G_i$. By the proof of [16, Proposition 1, §2.2], we see that each G_i is represented by the discrete quotient

$$k[\![X]\!]/\big([p^i](X)\big)$$

of $k[\![X]\!]$. Put $G_i^{(p^n)} := G_i \otimes_{k,\varphi^n} k$ for any $i \in \mathbb{N}$, where $n \in \mathbb{N}$, and φ is the Frobenius of k. Using the decomposition $p = VF = FV$, where $F : G_i^{(p^n)} \to G_i^{(p^{n+1})}$ is the Frobenius, and $V : G_i^{(p^{n+1})} \to G_i^{(p^n)}$ is the Verschiebung (cf. [20, Rem 5.21]), we obtain a natural transformation of functors:

$$\mathcal{F} : \underline{T}(G) = \varprojlim_p G_i \to \varprojlim_F G_i^{(p^{-n})}, \ (a_i)_{i\in\mathbb{N}} \mapsto (V^{i-1}a_i)_{i\in\mathbb{N}}.$$

Moreover, since multiplication by p is an isogeny on each G_i, we can write $F^m = pu_i$ for some $m \geq 1$, where $u_i : G_i^{(p^{-m})} \to G_i$ is an isogeny for any i (cf. [20, Proposition 5.25]). Denote the isogeny $G_i^{(p^{-mn})} \to G$ by u_i^n, and consider the natural transformation

$$\mathcal{G} : \varprojlim_{F^m} G_i^{(p^{-mn})} \to \varprojlim_p G_i, \ (a_i)_{i\in\mathbb{N}} \mapsto (u_i^{i-1}a_i)_{i\in\mathbb{N}}.$$

We also have the following natural transformation:

$$\mathcal{H} : \varprojlim_F G_i^{(p^{-n})} \to \varprojlim_{F^m} G_i^{(p^{-mn})}, \ (a_i)_{i\in\mathbb{N}} \mapsto (F^{(i-1)(1-m)}a_i)_{i\in\mathbb{N}}.$$

For each $i \in \mathbb{N}$, we then obtain

$$u_i^{i-1}F^{(i-1)(1-m)}V^{i-1} = u_i^{i-1}F^{m-im}F^{i-1}V^{i-1} = u_i^{i-1}F^{m-im}p^{i-1} = F^{m-im}F^{m(i-1)} = 1,$$

showing that the composition $\mathcal{G} \circ \mathcal{H} \circ \mathcal{F}$ is the identity. Also, for each $i \in \mathbb{N}$, we have

$$F^{(i-1)(1-m)}u_i^{i-1}V^{i-1} = F^{(i-1)(1-m)}(p^{-1}F^mV)^{i-1} = F^{(i-1)(1-m)}(F^{m-1})^{i-1} = 1,$$

i.e. $\mathcal{F} \circ \mathcal{G} \circ \mathcal{H} = 1$. Therefore, we have proved that $\mathcal{F}$ is an isomorphism, and it is enough to determine the representing algebra of $\varprojlim_F G_i^{(p^{-n})}$. But, for each i, there is an isomorphism $G_i \simeq G_i^{(p^{-n})}$ of affine schemes as k is perfect. Moreover, the map F on G_i is given on the variables by $X_j \mapsto X_j^p$ for each $j = 1, \ldots, d$ (cf. [20, Theorem 5.2]), so we conclude that the representing algebra of $\varprojlim_F G_i$ is

$$\varinjlim_F k[\![X]\!]/\big([p^i](X)\big) \simeq \varinjlim_n k[\![X^{1/p^n}]\!]/(X) \simeq k[\![X^{1/p^\infty}]\!]/(X),$$

as desired. $\square$

The following result was already observed in [14, Proposition 3.1.3.(iii)]. In fact, Scholze and Weinstein go the other way around, and deduce the result in the above proposition from the following statement (cf. [14, Proposition 3.3.1]).

Corollary 3.6. *The universal formal cover $\widetilde{G}$ of a connected p-divisible group is represented by*

$$k[\![X^{1/p^\infty}]\!] = k[\![X_1^{1/p^\infty}, \ldots, X_d^{1/p^\infty}]\!].$$

Proof. Since multiplication by p is an isogeny on G, as in the proof of the previous proposition, we have $F^m = pu$ for some $m \geq 1$, where $F : G^{(p^n)} \to G^{(p^{n+1})}$ is the Frobenius, and $u : G^{(p^{-m})} \to G$ is an isogeny. This induces a decomposition $F^m = pu'$ on $\underline{T}(G)$, so that $\widetilde{G} \simeq \underline{T}(G)[1/p]$ is represented by

$$\varprojlim_{p} k[\![X^{1/p^\infty}]\!]/(X) \simeq \varprojlim_{F^m} k[\![X^{1/p^\infty}]\!]/(X),$$

where F^m is given by $X_j \mapsto X_j^{p^m}$ on each variable X_j, $j = 1, \ldots, d$. But then, since $k[\![X^{1/p^\infty}]\!]$ is (X)-adically separated and complete, we get

$$\varprojlim_{F^m} k[\![X^{1/p^\infty}]\!]/(X) \simeq \varprojlim_{m} k[\![X^{1/p^\infty}]\!]/\big(X^{p^m}\big) \simeq k[\![X^{1/p^\infty}]\!].$$

$\square$

Remark 3.7. In [14, Proposition 3.1.3.(iii)], Scholze and Weinstein instead start with a connected p-divisible group G over k of dimension d, and proves that there is an isomorphism

$$\widetilde{G} = \varprojlim_{p} G \simeq \varprojlim_{F} G^{(p^n)}.$$

Observing that G, as a formal group, is represented by

$$\varprojlim_{i} k[\![X]\!]/\big([p^i](X)\big) \simeq k[\![X]\!],$$

where $(X) = (X_1, \ldots, X_d)$, they then show that the k-algebras

$$\varinjlim_{F} k[\![X]\!] = \varinjlim_{X_k \mapsto X_k^p} k[\![X]\!] = \varinjlim_{n} k[\![X^{1/p^n}]\!]$$

and $k[\![X^{1/p^\infty}]\!]$ represent the same functor; namely, $\widetilde{G}$.

Let G and H be p-divisible groups over k. Recall that $\widetilde{G}$ and $\widetilde{H}$ are group functors taking values in the category of $\mathbb{Q}_p$-vector spaces, so that $\mathrm{Hom}(\widetilde{G}, \widetilde{H})$ is a $\mathbb{Q}_p$-vector space. Therefore, the natural injection $\mathrm{Hom}(G, H) \to \mathrm{Hom}(\widetilde{G}, \widetilde{H})$, defined by $f \mapsto \tilde{f} := (f)_n$, induces a unique map

$$\mathrm{Hom}(G, H)[1/p] \to \mathrm{Hom}(\widetilde{G}, \widetilde{H}),\ f \otimes p^{-n} \mapsto p^{-n}\tilde{f},$$

which is clearly injective.

Moreover:

Proposition 3.8. *The universal formal cover functor on the category of p-divisible groups up to isogeny over k is fully faithful.*

Proof. To prove the proposition, we begin by justifying that it is enough to prove it for the categories of étale and connected p-divisible groups up to isogeny over k.

Let G and H be two p-divisible groups over k. Since k is perfect of characteristic $p > 0$, by the proof of [16, Proposition 4], any p-divisible group over k is isomorphic to the direct sum of its connected and étale parts, i.e. $G \simeq G^\circ \oplus G^{\text{ét}}$ and $H \simeq H^\circ \oplus H^{\text{ét}}$. But then, as both $\mathrm{Hom}(G^\circ, H^{\text{ét}})$ and $\mathrm{Hom}(G^{\text{ét}}, H^\circ)$ only contain the zero map, we get that

$$\mathrm{Hom}(G, H)[1/p] \simeq \mathrm{Hom}(G^\circ, H^\circ)[1/p] \oplus \mathrm{Hom}(G^{\text{ét}}, H^{\text{ét}})[1/p].$$

We claim that also $\mathrm{Hom}(\widetilde{G^\circ}, \widetilde{H^{\text{ét}}})$ and $\mathrm{Hom}(\widetilde{G^{\text{ét}}}, \widetilde{H^\circ})$ contain only the zero map. Assume, without loss of generality, that the action of Γ is trivial. Suppose first that we are given a map in $\mathrm{Hom}(\widetilde{G^\circ}, \widetilde{H^{\text{ét}}})$. Using Example 3.4 and Corollary 3.6, and , this gives us a map $F : \mathcal{C}(V, k) \to k[\![X^{1/p^\infty}]\!]$ of k-algebras, where $V = \widetilde{G}(\overline{k})$. We need to see that F factors through the augmentation

$$\varepsilon : \mathcal{C}(V, k) \to k, \ f \mapsto f(0),$$

of $\mathcal{C}(V, k)$. To this end, let $f \in \ker(\varepsilon)$, and consider the characteristic function g of the support of f. We have $f \cdot g = f$, where the multiplication in $\mathcal{C}(V, k)$ is defined pointwise. Since the support of f does not contain 0, we also get $g(0) = 0$. Moreover, g being an idempotent in $\mathcal{C}(V, k)$, it must be mapped to an idempotent in the maximal ideal of $k[\![X^{1/p^\infty}]\!]$, i.e. $F(g) = 0$, so that

$$F(f) = F(f \cdot g) = F(f)F(g) = 0,$$

which proves the claimed factorization.

