Invariant distributions on p-adic analytic groups

Jan Kohlhaase

Mathematisches Institut, Universität Münster, Einsteinstraße 62, D–48149 Münster, Germany; e-mail address: kohlhaaj@math.uni-muenster.de

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Abstract. Let p be a prime number, L a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, K a spherically complete extension field of L and G the group of L-rational points of a split reductive group over L. We derive several explicit descriptions of the center of the algebra $D(G,K)$ of locally analytic distributions on G with values in K. The main result is a generalization of an isomorphism of Harish-Chandra which connects the center of $D(G,K)$ with the algebra of Weyl-invariant, centrally supported distributions on a maximal torus of G. This isomorphism is supposed to play a role in the theory of locally analytic representations of G as studied by P. Schneider and J. Teitelbaum.

Introduction

Let p be a prime number, L a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, K a spherically complete extension field of L and G a locally L-analytic group of finite dimension with center Z and Lie algebra $\mathfrak{g}$.

The K-algebra $D(G,K)$ of locally analytic distributions on G plays a central role in the theory of locally analytic representations of G on locally convex K-vector spaces which was given a systematic treatment by P. Schneider and J. Teitelbaum (cf. [25] and [26]). Such representations appear in the cohomology of p-adic symmetric spaces (cf. [27]), as an important tool of M. Emerton's construction of the Eigencurve of Coleman-Mazur (cf. [13]) and, most recently, in C. Breuil's hypothetical p-adic Langlands program (cf. [5]).

This paper is devoted to the study of the center of the ring $D(G,K)$. Our approach relies on the observation that for locally analytic distributions on G there is a well-defined notion of support and that the support $supp(\delta)$ is a compact subset of G for any distribution $\delta \in D(G,K)$. It follows from the definition of the convolution product in $D(G,K)$ that any invariant distribution, i.e. any element of $D(G,K)^G$, is supported on a union of relatively compact conjugacy classes of G. If G is the group of L-rational points of a connected, reductive, linear algebraic group over L all of whose simple factors are L-isotropic (e.g. an L-split group) then the only such classes of G are the trivial ones, i.e. those belonging to the elements of Z (Sit's theorem). Therefore, we are led to the investigation of the K-algebra $D(G,K)_Z$ of centrally supported distributions on G.

If $\mathfrak{z}$ denotes the Lie algebra of Z then we let $U(\mathfrak{z},K)$ (resp. $U(\mathfrak{g},K)$) be the subalgebra of $D(Z,K)$ (resp. $D(G,K)$) consisting of distributions supported in the unit element. There is a natural continuous K-linear map

$$D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K) \longrightarrow D(G,K)_Z$$

of locally convex $D(Z,K)$-$U(\mathfrak{g},K)^{op}$-bimodules (here ι indicates the inductive tensor product topology). It is the main technical result of our work that under the assumption that K is discretely valued this map is a topological isomorphism (cf. Proposition 1.2.12). Its proof relies for one thing on certain compatibility conditions for global charts of small open subgroups of G and Z, respectively (cf. Proposition 1.3.5 and Corollary 1.3.6). On the other hand, we make extensive use of the fact that $D(G,K)$ is a K-Fréchet-Stein algebra (a notion introduced by P. Schneider and J. Teitelbaum) and a structure theorem of $D(G,K)$ as a module over $U(\mathfrak{g},K)$ after a certain completion process. The latter is due to H. Frommer who proved it for $\mathbb{Q}_p$ as a ground field. We generalize it to any finite extension $L|\mathbb{Q}_p$ (cf. Theorem 1.4.2).

G acts on $U(\mathfrak{g},K)$ and $D(G,K)_Z$. If G is an open subgroup of the group of L-rational points of a connected, algebraic group over L then we obtain a topological isomorphism

$$D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K)^G \longrightarrow D(G,K)_Z^G$$

of K-algebras (cf. Theorem 2.2.1). If moreover G satisfies the hypotheses of Sit's theorem then $D(G,K)^G = D(G,K)_Z^G$ and it remains to examine the *infinitesimal center* $U(\mathfrak{g},K)^G$.

Consider $\mathfrak{g}$ as an abelian locally L-analytic group and let $S(\mathfrak{g},K)$ be the subalgebra of $D(\mathfrak{g},K)$ consisting of distributions supported in $0 \in \mathfrak{g}$. $S(\mathfrak{g},K)$ and $U(\mathfrak{g},K)$ carry actions of G and $\mathfrak{g}$. We show that Duflo's famous isomorphism $S(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{g})^{\mathfrak{g}}$ extends to a topological isomorphism $S(\mathfrak{g},K)^{\mathfrak{g}} \to U(\mathfrak{g},K)^{\mathfrak{g}}$ of K-Fréchet algebras (cf. Proposition 2.1.5; $S(\mathfrak{g})$ and $U(\mathfrak{g})$ denote the symmetric and the universal enveloping algebra of $\mathfrak{g}$, respectively). If $\mathfrak{g}$ is split semisimple with split maximal toral subalgebra $\mathfrak{t}$ and corresponding Weyl group $\mathfrak{W}$ then $\mathfrak{W}$ naturally acts on the algebra $S(\mathfrak{t},K)$ of locally analytic distributions on $\mathfrak{t}$ supported in $0 \in \mathfrak{t}$. We show that the classical isomorphism $S(\mathfrak{g})^{\mathfrak{g}} \to S(\mathfrak{t})^{\mathfrak{W}}$ extends to a topological isomorphism $S(\mathfrak{g},K)^{\mathfrak{g}} \simeq S(\mathfrak{t},K)^{\mathfrak{W}}$ of K-algebras (cf. Theorem 2.1.6). It follows that

$$U(\mathfrak{g},K)^{\mathfrak{g}} \simeq S(\mathfrak{t},K)^{\mathfrak{W}}.$$

Even more is true: Just as $S(\mathfrak{t})^{\mathfrak{W}}$ is a polynomial ring in $n := \dim_L(\mathfrak{t})$ variables, $S(\mathfrak{t},K)^{\mathfrak{W}}$ is the algebra of holomorphic functions on the rigid analytic affine space $(\mathbb{A}_K^n)^{an}$ of dimension n over K (loc.cit.).

If G is the group of L-rational points of a connected, split reductive L-group $\mathbb{G}$ then the above results enable us to give two different, explicit descriptions of $D(G,K)^G$. Using results on the Fourier transform of Z obtained by M. Emerton, P. Schneider and J. Teitelbaum we deduce the existence of an explicitly computable quasi-Stein rigid analytic K-variety X_K and a continuous, injective homomorphism of K-algebras

$$D(G,K)^G \longrightarrow \mathcal{O}(X_K)$$

with dense image (cf. Corollary 2.3.3 and Remark 2.3.4). If $\mathbb{T}$ is a maximal L-split torus of $\mathbb{G}$, $T := \mathbb{T}(L)$ and $W := N_G(T)/T$ the corresponding Weyl group then we also construct a topological isomorphism

$$D(G,K)^G \simeq D(T,K)^W_Z$$

of separately continuous K-algebras extending Harish-Chandra's isomorphism $U(\mathfrak{g})^{\mathfrak{g}} \simeq S(\mathfrak{t})^W$ (cf. Theorem 2.4.2). Since the latter plays a fundamental role in the representation theory of the Lie algebra $\mathfrak{g}$ our extension is hoped to be of importance for the theory of locally analytic representations of the group G. We point out that in the theory of smooth representations – subsumed by the locally analytic theory – such an isomorphism does not exist.

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Conventions and notation. Throughout this paper p denotes a prime number and L a finite extension of $\mathbb{Q}_p$. Let $\mathfrak{o}_L$ be the ring of integers of L with maximal ideal $\mathfrak{m}_L$ and uniformizer π_L. We assume the valuation ω on L to be normalized such that $\omega(\pi_L) = 1$. Let further $e := \omega(p)$ be the ramification index of the extension $L|\mathbb{Q}_p$ and m its degree. The absolute value $|\cdot|$ of L corresponding to ω is assumed to be normalized through $|p| = p^{-1}$. We let K be a fixed spherically complete extension of L which for many results will have to be assumed to be discretely valued (cf. subsection 1.4, in particular). Let $\mathfrak{o}_K$ denote its ring of integers. We assume the absolute value $|\cdot|$ on K to extend the one on L. If V is a locally convex vector space over K then we let $V' := \mathrm{Hom}_K^{\mathrm{cont}}(V,K)$ denote the space of continuous functionals on V. We write V'_b for the locally convex K-vector space V' endowed with the topology of strong convergence. G will always be a locally L-analytic group of finite dimension d with center Z. The Lie algebra of Z will be denoted by $\mathfrak{z}$. We also fix an exponential map $exp: \mathfrak{g} \dashrightarrow G$ defined locally around zero on the Lie algebra $\mathfrak{g}$ of G.

1 Locally analytic distributions

1.1 Functoriality

Recall that a topological Hausdorff space M is called (strictly) paracompact if any open covering of M admits a locally finite refinement by (pairwise disjoint) open subsets. Let M be a paracompact, locally L-analytic manifold of finite dimension d. We note that in this situation M is automatically strictly paracompact (cf. [23], p. 35). The locally convex K-vector space $C^{an}(M,K)$ of locally analytic functions on M with values in K is the locally convex inductive limit

$$C^{an}(M,K) = \varinjlim_I \mathcal{F}_I(K)$$

where I runs through the inductive system of all "indices". An index I is a family of charts $\{(D_i,\varphi_i)\}_{i\in I}$ of M such that $(D_i)_{i\in I}$ is a covering of M by disjoint open subsets and such that each $\varphi_i(D_i)$ is an affinoid ball in L^d. Further,

$$\mathcal{F}_I(K) := \prod_{i \in I} \mathcal{F}_{\varphi_i}(K)$$

is the locally convex direct product of the K-Banach spaces $\mathcal{F}_{\varphi_i}(K)$ of functions $f : D_i \to K$ such that $f \circ \varphi_i^{-1}$ is a K-valued rigid analytic function on the affinoid ball $\varphi_i(D_i)$. The space of locally analytic distributions on M is the locally convex K-vector space

$$D(M, K) := C^{an}(M, K)'_b.$$

If $(M_i)_{i \in I}$ is a covering of M by disjoint open subsets M_i then there is a topological isomorphism

$$C^{an}(M, K) \simeq \prod_{i \in I} C^{an}(M_i, K) \tag{1.1}$$

dualizing to a topological isomorphism

$$D(M, K) \simeq \bigoplus_{i \in I} D(M_i, K) \tag{1.2}$$

(cf. [14], Korollar 2.2.4). If M is compact, then $C^{an}(M, K)$ is a K-vector space of compact type and, in particular, is reflexive (cf. [25], Lemma 2.1 and [22], Proposition 16.10). In this case $D(M, K)$ is a nuclear Fréchet space (cf. [25] Theorem 1.3).

There is an embedding $M \hookrightarrow D(M, K)$, sending $m \in M$ to the Dirac distribution $\delta_m := (f \mapsto f(m))$.

Lemma 1.1.1. *The subspace $K[M]$ of $D(M, K)$ generated by all Dirac distributions δ_m, $m \in M$, is dense.*

Choosing a covering $(M_i)_{i \in I}$ of M by disjoint compact open subsets, (1.1) shows that $C^{an}(M, K)$ is reflexive (cf. [22], Proposition 9.10 and Proposition 9.11). Hence the proof of Lemma 1.1.1 can be done as in [25], Lemma 3.1.

Let N, M be paracompact, locally L-analytic manifolds of finite dimension and $\varphi : N \to M$ be a morphism. φ defines a K-linear map $\varphi^* : C^{an}(M, K) \to C^{an}(N, K)$ via $\varphi^*(f) := f \circ \varphi$ for $f \in C^{an}(M, K)$. Using the definition of $C^{an}(M, K)$ and $C^{an}(N, K)$ via indices one can show that φ^* is continuous with respect to the locally convex topologies defined above (cf. [23], p. 65 or [14], Bemerkung 2.1.11). Thus, φ^* dualizes to a continuous K-linear map $\varphi_* : D(N, K) \to D(M, K)$.

Proposition 1.1.2. *Let $\varphi : N \to M$ be a closed embedding of paracompact, locally L-analytic manifolds of finite dimension. Then $\varphi^* : C^{an}(M, K) \to C^{an}(N, K)$ is a strict surjection and $\varphi_* : D(N, K) \to D(M, K)$ is a topological embedding.*

Proof: Let $f \in C^{an}(N, K)$ and $a \in N$. There is an open neighborhood U_a of a in N, an open neighborhood V_a of $\varphi(a)$ in M and a locally analytic manifold Z_a with the following properties: φ restricts to a morphism

$\varphi_a : U_a \to V_a$ and there is an isomorphism $g : V_a \to U_a \times Z_a$ such that $pr_{U_a} \circ g \circ \varphi_a = id_{U_a}$ (cf. [7], 5.7.1; here pr_{U_a} is the projection onto U_a). It follows that $f|U_a = \varphi_a^*((pr_{U_a} \circ g)^*(f|U_a)) \in im(\varphi_a^*)$.

Let C be a closed and open subset of M with $\varphi(N) \subseteq C \subseteq \cup_{a \in N} V_a$ (cf. [23], p. 37). Choose a refinement $(V_i)_{i \in I}$ of the open covering $(C \cap V_a)_{a \in N}$ of C consisting of disjoint open subsets V_i of C. For each $i \in I$ choose a point $a \in N$ such that $V_i \subseteq V_a$. There is a function $g_a \in C^{an}(V_a, K)$ such that $\varphi_a^*(g_a) = f|U_a$. Set $g_i := g_a|V_i \in C^{an}(V_i, K)$ and $g_{M \setminus C} := 0 \in C^{an}(M \setminus C, K)$. Then the family $g := (g_{M \setminus C}, (g_i)_{i \in I}) \in C^{an}(M, K)$ satisfies $\varphi^*(g) = f$, proving the surjectivity of φ^*.

If $(M_i)_{i \in I}$ is a covering of M by disjoint compact open subsets, $N_i := \varphi^{-1}(M_i)$ and $\varphi_i := \varphi|N_i$ for $i \in I$ then φ^* is open if and only if all φ_i^* are. Hence we may assume M and N to be compact.
In this case both $C^{an}(M, K)$ and $C^{an}(N, K)$ are locally convex K-vector spaces of compact type. In particular, they carry the locally convex final topology with respect to a countable family of BH-spaces. Therefore, the claim follows from [22], Proposition 8.8, and the surjectivity of φ^*.

If $(M_i)_{i \in I}$ and $(N_i)_{i \in I}$ are as above then φ_* is the direct sum of the maps $(\varphi_i)_* : D(N_i, K) \to D(M_i, K)$. Since φ_i^* is strict surjective and $(\varphi_i)_*$ is the corresponding dual map, $(\varphi_i)_*$ is a topological embedding according to [25], Proposition 1.2 (i). The same is then true for φ_* by [22], Lemma 5.3 (i). □

In the situation of Proposition 1.1.2 we will from now on write $D(N, K) \subseteq D(M, K)$ for the topological embedding $\varphi_* : D(N, K) \to D(M, K)$ of locally convex K-vector spaces.
If we assume $M = G$ to be a finite dimensional, locally L-analytic group then $D(G, K)$ carries the structure of a unital, associative K-algebra with separately continuous multiplication such that the natural inclusion $K[G] \hookrightarrow D(G, K)$ becomes a homomorphism of rings (cf. [25], section 2). It is explicitly given by

$$(1.3) \qquad (\delta \cdot \delta')(f) = \delta'(g' \mapsto \delta(g \mapsto f(gg')))$$

with $\delta, \delta' \in D(G, K)$ and $f \in C^{an}(G, K)$. If G_0 is an open subgroup of G then according to (1.2)

$$D(G, K) \simeq \bigoplus_{g \in G/G_0} D(g \cdot G_0, K) \simeq \bigoplus_{g \in G/G_0} \delta_g \cdot D(G_0, K).$$

If H is a closed locally L-analytic subgroup of G then the topological embedding $D(H, K) \subseteq D(G, K)$ is a homomorphism of algebras.

1.2 The notion of support

Definition 1.2.1. The support $supp(\delta)$ of a distribution $\delta \in D(M, K)$ is the complement of the largest open subset U of M such that $\delta(f) = 0$ for all $f \in C^{an}(M, K)$ with $supp(f) \subseteq U$. If C is a subset of M and $V \subseteq D(M, K)$ a subspace then we denote by V_C the subspace of all distributions $\delta \in V$ whose support is contained in C. Similarly, if W is a subspace of $C^{an}(M, K)$ then W_C denotes the subspace of all locally analytic functions $f \in W$ with $supp(f) \subseteq C$.

Remark 1.2.2. The existence of $supp(\delta)$ for $\delta \in D(M,K)$ follows from the strict paracompactness of M: Let U_1, U_2 be open subsets of M such that $\delta(f) = 0$ for all $f \in C^{an}(M,K)$ with $supp(f) \subseteq U_1$ or $supp(f) \subseteq U_2$, and let $f \in C^{an}(M,K)$ be supported on $U_1 \cup U_2$. There is a closed and open subset A of M with $supp(f) \subseteq A \subseteq U_1 \cup U_2$ (cf. [23], p. 37). Choose a refinement $(V_i)_{i\in I}$ of the covering $(U_1 \cap A, U_2 \cap A)$ of A consisting of disjoint open subsets V_i of A. Then $f|A \in C^{an}(A,K) = \prod_{i\in I} C^{an}(V_i,K)$, i.e. $f|A = (f_i)_{i\in I}$ with $f_i \in C^{an}(V_i,K)$ for all $i \in I$. Set $f^j := (f_i^j)_{i\in I}$, $j = 1,2$, with $f_i^1 := 0$ if $V_i \not\subseteq U_1 \cap A$ (i.e. $V_i \cap U_1 = \emptyset$), $f_i^1 := f_i$ if $V_i \subseteq U_1 \cap A$, $f_i^2 := 0$ if $V_i \subseteq U_1 \cap A$ and $f_i^2 := f_i$ if $V_i \not\subseteq U_1 \cap A$. Then $f^1, f^2 \in C^{an}(A,K)$ with $f^1 + f^2 = f|A$. Extending f^1, f^2 by zero outside of A we obtain functions $f^1, f^2 \in C^{an}(M,K)$ with $f^1 + f^2 = f$ and $supp(f^j) \subseteq U_j$, $j = 1,2$. By assumption $\delta(f) = \delta(f^1) + \delta(f^2) = 0$.

Remark 1.2.3. It follows from (1.2) that all locally analytic distributions on M are compactly supported, i.e. $supp(\delta)$ is a compact subset of M for all $\delta \in D(M,K)$.

If $M = G$ is a locally L-analytic group, $g \in G$ and $\delta \in D(G,K)$ then according to (1.3)

$$supp(\delta_g \cdot \delta) = g \cdot supp(\delta) \text{ and } supp(\delta \cdot \delta_g) = supp(\delta) \cdot g. \tag{1.4}$$

More generally we still have:

Lemma 1.2.4. *If $\delta_1, \delta_2 \in D(G,K)$ then $supp(\delta_1 \cdot \delta_2) \subseteq supp(\delta_1) \cdot supp(\delta_2)$.*

Proof: Let $g \in supp(\delta_1 \cdot \delta_2)$. Then for any open subgroup $H \subseteq G$ there is a function $f \in C^{an}(G,K)$ supported on gH with $(\delta_1\delta_2)(f) = \delta_2(h \mapsto \delta_1(R_h f)) \neq 0$ (here R_h is the right translation operator associated with h). Hence there are elements $\gamma_2 \in supp(\delta_2)$ and $h \in H$ such that $supp(\delta_1) \cap (supp(f) \cdot h^{-1} \cdot \gamma_2^{-1}) \neq \emptyset$. Since $supp(f) \subseteq gH$ there is $h' \in H$ and $\gamma_1 \in supp(\delta_1)$ such that $\gamma_1 = gh'h^{-1}\gamma_2^{-1}$, i.e. $g = \gamma_1\gamma_2 h(h')^{-1}$. It follows that $g \in supp(\delta_1) \cdot supp(\delta_2)$ because H is arbitrary and $supp(\delta_1) \cdot supp(\delta_2)$ is closed (even compact). $\square$

For a closed subset C of G the locally convex K-vector space $C^\omega_C(G,K)$ of generalized germs in C is the quotient space

$$C^\omega_C(G,K) := C^{an}(G,K)/C^{an}(G,K)_{G\setminus C} \tag{1.5}$$

(cf. [14], Definition 2.3.3). If C is compact then there is a topological isomorphism

$$C^\omega_C(G,K) = \varinjlim_U C^{an}(U,K)$$

with U running through the inductive system of open subsets of G containing C and transition maps defined by restriction of functions. In this case the inductive limit topology on $C^\omega_C(G,K)$ is Hausdorff. If $C = \{g\}$ is a singleton we write $C^\omega_g(G,K)$ instead of $C^\omega_{\{g\}}(G,K)$.

Lemma 1.2.5. *$C^{an}(G,K)_C$ is a closed subspace of $C^{an}(G,K)$ for any subset C of G. If C is closed then $D(G,K)_C$ is a closed subspace of $D(G,K)$ and there is a topological isomorphism*

$$D(G,K)_C \simeq C^\omega_C(G,K)'_b. \tag{1.6}$$

If C is compact then this is an isomorphism of nuclear K-Fréchet spaces.

Proof: Let C be a subset of G. As mentioned in [loc.cit.], section 2.3.1, $C^{an}(G,K)_C$ is equal to the intersection of the kernels of all continuous surjections $C^{an}(G,K) \twoheadrightarrow C^{\omega}_g(G,K)$, $g \in G \setminus C$, hence is closed in $C^{an}(G,K)$. If C is closed in G then $D(G,K)_C$ is the orthogonal space of $C^{an}(G,K)_{G\setminus C}$ with respect to the natural pairing

$$D(G,K) \times C^{an}(G,K) \to K$$

so that $D(G,K)_C$ is closed, as well. Further, the reflexivity of $D(G,K)$ implies by means of [6], IV.2.2 Corollary, that

$$(D(G,K)_C)'_b \simeq D(G,K)'_b / D(G,K)^{\circ}_C$$

where $D(G,K)^{\circ}_C$ denotes the orthogonal subspace of $D(G,K)_C$ with respect to the pairing $D(G,K)'_b \times D(G,K) \to K$. Since $C^{an}(G,K)$ is reflexive and $C^{an}(G,K)_{G\setminus C}$ is closed $D(G,K)^{\circ}_C \simeq C^{an}(G,K)^{\circ\circ}_{G\setminus C} = C^{an}(G,K)_{G\setminus C}$. It follows that

$$(D(G,K)_C)'_b \simeq C^{an}(G,K)/C^{an}(G,K)_{G\setminus C}.$$

If G_0 is a compact open subgroup of G then by (1.2) and [22], Lemma 5.3

$$D(G,K)_C = \oplus_{g\in G/G_0} D(gG_0,K)_{gG_0\cap C}$$

showing that $D(G,K)_C$ is reflexive ($D(gG_0,K)_{gG_0\cap C}$ is a closed subspace of the nuclear Fréchet space $D(gG_0,K)$). Thus, (1.6) follows. The last claim follows from $C^{\omega}_C(G,K)$ being of compact type if C is compact (cf. [14], Satz 2.3.2). □

Corollary 1.2.6. *If C is a closed subset of G such that $1 \in C$ and $C \cdot C \subseteq C$ then $D(G,K)_C$ is a closed subalgebra of $D(G,K)$. If in addition C is compact then $D(G,K)_C$ is a nuclear K-Fréchet algebra.* □

Remark 1.2.7. Let G_0 be a compact open subgroup of G. If H is a locally L-analytic subgroup of G and $H_0 := H \cap G_0$ then as seen above

$$D(G,K)_H = \bigoplus_{g\in G/G_0} D(gG_0,K)_{gG_0\cap H}$$

as locally convex K-vector spaces. Noting that $D(gG_0,K)_{gG_0\cap H} \neq 0$ if and only if $gG_0 \cap H \neq \emptyset$ we get

$$(1.7) \qquad D(G,K)_H = \bigoplus_{h\in H/H_0} \delta_h \cdot D(G_0,K)_{H_0}.$$

According to [14], Bemerkung 3.1.2 and Satz 3.3.4, the Lie algebra $\mathfrak{g}$ of G acts on $C^{an}(G,K)$ via continuous endomorphisms defined by

$$\mathfrak{x}(f)(g) := \frac{d}{dt} f(exp(-t\mathfrak{x})g)|_{t=0} \text{ for } \mathfrak{x} \in \mathfrak{g} \text{ and } f \in C^{an}(G,K).$$

This action extends to an action of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ on $C^{an}(G,K)$.

According to Lemma 1.2.5 and Corollary 1.2.6 $C^{\omega}_1(G,K)'_b \simeq D(G,K)_{\{1\}}$ is a K-Fréchet subalgebra of $D(G,K)$. Fixing an ordered L-basis $\mathfrak{X} = (\mathfrak{x}_1,\ldots,\mathfrak{x}_d)$

of $\mathfrak{g}$ the action of $U(\mathfrak{g})$ on $C^{an}(G,K)$ leads to the following explicit description of $C_1^\omega(G,K)'_b$ (cf. [25], Lemma 2.4):

$$C_1^\omega(G,K)'_b = \left\{ \sum_{\alpha\in\mathbb{N}^d} d_\alpha \mathfrak{X}^\alpha \,|\, d_\alpha \in K, \forall r>0 : \sup_\alpha |d_\alpha \cdot \alpha!| r^{-|\alpha|} < \infty \right\},$$

where for $\alpha = (\alpha_1,\ldots,\alpha_d)$ we set $|\alpha| := \alpha_1 + \ldots + \alpha_d$ and $\alpha! := \alpha_1! \cdot \ldots \cdot \alpha_d!$. Further, $\mathfrak{X}^\alpha := \mathfrak{x}_1^{\alpha_1} \cdots \mathfrak{x}_d^{\alpha_d}$ is viewed as a distribution via

$$\mathfrak{X}^\alpha(f) = ((-\mathfrak{x}_1)^{\alpha_1} \circ \ldots \circ (-\mathfrak{x}_d)^{\alpha_d}(f))(1) \text{ for } f \in C^{an}(G,K). \tag{1.8}$$

Finally, the Fréchet topology of $C_1^\omega(G,K)'_b$ is defined by the family of norms $(\nu'_r)_{r>0}$ with $\nu'_r(\sum_\alpha d_\alpha \mathfrak{X}^\alpha) := \sup_\alpha |d_\alpha \cdot \alpha!| r^{-|\alpha|}$.

Letting $(\mathfrak{z} \mapsto \dot{\mathfrak{z}})$ denote the unique anti-automorphism of $U(\mathfrak{g}) \otimes_L K$ extending multiplication by -1 on $\mathfrak{g}$, the natural homomorphism $(\mathfrak{z} \mapsto (f \mapsto \dot{\mathfrak{z}}(f)(1))) : U(\mathfrak{g}) \otimes_L K \to C_1^\omega(G,K)'_b$ of K-algebras is injective.

Proposition 1.2.8. *$U(\mathfrak{g}) \otimes_L K$ is dense in $C_1^\omega(G,K)'_b$. We have*

$$C_1^\omega(G,K)'_b = \left\{ \sum_\alpha d_\alpha \mathfrak{X}^\alpha \,|\, d_\alpha \in K, \forall r>0 : \sup_\alpha |d_\alpha| r^{-|\alpha|} < \infty \right\} \tag{1.9}$$

and the Fréchet topology of $C_1^\omega(G,K)'_b$ can be defined by the family of norms $(\nu_r)_{r>0}$ with $\nu_r(\sum_\alpha d_\alpha \mathfrak{X}^\alpha) := \sup_\alpha |d_\alpha| r^{-|\alpha|}$.

Proof: Since $|\alpha!| \le 1$ the right hand side of (1.9) is contained in $C_1^\omega(G,K)'_b$. Conversely, $|\alpha!|^{-1} \le p^{|\alpha|/(p-1)}$, so that if $\sup_\alpha |d_\alpha| r^{-|\alpha|} < \infty$ for all $r>0$ then also $\sup_\alpha |d_\alpha/\alpha!| r^{-|\alpha|} < \infty$ for all $r>0$. This proves the reverse inclusion as well as the fact that the two families of norms $(\nu'_r)_{r>0}$ and $(\nu_r)_{r>0}$ are equivalent. The density statement is clear. □

Remark 1.2.9. When working with $C_1^\omega(G,K)'_b$ we will henceforth use the description given by (1.9) and assume its topology to be defined by the family of norms $(\nu_r)_{r>0}$. To simplify notation we write $U(\mathfrak{g},K) := C_1^\omega(G,K)'_b$.

Lemma 1.2.10. *If C is a closed subset of G then the $U(\mathfrak{g},K)$-submodule of $D(G,K)_C$ generated by all Dirac distributions δ_c, $c \in C$, is dense.*

Proof: Let Δ be the closure of $\sum_{c\in C} \delta_c \cdot U(\mathfrak{g},K)$ in $D(G,K)$. It follows from Lemma 1.2.4 and Lemma 1.2.5 that $\Delta \subseteq D(G,K)_C$. We know that $C^{an}(G,K)/C^{an}(G,K)_{G\setminus C}$ is reflexive. Let ℓ be a continuous functional on $D(G,K)_C$ vanishing on Δ. By (1.5) and (1.6), ℓ corresponds to an element $\overline{f}$ of $C^{an}(G,K)/C^{an}(G,K)_{G\setminus C}$. To say ℓ vanishes on Δ is to say that any representative f of $\overline{f}$ in $C^{an}(G,K)$ vanishes in an open neighborhood of C. Hence $f \in C^{an}(G,K)_{G\setminus C}$, i.e. $\overline{f} = 0$, and $\Delta = D(G,K)_C$ by the Hahn-Banach theorem. □

Remark 1.2.11. Let B and C be locally convex K-vector spaces carrying separately continuous K-algebra structures with a common K-subalgebra A. If $B\hat{\otimes}_{K,\iota}C$ denotes the Hausdorff completion of the algebraic tensor product $B\otimes_{K,\iota}C$ endowed with its inductive tensor product topology then we let $B\hat{\otimes}_{A,\iota}C$

be the quotient of $B\hat{\otimes}_{K,\iota}C$ by the closure of the subspace generated by all elements of the form

$$ba \otimes c - b \otimes ac, \qquad a \in A, b \in B \text{ and } c \in C.$$

We endow $B\hat{\otimes}_{A,\iota}C$ with the corresponding quotient topology. If B and C are K-Fréchet spaces then the inductive and the projective tensor product topologies on $B \otimes_K C$ coincide. Therefore, we omit the ι from the notation and simply write $B\hat{\otimes}_K C$ and $B\hat{\otimes}_A C$.
Note that $B\hat{\otimes}_{A,\iota}C$ is naturally a B-C^{op}-bimodule (C^{op} being the K-algebra opposite to C). If A is contained in the centers of B and C then $B\hat{\otimes}_{A,\iota}C$ is naturally a module over $B \otimes_K C$ and even over $B \otimes_A C$.