Similarly, to see that $\mathrm{Hom}(\widetilde{G^{\text{ét}}}, \widetilde{H^\circ})$ is trivial, we must see that any map $k[\![X^{1/p^\infty}]\!] \to \mathcal{C}(V, k)$ of k-algebras factors through the augmentation $\varepsilon : X \mapsto 0$ of $k[\![X^{1/p^\infty}]\!]$. Let $f \in \ker(\varepsilon)$. As $f \in k[\![X^{1/p^\infty}]\!]$ has no constant term, it follows that $f(X)$ is topologically nilpotent. However, since k is a discrete field, and multiplication in $\mathcal{C}(V, k)$ is defined pointwise, it follows that $\mathcal{C}(V, k)$ has no nonzero topologically nilpotent elements. This implies $f(X) = 0$, and the desired factorization follows. If k is not algebraically closed, one can then pass to the Γ-invariants.

Now, since the universal formal cover functor (as a projective limit of formal groups) commutes with the finite direct sums, we obtain

$$\mathrm{Hom}(\widetilde{G}, \widetilde{H}) \simeq \mathrm{Hom}(\widetilde{G^\circ}, \widetilde{H^\circ}) \oplus \mathrm{Hom}(\widetilde{G^{\text{ét}}}, \widetilde{H^{\text{ét}}}).$$

Therefore, it follows that it is enough to prove the proposition for the categories of étale and connected p-divisible groups up to isogeny over k.

In order to do that, we need to prove that the map $\mathrm{Hom}(G, H)[1/p] \to \mathrm{Hom}(\widetilde{G}, \widetilde{H})$ defined above is surjective. Assume first that G and H are étale. Let $\widetilde{G} \to \widetilde{H}$ be a map between the universal formal covers $\widetilde{G}$ and $\widetilde{H}$. By Corollary 3.2, this gives a Γ-equivariant

$\mathbb{Q}_p$-linear map $f : \mathbb{Q}_p^{h_1} \to \mathbb{Q}_p^{h_2}$ of Γ-sets with $h_1, h_2 \in \mathbb{N}_{>0}$. Since the kernel of the composition $\mathbb{Q}_p^{h_1} \to \mathbb{Q}_p^{h_2} \to \mathbb{Q}_p^{h_2}/\mathbb{Z}_p^{h_2}$, where $\mathbb{Z}_p^{h_2}$ can be assumed to be Γ-stable in $\mathbb{Q}_p^{h_2}$ by choosing a suitable $\mathbb{Q}_p$-basis, contains $p^n\mathbb{Z}_p^{h_1}$ for some $n \in \mathbb{N}$ (being an open Γ-invariant subspace of $\mathbb{Q}_p^{h_1}$), we obtain an induced Γ-equivariant map $g : \mathbb{Q}_p^{h_1}/p^n\mathbb{Z}_p^{h_1} \to \mathbb{Q}_p^{h_2}/\mathbb{Z}_p^{h_2}$. Composing g with p^n, we get a map

$$p^n \circ g : \mathbb{Q}_p^{h_1}/p^n\mathbb{Z}_p^{h_1} \xrightarrow{g} \mathbb{Q}_p^{h_2}/\mathbb{Z}_p^{h_2} \xrightarrow{p^n} \mathbb{Q}_p^{h_2}/p^n\mathbb{Z}_p^{h_2}$$

in $\mathrm{Hom}(G, H)$. We would like to see that $p^n \circ g \otimes p^{-n}$ maps to f. But, it follows, by construction, that

$$p^{-n}\widetilde{(p^n \circ g)} = \widetilde{g} = f.$$

Now, assume that G and H are connected. Let $f \in \mathrm{Hom}(\widetilde{G}, \widetilde{H})$. Using Corollary 3.6, we assume that $\widetilde{G}$ is represented by $R = k[\![X^{1/p^\infty}]\!]$, and $\widetilde{H}$ by $S = k[\![Y^{1/p^\infty}]\!]$, and consider the map $\varphi : S \to R$ of Hopf algebras corresponding to f. Since H is a p-divisible formal group, we have that $[p](Y_i) = Y_i^{p^{h_i}}$ after a suitable change of variables for each $i = 1, \dots, d$, where $Y = (Y_1, \dots, Y_d)$ for some $d \geq 1$ (cf. [20, Theorem 5.2]). Thus, the composition $S \xrightarrow{p^n} S \xrightarrow{\varphi} R \to R/(X)$ factors through $S/(Y)$ for a large enough $n \in \mathbb{N}$ to give us a map $\overline{\varphi} : S/(Y) \to R/(X)$ of Hopf algebras. By Proposition 3.5, this corresponds to a map $\underline{T}(G) \to \underline{T}(H)$ between the Tate modules of G and H, which yields a unique map $g : G \to H$ in $\mathrm{Hom}(G, H)$ by Corollary 2.11. We are done if we show that $f = p^{-n}\tilde{g}$. But since $\widetilde{G} \simeq \underline{T}(G)[1/p]$ (and similarly for H), we see that the map $\tilde{g} : \widetilde{G} \to \widetilde{H}$ on the level of Hopf algebras is given by

$$\phi : \varprojlim_p S/(Y) \to \varprojlim_p R/(X).$$

Since the Hopf algebra structure on the quotients $R/(X)$ and $S/(Y)$ are induced by the Hopf algebras R and S, respectively, we get $\phi = \varphi \circ [p^n]$, or $\widetilde{g} = p^n f$, as required. $\square$

4 Formal p-adic vector spaces

Assume that k is a perfect field of characteristic $p > 0$. This is the chapter where we present most of our main results.

4.1 Definitions and examples

The notion of a formal $\mathbb{Q}_p$-vector space is defined in parallel with the definition of a formal group over k (cf. Definition 1.10):

Definition 4.1. *A formal $\mathbb{Q}_p$-vector space over k is a $\mathbb{Q}_p$-vector space object in the category of formal schemes over k, i.e. it is a representable functor from the category of prodiscrete k-algebras to the category of $\mathbb{Q}_p$-vector spaces.*

Let $G = \varinjlim_n G_n$ be a generalized p-divisible group over k. A fundamental example of a formal $\mathbb{Q}_p$-vector spaces is then given by the universal formal cover $\widetilde{G}$ of G. Recall that, in the previous chapter, we have seen that, as a functor on prodiscrete k-algebras, $\widetilde{G}(A)$ is a $\mathbb{Q}_p$-vector space for any prodiscrete k-algebra A. Moreover, by Proposition 3.1.(ii), we know that $\widetilde{G}$ is a formal group. Therefore, it follows that the universal formal cover of a p-divisible group over k is an example of formal $\mathbb{Q}_p$-vector spaces over k.

Definition 4.2. *An étale formal $\mathbb{Q}_p$-vector space over k is a formal $\mathbb{Q}_p$-vector space $\mathcal{F}$ such that its representing algebra $A = \varprojlim A_i$ is a projective limit of discrete indétale k-algebras.*

Example 4.3. Example 2.6 and Example 3.4 show that the representing algebra of the universal formal cover $\widetilde{G}$ of an étale p-divisible group G over k is

$$\varprojlim_p \mathcal{C}(T, \overline{k})^\Gamma \simeq \varprojlim_p \varinjlim_n \mathcal{C}(T/p^nT, \overline{k})^\Gamma,$$

where $T = \underline{T}(G)(\overline{k})$, showing that $\widetilde{G}$ is an étale formal $\mathbb{Q}_p$-vector space over k.

Example 4.4. Let $H = \mathrm{Spec}(A)$ be a p-adically separated and complete affine group scheme over k such that $H[p] = 0$. So we have that the induced map $A \xrightarrow{[p]} A$ is surjective (cf. Remark 1.6). A natural construction of formal $\mathbb{Q}_p$-vector spaces is then as follows: Consider $\mathcal{F} := H[1/p] = \varinjlim_p H$ as a formal scheme over k (cf. Remark 1.11.(ii)). Since $\mathrm{End}(H/p^nH)$ has a $\mathbb{Z}/p^n\mathbb{Z}$-module structure, $\mathrm{End}(H)$ acquires a natural $\mathbb{Z}_p$-module structure via

$$\varprojlim_n \mathbb{Z}/p^n\mathbb{Z} \to \varprojlim_n \mathrm{End}(H/p^nH) \to \mathrm{End}(\varprojlim_n H/p^nH) \simeq \mathrm{End}(H).$$

This implies that $\mathcal{F}$ is a functor that takes values in the category of $\mathbb{Q}_p$-vector spaces. As seen earlier, it is prorepresented by the prodiscrete k-algebra $\varprojlim_p A$, and can be extended to a representable functor on the category of prodiscrete k-algebras by letting it commute with projective limits.

A special class of formal $\mathbb{Q}_p$-vector spaces we are going to be interested in are the ones whose representing algebra is the ring $R = k[\![X_1^{1/p^\infty}, \dots, X_d^{1/p^\infty}]\!]$ of fractional formal power series, which we largely studied in §1.4. Observe that R is a local ring with the unique maximal ideal

$$\mathfrak{m}_R = (X^{1/p^\infty}) = (X_1^{1/p^\infty}, \dots, X_d^{1/p^\infty}) = \bigcup_{n \in \mathbb{N}} (X_1^{1/p^n}, \dots, X_d^{1/p^n}).$$

In particular, the prime spectrum of R is connected in the topological sense.