Let H be a closed, locally L-analytic subgroup of G and $\mathfrak{h}$ its Lie algebra. The multiplication map

$$D(H,K) \times U(\mathfrak{g},K) \longrightarrow D(G,K)_H \tag{1.10}$$

induces a continuous K-linear map

$$\mu : D(H,K)\hat{\otimes}_{U(\mathfrak{h},K),\iota}U(\mathfrak{g},K) \longrightarrow D(G,K)_H.$$

Proposition 1.2.12. *If K is discretely valued then μ is a topological isomorphism of $D(H,K)$-$U(\mathfrak{g},K)^{op}$-bimodules.*

Proof: In Corollary 1.3.6 and Corollary 1.4.3 we will prove that there is a compact open subgroup G_0 of G with the following properties: $D(G_0,K)$ is a K-Fréchet-Stein algebra with respect to a family of norms $||\cdot||_{\overline{r}}$, $r \in p^{\mathbb{Q}}$, $1/p < r < 1$, such that the completion $D_r(G_0,K)$ of $D(G_0,K)$ with respect to the norm $||\cdot||_{\overline{r}}$ is finitely generated and free as a module over the closure $U_r(\mathfrak{g},K)$ of $U(\mathfrak{g},K)$ in $D_r(G_0,K)$; if $H_0 := H \cap G_0$ then $D(H_0,K)$ is a K-Fréchet-Stein algebra with respect to the family of norms $||\cdot||_{\overline{r}}$ restricted to $D(H_0,K)$; for each r the closure $D_r(H_0,K)$ of $D(H_0,K)$ in $D_r(G_0,K)$ is finitely generated and free as a module over the closure $U_r(\mathfrak{h},K)$ of $U(\mathfrak{h},K)$ in $D_r(H_0,K)$; $U_r(\mathfrak{g},K)$ and $U_r(\mathfrak{h},K)$ are noetherian K-Banach algebras.

Lemma 1.2.13. *If $(V_i)_{i\in I}$ and W are Hausdorff locally convex K-vector spaces then there is a topological isomorphism*

$$(\bigoplus_{i\in I} V_i)\hat{\otimes}_{K,\iota}W \simeq \bigoplus_{i\in I}(V_i\hat{\otimes}_{K,\iota}W).$$

Proof: This is a straightforward generalization of [18], I.3.1 Proposition 14.I, to the non-archimedean setting. □

By (1.7), Lemma 1.2.13 and [22], Lemma 5.3, it suffices to show that the map

$$D(H_0,K)\hat{\otimes}_{U(\mathfrak{h},K)}U(\mathfrak{g},K) \longrightarrow D(G_0,K)_{H_0}$$

is a topological isomorphism. We again denote it by μ.
Let $r \in p^{\mathbb{Q}}$ with $1/p < r < 1$. The multiplication in $D_r(G_0,K)$ induces a continuous K-linear map

$$\mu_r : D_r(H_0,K) \otimes_K U_r(\mathfrak{g},K) \longrightarrow D_r(G_0,K)_{H_0};$$

here $D_r(G_0,K)_{H_0}$ denotes the closure of $D(G_0,K)_{H_0}$ in $D_r(G_0,K)$. In the proof of Corollary 1.4.3 we will show that $D_r(G_0,K)_{H_0}$ is free and finitely generated as a module over $U_r(\mathfrak{g},K)$ and has a basis $(\mathbf{b}^\alpha)_{\alpha\in A'}$ in $K[H_0]$ which is simultaneously a basis of the free $U_r(\mathfrak{h},K)$-module $D_r(H_0,K)$. Hence μ_r induces a continuous K-linear bijection

$$D_r(H_0,K)\otimes_{U_r(\mathfrak{h},K)} U_r(\mathfrak{g},K) \longrightarrow D_r(G_0,K)_{H_0}. \tag{1.11}$$

$D_r(H_0,K)$ and $U_r(\mathfrak{g},K)$ are complete normed modules over the noetherian K-Banach algebra $U_r(\mathfrak{h},K)$. Further, $D_r(H_0,K)$ is a finitely generated, free $U_r(\mathfrak{h},K)$-module and therefore topologically isomorphic to a direct sum of copies of $U_r(\mathfrak{h},K)$ (cf. [26], Proposition 2.1 (iii)). A straightforward generalization to the non-commutative setting of [2], 2.1.7 Proposition 6, shows that $D_r(H_0,K)\otimes_{U_r(\mathfrak{h},K)} U_r(\mathfrak{g},K)$ is a complete normed space with respect to the tensor product norm. By the open mapping theorem (1.11) is a topological isomorphism. In addition,

$$\begin{aligned} D_r(H_0,K)\otimes_{U_r(\mathfrak{h},K)} U_r(\mathfrak{g},K) &= (D_r(H_0,K)\otimes_K U_r(\mathfrak{g},K))/\ker\mu_r \\ &\simeq (D_r(H_0,K)\hat{\otimes}_K U_r(\mathfrak{g},K))/\overline{\ker\mu_r} \end{aligned}$$

where $\overline{\ker\mu_r}$ is the closure of $\ker\mu_r$ in $D_r(H_0,K)\hat{\otimes}_K U_r(\mathfrak{g},K)$. Thus, we obtain a short exact sequence of strict continuous K-linear maps between Banach spaces

$$0 \longrightarrow \overline{\ker\mu_r} \longrightarrow D_r(H_0,K)\hat{\otimes}_K U_r(\mathfrak{g},K) \longrightarrow D_r(G_0,K)_{H_0} \longrightarrow 0.$$

Recall that $U := \ker(D(H_0,K)\hat{\otimes}_K U(\mathfrak{g},K) \longrightarrow D(H_0,K)\hat{\otimes}_{U(\mathfrak{h},K)} U(\mathfrak{g},K))$ is the closure of the subspace of $D(H_0,K)\hat{\otimes}_K U(\mathfrak{g},K)$ generated by all elements of the form

$$\lambda\mathfrak{y}\otimes\mathfrak{x} - \lambda\otimes\mathfrak{y}\mathfrak{x} \text{ with } \lambda\in D(H_0,K), \mathfrak{y}\in U(\mathfrak{h},K) \text{ and } \mathfrak{x}\in U(\mathfrak{g},K).$$

Since by (1.11) the kernel of μ_r is the vector space generated by all elements of the form

$$\lambda\mathfrak{y}\otimes\mathfrak{x} - \lambda\otimes\mathfrak{y}\mathfrak{x} \text{ with } \lambda\in D_r(H_0,K), \mathfrak{y}\in U_r(\mathfrak{h},K) \text{ and } \mathfrak{x}\in U_r(\mathfrak{g},K)$$

$U\subseteq\overline{\ker\mu_r}$ is dense for all r. Therefore, the system $(\overline{\ker\mu_r})$ with $r\in p^{\mathbb{Q}}$ and $1/p<r<1$ satisfies the Mittag-Leffler property as formulated in [17], 13.2.4. By [loc.cit], 13.2.2, we obtain an exact sequence

$$0 \longrightarrow U = \varprojlim_r \overline{\ker\mu_r} \longrightarrow D(H_0,K)\hat{\otimes}_K U(\mathfrak{g},K) \longrightarrow D(G_0,K)_{H_0} \longrightarrow 0,$$

because

$$\varprojlim_r (D_r(H_0,K)\hat{\otimes}_K U_r(\mathfrak{g},K)) \simeq (\varprojlim_r D_r(H_0,K))\hat{\otimes}_K (\varprojlim_r U_r(\mathfrak{g},K))$$

(cf. [12], Proposition 1.1.29). It induces a continuous K-linear bijection

$$D(H_0,K)\hat{\otimes}_{U(\mathfrak{h},K)} U(\mathfrak{g},K) \longrightarrow D(G_0,K)_{H_0}$$

which is a topological isomorphism by the open mapping theorem. That it coincides with μ is clear from the fact that for each r the restriction of μ_r to $D(H_0,K)\otimes_K U(\mathfrak{g},K)$ is induced by the multiplication in $D(G_0,K)$. □

Remark 1.2.14. Assume there is a compact open subgroup G_0 of G and a closed locally L-analytic subgroup C_0 of G_0 such that $G_0 = H_0 \times C_0$ as locally L-analytic groups with $H_0 := H \cap G_0$. Then the above proposition can be proved without any allusion to Fréchet-Stein structures and simplifies in the following manner: According to Proposition A.3 and Remark A.4 of [28] there is a topological isomorphism

$$D(H_0, K)\hat{\otimes}_K D(C_0, K) \longrightarrow D(G_0, K)$$

induced by multiplication. It follows from Lemma 1.2.10 and [22], Corollary 17.5 (ii) and Proposition 19.10 (i), that the preimage of $D(G_0, K)_{H_0}$ under this map is $D(H_0, K)\hat{\otimes}_K U(\mathfrak{c}, K)$ where $\mathfrak{c}$ is the Lie algebra of C_0. Hence we obtain from Lemma 1.2.13 that

$$D(G, K)_H \simeq D(H, K)\hat{\otimes}_{K,\iota} U(\mathfrak{c}, K).$$

1.3 Restriction of the base field

Let $L_0|\mathbb{Q}_p$ be an extension of fields with $L_0 \subseteq L$ and let $R^{L|L_0}$ be the functor "*restriction of the base field from L to L_0*" from the category of paracompact locally L-analytic manifolds to the category of locally analytic manifolds of the same type over L_0 (cf. [7], 5.14).

There is a natural embedding

$$\tau : C^{an}(G, K) \longrightarrow C^{an}(R^{L|L_0}G, K)$$

mapping $C^{an}(G, K)$ homeomorphically onto its closed image (cf. [24], Lemma 1.2).

Lemma 1.3.1. *The dual map $\tau' : D(R^{L|L_0}G, K) \to D(G, K)$ is a strict surjection and a homomorphism of K-algebras.*

Proof: Since τ' restricts distributions on $R^{L|L_0}G$ to the subspace $C^{an}(G, K)$ of $C^{an}(R^{L|L_0}G, K)$ it is clear that τ' is a homomorphism of K-algebras. To show the surjectivity we may assume G to be compact. But then τ is a topological embedding of spaces of compact type so that the claim follows from [25], Proposition 1.2 (i). □

Consider the ideal $I := \ker(\tau')$ of $D(R^{L|L_0}G, K)$. It is the orthogonal subspace of $C^{an}(G, K)$ with respect to the natural pairing

$$D(R^{L|L_0}G, K) \times C^{an}(R^{L|L_0}G, K) \longrightarrow K.$$

Since $D(R^{L|L_0}G, K)$ is reflexive we obtain by means of [6], IV.2.2 Corollary, that I'_b is topologically isomorphic to $C^{an}(R^{L|L_0}G, K)/C^{an}(G, K)$. The topological isomorphism $I \simeq \oplus_{g \in G/G_0} \ker((\tau|C^{an}(gG_0, K))')$ for a compact open subgroup G_0 of G shows that I itself is reflexive. Thus, there is a topological isomorphism

$$I \simeq (C^{an}(R^{L|L_0}G, K)/C^{an}(G, K))'_b. \tag{1.12}$$

In order to give an explicit description of the locally L-analytic functions inside $C^{an}(R^{L|L_0}G, K)$ we follow the arguments given in section 1 of [24]. If we write $\mathfrak{g}_{L_0}$ for $\mathfrak{g}$ viewed as a Lie algebra over L_0 then $\mathfrak{g}_{L_0}$ can be identified with the Lie algebra of $R^{L|L_0}G$.

Lemma 1.3.2. *$C^{an}(G,K)$ is the closed subspace of all those functions $f \in C^{an}(R^{L|L_0}G,K)$ for which $(t\mathfrak{x})(f) = t \cdot \mathfrak{x}(f)$ for all $t \in L$ and all $\mathfrak{x} \in \mathfrak{g}_{L_0}$.*

Proof: If we let W be the subspace of $C^{an}(R^{L|L_0}G,K)$ consisting of all functions with the above property then $C^{an}(G,K) \subseteq W$. Let $f \in W$. If $\mathfrak{x},\mathfrak{y} \in \mathfrak{g}$ and $t \in L$ then

$$\begin{aligned}(t\mathfrak{x})(\mathfrak{y}(f)) &= \mathfrak{y}((t\mathfrak{x})(f)) + [t\mathfrak{x},\mathfrak{y}](f) = \mathfrak{y}(t\cdot\mathfrak{x}(f)) + (t\cdot[\mathfrak{x},\mathfrak{y}])(f)\\ &= t\cdot\mathfrak{y}(\mathfrak{x}(f)) + t\cdot[\mathfrak{x},\mathfrak{y}](f) = t\cdot\mathfrak{x}(\mathfrak{y}(f))\end{aligned}$$

shows that W is $\mathfrak{g}_{L_0}$-invariant. Therefore, the proof of [loc.cit.], Lemma 1.1, generalizes to the non-commutative setting in the following manner: Fix an L-basis $\mathfrak{X} = (\mathfrak{x}_1,\ldots,\mathfrak{x}_d)$ of $\mathfrak{g}$. Choose an orthonormal basis $(v_1,\ldots,v_n)$ of L as a vector space over L_0 and put $\mathfrak{Y} := (v_1\mathfrak{x}_1, v_2\mathfrak{x}_1,\ldots,v_n\mathfrak{x}_d)$. The corresponding system θ_{L_0} of canonical coordinates of the second kind is defined by

$$\theta_{L_0}(\sum_{i,j} t_{ij}v_i\mathfrak{x}_j) := exp(t_{11}v_1\mathfrak{x}_1)exp(t_{21}v_2\mathfrak{x}_1)\cdots exp(t_{nd}v_n\mathfrak{x}_d)$$

for t_{ij} sufficiently close to zero in L_0 (cf. [4], III.4.3 Proposition 3). Given $g \in R^{L|L_0}G$ we have the expansion

$$(R_g f \circ \theta_{L_0})(\sum_{i,j} t_{ij}v_i\mathfrak{x}_j) = \sum_{\beta\in\mathbb{N}^n\times\mathbb{N}^d} c_\beta \mathbf{t}^\beta$$

converging for all t_{ij} near zero in L_0; here $c_\beta \in K$, $\mathbf{t}^\beta := \prod_{i,j} t_{ij}^{\beta_{ij}}$ and R_g is the right translation operator associated with g. Letting $\mathfrak{Y}^\beta(R_g f) := (v_1\mathfrak{x}_1)^{\beta_{11}} \circ (v_2\mathfrak{x}_1)^{\beta_{21}} \circ\cdots\circ (v_n\mathfrak{x}_d)^{\beta_{nd}}(R_g f)$ it follows from the remarks after Lemma 4.7.2 of [14] that

$$c_\beta = \frac{(-1)^{|\beta|}}{\beta!}\mathfrak{Y}^\beta(R_g f)(1) = \frac{(-1)^{|\beta|}}{\beta!}\mathfrak{Y}^\beta(f)(g)$$

for all $\beta \in \mathbb{N}^n \times \mathbb{N}^d$ where $|\beta|$ and $\beta!$ are as in subsection 1.2. Letting $\varphi(\beta) := (\alpha_1,\ldots,\alpha_d)$ with $\alpha_j := \beta_{1j} + \ldots + \beta_{nj}$, $b_{\varphi(\beta)} := c_{(\alpha_1,0,\ldots,\alpha_2,0,\ldots,\alpha_d,0,\ldots)}$ and $\mathfrak{X}^{\varphi(\beta)}(R_g f) := \mathfrak{x}_1^{\alpha_1}\circ\cdots\circ\mathfrak{x}_d^{\alpha_d}(R_g f)$ we deduce

$$\mathfrak{Y}^\beta(f)(g) = \prod_{i=1}^n v_i^{\beta_{i1}+\ldots+\beta_{id}} \cdot \mathfrak{X}^{\varphi(\beta)}(f)(g)$$

from the assumption on f and the $\mathfrak{g}_{L_0}$-invariance of W. Thus

$$c_\beta = b_{\varphi(\beta)}\frac{\varphi(\beta)!}{\beta!}\prod_{i=1}^n v_i^{\beta_{i1}+\ldots+\beta_{id}}$$

for all β. Since this is precisely the relation given in the proof of [24], Lemma 1.1, we may conclude that f is locally L-analytic at g. □

The proof of the following lemma uses the same Hahn-Banach argument as the proof of Lemma 1.2.10.

Lemma 1.3.3. *If $J := I \cap (U(\mathfrak{g}_{L_0}) \otimes_{L_0} K)$ then the vector space $\sum_{g\in G}\delta_g \cdot J$ is dense in I.* □

Lemma 1.3.4. *Let $C \subseteq G$ be a closed subset, considered also as a subset of $R^{L|L_0}G$. Then the image of $D(R^{L|L_0}G, K)_C$ under τ' is dense in $D(G, K)_C$.*

Proof: That $\tau'(D(R^{L|L_0}G, K)_C)$ is contained in $D(G, K)_C$ follows from

$$C^{an}(G, K)_{G \backslash C} = C^{an}(R^{L|L_0}G_0, K)_{G \backslash C} \cap C^{an}(G, K).$$

The same equation shows that τ induces a continuous injection

$$C^{an}(G, K)/C^{an}(G, K)_{G \backslash C} \hookrightarrow C^{an}(R^{L|L_0}G, K)/C^{an}(R^{L|L_0}G, K)_{R^{L|L_0}G \backslash C}.$$

We know from the proof of Lemma 1.2.10 that the locally convex K-vector spaces on both sides are reflexive so that as a consequence of the Hahn-Banach Theorem the dual map $\tau' : D(R^{L|L_0}G, K)_C \to D(G, K)_C$ has to have dense image. □

We recall the following basic definitions (cf. [9], Part I): If G is a pro-p group set $P_1(G) := G$ and $P_{i+1}(G) := \overline{P_i^p[P_i(G), G]}$ for $i \geq 1$. Here $P_i(G)^p[P_i(G), G]$ denotes the subgroup of G generated by the pth powers of elements of $P_i(G)$ and by all commutators $[a, b]$ with $a \in P_i(G)$ and $b \in G$; $\overline{X}$ denotes the topological closure of a subset X of G. A pro-p group G is called powerful if p is odd and $G/\overline{G^p}$ is abelian or if $p = 2$ and $G/\overline{G^4}$ is abelian. A pro-p group G is called uniform if it is topologically finitely generated, powerful and if $(P_i(G) : P_{i+1}(G)) = (G : P_2(G))$ for all $i \geq 1$.
One of the most fundamental properties of a uniform pro-p group G is given by the following theorem ([loc.cit], Theorem 4.9): If $(a_1, \ldots, a_d)$ is a system of topological generators of G with $d = \dim G$ then every element has a unique expression of the form $a_1^{\lambda_1} \cdots a_d^{\lambda_d}$ with $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}_p$. The resulting bijection $\mathbb{Z}_p^d \simeq G$ is a homeomorphism. In this way, uniform pro-p groups turn out to be the fundamental examples of locally $\mathbb{Q}_p$-analytic groups ([loc.cit.], Theorem 8.32).

Assume $L_0 = \mathbb{Q}_p$. For further applications we need the following technical results:

Proposition 1.3.5. *Let G be a locally L-analytic group. Then there is an open subgroup G_0 of G and a $\mathbb{Z}_p$-lattice $\Lambda \subset \mathfrak{g}_{\mathbb{Q}_p}$ with the following properties:*

i) there is an L-basis $(\mathfrak{x}_1, \ldots, \mathfrak{x}_d)$ of $\mathfrak{g}$ and a $\mathbb{Z}_p$-basis $(v_1, \ldots, v_m)$ of $\mathfrak{o}_L$ such that $(v_1\mathfrak{x}_1, \ldots, v_m\mathfrak{x}_d)$ is a $\mathbb{Z}_p$-basis of Λ;

ii) the corresponding canonical coordinates of the second kind give a well defined isomorphism $\theta_{\mathbb{Q}_p} : \Lambda \to R^{L|\mathbb{Q}_p}G_0$ of locally $\mathbb{Q}_p$-analytic manifolds;

iii) $R^{L|\mathbb{Q}_p}G_0$ is a uniform pro-p group.

Proof: Let $(\mathfrak{x}_1, \ldots, \mathfrak{x}_d)$ be an L-basis of $\mathfrak{g}$ and θ_L the corresponding system of canonical coordinates of the second kind. Since θ_L is étale at $0 \in \mathfrak{g}$ we may choose an open subgroup G' of G and an open neighborhood U of zero in $\mathfrak{g}$ such that $\theta_L : U \to G'$ is an isomorphism of locally L-analytic manifolds. Let Φ_L be its inverse. According to [4], III.7.3 Théorème 4 and its proof there is $\lambda \in L^*$ such that $\oplus_i \mathfrak{m}_L \mathfrak{x}_i \subseteq \lambda \cdot \Phi_L(G') = \lambda \cdot U$ and the group structure on $\oplus_i \lambda^{-1} \mathfrak{m}_L \mathfrak{x}_i$ obtained by transport of structure from G' is given by formal power series with coefficients in $\mathfrak{o}_L$.

If p is odd set $\Lambda := \oplus_i \lambda^{-1}\mathfrak{m}_L^e \mathfrak{x}_i$ and $\Lambda := \oplus_i \lambda^{-1}\mathfrak{m}_L^{2e}\mathfrak{x}_i$ otherwise. By [loc.cit.], III.7.4 Proposition 5, $G_0 := \theta_L(\Lambda)$ is an open subgroup of G. Choosing a $\mathbb{Z}_p$-basis $(v_1, \ldots, v_m)$ of $\mathfrak{o}_L$ the canonical coordinates of the second kind

$$\theta_{\mathbb{Q}_p} : \mathfrak{g}_{\mathbb{Q}_p} \dashrightarrow R^{L|\mathbb{Q}_p} G$$

corresponding to the decomposition $\mathfrak{g}_{\mathbb{Q}_p} = \oplus_{i,j}\mathbb{Q}_p \lambda^{-1} v_j \mathfrak{x}_i$ coincide with θ_L as $[v_i\mathfrak{x}_j, v_k\mathfrak{x}_j] = 0$ in $\mathfrak{g}_L$ and because of the properties of the exponential map. Since $\mathfrak{m}_L^e = p\mathfrak{o}_L$ (resp. $4\mathfrak{o}_L$ if $p = 2$) (i) and (ii) are proved if for (i) we choose $(\lambda^{-1}p\mathfrak{x}_i)$ as an L-basis of $\mathfrak{g}$ (resp. $(\lambda^{-1}4\mathfrak{x}_i)$ if $p = 2$).

It remains to show that $\theta_{\mathbb{Q}_p}(\Lambda) = R^{L|\mathbb{Q}_p} G_0$ is a uniform pro-p group. According to [9], Theorem 8.31, we only need to show that $R^{L|\mathbb{Q}_p} G_0$ is a standard group in the sense of [loc.cit.], Definition 8.22. This follows directly from the construction. □

If H is a closed, uniform subgroup of a uniform pro-p group G then we say that H is compatible with G if there is a basis of topological generators of H that can be extended to a basis of topological generators of G.

Corollary 1.3.6. *Let G be a locally L-analytic group and H a closed locally L-analytic subgroup. Then there is an open subgroup G_0 of G as in Proposition 1.3.5 such that $H_0 := H \cap G_0$, as an open subgroup of H, satisfies conditions (i) – (iii) of Proposition 1.3.5 and $R^{L|\mathbb{Q}_p} H_0$ is compatible with $R^{L|\mathbb{Q}_p} G_0$.*

Proof: Extend an L-basis $(\mathfrak{x}_1, \ldots, \mathfrak{x}_j)$ of the Lie algebra $\mathfrak{h}$ of H to an L-basis $(\mathfrak{x}_1, \ldots, \mathfrak{x}_d)$ of $\mathfrak{g}$, $j \leq d$. We may assume U and G' from the proof of Proposition 1.3.5 to satisfy $\Phi_L(H \cap G') \subseteq \mathfrak{h}$. Starting with G' define $\Lambda \subseteq U$ and $G_0 \subseteq G'$ as before. Then $\Lambda' := \Lambda \cap \mathfrak{h}$ is an open neighborhood of 0 in $\mathfrak{h}$ and a direct summand of Λ. Therefore, the restriction of θ_L from Λ to Λ' is an isomorphism $\Lambda' \to H_0 := G_0 \cap H$ of locally L-analytic manifolds. It follows as above that H_0 satisfies conditions (i) – (iii) of Proposition 1.3.5 with respect to Λ'. By definition, Λ (resp. Λ') gives rise to the basis of topological generators $(exp(v_k\mathfrak{x}_i))$, $1 \leq k \leq m$, $1 \leq i \leq d$, (resp. $1 \leq k \leq m$, $1 \leq i \leq j$) of $R^{L|\mathbb{Q}_p} G_0$ (resp. $R^{L|\mathbb{Q}_p} H_0$). Thus, $R^{L|\mathbb{Q}_p} G_0$ and $R^{L|\mathbb{Q}_p} H_0$ are compatible. □

1.4 Explicit Fréchet-Stein structures

The notion of a K-Fréchet-Stein algebra was first introduced by P. Schneider and J. Teitelbaum (cf. [26], section 3): A K-Fréchet algebra A is called a K-Fréchet-Stein algebra if there is a sequence $q_1 \leq q_2 \leq \ldots$ of continuous algebra seminorms on A defining its Fréchet topology such that for all $n \in \mathbb{N}$ the Hausdorff completion A_{q_n} of A with respect to q_n is a (left) noetherian K-Banach algebra and a flat $A_{q_{n+1}}$-module via the natural map $A_{q_{n+1}} \to A_{q_n}$. In this subsection we will assume K to be discretely valued.

Let G_0 be a uniform pro-p group with a basis $(a_1, \ldots, a_d)$ of topological generators. Putting $b_i := a_i - 1$ and $\mathbf{b}^\alpha := b_1^{\alpha_1} \ldots b_d^{\alpha_d}$ in $K[G_0]$ for a multi-index

$\alpha \in \mathbb{N}^d$ $D(G_0, K)$ admits the explicit description

$$D(G_0, K) = \left\{ \sum_\alpha d_\alpha \mathbf{b}^\alpha \mid d_\alpha \in K,\ \forall\, 0 < r < 1 : \sup_\alpha |d_\alpha| r^{\tau\alpha} < \infty \right\}$$

(loc.cit. section 4). Here $\tau\alpha = \sum \tau_i \alpha_i$ with rational numbers τ_i depending on the structure of G_0 as a p-valued group. The Fréchet topology of $D(G_0, K)$ can be defined by the family of norms $(||\cdot||_r)_{0<r<1}$ given by

$$\left|\left| \sum_\alpha d_\alpha \mathbf{b}^\alpha \right|\right|_r := \sup_\alpha |d_\alpha| r^{\tau\alpha}.$$

The norms $||\cdot||_r$ are independent of the choice of a basis $(a_1, \ldots, a_d)$ of topological generators. If we let $D_r(G_0, K) = \{\sum_\alpha d_\alpha \mathbf{b}^\alpha \mid \lim_{|\alpha|\to\infty} |d_\alpha| r^{\tau\alpha} = 0\}$ be the completion of $D(G_0, K)$ with respect to the norm $||\cdot||_r$ then

$$D(G_0, K) = \varprojlim_r D_r(G_0, K)$$

as K-Fréchet spaces. We summarize some of the main results of [26] in the following theorem (loc.cit. Theorem 4.5 and Theorem 4.9):

Theorem (Schneider-Teitelbaum)**.** *If K is discretely valued, $r \in p^{\mathbb{Q}}$ and $1/p < r < 1$ then the algebra structure of $D(G_0, K)$ extends to $D_r(G_0, K)$ making it a K-Banach algebra with multiplicative norm $||\cdot||_r$. Moreover, for any two real numbers $r, r' \in p^{\mathbb{Q}}$ with $1/p < r' < r < 1$ the natural inclusion $D_r(G_0, K) \hookrightarrow D_{r'}(G_0, K)$ is a flat map of noetherian rings. In other words: $D(G_0, K)$ is a K-Fréchet-Stein algebra with respect to the family of norms $||\cdot||_r$, $r \in p^{\mathbb{Q}}$, $1/p < r < 1$.*

For $0 < r < 1$ we let $U_r(\mathfrak{g}, K)$ be the closure of $U(\mathfrak{g}, K)$ in $D_r(G_0, K)$ with respect to the norm $||\cdot||_r$. A careful analysis of orthogonal bases (cf. [16], section 1) leads to the following result (loc.cit. 1.4 Lemma 3, Corollaries 1, 2 and 3):

Theorem (Frommer)**.** *If $r \in p^{\mathbb{Q}}$ and $1/p < r < 1$ then $U_r(\mathfrak{g}, K)$ is a noetherian subalgebra of $D_r(G_0, K)$. In fact, there are integers $\ell_i > 0$ depending on r such that $D_r(G_0, K)$ is free as a (right) module over $U_r(\mathfrak{g}, K)$ with basis consisting precisely of those $\mathbf{b}^\alpha \in K[G_0]$ for which $0 \le \alpha_i < \ell_i$ for all $i = 1, \ldots, d$. Further, $U_r(\mathfrak{g}, K)$ is equal to the algebra*

$$U_r(\mathfrak{g}, K) = \left\{ \sum_\alpha d_\alpha \mathfrak{X}^\alpha \mid d_\alpha \in K,\ \lim_{|\alpha|\to\infty} |d_\alpha| ||\mathfrak{X}^\alpha||_r = 0 \right\},$$

where $\mathfrak{X}$ is the $\mathbb{Q}_p$-basis $(\mathfrak{x}_i := \log(1 + b_i))_{1\le i\le d}$ of $\mathfrak{g}$. The norm $||\cdot||_r$ can be computed via $||\sum_\alpha d_\alpha \mathfrak{X}^\alpha||_r = \sup_\alpha |d_\alpha| ||\mathfrak{X}^\alpha||_r$.