Motivated by this fact and for an ease of presentation, we define:

Definition 4.5. *A formal $\mathbb{Q}_p$-vector space over k that is represented by R is called a connected formal $\mathbb{Q}_p$-vector space of dimension d.*

Example 4.6. It follows, by Corollary 3.6, that the universal formal cover $\widetilde{G}$ of a connected p-divisible group is a connected formal $\mathbb{Q}_p$-vector space of the kind described in Example 4.4 above.

Thus, we have seen that the universal formal cover $\widetilde{G}$ of a p-divisible group G over a perfect field k of characteristic $p > 0$ constitutes an example of formal $\mathbb{Q}_p$-vector spaces over k, namely étale and connected formal $\mathbb{Q}_p$-vector spaces. We are fundamentally interested in the questions in the reverse direction:

(1) Given an étale formal $\mathbb{Q}_p$-vector space $\mathcal{F}$, is there a (generalized) p-divisible group G such that $\widetilde{G} \simeq \mathcal{F}$?

(2) Given a connected formal $\mathbb{Q}_p$-vector space $\mathcal{F}$ of dimension $d \geq 1$, is there a (generalized) p-divisible group G over k such that $\widetilde{G} \simeq \mathcal{F}$?

4.2 Étale formal vector spaces

Let $\mathcal{F}$ be an étale formal $\mathbb{Q}_p$-vector space. Let $A = \varprojlim_i A_i$, where each A_i is an indétale k-algebra, be the representing algebra of $\mathcal{F}$. Recall that, under the equivalent assumptions of Lemma 1.8 on the indétale k-algebras A_i, Corollary 1.9 ensures the existence of a locally profinite space X with a continuous Γ-action such that $A \simeq \mathcal{C}(X, \overline{k})^\Gamma$ as prodiscrete spaces.

We give a conditional answer to Question (1) we posed above:

Proposition 4.7. *Let the notation be as in the above paragraph. Assume that*

(i) the transition maps $A_j \to A_i$ for $i \leq j$ satisfy the equivalent assumptions of Lemma 1.8,

(ii) the topology introduced in Corollary 1.9 makes $\mathcal{F}(\overline{k}) = X$ into a topological $\mathbb{Q}_p$-vector space.

Then, there is an étale p-divisible group G over k such that $\widetilde{G} \simeq \mathcal{F}$.

Proof. We know, by the results in [17, §2], that the locally profinite topological space X admits a compact open subgroup X_0. Since the $\mathbb{Q}_p$-action on X is continuous by assumption, X_0 is a compact $\mathbb{Z}_p$-module. Moreover, if $x \in X$, then $p^n x \in X_0$ for a sufficiently large n. This implies that $X = X_0[1/p]$, and hence, X is a finite dimensional $\mathbb{Q}_p$-vector space.

Noting that the Γ-action on $X = \mathcal{F}(\overline{k})$ is $\mathbb{Q}_p$-linear, choose a Γ-stable $\mathbb{Z}_p$-lattice $T \subseteq X$. By Corollary 3.2, there is a corresponding étale p-divisible group G over k with $\underline{T}(G)(\overline{k}) \simeq T$. But then, $\underline{T}(G)$ is represented by $\mathcal{C}(T, \overline{k})^{\Gamma}$, and hence the universal formal cover $\widetilde{G} \simeq \underline{T}(G)[1/p]$ by

$$\varprojlim_{p} \mathcal{C}(T, \overline{k})^{\Gamma} \simeq \mathcal{C}(\varinjlim_{p} T, \overline{k})^{\Gamma} = \mathcal{C}(X, \overline{k})^{\Gamma},$$

as required. □

Now, in order to generalize the previous construction beyond étale p-divisible groups, let V be an arbitrary (i.e. possibly infinite dimensional) $\mathbb{Q}_p$-Banach space with a continuous $\mathbb{Q}_p$-linear action of Γ. Then, there is a bounded open lattice U of V that is Γ-invariant (cf. [13, Remark 18.2]). We consider the inverse system

$$W_n := (V/p^n U)_{n \in \mathbb{N}}$$

of discrete Γ-sets. Write each $W_n \simeq \varinjlim_{m \in \mathbb{N}} W_{nm}$ as an inductive limit of its finite discrete Γ-stable subgroups W_{nm}. Each Γ-set W_n then gives rise to a formal group G_n over k that is represented by the proétale k-algebra

$$\mathcal{C}(W_n, \overline{k})^{\Gamma} \simeq \mathcal{C}\big(\varinjlim_{m} W_{nm}, \overline{k}\big)^{\Gamma} \simeq \varprojlim_{m} \mathcal{C}(W_{nm}, \overline{k})^{\Gamma}$$

endowed with the projective limit topology (cf. [5, Chapter 1, §7.1]). Note, in particular, that this implies that $\mathcal{C}(W_n, \overline{k})^{\Gamma}$ is a separated and complete linearly topologized k-vector space by Proposition 1.4.(ii). Setting now

$$\underline{V} := \varprojlim_{n} G_n,$$

and viewing it as a group functor on the category of prodiscrete k-algebras, we have the following elementary observation:

Lemma 4.8. *Let the notation be as above.*

(i) The canonical Γ-invariant map $V \to \varprojlim_n W_n$ is a topological isomorphism.

(ii) There is a Γ-invariant isomorphism $V \simeq \underline{V}(\overline{k})$.

Proof. For part (i), note that the topology of the Banach space V is generated by the subsets of the form $v + p^n U$ with $v \in V$ (cf. [13, Proposition 4.1]), so that the map $V \to W_n$ is continuous for each n, where the space on the right is discrete. Hence, the map $V \to \varprojlim_n W_n$ is continuous. Moreover, note that $(p^n U)_{n\in\mathbb{N}}$ is a basis of open neighborhoods of zero in V, and that the image of any $p^n U$ is the kernel of the projection $\varprojlim_n W_n \to V/p^n U$ as the canonical map $V \to \varprojlim_n W_n$ is a bijection. Hence, the image of $p^n U$ is open, showing that the map is open.

For part (ii), the canonical isomorphism $G_n(\overline{k}) \simeq W_n$, for each $n \in \mathbb{N}$, gives, together with part (i), that

$$\underline{V}(\overline{k}) \simeq \varprojlim_n G_n(\overline{k}) \simeq \varprojlim_n W_n \simeq V$$

is a Γ-invariant isomorphism. $\square$

Observe that the functor $\underline{V}$, as a projective limit of formal groups, need not be representable for purely formal reasons. However, by applying the construction in §1.3, we will see that the space $C^u(V, \overline{k})^\Gamma$ of uniformly continuous maps from V to $\overline{k}$ that commute with the action of Γ, topologized with the strict k-linear inductive limit (cf. §1.3), makes $\underline{V}$ into a functor that satisfies some representability-like properties.

In fact, since $(p^n U)_{n\in\mathbb{N}}$ forms a basis of open neighborhoods of zero in V, any uniformly continuous map $V \to \overline{k}$ factors through W_n for some $n \in \mathbb{N}$, yielding an isomorphism

$$\mathcal{C}^u(V, \overline{k})^\Gamma \simeq \varinjlim_n \mathcal{C}(W_n, \overline{k})^\Gamma.$$

As, for each $n \in \mathbb{N}$, the map $\mathcal{C}(W_n, \overline{k}) \to \mathcal{C}(W_{n+1}, \overline{k})$ is injective, we view $C^u(V, \overline{k})^\Gamma$ as a filtered union of linearly topologized k-vector spaces $\mathcal{C}(W_n, \overline{k})^\Gamma$. Endow now $C^u(V, \overline{k})^\Gamma$ with the strict k-linear inductive limit topology as in Definition 1.12.

Note that although, by Lemma 1.15, the space $C^u(V, \overline{k})^\Gamma$ is Hausdorff and complete, it is not a linearly topologized ring: it has a basis of open neighborhoods of zero consisting of k-vector subspaces, but not of ideals. In particular, $C^u(V, \overline{k})^\Gamma$ is not a prodiscrete k-algebra; however, we still have:

Lemma 4.9. *Let A be a discrete k-algebra. Then,*

$$\underline{V}(A) \simeq \mathrm{Hom}_k^{cont}\big(C^u(V, \overline{k})^\Gamma, A\big).$$

Proof. By Lemma 1.15.(i) and the construction of the strict k-linear inductive limit topology on $C^u(V, \overline{k})^\Gamma$, a k-linear map $C^u(V, \overline{k})^\Gamma \to A$ is continuous if and only if its restriction $\mathcal{C}(W_n, \overline{k})^\Gamma \to A$ is continuous for all $n \in \mathbb{N}$. In particular, this holds for homomorphisms of k-algebras, giving that

$$\underline{V}(A) = \varprojlim_n G_n(A) \simeq \varprojlim_n \mathrm{Hom}_k^{\mathrm{cont}}\left(\mathcal{C}(W_n, \overline{k})^\Gamma, A\right) \simeq \mathrm{Hom}_k^{\mathrm{cont}}\left(\varinjlim_n \mathcal{C}(W_n, \overline{k})^\Gamma, A\right)$$
$$\simeq \mathrm{Hom}_k^{\mathrm{cont}}\left(C^u(V, \overline{k})^\Gamma, A\right).$$

□

Let again G be an étale p-divisible group over k. Consider the open Γ-invariant $\mathbb{Z}_p$-lattice T of $\widetilde{G}(\overline{k})$ that satisfies $T = \underline{T}(G)(\overline{k})$. For each $n \in \mathbb{N}$, consider the formal group corresponding to the discrete Γ-set $\widetilde{G}(\overline{k})/p^nT$, and form the projective limit

$$\underline{\widetilde{G}(\overline{k})} := \varprojlim_n \mathrm{Spf}\left(\mathcal{C}\big(\widetilde{G}(\overline{k})/p^nT, \overline{k}\big)^\Gamma\right),$$

viewed as a functor on the category of prodiscrete k-algebras. We obtain:

Proposition 4.10. *Let G be an étale p-divisible group of height $h \geq 1$ over k. We have an isomorphism*

$$\underline{\widetilde{G}(\overline{k})} \simeq \varprojlim_p G$$

as functors on the category of prodiscrete k-algebras. Moreover, for any prodiscrete k-algebra A, we have

$$\widetilde{G}(A) = \mathrm{Hom}_k^{cont}\left(C^u(\mathbb{Q}_p^h, \overline{k})^\Gamma, A\right).$$

Proof. For the first assertion, it is enough to see that we have an isomorphism

$$\varinjlim_n \mathcal{C}\big(\widetilde{G}(\overline{k})/p^nT, \overline{k}\big)^\Gamma \simeq \varinjlim_p \mathcal{C}(G(\overline{k}), \overline{k})^\Gamma$$

of inductive systems. To this end, note that, as in the left hand side, the transition maps in the inductive limit on the right hand side are injective because they are induced from multiplication by p on $G(\overline{k})$. Hence, it is isomorphic to the filtered union

$$\bigcup_{n\in\mathbb{N}} p^n\mathcal{C}(G(\overline{k}), \overline{k})^\Gamma$$

of the images $p^n\mathcal{C}(G(\overline{k}), \overline{k})^\Gamma$ of the function space $\mathcal{C}(G(\overline{k}), \overline{k})^\Gamma$ under the multiplication by p. But we have an isomorphism

$$\mathcal{C}\big(\widetilde{G}(\overline{k})/p^nT, \overline{k}\big) \simeq p^n\mathcal{C}(G(\overline{k}), \overline{k})$$

for each $n \in \mathbb{N}$, which justifies the claim. Endowing the inductive limit $\varinjlim_p \mathcal{C}(G(\overline{k}), \overline{k})^\Gamma$ with the strict k-linear inductive limit topology, the last assertion follows by the above lemma. □

Remark 4.11. (i) The fact that the representing algebra of $\widetilde{G}$ is $\mathcal{C}(\mathbb{Q}_p^h, \overline{k})^\Gamma$ when G is an étale p-divisible group of height $h \geq 1$ over k (cf. Example 3.4) should not lead to confusion since, although the k-algebras $C^u(V, \overline{k})^\Gamma$ and $\mathcal{C}(V, \overline{k})^\Gamma$ of function spaces do not generally agree (even when V is a finite dimensional $\mathbb{Q}_p$-Banach space, e.g. $V = \mathbb{Q}_p^h$), the functors $\mathrm{Hom}_k^{\mathrm{cont}}(C^u(\mathbb{Q}_p^h, \overline{k})^\Gamma, \bullet)$ and $\mathrm{Hom}_k^{\mathrm{cont}}(\mathcal{C}(\mathbb{Q}_p^h, \overline{k}^\Gamma), \bullet)$ are still identical on the category of prodiscrete k-algebras.

(ii) As the new approach we offered above has no finiteness condition on the dimension of the $\mathbb{Q}_p$-Banach space V, it somehow generalizes the classical finite dimensional picture in the following sense: As opposed to the case where V is finite dimensional, so that $\underline{V}$ is a formal scheme with the representing k-algebra $\mathcal{C}(V, \bar{k})^\Gamma$, in case V is infinite dimensional, the k-algebra $\mathcal{C}^u(V, \overline{k})^\Gamma$ is no longer prodiscrete; nonetheless, we still have Lemma 4.9, which practically allows us to compute $\underline{V}$ in terms of the continuous Hom functor in the category of k-algebras, though this is technically no representability.

(iii) This approach allows us to meaningfully extend the definition of étaleness to the generalized p-divisible groups (cf. Remark 2.16), and has a potential to yield a classificaion of the universal formal cover of étale generalized p-divisible groups. Indeed, we could define an étale generalized p-divisible group over k as a generalized p-divisible group $G = \varinjlim_n G_n$ such that each G_n is a proétale group scheme over k. Letting then V be a $\mathbb{Q}_p$-Banach space with a continuous action of Γ, one could reasonably claim that there is an étale generalized p-divisible group G over k such that $\widetilde{G} \simeq \underline{V}$, and that, if V is finite dimensional, then G is indeed a p-divisible group over k.

4.3 Connected formal vector spaces

In this section, we attempt to answer Question (2) we asked at the end of §4.1 in two cases: $d = 1$ and $d \geq 2$. Throughout the section, assume given a connected formal $\mathbb{Q}_p$-vector space $\mathcal{F} = \mathrm{Spf}(R)$, where $R = k[\![X_1^{1/p^\infty}, \ldots X_d^{1/p^\infty}]\!]$ with $d \geq 1$.

Our general strategy is to find suitable Hopf ideals $\mathcal{I}$ when $d = 1$ (resp. when $d > 1$) of R such that the affine subgroup $\mathrm{Spec}(R/\mathcal{I})$ is a Tate k-group (resp. generalized Tate k-group) T, and then make use of Corollary 2.11 (resp. Proposition 2.10) to deduce that there is a corresponding p-divisible group G (resp. generalized p-divisible group) over k such that $\underline{T}(G) = T$, i.e. that T is the Tate module of G. Next, we show $T[1/p] \simeq \mathcal{F}$ unconditionally (resp. under a mild continuity condition), which shall imply, by Proposition 3.1.(ii), that $\mathcal{F}$ is isomorphic to the universal formal cover of the p-divisible group (resp. generalized p-divisible group) G over k.

4.3.1 One-dimensional case

Assume $d = 1$. We denote by $G(X, Y) \in k[\![X^{1/p^\infty}, Y^{1/p^\infty}]\!]$ the perfect formal group law of $\mathcal{F}$, and by $\iota(X) \in k[\![X^{1/p^\infty}]\!]$ the inverse of X (cf. §1.4). We write

$$G(X,Y) = \sum_{i,j\in\mathbb{N}[1/p]} a_{ij} X^i Y^j \qquad \text{and} \qquad \iota(X) = \sum_{i\in\mathbb{N}[1/p]} a_i X^i.$$

The following lemmas are particular to the case $d = 1$.

Lemma 4.12. *We have* $G(X,Y) = X + Y \bmod (XY)$*.*

Proof. Let $\ell_0 := \min\{i > 0 \mid \exists j\ a_{ij} \neq 0\}$, and let $\ell_0' := \min\{j > 0 \mid \exists i\ a_{ij} \neq 0\}$. Observe that Definition 1.20.(i) allows us to write

$$G(X,Y) = X + Y + \sum_{i,j\in\mathbb{N}[1/p]\setminus 0} a_{ij} X^i Y^j.$$

By Definition 1.20.(ii), it follows that $\ell_0 = \ell_0'$, so that we can write

$$G(X,Y) = X + Y + X^{\ell_0} Y^{\ell_0} \cdot u,$$

with $u = u(X,Y)$ satisfying $u(0,0) = a_{\ell_0\ell_0'} \neq 0$. Note that, if $\ell_0 = 1$, we are done. Suppose, for a contradiction, that $\ell_0 < 1$. Write $\ell_0 = m_0/p^{r_0}$ with $m_0, r_0 \in \mathbb{N}_{>0}$, and $p \nmid m_0$. Consider the ideal

$$\mathcal{I} := \left\{ \sum_{i,j,h\in\mathbb{N}[1/p]} a_{ijh} X^i Y^j Z^h \mid i > \ell_0/p^{r_0} \text{ or } j \geq \ell_0 \right\} \subseteq k[\![X^{1/p^\infty}, Y^{1/p^\infty}, Z^{1/p^\infty}]\!].$$

Since $1 > \ell_0 > \ell_0/p^{r_0}$, we have

$$G(X, G(Y,Z)) = X + G(Y,Z) + X^{\ell_0} G(Y,Z)^{\ell_0} u\big(X, G(Y,Z)\big) \equiv G(Y,Z) \bmod \mathcal{I}.$$

On the other hand, computing $G(G(X,Y),Z)$ modulo $\mathcal{I}$,

$$\begin{aligned}
G(G(X,Y),Z) &= X + Y + X^{\ell_0}Y^{\ell_0}u + Z + G(X,Y)^{\ell_0} Z^{\ell_0} u\big(G(X,Y),Z\big) \\
&\equiv Z + G(X,Y)^{\ell_0} Z^{\ell_0} u\big(G(X,Y),Z\big) \\
&\equiv Z + (X^{1/p^{r_0}} + Y^{1/p^{r_0}} + X^{\ell_0/p^{r_0}} Y^{\ell_0/p^{r_0}} u^{1/p^{r_0}})^{m_0} Z^{\ell_0} u\big(G(X,Y),Z\big) \\
&\equiv Z + \left(\sum_{j=0}^{m_0} \binom{m_0}{j} Y^{(m_0-j)/p^{r_0}} X^{j\ell_0/p^{r_0}} Y^{j\ell_0/p^{r_0}} u^{j/p^{r_0}} \right) Z^{\ell_0} u\big(G(X,Y),Z\big) \\
&\equiv Z + m_0 X^{\ell_0/p^{r_0}} Y^{(\ell_0+m_0-1)/p^{r_0}} u^{1/p^{r_0}} Z^{\ell_0} u\big(G(X,Y),Z\big),
\end{aligned}$$

where the last two congruences follow from the facts that $1/p^{r_0} > \ell_0/p^{r_0}$, and that $2\ell_0/p^{r_0} > \ell_0/p^{r_0}$, respectively. But then, using Definition 1.20.(iii), we obtain

$$Z + m_0 X^{\ell_0/p^{r_0}} Y^{(\ell_0+m_0-1)/p^{r_0}} u^{1/p^{r_0}} Z^{\ell_0} u\big(G(X,Y),Z\big) - G(Y,Z) \in \mathcal{I}.$$