Using compatible uniform pro-p groups we can slightly extend this result:

Corollary 1.4.1. *Let G_0 be a uniform pro-p group with closed, compatible uniform subgroup H_0. Then $D(H_0, K)$ is a K-Fréchet-Stein algebra with respect to the family of norms $||\cdot||_r$, $r \in p^{\mathbb{Q}}$, $1/p < r < 1$, restricted to $D(H_0, K)$. The conclusions of Frommer's theorem hold for $D(H_0, K)$. If $r \in p^{\mathbb{Q}}$ is a real number with $1/p < r < 1$ then the closure $D_r(G_0, K)_{H_0}$ of $D(G_0, K)_{H_0}$ in $D_r(G_0, K)$ is a finitely generated, free $U_r(\mathfrak{g}, K)$-module possessing a basis contained in $K[H_0]$.*

Proof: Choose a basis $(a_1, \dots, a_d)$ of topological generators of G_0 such that $(a_1, \dots, a_j)$ is a basis of topological generators of H_0, $j := \dim H_0 \le d$. Clearly, $D(H_0, K)$ is a K-Fréchet-Stein algebra with respect to the restricted norms $||\cdot||_r$, $r \in p^{\mathbb{Q}}$, $1/p < r < 1$, if H_0 is viewed as a p-valued group with respect to the valuation coming from G_0. It is also clear that Frommer's theorem applies to $D(H_0, K)$. Fix $r \in p^{\mathbb{Q}}$ with $1/p < r < 1$. Let $A \subset \mathbb{N}^d$ be the set of all multi-indices satisfying $0 \le \alpha_i < \ell_i$ for all i and $A' \subseteq A$ be the subset of all α such that $\alpha_{j+1} = \ldots = \alpha_d = 0$. If $\mathfrak{h}$ denotes the Lie algebra of H then $(\mathbf{b}^\alpha)_{\alpha \in A'}$ is a basis of the free $U_r(\mathfrak{h}, K)$-module $D_r(H_0, K)$: The proof of [16], 1.4 Lemma 3, shows that writing $\mathfrak{x}_i = \log(1 + b_i) = \sum_{n \ge 1} (-1)^{n+1} b_i^n / n$ one can choose

$$\ell_i = \max\{m \ge 1 \mid \sup_{n \ge 1} |1/n| r^{n\tau_i} = |1/m| r^{m\tau_i}\}.$$

Hence for $1 \le i \le j$ the integers ℓ_i do not depend on whether we consider b_i as an element of $K[G_0]$ or $K[H_0]$.

If D denotes the free $U_r(\mathfrak{g}, K)$-submodule of $D_r(G_0, K)$ generated by $(\mathbf{b}^\alpha)_{\alpha \in A'}$ then $D \subseteq D_r(G_0, K)_{H_0}$. Conversely, D contains $D_r(H_0, K)$ and $U_r(\mathfrak{g}, K)$ and thereby a dense subspace of $D_r(G_0, K)_{H_0}$ (cf. Lemma 1.2.10). According to [26], Proposition 2.1 (ii), D is closed. Hence $D = D_r(G_0, K)_{H_0}$. □

We are now going to extend Frommer's theorem and Corollary 1.4.1 to the case of a finite extension $L|\mathbb{Q}_p$. Recall that if A is a K-Fréchet-Stein algebra with respect to a sequence $(q_n)_{n \ge 1}$ of continuous algebra seminorms and if I is a closed ideal of A then according to [26], Proposition 3.7, A/I is a K-Fréchet-Stein algebra with respect to the sequence $(\overline{q_n})_{n \ge 1}$ of residue norms $\overline{q_n}$. It follows that if G_0 is a locally L-analytic group such that $R^{L|\mathbb{Q}_p} G_0$ is uniform pro-p then $D(G_0, K)$ is a K-Fréchet-Stein algebra (loc.cit. Theorem 5.1). Namely, $D(G_0, K)$ is topologically isomorphic to the quotient of $D(R^{L|\mathbb{Q}_p} G_0, K)$ by $I := \ker(\tau')$ (cf. Lemma 1.3.1).

For $1/p < r < 1$ we denote by $||\cdot||_{\overline{r}}$ the residue norm on $D(G_0, K)$ induced by $||\cdot||_r$. The completion of $D(G_0, K)$ with respect to $||\cdot||_{\overline{r}}$ is denoted by $D_r(G_0, K)$. Let I_r be the closure of I in $D_r(R^{L|\mathbb{Q}_p} G_0, K)$ and consider the projection

$$\tau_r : D_r(R^{L|\mathbb{Q}_p} G_0, K) \longrightarrow D_r(R^{L|\mathbb{Q}_p} G_0, K)/I_r.$$

According to the proof of [26], Proposition 3.7, we have

$$D_r(G_0, K) = D_r(R^{L|\mathbb{Q}_p} G_0, K)/I_r. \tag{1.13}$$

As before we let $U_r(\mathfrak{g}, K)$ (resp. $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$) denote the closure of $U(\mathfrak{g}, K)$ (resp. $U(\mathfrak{g}_{\mathbb{Q}_p}, K)$) in $D_r(G_0, K)$ (resp. $D_r(R^{L|\mathbb{Q}_p} G_0, K)$). Set further $J_r := I_r \cap U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$.

Theorem 1.4.2. *Let G be a locally L-analytic group and G_0 as in Proposition 1.3.5. If $r \in p^{\mathbb{Q}}$ with $1/p < r < 1$ then $D_r(G_0, K)$ is a free, finitely generated module over the noetherian subalgebra $U_r(\mathfrak{g}, K)$ with the same basis in $K[G_0]$ as in Frommer's theorem applied to $R^{L|\mathbb{Q}_p} G_0$. Further, there is an L-basis $\mathfrak{X}$ of $\mathfrak{g}$ and a norm $\nu_{\overline{r}}$ on $U_r(\mathfrak{g}, K)$ equivalent to $||\cdot||_{\overline{r}}$ such that*

$$U_r(\mathfrak{g}, K) = \left\{ \sum_\alpha d_\alpha \mathfrak{X}^\alpha \mid d_\alpha \in K,\ \lim_{|\alpha| \to \infty} |d_\alpha| \nu_{\overline{r}}(\mathfrak{X}^\alpha) = 0 \right\}.$$

The norm $\nu_{\overline{r}}$ can be computed via $\nu_{\overline{r}}(\sum_\alpha d_\alpha \mathfrak{X}^\alpha) = \sup_\alpha |d_\alpha| \nu_{\overline{r}}(\mathfrak{X}^\alpha)$.

Proof: Let $(\mathbf{b}^\alpha)_{\alpha \in A}$ be the $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$-basis of $D_r(R^{L|\mathbb{Q}_p} G_0, K)$ considered before and D the (right) $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$-submodule $D := \oplus_{\alpha \in A} \mathbf{b}^\alpha J_r$. Since I_r is an ideal of $D_r(R^{L|\mathbb{Q}_p} G_0, K)$ containing J_r, we naturally have $D \subseteq I_r$. On the other hand, D contains a dense subspace of I_r according to Lemma 1.3.3 since $J := I \cap (U(\mathfrak{g}_{\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} K) \subseteq J_r$. Since D is closed according to [26], Proposition 2.1 (ii), we also have $I_r \subseteq D$. Hence $D = I_r$.
It follows from (1.13) and Frommer's theorem that there is an isomorphism

$$D_r(G_0, K) \simeq \oplus_{\alpha \in A} \mathbf{b}^\alpha (U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)/J_r)$$

of (right) $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$-modules. It becomes topological if $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)/J_r$ carries the (Banach) quotient topology (cf. [26], Proposition 2.1). In particular, the image of $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$ under τ_r is closed. According to Lemma 1.3.4 it contains a dense subspace of $U_r(\mathfrak{g}, K)$ whence there is a topological isomorphism

$$U_r(\mathfrak{g}, K) \simeq U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)/J_r. \tag{1.14}$$

This proves the first statement of the theorem. We claim that the assertions concerning the explicit description of $U_r(\mathfrak{g}, K)$ hold if we equip $U_r(\mathfrak{g}, K)$ with the residue norm $\nu_{\overline{r}}$ coming from (1.14).

According to Proposition 1.3.5 there is an L-basis $\mathfrak{X} = (\mathfrak{x}_1, \ldots, \mathfrak{x}_d)$ of $\mathfrak{g}$ and a $\mathbb{Z}_p$-basis $(v_1, \ldots, v_m)$ of $\mathfrak{o}_L$ such that the family $\mathfrak{Y} := (v_i \mathfrak{x}_j)_{i,j}$ gives rise to the set of topological generators $(exp(v_i \mathfrak{x}_j))_{i,j}$ of $R^{L|\mathbb{Q}_p} G_0$. By Frommer's theorem

$$U_r(\mathfrak{g}_{\mathbb{Q}_p}, K) = \left\{ \sum_\beta c_\beta \mathfrak{Y}^\beta \mid \lim_{|\beta| \to \infty} |c_\beta| ||\mathfrak{Y}^\beta||_r = 0 \right\}$$

with multiplicative norm $|| \sum_\beta c_\beta \mathfrak{Y}^\beta ||_r = \sup_\beta |c_\beta| ||\mathfrak{Y}^\beta||_r$. If $\beta = (\beta_{ij}) \in \mathbb{N}^m \times \mathbb{N}^d$ let $\varphi(\beta) := (\sum_{i=1}^m \beta_{ij})_{1 \le j \le d} \in \mathbb{N}^d$. For any β with $\varphi(\beta) = \alpha$ we have $\tau'(\mathfrak{Y}^\beta) = \prod_{i,j} v_j^{\beta_{ij}} \mathfrak{X}^\alpha$ and $|\alpha| = |\beta|$. Since τ_r continuously extends τ' we have

$$\tau_r(\sum_\beta c_\beta \mathfrak{Y}^\beta) = \sum_\beta \tau_r(c_\beta \mathfrak{Y}^\beta) = \sum_{\alpha \in \mathbb{N}^d} \left(\sum_{\varphi(\beta) = \alpha} c_\beta \prod_{i,j} v_i^{\beta_{ij}} \right) \mathfrak{X}^\alpha$$

and also

$$\left| \sum_{\varphi(\beta)=\alpha} c_\beta \prod_{i,j} v_i^{\beta_{ij}} \right| \nu_{\overline{r}}(\mathfrak{X}^\alpha) \le \max_{\varphi(\beta)=\alpha} |c_\beta| ||\mathfrak{Y}^\beta||_r \to 0 \text{ as } |\alpha| \to \infty.$$

Therefore, $U_r(\mathfrak{g}, K) \subseteq \{\sum_\alpha d_\alpha \mathfrak{X}^\alpha \mid \lim_{|\alpha| \to \infty} |d_\alpha| \nu_{\overline{r}}(\mathfrak{X}^\alpha) = 0\}$. The converse inclusion is clear.

We claim that J is dense in J_r. Note first that J is dense in $I \cap U(\mathfrak{g}_{\mathbb{Q}_p}, K)$: If $\delta = \sum_\beta c_\beta \mathfrak{Y}^\beta \in U(\mathfrak{g}_{\mathbb{Q}_p}, K)$ then by (1.9) $\lim_{|\beta| \to \infty} |c_\beta| \rho^{-|\beta|} = 0$ for all $\rho > 0$. Hence $\tau'(\delta) = \sum_\alpha (\sum_{\varphi(\beta)=\alpha} c_\beta \prod_{i,j} v_i^{\beta_{ij}}) \mathfrak{X}^\alpha$ converges in $U(\mathfrak{g}, K)$. If $\delta \in I \cap U(\mathfrak{g}_{\mathbb{Q}_p}, K)$

then due to uniqueness in $U(\mathfrak{g},K)$ we have $\sum_{\varphi(\beta)=\alpha} c_\beta \prod_{i,j} v_i^{\beta_{ij}} = 0$ and hence $\sum_{\varphi(\beta)=\alpha} c_\beta \mathfrak{Y}^\beta \in J$ for all α. Now $(\sum_{|\alpha|\leq N} \sum_{\varphi(\beta)=\alpha} c_\beta \mathfrak{Y}^\beta)_{N\geq 0}$ converges to δ as $N \to \infty$, proving the claim.
To see that $I \cap U(\mathfrak{g}_{\mathbb{Q}_p},K)$ is dense in J_r we note that as a direct consequence of Frommer's theorem $U(\mathfrak{g}_{\mathbb{Q}_p},K)$ is a K-Fréchet-Stein algebra with respect to the norms $||\cdot||_r$. As a closed ideal $I \cap U(\mathfrak{g}_{\mathbb{Q}_p},K)$ is a coadmissible module over $U(\mathfrak{g}_{\mathbb{Q}_p},K)$. Since J is dense in $I \cap U(\mathfrak{g}_{\mathbb{Q}_p},K)$ we know from Theorem A (cf. [26], section 3) that the corresponding coherent sheaf is given by the $U_r(\mathfrak{g}_{\mathbb{Q}_p},K)$-ideals J'_r where J'_r is the closure of J in $U_r(\mathfrak{g}_{\mathbb{Q}_p},K)$. The same reasoning as above shows that $I_r = \oplus_{\alpha\in A} \mathbf{b}^\alpha J'_r$. Since also $I_r = \oplus_{\alpha\in A}\mathbf{b}^\alpha J_r$ and $J'_r \subseteq J_r$ we obtain $J'_r = J_r$.

Let us now prove the last assertion on $\nu_{\overline{r}}$. Assume $\delta = \sum_\alpha d_\alpha \mathfrak{X}^\alpha \in U_r(\mathfrak{g},K)$, i.e. $\lim_{|\alpha|\to\infty} |d_\alpha|\nu_{\overline{r}}(\mathfrak{X}^\alpha) = 0$. Let $\varepsilon > 0$ be given and choose $N \in \mathbb{N}$ so large that
$$\sup_{|\alpha|\leq N} |d_\alpha|\nu_{\overline{r}}(\mathfrak{X}^\alpha) = \sup_\alpha |d_\alpha|\nu_{\overline{r}}(\mathfrak{X}^\alpha) \text{ and } \nu_{\overline{r}}(\sum_{|\alpha|>N} d_\alpha\mathfrak{X}^\alpha) \leq \varepsilon.$$
Note that the preimage of $\sum_{|\alpha|\leq N} d_\alpha \mathfrak{X}^\alpha$ under τ_r contains elements in $U(\mathfrak{g}_{\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} K$. By our above claim there is an element $\sum_\beta c_\beta \mathfrak{Y}^\beta \in U(\mathfrak{g}_{\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} K$ mapping to $\sum_{|\alpha|\leq N} d_\alpha\mathfrak{X}^\alpha$ under τ_r such that
$$\nu_{\overline{r}}(\sum_{|\alpha|\leq N} d_\alpha \mathfrak{X}^\alpha) \geq ||\sum_\beta c_\beta \mathfrak{Y}^\beta||_r - \varepsilon.$$
Uniqueness in $U(\mathfrak{g}) \otimes_L K$ implies that $\tau_r(\sum_{\varphi(\beta)=\alpha} c_\beta \mathfrak{Y}^\beta) = d_\alpha \mathfrak{X}^\alpha$ for all α with $|\alpha| \leq N$. Therefore,
$$||\sum_\beta c_\beta \mathfrak{Y}^\beta||_r = \sup_\beta |c_\beta|||\mathfrak{Y}^\beta||_r \geq \sup_{|\alpha|\leq N} \left\{ \sup_{\varphi(\beta)=\alpha} |c_\beta|||\mathfrak{Y}^\beta||_r \right\} \geq \sup_\alpha |d_\alpha|\nu_{\overline{r}}(\mathfrak{X}^\alpha).$$
Hence for all $\varepsilon > 0$
$$\max\{\varepsilon, \nu_{\overline{r}}(\delta)\} \geq \nu_{\overline{r}}(\sum_{|\alpha|\leq N} d_\alpha\mathfrak{X}^\alpha) \geq \sup_\alpha |d_\alpha|\nu_{\overline{r}}(\mathfrak{X}^\alpha) - \varepsilon,$$
i.e. $\nu_{\overline{r}}(\delta) \geq \sup_\alpha |d_\alpha|\nu_{\overline{r}}(\mathfrak{X}^\alpha)$. As one always has $\nu_{\overline{r}}(\delta) \leq \sup_\alpha |d_\alpha|\nu_{\overline{r}}(\mathfrak{X}^\alpha)$, this finishes the proof. □