Since $u(0,0) = a_{\ell_0\ell_0} \neq 0$, and $m_0 \in k^\times$, we get that $m_0 a_{\ell_0\ell_0}^{1/p^{r_0}} a_{\ell_0\ell_0} \in k^\times$, and the summand

$$m_0 a_{\ell_0\ell_0}^{1/p^{r_0}} a_{\ell_0\ell_0} X^{\ell_0/p^{r_0}} Y^{(\ell_0+m_0-1)/p^{r_0}} Z^{\ell_0}$$

of the above sum is the unique such with the smallest positive X-power. Hence, it must belong to the ideal $\mathcal{I}$, contradicting the definition of $\mathcal{I}$. Thus, $\ell_0 \geq 1$, as needed. $\square$

Corollary 4.13. *There exists an $h \in \mathbb{N}_{>0}$ such that $[p](X) = X^{p^h} + f(X)$, where the X-order of $f \in k[\![X^{1/p^\infty}]\!]$ is strictly bigger than p^h.*

Proof. By Proposition 1.21, we have

$$[p](X) = \underbrace{G(X, G(X, G(X, \cdots)))}_{G \text{ applied p times}}.$$

Since $p = 0$ in k, this implies, by the above lemma, that

$$[p](X) = a_m X^m + f(X),$$

where $a_m \neq 0$, $m > 1$, and the X-order of $f(X) \in k[\![X^{1/p^\infty}]\!]$ is strictly bigger than m. We want to see that $m = p^h$ for some $h \in \mathbb{Z}_{>0}$. Indeed, as $[p]$ defines an automorphism of $k[\![X^{1/p^\infty}]\!]$, there is $g(X) \in k[\![X^{1/p^\infty}]\!]$ such that $g\big([p](X)\big) = X$. Write

$$g(X) = \sum_{i=0}^{\infty} a_{m(i)} X^{m(i)} \in k[\![X^{1/p^\infty}]\!]$$

with $a_{m(0)} \neq 0$, $m(i) \in \mathbb{N}[1/p]$, and $m(i) < m(i+1)$ for all $i \in \mathbb{N}$. Since $m(0)$ is the smallest among the indices $m(i) \in \mathbb{N}[1/p]$, and $a_{m(0)} \neq 0$, we have

$$\begin{aligned} g\big([p](X)\big) = \sum_{i=0}^{\infty} (a_m X^m + f(X))^{m(i)} &= a_{m(0)}\big(a_m X^m + f(X)\big)^{m(0)} \\ &\quad + \sum_{i=1}^{\infty} (a_m X^m + f(X))^{m(i)} \\ &= a_{m(0)} a_m X^{mm(0)} + h(X) \\ &= X, \end{aligned}$$

where the X-order of $h(X) \in k[\![X^{1/p^\infty}]\!]$ is strictly bigger than $mm(0)$. Thus, $mm(0) = 1$ in $\mathbb{N}[1/p]$, from which we get that $m = p^h$ for some $h \in \mathbb{Z}_{>0}$. $\square$

Lemma 4.14. *We have*

$$\iota(X) = -X + \sum_{\mathbb{N}[1/p]\ni i>1} a_i X^i \in k[\![X^{1/p^\infty}]\!].$$

Proof. Write

$$\iota(X) = \sum_{i=0}^{\infty} a_{m(i)} X^{m(i)}$$

with $a_{m(0)} \neq 0$, and $m(i) < m(i+1)$ for all $i \in \mathbb{N}$. Use the equality $\iota(\iota(X)) = X$ to obtain

$$\iota(\iota(X)) = a_{m(0)}^{m(0)+1} X^{m(0)^2} + f(X) = X,$$

where the X-order of $f(X)$ is strictly bigger than $m(0)^2$. We get $a_{m(0)}^{m(0)+1} = 1$, and $m(0)^2 = 1$. Now, writing

$$\iota(X) = a_1 X + \sum_{i=1}^{\infty} a_{m(i)} X^{m(i)}$$

with $m(i) > 1$ for all $i \geq 1$, and, using the equality $G(X, \iota(X)) = 0$ together with Lemma 4.12, we get

$$G(X, \iota(X)) = X + a_1 X + \sum_{i=1}^{\infty} a_{m(i)} X^{m(i)} + \sum_{i,j\geq 1} a_{ij} X^i \big(\iota(X)\big)^j = 0$$

with $a_{ij} \in k$ for all $i, j \in \mathbb{N}[1/p]$. Thus, $(1 + a_1)X$ is the unique term with the smallest X-order in the above expression, and we get $1 + a_1 = 0$, as required. $\square$

We therefore obtain:

Corollary 4.15. *The ideal (X) is a Hopf ideal of $R = k[\![X^{1/p^\infty}]\!]$. Put another way, the scheme*

$$\operatorname{Spec}\big(R/(X)\big) = \operatorname{Spec}\big(k[\![X^{1/p^\infty}]\!]/(X)\big)$$

defines an affine subgroup of $\operatorname{Spf}(R)$.

Proof. Let μ, ι, ε be respectively the comultiplication, coinverse, and augmentation of the topological Hopf algebra $R = k[\![X^{1/p^\infty}]\!]$. Let $\mathcal{I} = (X)$. By definition, we have $\varepsilon(\mathcal{I}) = 0$. Let

$$f(X) = \sum_{i\in\mathbb{N}[1/p]} a_i X^i \in \mathcal{I}$$

be arbitrary. Then, clearly, by Lemma 4.14, we get

$$\iota(f) = \sum_{i\in\mathbb{N}[1/p]} a_i \iota(X)^i = \sum_{i\in\mathbb{N}[1/p]} a_i \left(-X + \sum_{j\in\mathbb{N}[1/p]} a_j X^j \right)^i \in \mathcal{I}$$

as $i, j \geq 1$. Also, we see that $\mu(f)$ is mapped to 0 under the reduction map

$$k[\![X^{1/p^\infty}]\!]\hat{\otimes}k[\![Y^{1/p^\infty}]\!] \to k[\![X^{1/p^\infty}]\!]/(X)\hat{\otimes}k[\![Y^{1/p^\infty}]\!]/(Y)$$

since, by Lemma 4.12,

$$\begin{aligned}\mu(f) = \sum_{i\in\mathbb{N}[1/p]} a_i\mu(X)^i &= \sum_{i\in\mathbb{N}[1/p]} G(X,Y)^i \\ &= \sum_{i\in\mathbb{N}[1/p]} a_i\left(X + Y + \sum_{j,k\in\mathbb{N}[1/p]} a_{jk}X^jY^k\right)\end{aligned}$$

as $i, j, k \geq 1$. $\square$

Put

$$H := \operatorname{Spec}\big(R/(X)\big) = \operatorname{Spec}\big(k[\![X^{1/p^\infty}]\!]/(X)\big),$$

and

$$H/p^nH := \operatorname{coker}(H \xrightarrow{p^n} H).$$

Then:

Lemma 4.16. *Let A_n be the representing algebra of H/p^nH. We have*

$$A_n = k[\![X^{1/p^{nh}}]\!]/(X).$$

In particular, H/p^nH is a finite affine group scheme over k of order p^{nh} for each $n \in \mathbb{N}$. Here, h is the positive integer whose existence was shown in Corollary 4.13 above.