Corollary 1.4.3. *Let G be a locally L-analytic group, H a closed locally L-analytic subgroup and G_0 as in Corollary 1.3.6. If $H_0 := H \cap G_0$ then $D(H_0,K)$ is a K-Fréchet-Stein algebra with respect to the family of norms $||\cdot||_{\overline{r}}$, $r \in p^{\mathbb{Q}}$, $1/p < r < 1$, restricted from $D(G_0,K)$ to $D(H_0,K)$. The conclusions of Theorem 1.4.2 hold for $D(H_0,K)$. If $r \in p^{\mathbb{Q}}$ is a real number with $1/p < r < 1$ then the closure $D_r(G_0,K)_{H_0}$ of $D(G_0,K)_{H_0}$ in $D_r(G_0,K)$ is a finitely generated, free $U_r(\mathfrak{g},K)$-module with the same basis in $K[H_0]$ as in Corollary 1.4.1 applied to the pair $(R^{L|\mathbb{Q}_p}G_0, R^{L|\mathbb{Q}_p}H_0)$.*

Proof: Since $R^{L|\mathbb{Q}_p}H_0$ is compatible with $R^{L|\mathbb{Q}_p}G_0$ we know from Corollary 1.4.1 that $D(R^{L|\mathbb{Q}_p}H_0,K)$ is a K-Fréchet-Stein algebra with respect to the family of norms $||\cdot||_r$, $r \in p^{\mathbb{Q}}$, $1/p < r < 1$, obtained by restriction from $D(R^{L|\mathbb{Q}_p} G_0, K)$. The commutativity of the diagram

$$\begin{array}{ccc} D(R^{L|\mathbb{Q}_p} H_0, K) & \hookrightarrow & D(R^{L|\mathbb{Q}_p} G_0, K) \\ \downarrow & & \downarrow \tau' \\ D(H_0, K) & \hookrightarrow & D(G_0, K) \end{array}$$

shows that the kernel of the left vertical arrow is $I' := I \cap D(R^{L|\mathbb{Q}_p} H_0, K)$. Applying Theorem 1.4.2 to H_0 shows that if we let I'_r be the closure of I' in $D_r(R^{L|\mathbb{Q}_p} H_0, K)$ then $D(H_0, K)$ is a K-Fréchet-Stein algebra with respect to the corresponding quotient norms and

$$D_r(H_0, K) = D_r(R^{L|\mathbb{Q}_p} H_0, K)/I'_r$$

(cf. (1.13) applied to H_0). Recall that we have

$$D_r(R^{L|\mathbb{Q}_p} G_0, K) = \bigoplus_{\alpha \in A} \mathbf{b}^\alpha U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$$

as K-Banach spaces and similarly

$$D_r(R^{L|\mathbb{Q}_p} H_0, K) = \bigoplus_{\alpha \in A'} \mathbf{b}^\alpha U_r(\mathfrak{h}_{\mathbb{Q}_p}, K)$$

with $A' \subseteq A$ (cf. Corollary 1.4.1 and its proof). Moreover, we know from the proof of Theorem 1.4.2 that $I_r = \oplus_{\alpha \in A} \mathbf{b}^\alpha J_r$ with $J_r := I_r \cap U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$ and similarly $I'_r = \oplus_{\alpha \in A'} \mathbf{b}^\alpha (I'_r \cap U_r(\mathfrak{h}_{\mathbb{Q}_p}, K))$. It follows that $I'_r = I_r \cap D_r(R^{L|\mathbb{Q}_p} H_0, K)$ and hence that

$$D_r(H_0, K) = D_r(R^{L|\mathbb{Q}_p} H_0, K)/(I_r \cap D_r(R^{L|\mathbb{Q}_p} H_0, K)). \tag{1.15}$$

We need to show that the image of $D_r(R^{L|\mathbb{Q}_p} H_0, K)$ under τ_r is closed.
Making use of the above direct sum decompositions it suffices to show that the image of $U_r(\mathfrak{h}_{\mathbb{Q}_p}, K)$ under τ_r is closed. We make use of the notation introduced earlier: By construction we may assume $\mathfrak{X}' := (\mathfrak{x}_1, \ldots, \mathfrak{x}_j)$, $1 \leq j := \dim H_0 \leq d$, to be an L-basis of $\mathfrak{h}$. Since $U_r(\mathfrak{g}, K) = \{\sum_\alpha d_\alpha \mathfrak{X}^\alpha \mid \lim_{|\alpha| \to \infty} |d_\alpha| \nu_{\overline{r}}(\mathfrak{X}^\alpha) = 0\}$ with $\nu_{\overline{r}}(\sum_\alpha d_\alpha \mathfrak{X}^\alpha) = \sup_\alpha |d_\alpha| \nu_{\overline{r}}(\mathfrak{X}^\alpha)$, a straightforward calculation shows that

$$\tau_r(U_r(\mathfrak{h}_{\mathbb{Q}_p}, K)) = \{\sum_{\alpha \in \mathbb{N}^j} d_\alpha (\mathfrak{X}')^\alpha \mid \lim_{|\alpha| \to \infty} |d_\alpha| \nu_{\overline{r}}((\mathfrak{X}')^\alpha) = 0\}$$

which is a closed subspace of $U_r(\mathfrak{g}, K)$.

According to the proof of Corollary 1.4.1 there is a finite basis $(\mathbf{b}^\alpha)_{\alpha \in A}$ of the free $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$-module $D_r(R^{L|\mathbb{Q}_p} G_0, K)$ and a subset $A' \subseteq A$ such that $(\mathbf{b}^\alpha)_{\alpha \in A'}$ is a basis of the free, finitely generated $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$-module $D_r(R^{L|\mathbb{Q}_p} G_0, K)_{H_0}$. It follows from the decomposition $I_r = \oplus_{\alpha \in A} \mathbf{b}^\alpha J_r$ that $I_r \cap D_r(R^{L|\mathbb{Q}_p} G_0, K)_{H_0} = \oplus_{\alpha \in A'} \mathbf{b}^\alpha J_r$. Thus, by (1.14)

$$D_r(R^{L|\mathbb{Q}_p} G_0, K)_{H_0}/(I_r \cap D_r(R^{L|\mathbb{Q}_p} G_0, K)_{H_0}) \simeq \oplus_{\alpha \in A'} \mathbf{b}^\alpha U_r(\mathfrak{g}, K). \tag{1.16}$$

In particular, the image of $D_r(R^{L|\mathbb{Q}_p}G_0, K)_{H_0}$ under τ_r is closed. It follows by means of Lemma 1.3.4 and (1.13) that the left hand side of (1.16) is topologically isomorphic to $D_r(G_0, K)_{H_0}$.
Note that by Theorem 1.4.2 $(\mathbf{b}^\alpha)_{\alpha\in A'}$ is also a basis of the free $U_r(\mathfrak{h}, K)$-module $D_r(H_0, K)$ and the free $U_r(\mathfrak{h}_{\mathbb{Q}_p}, K)$-module $D_r(R^{L|\mathbb{Q}_p}H_0, K)$. □

Corollary 1.4.4. *If $L_0|\mathbb{Q}_p$ and $L|L_0$ are finite extensions of fields and if G is a locally L-analytic group then the natural homomorphism*

$$D(R^{L|L_0}G, K)\hat{\otimes}_{U(\mathfrak{g}_{L_0},K),\iota}U(\mathfrak{g}, K) \longrightarrow D(G, K)$$

of $D(R^{L|L_0}G, K)$-$U(\mathfrak{g}, K)^{op}$-bimodules is a topological isomorphism.

Proof: Let G_0 be an open subgroup of G as in Proposition 1.3.5. Using $D(G, K) = \oplus_{g\in G/G_0}\delta_g \cdot D(G_0, K)$ (resp. with $R^{L|L_0}G$ and $R^{L|L_0}G_0$) it suffices to show that the map

$$D(R^{L|L_0}G_0, K)\hat{\otimes}_{U(\mathfrak{g}_{L_0},K)}U(\mathfrak{g}, K) \longrightarrow D(G_0, K) \tag{1.17}$$

is a topological isomorphism. One easily verifies that also $R^{L|L_0}G_0$ satisfies conditions (i) – (iii) of Proposition 1.3.5 (replacing L by L_0) so that according to Theorem 1.4.2 the modules $D_r(R^{L|L_0}G_0, K)$, resp. $D_r(G_0, K)$, are finitely generated and free over the noetherian Banach algebras $U_r(\mathfrak{g}_{L_0}, K)$, resp. $U_r(\mathfrak{g}, K)$, with a common basis $(\mathbf{b}^\alpha)_{\alpha\in A}$. It follows that the base change map

$$D_r(R^{L|L_0}G_0, K) \otimes_{U_r(\mathfrak{g}_{L_0},K)} U_r(\mathfrak{g}, K) \longrightarrow D_r(G_0, K)$$

is an isomorphism of $D_r(R^{L|L_0}G_0, K)$-$U_r(\mathfrak{g}, K)^{\mathrm{op}}$-bimodules. As in Proposition 1.2.12 one shows that it is bi-continuous and that we may pass to the projective limit in order to obtain that (1.17) is a topological isomorphism. □

The same line of proof gives:

Corollary 1.4.5. *Let $L_0|\mathbb{Q}_p$ and $L|L_0$ be finite extensions of fields and G be a locally L-analytic group. If H is a closed, locally L-analytic subgroup of G then the map $\tau' : D(R^{L|L_0}G, K)_H \to D(G, K)_H$ is surjective.* □

2 Invariant distributions

G acts on itself via conjugation inducing an action by continuous automorphisms on the space $C^{an}(G, K)$ of locally analytic functions on G. The contragredient action on $D(G, K)$ is explicitly given by $(g * \delta)(f) = \delta(h \mapsto f(ghg^{-1})) = (\delta_g\delta\delta_{g^{-1}})(f)$ for $g \in G$, $\delta \in D(G, K)$ and $f \in C^{an}(G, K)$, i.e.

$$g * \delta = \delta_g\delta\delta_{g^{-1}}. \tag{2.1}$$

We call a distribution $\delta \in D(G, K)$ *invariant* if $g * \delta = \delta$ for all $g \in G$. If U is a G-invariant subspace of $D(G, K)$ we denote by U^G the subspace of all invariant distributions contained in U.

The separate continuity of the multiplication together with the density of $K[G]$ in $D(G, K)$ imply by means of (2.1) that the subspace $D(G, K)^G$ of all invariant distributions on G coincides with the center of the ring $D(G,K)$.

For later use we introduce the subspace

$$D^{pt}(G,K) \quad := \quad \sum_{g \in G} \delta_g \cdot (U(\mathfrak{g}) \otimes_L K)$$

of $D(G,K)$. It is the space of all point distributions in the sense of [7], 13.2.1.

2.1 The infinitesimal center

Viewing $\mathfrak{g}$ as an abelian locally L-analytic group the space $C_0^\omega(\mathfrak{g},K)$ is defined as in (1.5). The exponential map *exp* induces a topological isomorphism

$$exp^* : C_1^\omega(G,K) \xrightarrow{\sim} C_0^\omega(\mathfrak{g},K)$$

which does not depend on the choice of *exp* (cf. the remark following III.4.3 Définition 1 of [4]). Dualizing, we obtain a topological isomorphism

$$exp_* : C_0^\omega(\mathfrak{g},K)'_b \xrightarrow{\sim} U(\mathfrak{g},K) = C_1^\omega(G,K)'_b$$

of locally convex vector spaces which for $\delta \in C_0^\omega(\mathfrak{g},K)'_b$ and $[f] \in C_1^\omega(G,K)$ is explicitly given by

$$(exp_*\delta)([f]) = \delta(exp^*[f]) = \delta([\mathfrak{x} \mapsto f(exp(\mathfrak{x}))]).$$

Here $[f]$ denotes the germ in 1 of a locally analytic function f defined in an open neighborhood of $1 \in G$.

Viewing $\mathfrak{g}$ as its own Lie algebra Proposition 1.2.8 shows that

$$C_0^\omega(\mathfrak{g},K)'_b = \left\{ \sum_\alpha d_\alpha \mathfrak{X}^\alpha \mid d_\alpha \in K, \forall r > 0 : \sup |d_\alpha| r^{-|\alpha|} < \infty \right\}$$

in terms of power series with commutative multiplication. Since the symmetric algebra $S(\mathfrak{g}) \otimes_L K$ of $\mathfrak{g}$ is dense in $C_0^\omega(\mathfrak{g},K)'_b$ we prefer to change notation and write $S(\mathfrak{g},K)$ instead of $C_0^\omega(\mathfrak{g},K)'_b$.

The action of G on $C^{an}(G,K)$ by conjugation descends to $C_1^\omega(G,K)$ (cf. (1.5)) which is a locally analytic G-representation in the sense of [25], section 3: if G_0 is a compact open subgroup of G then the natural projection $C^{an}(G,K) \to C_1^\omega(G,K)$ factors G_0-equivariantly through $C^{an}(G_0,K)$. By [14], Satz 3.3.4, the G_0-action on $C^{an}(G_0,K)$ is locally analytic whence so is the G_0-action on the barrelled quotient $C_1^\omega(G,K) = C_1^\omega(G_0,K)$ (cf. [12], Lemma 3.6.14). Since G_0 is open in G the claim follows.
Similarly, the action of G on $\mathfrak{g}$ via the adjoint representation *Ad* induces an action on $C_0^\omega(\mathfrak{g},K)$. Using the formula $g \cdot exp(\mathfrak{x}) \cdot g^{-1} = exp(Ad(g)(\mathfrak{x}))$ for $g \in G$ and all $\mathfrak{x}$ in a neighborhood of zero in $\mathfrak{g}$ depending on g (cf. [4], III.4.4 Corollaire 3) one deduces that exp^* is G-equivariant.

Recall that if $n \in \mathbb{N}$, $\mathfrak{y}_1, \ldots, \mathfrak{y}_n \in \mathfrak{g}$ and $\mathfrak{y}_1 \cdots \mathfrak{y}_n$ is their product in $S(\mathfrak{g})$ then the symmetrization map $sym : S(\mathfrak{g}) \to U(\mathfrak{g})$ is defined by

$$sym(\mathfrak{y}_1 \cdots \mathfrak{y}_n) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathfrak{y}_{\sigma(1)} \cdots \mathfrak{y}_{\sigma(n)}$$

through L-linear continuation. Here $\mathfrak{S}_n$ denotes the symmetric group on n letters.

Proposition 2.1.1. *$exp^* : C_1^\omega(G,K) \to C_0^\omega(\mathfrak{g},K)$ is an isomorphism of locally analytic G-representations on locally convex K-vector spaces of compact type. The corresponding dual map $exp_* : S(\mathfrak{g},K) \to U(\mathfrak{g},K)$ is an isomorphism of separately continuous (left) $D(G,K)$-modules. Its restriction to $S(\mathfrak{g}) \otimes_L K$ coincides with $sym \otimes id$ and maps isomorphically onto $U(\mathfrak{g}) \otimes_L K$. Further, if the $D(G,K)$-actions on $S(\mathfrak{g},K)$ and $U(\mathfrak{g},K)$ are denoted by $*$ then the following formulae hold:*

i) $\mathfrak{x} * \mathfrak{y} = [\mathfrak{x}, \mathfrak{y}]$ for all $\mathfrak{x}, \mathfrak{y} \in \mathfrak{g}$ where $\mathfrak{x}$ is considered as an element of $D(G,K)$ and $\mathfrak{y}, [\mathfrak{x}, \mathfrak{y}]$ as elements of $S(\mathfrak{g},K)$ (or $U(\mathfrak{g},K)$);

ii) $\mathfrak{x} * \delta = \mathfrak{x} \cdot \delta - \delta \cdot \mathfrak{x}$ in $U(\mathfrak{g},K)$ for all $\mathfrak{x} \in \mathfrak{g}$ and $\delta \in U(\mathfrak{g},K)$;

iii) $\mathfrak{x} * (\delta_1 \cdots \delta_n) = (\mathfrak{x} * \delta_1)\delta_2 \cdots \delta_n + \ldots + \delta_1 \cdots \delta_{n-1}(\mathfrak{x} * \delta_n)$ for all $\mathfrak{x} \in \mathfrak{g}$ and $\delta_1, \ldots, \delta_n \in S(\mathfrak{g},K)$.