Proof. Using Lemma 4.12, write

$$G(X,Y) = X + Y + XYg(X,Y)$$

with $g(X,Y) \in k[\![X^{1/p^\infty}, Y^{1/p^\infty}]\!]$. Fix $n \in \mathbb{N}$. To show that $k[\![X^{1/p^{nh}}]\!]/(X) \subseteq A_n$, it is enough to see that $X^{1/p^{nh}} + (X) \in A_n$, i.e. that

$$G\big(X, [p^n](Y)\big)^{1/p^{nh}} \equiv X^{1/p^{nh}} \bmod (X,Y).$$

But, by Corollary 4.13, we have $[p^n](Y)^{1/p^{nh}} \equiv 0 \bmod (Y)$, which implies

$$\begin{aligned}G\big(X,[p^n](Y)\big)^{1/p^{nh}} &\equiv X^{1/p^{nh}} + [p^n](Y)^{1/p^{nh}} + X^{1/p^{nh}}[p^n](Y)^{1/p^{nh}}g\big(X,[p^n](Y)\big)^{1/p^{nh}} \\ &\equiv X^{1/p^{nh}} \bmod (Y).\end{aligned}$$

For the converse, let $f \in k[\![X^{1/p^\infty}]\!]$ such that $f\big(G(X,[p^n](Y))\big) \equiv f(X) \bmod (X,Y)$. We may assume $f(0) = 0$. Write

$$f(X) = \sum_{\ell\in\mathbb{N}[1/p]} a_\ell X^\ell.$$

We need to show that $f(X) \in k[\![X^{1/p^{nh}}]\!]$ modulo (X). To this end, let

$$r_0 := \max\{-v_p(\ell) \mid \ell < 1 \text{ and } a_\ell \neq 0\},$$

and

$$\ell_0 := \min\{\ell \mid \ell < 1,\ a_\ell \neq 0 \text{ and } -v_p(\ell) = r_0\},$$

where v_p denotes the p-adic valuation. Put $m_0 := \ell_0 p^{r_0}$, so that $m_0 \in \mathbb{Z}_{>0}$ with $p \nmid m_0$. Observe that we are done if we show that $r_0 \leq nh$. So suppose, for a contradiction, that $r_0 > nh$. We use Lemma 4.12 and Corollary 4.13 to write

$$G\big(X, [p^n](Y)\big) = X + Y^{p^{nh}} \cdot u$$

with $u \in k[\![X^{1/p^\infty}, Y^{1/p^\infty}]\!]$, so that u is of the form $1+F$ for some $F \in k[\![X^{1/p^\infty}, Y^{1/p^\infty}]\!]$ with $F(0,0) = 0$. For $\ell \in \mathbb{N}[1/p]$ with $\ell < 1$, and $a_\ell \neq 0$, write $\ell = m/p^r$ with $p \nmid m$, and consider the ideal

$$\mathcal{I} := \left\{ \sum_{\substack{i > (m_0-1)/p^{r_0} \\ \text{or} \\ j > p^{nh-r_0}}} a_{ij} X^i Y^j \right\} \subseteq k[\![X^{1/p^\infty}, Y^{1/p^\infty}]\!].$$

From the expansion

$$\begin{aligned} G\big(X, [p^n](Y)\big)^\ell &= (X + Y^{p^{nh}} u)^\ell = (X^{1/p^r} + Y^{p^{nh-r}} u^{1/p^r})^m \\ &= X^\ell + \sum_{j=1}^{m} \binom{m}{j} X^{(m-j)/p^r} Y^{jp^{nh-r}} u^{j/p^r}, \end{aligned}$$

it follows that

$$G\big(X, [p^n](Y)\big)^\ell \equiv \begin{cases} X^\ell \bmod \mathcal{I} & \text{if } \ell \neq \ell_0 \\ X^\ell + m_0 X^{(m_0-1)/p^{r_0}} Y^{p^{nh-r_0}} u^{1/p^{r_0}} \bmod \mathcal{I} & \text{otherwise} \end{cases}.$$

Indeed, assume $\ell \neq \ell_0$. If $r < r_0$, then

$$\begin{aligned} G\big(X, [p^n](Y)\big)^\ell = (X^{1/p^r} + Y^{p^{nh-r}} u^{1/p^r})^m &= X^\ell + \sum_{j=1}^{m} \binom{m}{j} X^{(m-j)/p^r} Y^{jp^{nh-r}} u^{j/p^r} \\ &\equiv X^\ell \bmod \mathcal{I} \end{aligned}$$

as $Y^{jp^{nh-r}} \in \mathcal{I}$ for all $j \geq 1$ using $r < r_0$. If $r = r_0$, and $m > m_0$, then we get

$$G\big(X, [p^n](Y)\big)^\ell \equiv X^\ell \bmod \mathcal{I},$$

using $m > m_0$ for the summand at $j = 1$. Therefore,

$$\begin{aligned} f\big(G(X,[p^n](Y))\big) &= \sum_{\ell\in\mathbb{N}[1/p]} a_\ell G\big(X,[p^n](Y)\big)^\ell \\ &\equiv \sum_{\ell<1} a_\ell G\big(X,[p^n](Y)\big)^\ell \bmod (X,Y) \\ &\equiv \sum_{1>\ell\neq\ell_0} a_\ell G\big(X,[p^n](Y)\big)^\ell + a_{\ell_0} G\big(X,[p^n](Y)\big)^{\ell_0} \bmod (X,Y) \\ &\equiv \sum_{1>\ell\neq\ell_0} a_\ell X^\ell \\ &+ a_{\ell_0}(X^{\ell_0} + m_0 X^{(m_0-1)/p^{r_0}} Y^{p^{nh-r_0}} u^{1/p^{r_0}}) \bmod (X,Y) + \mathcal{I} \\ &\equiv f(X) + a_{\ell_0} m_0 X^{(m_0-1)/p^{r_0}} Y^{p^{nh-r_0}} u^{1/p^{r_0}} \bmod (X,Y) + \mathcal{I} \\ &\equiv f(X) + a_{\ell_0} m_0 X^{(m_0-1)/p^{r_0}} Y^{p^{nh-r_0}} \bmod (X,Y) + \mathcal{I}, \end{aligned}$$

where the last equivalence follows from the definition of u. However, we have $X \in \mathcal{I}$ because $(m_0 - 1)/p^{r_0} < \ell_0 < 1$, and the assumption $r_0 > nh$ implies that also $Y \in \mathcal{I}$. Hence, $(X, Y) \in \mathcal{I}$. Moreover, by assumption, we have $f\big(G(X,[p^n](Y))\big) - f(X) \in \mathcal{I}$, which implies that

$$a_{\ell_0} m_0 X^{(m_0-1)/p^{r_0}} Y^{p^{nh-r_0}} \in \mathcal{I},$$

in contradiction to the definition of $\mathcal{I}$ as $a_{\ell_0} m_0 \in k^\times$. Thus, we conclude that $r_0 \leq nh$, and the result follows. □

Corollary 4.17. *The affine group scheme H is a Tate k-group.*

Proof. By the above lemma, it remains to see that H is p-adically separated and complete, and that $H[p] = 0$. Again, by the above lemma, to prove the isomorphism $H \simeq \varprojlim H/p^n H$ amounts to proving that the canonical injection

$$\varinjlim_n k[\![X^{1/p^{nh}}]\!]/(X) \hookrightarrow k[\![X^{1/p^\infty}]\!]/(X)$$

is an isomorphism. But since $h \geq 1$, the map is also surjective. To show that $H[p] = 0$, notice that, by definition of $\mathcal{F}$, the map on R induced by multiplication by p is bijective. Hence, the induced map on $R/(X)$ is surjective, which implies $H[p] = 0$. □

The following corollary finally gives an answer to Question (2) when $d = 1$:

Corollary 4.18. *Any connected formal $\mathbb{Q}_p$-vector space $\mathcal{F} = \operatorname{Spf}\big(k[\![X^{1/p^\infty}]\!]\big)$ of dimension $d = 1$ comes from a (connected) p-divisible group over k, in the sense that there is a connected p-divisible group $\mathcal{G}$ over k whose universal formal cover is isomorphic to $\mathcal{F}$.*

Proof. By Corollary 2.11, we know that there is a p-divisible group $\mathcal{G}$ such that $\underline{T}(\mathcal{G}) \simeq H$. Since $G[p^n] = \underline{T}(G)/p^n\underline{T}(G)$ is represented by $k[\![X^{1/p^{nh}}]\!]/(X)$ (cf. Lemma 4.16), a local ring, we see that G is connected. Moreover, by the proof of Corollary 3.5, we have $\varinjlim_p H \simeq \varinjlim_p \underline{T}(\mathcal{G}) \simeq \mathcal{F}$. But then, by Proposition 3.1.(ii), we have $\varinjlim_p \underline{T}(\mathcal{G}) = \underline{T}(\mathcal{G})[1/p] \simeq \widetilde{\mathcal{G}}$, and thus, $\widetilde{\mathcal{G}} \simeq \mathcal{F}$. $\square$

4.3.2 Case of several variables

As an explicit example of the construction given in Proposition 1.25, we begin by presenting a perfect formal group law

$$E = (E_1, E_2) \in k[\![U_1^{1/p^\infty}, X_1^{1/p^\infty}, V_1^{1/p^\infty}, Y_1^{1/p^\infty}]\!]^2$$

of dimension two represented by $R := k[\![U_1^{1/p^\infty}, X_1^{1/p^\infty}]\!]$ that is isomorphic to a (classical) formal group law such that the ideal (U_1, X_1) is not a Hopf ideal.

Indeed, consider the multiplicative group $\mathbb{G}_m = X_1 + Y_1 + X_1Y_1 \in k[\![X_1, Y_1]\!]$ and the additive group $\mathbb{G}_a = U_1 + V_1 \in k[\![U_1, V_1]\!]$. Let $f(X_1) := X_1^{1/p} \in k[\![X_1^{1/p^\infty}]\!]$, so that, with the notation of Proposition 1.25, we have

$$\Delta f(X_1, Y_1) = -X_1^{1/p}Y_1^{1/p} \in k[\![X_1^{1/p^\infty}, Y_1^{1/p^\infty}]\!].$$

Define

$$\begin{aligned} E(U_1, X_1, V_1, Y_1) : &= \big(E_1(U_1, X_1, V_1, Y_1), E_2(U_1, X_1, V_1, Y_1)\big) \\ &= (U_1 +_{\mathbb{G}_a} V_1 +_{\mathbb{G}_a} \Delta f(X_1, Y_1), X_1 +_{\mathbb{G}_m} Y_1) \\ &= (U_1 + V_1 - X_1^{1/p}Y_1^{1/p}, X_1 + Y_1 + X_1Y_1). \end{aligned}$$

Then, E is a perfect formal group law over k of dimension two, and the map

$$i(U_1, X_1) = (U_1 + X_1^{1/p}, X_1) : \mathbb{G}_a \times \mathbb{G}_m \to E$$

is an isomorphism of perfect formal group laws with inverse

$$i^{-1}(U_1, X_1) = (U_1 - X_1^{1/p}, X_1).$$

Observe, however, that the ideal (U_1, X_1) is not a Hopf ideal for $k[\![U_1^{1/p^\infty}, X_1^{1/p^\infty}]\!]$. Indeed, letting μ be the comultiplication of the perfect formal group law E, we see that

$$\begin{aligned} \mu(U_1 + X_1) &= E_1(U_1, X_1, V_1, Y_1) + E_2(U_1, X_1, V_1, Y_1) \\ &= U_1 + V_1 + X_1^{1/p}Y_1^{1/p} + X_1 + Y_1 + X_1Y_1 \\ &\notin (U_1, X_1, V_1, Y_1). \end{aligned}$$

Therefore, we have constructed the desired perfect formal group law.

Notice that, taking up the notation from Proposition 1.25, we have

$$U +_G V -_G \Delta f(X,Y) \equiv \Delta f(X,Y) \bmod (U,V)$$

by the axioms of the (classical) formal group law G. Therefore, as a general strategy, by choosing $f(X) \in k[\![X^{1/p^\infty}]\!]^n$ suitably such that $\Delta f(X,Y) \notin (X,Y)$, we can construct perfect formal group laws over k for which the ideal generated by the variables is not a Hopf ideal.

This can also be achieved for perfect formal group laws associated with formal $\mathbb{Q}_p$-vector spaces as the following result shows:

Corollary 4.19. *The result in Corollary 4.15 fails in the higher dimensional case. In other words, there are connected formal $\mathbb{Q}_p$-vector spaces represented by $k[\![X^{1/p^\infty}]\!]$ with $d \geq 2$ such that the ideal $(X) = (X_1, \ldots, X_d)$ generated by the variables is not a Hopf ideal of $k[\![X^{1/p^\infty}]\!]$.*

Proof. We directly give an example for $d = 2$ and $p \geq 3$. Let $F \in k[\![X,Y]\!]$ and $G \in k[\![U,V]\!]$ be the p-divisible formal group laws over k corresponding to the (connected) p-divisible group μ_∞ that is represented by $k[\![X]\!]$ and $k[\![U]\!]$, respectively (cf. [16, Proposition 1]). Then, $G \times F \in k[\![U,X,V,Y]\!]^2$ is a p-divisible formal group law over k. Choosing $f(X) = X^{2/p}$, we obtain

$$\begin{aligned}
\Delta f(X,Y) &= X^{2/p} +_G Y^{2/p} -_G f(X+Y+XY) \\
&= X^{2/p} + Y^{2/p} + (XY)^{2/p} -_G (X+Y+XY)^{2/p} \\
&= X^{2/p} + Y^{2/p} + (XY)^{2/p} +_G \sum_{m=1}^{\infty} \big(-(X+Y+XY)^{2/p}\big)^m \\
&= X^{2/p} + Y^{2/p} + (XY)^{2/p} + \sum_{m=1}^{\infty} \big(-(X+Y+XY)^{2/p}\big)^m \\
&\quad + \sum_{m=1}^{\infty} \big(X^{2/p} + Y^{2/p} + (XY)^{2/p}\big)\big(-(X+Y+XY)^{2/p}\big)^m \\
&= -2(XY)^{1/p} - 2(XY)^{1/p}(X^{1/p} + Y^{1/p}) + \sum_{m=2}^{\infty} \big(-(X+Y+XY)^{2/p}\big)^m \\
&\quad + \sum_{m=1}^{\infty} \big(X^{2/p} + Y^{2/p} + (XY)^{2/p}\big)\big(-(X+Y+XY)^{2/p}\big)^m \\
&\notin (X,Y)
\end{aligned}$$

as $-2(XY)^{1/p}$ survives the sum, but is not an element of the ideal (X,Y). According to Proposition 1.25, the formal group $G \times F$ is then isomorphic to the perfect formal group law

$$E = (E_1, E_2) = (U +_G V -_G \Delta f(X,Y), X +_F Y),$$

for which the ideal (U, X) is not a Hopf ideal. As $G \times F$ is a p-divisible formal group law, we get that E is a connected formal $\mathbb{Q}_p$-vector space, and hence, the result follows.

$\square$

Nonetheless, we show that any connected formal $\mathbb{Q}_p$-vector space $\mathcal{F}$ of dimension $d \geq 1$ at least comes from a generalized p-divisible group. The idea is to construct finitely generated open Hopf ideals different from (X). Unless in the case $d = 1$, however, we are not able to establish the finiteness property analogous to Lemma 4.16.

We have the same setup as in the last subsection. Consider the connected formal $\mathbb{Q}_p$-vector space $\mathcal{F}$ of dimension $d \geq 1$. Let the comultiplication $\Delta : R \to R\hat{\otimes}_k R$ on R be given by the perfect formal group law

$$G = (G_1, \ldots, G_d) \in k[\![X^{1/p^\infty}, Y^{1/p^\infty}]\!]^d,$$

and denote the coinverse by $\iota = (\iota_1, \ldots, \iota_d)$. Note that then

$$\Delta(X_i) = G_i(X, Y) \;\text{ and }\; \iota(X_i) = \iota_i(X, Y)$$

for any $i = 1, \ldots, d$. Moreover, by a common abuse of notation, we define $\Delta(X)^\alpha = G(X, Y)^\alpha$ for any $\alpha \in \mathbb{N}[1/p]^d$ by the formula

$$\Delta(X)^\alpha := \Delta(X_1)^{\alpha_1} \cdots \Delta(X_d)^{\alpha_d} = G_1(X, Y)^{\alpha_1} \ldots G_d(X, Y)^{\alpha_d} = G(X, Y)^\alpha.$$

Write, for any $i = 1, \ldots, d$,

$$G_i(X, Y) = \sum_{\alpha, \beta \in \mathbb{N}[1/p]^d} c^{(i)}_{\alpha\beta} X^\alpha Y^\beta = \sum_{\beta \in \mathbb{N}[1/p]^d} f^{(i)}_\beta(X) Y^\beta,$$

where

$$f^{(i)}_\beta(X) = \sum_{\alpha \in \mathbb{N}[1/p]^d} c^{(i)}_{\alpha\beta} X^\alpha$$

with $c^{(i)}_{\alpha\beta} \in k$ for all $\alpha, \beta \in \mathbb{N}[1/p]^d$. For any $\alpha \in \mathbb{N}[1/p]^d$, set $\max(\alpha) := \max\{\alpha_1, \ldots, \alpha_d\}$, and consider the ideal

$$I := \left(f^{(i)}_\beta(X)\right)_{\substack{i=1,\ldots,d \\ \max(\beta)<1}} \subseteq R.$$

Note that the ideal $I \subseteq R$ is not the unit ideal. Indeed, the generator $f^{(i)}_\beta(X)$ for any $i = 1, \ldots, d$ must lie in the maximal ideal $\mathfrak{m}_R$ of R for otherwise, if there was β_0 among the indices β with $\max(\beta) < 1$ such that $f^{(i)}_{\beta_0}(X) \in k^\times$, then the equality

$$G_i(0, Y) = \sum_\beta f^{(i)}_\beta(0) Y^\beta = f^{(i)}_{\beta_0}(0) Y^{\beta_0} + \sum_{\beta \neq \beta_0} f^{(i)}_\beta(0) Y^\beta = Y_i$$

would not hold.

For an easier readibility, in what follows, we will be dropping the index set $\mathbb{N}[1/p]^d$ of the sums from the notation.

Proposition 4.20. *(i) The ideal I is finitely generated and open, hence is an ideal of definition of R for its (X)-adic topology.*

(ii) The ideal I is a topological coideal; i.e., $\Delta(I) \subseteq I\hat{\otimes}_k R + R\hat{\otimes}_k I$.

Proof. To prove part (i), note first that, for all $i = 1, \dots, d$, we have $X_i = f_0^{(i)}(X) \in I$, so that $(X) \subseteq I$, and I is open in R. Now fix $i = 1, \dots, d$. To prove that I is finitely generated, we will see that, for all but finitely many β with $\max(\beta) < 1$, we have

$$f_\beta^{(i)}(X) \in (X) = (f_0^{(i)}(X))_{i=1,\dots,d}.$$

Indeed, if $\max(\alpha), \max(\beta) < 1$, then $|\alpha| + |\beta| < 2d$. But it follows, from Lemma 1.17, that the set

$$\left\{(\alpha,\beta) \in \mathbb{N}[1/p]^{2d} \mid |\alpha| + |\beta| < 2d \text{ and } c_{\alpha\beta}^{(i)} \neq 0 \text{ for some } i \in \mathbb{N}\right\}$$

is finite. This proves the first part of the proposition.