Proof: The first statement follows from what was said above. By general principles the dual map $exp_* : S(\mathfrak{g},K) \to U(\mathfrak{g},K)$ is a topological isomorphism of nuclear Fréchet spaces carrying separately continuous $D(G,K)$-module structures for which exp_* is a homomorphism (cf. [25], Corollary 3.3). For the statement about the restriction of exp_* to $S(\mathfrak{g}) \otimes_L K$ confer [4], III.4.3 Théorème 4 and II.1.5 Proposition 9.

For $g \in G$, $\mathfrak{y} \in \mathfrak{g}$ and $f \in C^{an}(G,K)$ we have

$$\begin{aligned} (g * \mathfrak{y})(f) &= \mathfrak{y}(g^{-1} * f) = \frac{d}{dt} f(g \cdot exp(t\mathfrak{y}) \cdot g^{-1})|_{t=0} \\ &= \frac{d}{dt} f(exp(tAd(g)(\mathfrak{y})))|_{t=0} = Ad(g)(\mathfrak{y})(f) \end{aligned}$$

showing that $\mathfrak{g} \otimes_L K$ carries the structure of a $D(G,K)$-submodule of $U(\mathfrak{g},K)$ coming from the adjoint representation of G on $\mathfrak{g}$. By [4], III.3.12 Proposition 44, we have $\mathfrak{x} * \mathfrak{y} = d/dt(Ad(exp(t\mathfrak{x}))(\mathfrak{y}))_{t=0} = ad(\mathfrak{x})(\mathfrak{y}) = [\mathfrak{x}, \mathfrak{y}]$. Note that if V is a Banach space then the notion of a locally analytic G-representation as given in [25], section 3, coincides with the notion of an analytic Banach space representation in the sense of Bourbaki (cf. [14], Korollar 3.1.9).

By [4], III.3.11 Proposition 41 and (i) we have

$$\mathfrak{x} * (\prod_i \mathfrak{y}_i) = [\mathfrak{x}, \mathfrak{y}_1]\mathfrak{y}_2 \cdots \mathfrak{y}_n + \ldots + \mathfrak{y}_1 \cdots \mathfrak{y}_{n-1}[\mathfrak{x}, \mathfrak{y}_n]$$

for all $\mathfrak{x} \in \mathfrak{g}$. Since $[\mathfrak{x}, \mathfrak{y}_i] = \mathfrak{x}\mathfrak{y}_i - \mathfrak{y}_i\mathfrak{x}$ in $U(\mathfrak{g})$ we obtain (ii). The statements on $S(\mathfrak{g},K)$ are proved analogously. □

If $\delta = \sum_\alpha d_\alpha \mathfrak{X}^\alpha \in S(\mathfrak{g}, K)$ or $U(\mathfrak{g}, K)$ and $n \geq 0$ then we let $\delta^{\leq n} := \sum_{|\alpha| \leq n} d_\alpha \mathfrak{X}^\alpha$ and $\delta^{>n} := \sum_{|\alpha|>n} d_\alpha \mathfrak{X}^\alpha$. Note that if $g \in G$ then $g * \delta^{\leq n}$ is of degree $\leq n$ for every $n \in \mathbb{N}$. This follows from writing $g * \mathfrak{x}_i = \sum_j a_j \mathfrak{x}_j$, $a_j \in L$, and noting that by (2.1)

$$g * (\lambda \cdot \prod_i \mathfrak{x}_i^{\alpha_i}) = \lambda \cdot \prod_i (g * \mathfrak{x}_i)^{\alpha_i}.$$

In particular, G acts on $S(\mathfrak{g}) \otimes_L K$ and $U(\mathfrak{g}) \otimes_L K$.

Proposition 2.1.2. *$U(\mathfrak{g})^{\mathfrak{g}} \otimes_L K$ and $U(\mathfrak{g})^G \otimes_L K$ are dense in $U(\mathfrak{g}, K)^{\mathfrak{g}}$ and $U(\mathfrak{g}, K)^G$, respectively.*

Proof: Since exp_* is equivariant for the actions of $\mathfrak{g}$ and G we may equally well show that $S(\mathfrak{g})^{\mathfrak{g}} \otimes_L K$ and $S(\mathfrak{g})^G \otimes_L K$ are dense in $S(\mathfrak{g}, K)^{\mathfrak{g}}$ and $S(\mathfrak{g}, K)^G$, respectively. If $\delta \in S(\mathfrak{g}, K)$ is homogeneous of degree n then it follows from Proposition 2.1.1 that for $\mathfrak{x} \in \mathfrak{g}$ either $\mathfrak{x} * \delta = 0$ or $\mathfrak{x} * \delta$ is again homogeneous of degree n (write $[\mathfrak{x}, \mathfrak{x}_i] = \sum_j a_j \mathfrak{x}_j$ for $\mathfrak{x} \in \mathfrak{g}$, $a_j \in L$). We have seen above that similarly $g * \delta$ will again be homogeneous of degree n. Thus, if $\delta \in S(\mathfrak{g}, K)^{\mathfrak{g}}$ (resp. $S(\mathfrak{g}, K)^G$) then also $\delta^{\leq n}$ and $\delta^{>n}$ are $\mathfrak{g}$-invariant (resp. G-invariant). Since $\delta^{\leq n} \in S(\mathfrak{g}) \otimes_L K$ and $\delta^{\leq n} \to \delta$ for $n \to \infty$, the assertion follows. □

Remark 2.1.3. If G is an open subgroup of the group of L-rational points of a connected algebraic group over L then [25], Proposition 3.7, shows that $U(\mathfrak{g})^{\mathfrak{g}} \otimes_L K = U(\mathfrak{g})^G \otimes_L K$. According to Proposition 2.1.2 $U(\mathfrak{g}, K)^{\mathfrak{g}} = U(\mathfrak{g}, K)^G$. Similarly, $S(\mathfrak{g}, K)^{\mathfrak{g}} = S(\mathfrak{g}, K)^G$ in this case.

Remark 2.1.4. Let ν denote a norm on $S(\mathfrak{g}) \otimes_L K$ with respect to which the action of G (resp. $\mathfrak{g}$) is continuous. If the completion $S_\nu(\mathfrak{g}, K)$ of $S(\mathfrak{g}) \otimes_L K$ with respect to ν has the explicit description $\{\sum_\alpha d_\alpha \mathfrak{X}^\alpha \mid \lim_{|\alpha| \to \infty} |d_\alpha| \nu(\mathfrak{X}^\alpha) = 0\}$ with

$$\nu(\sum_\alpha d_\alpha \mathfrak{X}^\alpha) = \sup_\alpha |d_\alpha| \nu(\mathfrak{X}^\alpha),$$

then the above proof shows that $S(\mathfrak{g})^G \otimes_L K$ and $S(\mathfrak{g})^{\mathfrak{g}} \otimes_L K$ are even dense in $S_\nu(\mathfrak{g}, K)^G$ and $S_\nu(\mathfrak{g}, K)^{\mathfrak{g}}$, respectively.

In general, the restriction of exp_* to $S(\mathfrak{g}, K)^{\mathfrak{g}}$ is not an isomorphism of algebras although both $S(\mathfrak{g}, K)^{\mathfrak{g}}$ and $U(\mathfrak{g}, K)^{\mathfrak{g}}$ are commutative. Making use of a construction by M. Duflo we will show, however, that one does obtain an isomorphism

$$\eta : S(\mathfrak{g}, K)^{\mathfrak{g}} \to U(\mathfrak{g}, K)^{\mathfrak{g}}$$

of K-algebras if exp_* is suitably normalized. This result is similar to the conjecture of Kashiwara and Vergne for real Lie groups (cf. [1]) involving, however, distributions on germs of functions rather than germs of distributions.

Recall the following construction (cf. [10], p. 55): let k be a field of characteristic zero and $\mathfrak{h}$ a Lie algebra of finite dimension over k. Choosing dual k-bases $(\mathfrak{x}_1, \ldots, \mathfrak{x}_d)$ and $(\mathfrak{x}_1^*, \ldots, \mathfrak{x}_d^*)$ of $\mathfrak{h}$ and $\mathfrak{h}^*$, respectively, we identify $S(\mathfrak{h})$ with the algebra of polynomial functions on $\mathfrak{h}^*$ and $S(\mathfrak{h}^*)$ with the algebra of differential operators with constant coefficients on $\mathfrak{h}^*$ (denoting by $D(q)$ the operator defined by an element $q \in S(\mathfrak{h}^*)$). The completion $\hat{S}(\mathfrak{h}^*)$ of $S(\mathfrak{h}^*)$ with respect to the topology defined by the maximal ideal $(\mathfrak{x}_1^*, \ldots, \mathfrak{x}_d^*)$ may be identified with the algebra of formal power series in the variables $\mathfrak{x}_i^*$ over k. If $f \in S(\mathfrak{h})$ is given and the first non-zero coefficient of $q \in S(\mathfrak{h}^*)$ appears in sufficiently high order then $D(q)(f) = 0$. Hence for $q \in \hat{S}(\mathfrak{h}^*)$ one can define $D(q)(f)$ by continuity and set $\langle q, f\rangle := D(q)(f)(0)$. This identifies $S(\mathfrak{h})$ with the space $\hat{S}(\mathfrak{h}^*)'$ of continuous functionals on $\hat{S}(\mathfrak{h}^*)$.

If $S(\mathfrak{h})$ is identified with the algebra of constant coefficient differential operators on $\mathfrak{h}$ and $f \in S(\mathfrak{h})$ then we let $D^*(f)$ be the corresponding operator. $D^*(f)$ is an endomorphism of $\hat{S}(\mathfrak{h}^*)$. If $q \in \hat{S}(\mathfrak{h}^*)$ is a power series we let $q(0)$ be its constant term. According to the remarks preceding Lemme II.2 of [loc.cit.] we have

$$D^*(f)(q)(0) = D(q)(f)(0) = \langle q, f\rangle \tag{2.2}$$

for all $q \in \hat{S}(\mathfrak{h}^*)$ and $f \in S(\mathfrak{h})$.

Let $ad(\mathfrak{X}) \in M_d(k[\mathfrak{x}_1^*, \ldots, \mathfrak{x}_d^*])$ be the matrix $ad(\mathfrak{X}) := \sum_i \mathfrak{x}_i^* A_i$ where $A_i \in M_d(k)$ represents $ad(\mathfrak{x}_i) \in \mathrm{End}_k(\mathfrak{h})$ with respect to the k-basis $(\mathfrak{x}_1, \ldots, \mathfrak{x}_d)$ of $\mathfrak{h}$. If $B_{2n} \in \mathbb{Q}$ denote the Bernoulli numbers of even degree and $exp(t) \in \mathbb{Q}[[t]]$ is the usual exponential series then the formula

$$\begin{aligned} q = q(\mathfrak{x}_1^*, \ldots, \mathfrak{x}_d^*) &:= \det\left(\frac{exp(ad(\mathfrak{X})/2) - \ exp(-ad(\mathfrak{X})/2)}{ad(\mathfrak{X})}\right)^{1/2} \\ &= \ exp(\sum_{n=1}^{\infty} \frac{B_{2n}}{4n(2n)!} tr[ad(\mathfrak{X})^{2n}]) \end{aligned} \tag{2.3}$$

defines a formal power series in the indeterminates $\mathfrak{x}_i^*$ with coefficients in k, i.e. an element of $\hat{S}(\mathfrak{h}^*)$ (for the second formula cf. [1]). One of the main results of [11] is the following theorem (loc.cit. Théorème 2):

Theorem (Duflo)**.** *If $\mathfrak{h}$ is a finite dimensional Lie algebra over a field k of characteristic zero then the normalized symmetrization map*

$$\eta := sym \circ D(q) : S(\mathfrak{h})^{\mathfrak{h}} \to U(\mathfrak{h})^{\mathfrak{h}}$$

is a well-defined isomorphism of k-algebras.

It is known that in the case of Lie algebras $\mathfrak{h}$ over the fields $k = \mathbb{R}$ or $\mathbb{C}$, the formal power series q defines an analytic function around 0 in $\mathfrak{h}$. This is also true for the Lie algebra $\mathfrak{g}$ over the non-archimedean field L:

Proposition 2.1.5. *The formal power series q defines an analytic function in a neighborhood of* 0 *in $\mathfrak{g}$. If we let $[q] \in C_0^\omega(\mathfrak{g}, K)$ denote its germ in* 0 *then the normalized exponential map $\eta : S(\mathfrak{g}, K) \to U(\mathfrak{g}, K)$ defined by*

$$\eta(\delta)([f]) := \delta([q] \cdot exp^*([f])) \ \text{for}\ \delta \in S(\mathfrak{g}, K) \ \text{and}\ [f] \in C_1^\omega(G, K),$$

restricts to a topological isomorphism of K-Fréchet algebras

$$\eta : S(\mathfrak{g}, K)^{\mathfrak{g}} \xrightarrow{\sim} U(\mathfrak{g}, K)^{\mathfrak{g}}.$$

Proof: Using the estimates $|n!| \geq p^{-n/(p-1)}$ and $|B_{2n}| \leq p$ (cf. [21], Lemma 5.3.1 and Corollary 5.5.5) it is straightforward to show that q defines an analytic function in a neighborhood of zero in $\mathfrak{g}$.

The normalized exponential map $\eta : S(\mathfrak{g}, K) \to U(\mathfrak{g}, K)$ is a topological isomorphism of K-Fréchet spaces: Note that $q(0) = 1$ so that $[q]$ is invertible in $C_0^\omega(\mathfrak{g}, K)$. If $\delta \in S(\mathfrak{g})$ and $[p] \in C_0^\omega(\mathfrak{g}, K)$ is represented by a formal power series $p \in \hat{S}(\mathfrak{g}^*)$ then by (2.2) and [10], Lemme II.1,

$$\begin{aligned} \delta([q] \cdot [p]) &= D^*(\delta)(qp)(0) = D(qp)(\delta)(0) = \langle qp, \delta \rangle \\ &= \langle p, D(q)(\delta) \rangle = D(q)(\delta)([p]). \end{aligned}$$

Since the restriction of *exp*$_*$ to $S(\mathfrak{g}) \otimes_L K$ coincides with *sym* (cf. Proposition 2.1.1) it follows that $\eta | S(\mathfrak{g}, K)^{\mathfrak{g}}$ extends Duflo's isomorphism. Since by Proposition 2.1.2 $S(\mathfrak{g})^{\mathfrak{g}} \otimes_L K$ (resp. $U(\mathfrak{g})^{\mathfrak{g}} \otimes_L K$) is dense in $S(\mathfrak{g}, K)^{\mathfrak{g}}$ (resp. $U(\mathfrak{g}, K)^{\mathfrak{g}}$) it follows that η is an isomorphism of algebras onto $U(\mathfrak{g}, K)^{\mathfrak{g}}$. □

We are now going to explicitly compute $U(\mathfrak{g}, K)^{\mathfrak{g}}$ in the case that $\mathfrak{g}$ is semisimple and contains a split maximal toral subalgebra $\mathfrak{t}$ (cf. [8], 1.9.10). The Weyl group $\mathfrak{W} = \mathfrak{W}(\mathfrak{g}, \mathfrak{t})$ acts on $\mathfrak{t}^*$ by L-linear endomorphisms and dually on $\mathfrak{t}$. Thus, $\mathfrak{W}$ acts continuously on $C^{an}(\mathfrak{t}, K)$. Since the subspace $C^{an}(\mathfrak{t}, K)_{\mathfrak{t} \setminus \{0\}}$ is $\mathfrak{W}$-invariant $\mathfrak{W}$ acts on the quotient $C_0^\omega(\mathfrak{t}, K)$ and hence on $S(\mathfrak{t}, K)$.