For part (ii), fix again $i = 1, \dots, d$, and write, for $\alpha \in \mathbb{N}[1/p]^d$,

$$G(Y,Z)^\alpha = \sum_{\gamma,\delta} c_{\gamma\delta}^{(\alpha)} Y^\gamma Z^\delta$$

with $c_{\gamma\delta}^{(\alpha)} \in k$, so that

$$G_i(X, G(Y,Z)) = \sum_\alpha f_\alpha^{(i)}(X) G(Y,Z)^\alpha = \sum_{\alpha,\gamma,\delta} f_\alpha^{(i)}(X) c_{\gamma\delta}^{(\alpha)} Y^\gamma Z^\delta,$$

where

$$f_\alpha^{(i)}(X) = \sum_\beta c_{\alpha\beta}^{(i)} X^\beta.$$

On the other hand, we have

$$G_i(G(X,Y),Z) = \sum_\beta f_\beta^{(i)}(G(X,Y)) Z^\beta = \sum_\beta \Delta(f_\beta^{(i)}(X)) Z^\beta.$$

Fixing $\beta \in \mathbb{N}[1/p]^d$ with $\max(\beta) < 1$, and comparing the coefficients of Z^β in

$$G_i(X, G(Y,Z)) = G_i(G(X,Y),Z)$$

above, we obtain an alternative expression for $\Delta(f_\beta^{(i)})$:

$$\Delta(f_\beta^{(i)}(X)) = \sum_{\alpha,\gamma} f_\alpha^{(i)}(X) c_{\gamma\beta}^{(\alpha)} Y^\gamma = \sum_{\substack{\alpha,\gamma \\ \max(\alpha)<1}} f_\alpha^{(i)}(X) c_{\gamma\beta}^{(\alpha)} Y^\gamma + \sum_{\substack{\alpha,\gamma \\ \max(\alpha)\geq 1}} f_\alpha^{(i)}(X) c_{\gamma\beta}^{(\alpha)} Y^\gamma,$$

i.e. it is enough to prove that the above sum is in $I\hat{\otimes}_k R + R\hat{\otimes}_k I$. Observe that, by definition, the sum with $\max(\alpha) < 1$ is an element of $I\hat{\otimes}_k R$. We claim that the sum with $\max(\alpha) \geq 1$ is an element of $R\hat{\otimes}_k I$. To prove this, we will show that

$$\sum_\gamma c^{(\alpha)}_{\gamma\beta} Y^\gamma \in I \subseteq k[\![Y^{1/p^\infty}]\!].$$

Indeed, as $\max(\alpha) \geq 1$, we first choose $1 \leq j \leq d$ with $\alpha_j \geq 1$, and write

$$G(Y,Z)^\alpha = G_j(Y,Z)G(Y,Z)^{\alpha-e_j} = \Big(\sum_\theta f^{(j)}_\theta(Y)Z^\theta\Big)\Big(\sum_{\gamma'} g_{\gamma'}(Y)Z^{\gamma'}\Big),$$

where $f^{(j)}_\theta(Y), g_{\gamma'}(Y) \in k[\![Y^{1/p^\infty}]\!]$. Observing that the sum $\sum_\gamma c^{(\alpha)}_{\gamma\beta}Y^\gamma$ is the coefficient of Z^β in $G(Y,Z)^\alpha$, we get

$$\sum_\gamma c^{(\alpha)}_{\gamma\beta}Y^\gamma = \sum_{\theta+\gamma'=\beta} f^{(j)}_\theta(Y) g_{\gamma'}(Y).$$

But then, $\max(\beta) < 1$, and $\theta + \gamma' = \beta$ imply $\max(\theta) < 1$, so that

$$f^{(j)}_\theta(Y) \in I = \big(f^{(j)}_\theta(Y)\big)_{\substack{j=1,\dots,d\\ \max(\theta)<1}},$$

as desired. □

Corollary 4.21. *The ideal $\mathcal{I} = I + \iota(I)$ is a finitely generated ideal of definition of R, and is a topological Hopf ideal.*

Proof. Note that

$$\iota(\mathcal{I}) = \iota(I + \iota(I)) = \iota(I) + \iota(\iota(I)) = \iota(I) + I = \mathcal{I}.$$

Also,

$$\Delta(\mathcal{I}) = \Delta(I + \iota(I)) = \Delta(I) + \Delta(\iota(I)),$$

where $\Delta(I) \in I\hat{\otimes}_k R + R\hat{\otimes}_k I$, and $\Delta(\iota(I)) = (\iota \otimes \iota)(\Delta(I)) \subseteq \iota(I)\hat{\otimes}_k R + R\hat{\otimes}_k \iota(I)$. Thus,

$$\Delta(I) + \Delta(\iota(I)) \subseteq \mathcal{I}\hat{\otimes}_k R + R\hat{\otimes}_k \mathcal{I},$$

as needed. □

Thus, the ideal $\mathcal{I}$ defines a subgroup scheme $H := \operatorname{Spec}(R/\mathcal{I})$ of $\mathcal{F}$. Note that, as multiplication by p on $\mathcal{F}$ is bijective, the induced map on $R/\mathcal{I}$ is surjective, so that $H[p] = 0$. The following proposition shows that H is also p-adically separated and complete. This gives that it is a Tate k-group, and thus corresponds to a generalized p-divisible group G over k by Proposition 2.10.

Proposition 4.22. *The affine group scheme H is a projective limit of finite connected group schemes of p-power order. In particular, $H \simeq \varprojlim_n H/p^n H$ is p-adically separated and complete.*

Proof. Let $A = R/\mathcal{I}$. As R is local, so is A. Since $(X) \subseteq \mathcal{I} \subsetneq (X^{1/p^\infty})$, the maximal ideal $\mathfrak{m}_A$ of A consists of nilpotent elements. By [18, 2nd Theorem in §3.3], A is a filtered union of finitely generated Hopf subalgebras A_i over k. Note that we can then write $A_i = k[a_1, \dots, a_n]$, where $a_j = \lambda_j + b_j$ with $\lambda_j \in k$, and $b_n \in \mathfrak{m}_A$, which implies $A_i = k[b_1, \dots, b_n]$, and hence, it is a finite dimensional algebra as b_i is nilpotent for all $i = 1, \dots, n$. Thus, A is local, and $H_i := \operatorname{Spec}(A_i)$ is a finite connected group scheme over k. Moreover, by [18, 1st Corollary in §14.4], H_i is of p-power order, and we have showed that A is a projective limit of finite connected group schemes of p-power order.

To prove that H is p-adically separated and complete, we need to see that $A = \varinjlim_n A_n$, where A_n is the representing algebra of $H/p^n H = \operatorname{coker}(H \xrightarrow{p^n} H)$. In other words, for any $a \in A$, we need to see that there is $n \in \mathbb{N}$ such that

$$1 \otimes a = \big([p^n] \otimes \mathrm{id}_A\big) \circ \Delta(a),$$

where $[p^n] : A \to A$ is the comorphism of multiplication by p on H, and Δ is the comultiplication of A (cf. §1.1). But, choosing i with $a \in A_i$, and n such that p^n is the order of $H_i = \operatorname{Spec}(A_i)$, we have that $[p^n]$ factors through the augmentation $\varepsilon : X \mapsto 0$ of A_i (see [10, Proposition 13.26]), so that the above equality holds by the axioms of Hopf algebras. $\square$

Since each variable X_i is topologically nilpotent in $R = k[\![X_1^{1/p^\infty}, \dots X_d^{1/p^\infty}]\!]$, it follows that, when $d = 1$ (in which case R is p-divisible, and hence, by Corollary 4.18, is isomorphic to a formal power series ring in one variable), the multiplication by p on R is given by $[p](X) = X^{p^n}$ for a suitable choice of X, for some $n \geq 1$ (cf. [20, Theorem 5.2]). Thus, we see that, in this case, $[p](X)$ tends to 0 as n tends to infinity.

Although we do not know if that condition holds in case $d > 1$, by imposing it as a natural assumption, we finally obtain:

Corollary 4.23. *Assume that, for any $f \in \mathfrak{m}_R$, we have*

$$\lim_{n \to \infty} [p^n](f) = 0.$$

Then, we have $\mathcal{F} \simeq H[1/p]$. Moreover, there is a generalized p-divisible group G over k with $\widetilde{G} \simeq \mathcal{F}$.

Proof. By assumption, we see that $\big([p^n](\mathcal{I})\big)_{n \in \mathbb{N}}$ is a basis of open neighborhoods of zero consisting of ideals. This yields

$$R \simeq \varprojlim_n R/[p^n](\mathcal{I}) \simeq \varprojlim_p R/\mathcal{I}.$$

In other words, $\mathcal{F} = \mathrm{Spf}(R) = \varinjlim_p H = H[1/p]$. It also follows, by Proposition 3.1.(ii), that $\mathcal{F}$ is isomorphic to the universal formal cover of the generalized p-divisible group G that corresponds to the generalized Tate k-group H under the equivalence by Proposition 2.10. $\square$

Remark 4.24. Assume that $\mathcal{F}$ is a connected formal $\mathbb{Q}_p$-vector space over k of dimension d. Then, Lemma 1.18 and Lemma 1.19 imply that, for any prodiscrete k-algebra A, we have $\mathcal{F}(A) = \big(\overset{\infty}{A^\flat}\big)^d$ as sets. Endowing this with the topology induced from $A^\flat$, it is natural to assume that this makes $\mathcal{F}$ a functor into the category of *topological* $\mathbb{Q}_p$-vector spaces over k. An analogous condition is imposed by Weinstein in [19, Definition 2.1.1]. It ensures that the continuity assumption and the conclusion of Corollary 4.23 are satisfied.

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