Theorem 2.1.6. *If $\mathfrak{g}$ is split semisimple with $\mathfrak{t}$ and $\mathfrak{W}$ as above then there are isomorphisms*

$$U(\mathfrak{g}, K)^{\mathfrak{g}} \simeq S(\mathfrak{t}, K)^{\mathfrak{W}} \simeq \mathcal{O}((\mathbb{A}_K^n)^{an})$$

of K-Fréchet algebras with $n := \dim_L(\mathfrak{t})$. Here $\mathcal{O}((\mathbb{A}_K^n)^{an})$ is the K-Fréchet algebra of holomorphic functions on the rigid analytic affine space $(\mathbb{A}_K^n)^{an}$ of dimension n over K.

In order to construct the above isomorphisms we need some preparation. Let k be a field which is complete with respect to a non-trivial, non-archimedean valuation and $(\cdot)^{an}$ be the rigid analytification functor on the category of k-schemes which are locally of finite type.

Proposition 2.1.7. *Let X be an affine scheme of finite type over k, Γ a finite group of k-automorphisms of X and $\pi : X \to X/\Gamma$ be the canonical quotient map. The presheaf $\mathcal{F}$ on $(X/\Gamma)^{an}$ defined by $\mathcal{F}(U) := \mathcal{O}_{X^{an}}((\pi^{an})^{-1}(U))^\Gamma$ is an $\mathcal{O}_{(X/\Gamma)^{an}}$-submodule of $\pi_*^{an}\mathcal{O}_{X^{an}}$ via the natural map $(\pi^{an})^\# : \mathcal{O}_{(X/\Gamma)^{an}} \to \pi_*^{an}\mathcal{O}_{X^{an}}$. In fact, $(\pi^{an})^\#$ is an isomorphism onto $\mathcal{F}$.*

Proof: With π also π^{an} is surjective and we have the following commutative diagram of locally G-ringed spaces:

$$(2.4) \qquad \begin{array}{ccc} X^{an} & \xrightarrow{\pi^{an}} & (X/\Gamma)^{an} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\pi} & X/\Gamma. \end{array}$$

We see that if $U \subseteq (X/\Gamma)^{an}$ is admissible open then $V := (\pi^{an})^{-1}(U) \subseteq X^{an}$ is admissible open and Γ-invariant. Thus, Γ acts on $\mathcal{O}_{X^{an}}(V)$ so that the presheaf

$\mathcal{F}$ is well-defined. It is straightforward to check that it is in fact a sheaf of $\mathcal{O}_{X^{an}}$-modules. By [2], 9.4.1 Proposition 2, it suffices to prove the claim for an admissible open covering $(U_i)_{i\in I}$ of $(X/\Gamma)^{an}$. By construction, $(X/\Gamma)^{an}$ admits a countable, ascending covering by open affinoid subdomains $U_i := \mathrm{Sp}(B_i)$, $i \in \mathbb{N}$. Setting $A := \mathcal{O}_X(X)$ and $B := A^\Gamma$ the algebra $A_i := B_i \otimes_B A$ is finite over B_i and hence k-affinoid. In fact, the maps $A_{i+1} \to A_i$ induced by $B_{i+1} \to B_i$ define $(\mathrm{Sp}(A_i))_{i\in I}$ as an admissible covering of X^{an} with $(\pi^{an})^{-1}(U_i) = \mathrm{Sp}(A_i)$ (this is the way one shows that with π also its analytification is finite). Since B_i is flat over B (cf. [20], Satz 2.1) we have

$$A_i^\Gamma = (B_i \otimes_B A)^\Gamma = B_i \otimes_B A^\Gamma = B_i$$

(cf. [3], I.2.3 Remark 2 and I.2.6 Remark 1). □

Remark 2.1.8. It follows from (2.4) that the underlying point space of $(X/\Gamma)^{an}$ is the set theoretical quotient of X^{an} modulo Γ. According to Proposition 2.1.7 the structure sheaf of $(X/\Gamma)^{an}$ is given by $\mathcal{O}_{(X/\Gamma)^{an}}(U) = \mathcal{O}_{X^{an}}((\pi^{an})^{-1}(U))^\Gamma$ so that $(X/\Gamma)^{an}$ can be identified with the rigid analytic quotient X^{an}/Γ whose existence is claimed (but not proved) in [15], 6.4.

Proof of Theorem 2.1.6: Let $\mathbf{t} = (t_1, \ldots, t_n)$ be an L-basis of $\mathfrak{t}$ considered also as a K-basis of $\mathfrak{t}\otimes_L K$. Proposition 1.2.8 shows that there is a topological isomorphism $S(\mathfrak{t}, K) \to \mathcal{O}((\mathbb{A}^n_K)^{an})$ of K-Fréchet algebras identifying the subalgebra $S(\mathfrak{t})\otimes_L K$ with the polynomial algebra $K[t_1, \ldots, t_n]$ in the variables t_i, i.e. with the algebra of regular functions on the affine space $\mathbb{A}^n_K$ of dimension n over K. There is a family $\mathbf{s} = (s_1, \ldots, s_n)$ of n algebraically independent, homogeneous elements in $(S(\mathfrak{t}) \otimes_L K)^{\mathfrak{W}}$ such that the inclusion homomorphism

$$\varphi : K[s_1, \ldots, s_n] \longrightarrow (S(\mathfrak{t}) \otimes_L K)^{\mathfrak{W}}$$

is an isomorphism (cf. [8], 11.1.14). According to Proposition 2.1.7 it extends to an isomorphism

$$\varphi : \mathcal{O}((\mathbb{A}^n_K)^{an}) \longrightarrow S(\mathfrak{t}, K)^{\mathfrak{W}}$$

of K-algebras. If $c \in K^*$ with $|c| > 1$ and $i \in \mathbb{N}$ we denote by $|\cdot|_i$ the norm on the left hand side for which the family $(\mathbf{s}^\alpha)_{\alpha\in\mathbb{N}^n}$ is orthogonal with $|s_j^{\alpha_j}|_i = |c^i|^{\alpha_j}$. Similarly, $\nu_i := \nu_{|c^{-i}|}$ is the multiplicative norm on $S(\mathfrak{t}, K)$ for which $(\mathbf{t}^\alpha)_{\alpha\in\mathbb{N}^n}$ is orthogonal with $\nu_i(t_j^{\alpha_j}) = |c^i|^{\alpha_j}$. We view $S(\mathfrak{t}, K)^{\mathfrak{W}}$ as a (closed) subspace of $S(\mathfrak{t}, K)$. Given $i \in \mathbb{N}$ choose $i_0 \in \mathbb{N}$ such that $\max_j\{\nu_i(\varphi(s_j))\} \le |c^{i_o}|$. Then

$$\nu_i(\varphi(\sum_\alpha d_\alpha \mathbf{s}^\alpha)) \le |\sum_\alpha d_\alpha \mathbf{s}^\alpha|_{i_0},$$

so that φ is continuous and in fact a topological isomorphism due to the open mapping theorem.

Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$ and choose an eigenvector X_α of α in $\mathfrak{g}$ for any $\alpha \in \Phi$. Extend $\mathbf{t}$ to the L-basis $\mathfrak{X} = (t_1, \ldots, t_n, (X_\alpha)_{\alpha\in\Phi})$ of $\mathfrak{g}$ and let $\overline{J}$ be the closed ideal of $S(\mathfrak{g}, K)$ generated by $\{X_\alpha\}_{\alpha\in\Phi}$. The explicit descriptions of $S(\mathfrak{g}, K)$ and $S(\mathfrak{t}, K)$ show that

$$S(\mathfrak{g}, K) = S(\mathfrak{t}, K) \oplus \overline{J}$$

first as abstract vector spaces but then also topologically due to the open mapping theorem. We claim that the induced continuous, surjective homomorphism $S(\mathfrak{g},K) \to S(\mathfrak{t},K)$ of K-algebras restricts to a topological isomorphism $\theta : S(\mathfrak{g},K)^{\mathfrak{g}} \xrightarrow{\sim} S(\mathfrak{t},K)^{\mathfrak{W}}$. By the open mapping theorem we only need to show that θ is bijective. If $J := S(\mathfrak{g}) \cap \overline{J}$ then $S(\mathfrak{g}) = S(\mathfrak{t}) \oplus J$ and the corresponding projection $S(\mathfrak{g}) \to S(\mathfrak{t})$ restricts to an isomorphism

$$S(\mathfrak{g})^{\mathfrak{g}} \simeq S(\mathfrak{t})^{\mathfrak{W}} \tag{2.5}$$

of algebras (cf. [8], Théorème 7.3.7).

As in the proof of Proposition 2.1.2 one sees that if δ is an element of $S(\mathfrak{g},K)^{\mathfrak{g}}$ (resp. $\overline{J}$) then both $\delta^{\leq n}$ and $\delta^{>n}$ are elements of $S(\mathfrak{g},K)^{\mathfrak{g}}$ (resp. $\overline{J}$). Since $(S(\mathfrak{g})^{\mathfrak{g}} \otimes_L K) \cap (J \otimes_L K) = 0$ it follows that $S(\mathfrak{g},K)^{\mathfrak{g}} \cap \overline{J} = 0$.

Let $\tau \in S(\mathfrak{t},K)^{\mathfrak{W}}$. It follows from (1.3) that for $\mathfrak{x}_1, \mathfrak{x}_2 \in S(\mathfrak{t}) \otimes_L K$ and $w \in \mathfrak{W}$

$$w \cdot (\mathfrak{x}_1 \cdot \mathfrak{x}_2) = (w \cdot \mathfrak{x}_1) \cdot (w \cdot \mathfrak{x}_2).$$

Thus, the homogeneous components τ_k of τ of degree k with respect to the variables $\mathbf{t}$ are $\mathfrak{W}$-invariant for all $k \geq 0$. Write $\tau_k = \sum_\alpha d_\alpha(k)\mathbf{s}^\alpha$ and let $\xi_1, \ldots, \xi_n \in S(\mathfrak{g})^{\mathfrak{g}}$ be preimages of $s_1, \ldots, s_n$ under the map (2.5). Then $\gamma_k := \sum_\alpha d_\alpha(k)\xi^\alpha \in S(\mathfrak{g})^{\mathfrak{g}} \otimes_L K$ maps to τ_k and we need to show that the series $\sum_k \gamma_k$ converges in $S(\mathfrak{g},K)$. Note that the Fréchet topology on $S(\mathfrak{g},K)$ can be defined by a family of multiplicative norms $(\nu_i)_{i\in\mathbb{N}}$ extending the norms ν_i on $S(\mathfrak{t},K)$ because $\mathfrak{X}$ extends the L-basis $\mathbf{t}$ of $\mathfrak{t}$ (cf. Proposition 1.2.8). Since φ^{-1} is continuous we have $\lim_{k\to\infty} |\varphi^{-1}(\tau_k)|_i = 0$ for all $i \in \mathbb{N}$. Given $i \in \mathbb{N}$, choose $i_0 \in \mathbb{N}$ such that $\max_j\{\nu_i(\xi_j)\} \leq |c^{i_0}|$. Then

$$\begin{aligned} \nu_i(\gamma_k) &\leq \sup_\alpha |d_\alpha(k)|\nu_i(\xi^\alpha) \leq \sup_\alpha |d_\alpha(k)||c^{i_0}|^{|\alpha|} \\ &= |\sum_\alpha d_\alpha(k)\mathbf{s}^\alpha|_{i_0} = |\varphi^{-1}(\tau_k)|_{i_0} \to 0 \end{aligned}$$

as $k \to \infty$. Composing θ with the inverse of Duflo's isomorphism we obtain the isomorphism $\xi := \theta \circ \eta^{-1} : U(\mathfrak{g},K)^{\mathfrak{g}} \to S(\mathfrak{t},K)^{\mathfrak{W}}$ of K-Fréchet algebras. □

2.2 Centrally supported invariant distributions

The conjugation action of G on $D(G,K)$ restricts to $U(\mathfrak{g},K)$, $D(Z,K)$ and $D(G,K)_Z$, inducing an action on $D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K)$.

Theorem 2.2.1. *If K is discretely valued and G is an open subgroup of the group of L-rational points of a connected, algebraic group defined over L then there are K-linear topological isomorphisms*

$$D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K)^G \simeq (D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K))^G \simeq D(G,K)_Z^G$$

of separately continuous K-algebras induced by multiplication in $D(G,K)_Z^G$. In particular, the subspace $D^{pt}(G,K)_Z^G$ of centrally supported invariant point distributions is dense in $D(G,K)_Z^G$.

Proof: We endow $D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K)^G$ and $(D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K))^G$ with the $D(Z,K)\otimes_{U(\mathfrak{z},K)}U(\mathfrak{g},K)^G$-module actions of Remark 1.2.11. Since $D(Z,K)$ and $U(\mathfrak{g},K)^G$ are contained in the center of $D(G,K)$ it is clear that the maps

$$\begin{array}{rcl} D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K)^G & \longrightarrow & D(G,K)_Z^G \\ (D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K))^G & \longrightarrow & D(G,K)_Z^G \end{array}$$

induced by multiplication are homomorphisms of $D(Z,K)\otimes_{U(\mathfrak{z},K)}U(\mathfrak{g},K)^G$-modules. If we can show them to be topological isomorphisms then, by the density of $D(Z,K)\otimes_{U(\mathfrak{z},K)}U(\mathfrak{g},K)^G$ in the space $D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K)^G$, both $D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K)^G$ and $(D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K))^G$ carry unique K-algebra structures extending the action of $D(Z,K)\otimes_{U(\mathfrak{z},K)}U(\mathfrak{g},K)^G$ and for which the above maps are homomorphisms.

The $D(Z,K)$-$U(\mathfrak{g},K)^{op}$-bimodule isomorphism

$$\mu : D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K) \longrightarrow D(G,K)_Z$$

of Proposition 1.2.12 is G-equivariant by definition of the respective G-actions. This gives the second isomorphism of the theorem.

If G_0 is a compact open subgroup of G and $Z_0 := G_0\cap Z$ then, using Lemma 1.2.13, it suffices to show that the map

$$D(Z_0,K)\hat{\otimes}_{U(\mathfrak{z},K)}U(\mathfrak{g},K)^G \longrightarrow D(G_0,K)_{Z_0}^G$$

induced by multiplication is a topological isomorphism.

According to [4], III.7.2 Proposition 3, there are compact open subgroups $\Lambda_\mathfrak{g}$ and G_0 of $\mathfrak{g}$ and G, respectively, such that $\Lambda_\mathfrak{g}$ lies in the domain of the exponential map and $exp : \Lambda_\mathfrak{g} \to G_0$ is an isomorphism of locally L-analytic manifolds. In fact, $\Lambda_\mathfrak{g}$ may be chosen to be contained in any open neighborhood of zero in $\mathfrak{g}$. If therefore $\Lambda_\mathfrak{z} := \Lambda_\mathfrak{g}\cap\mathfrak{z}$ and $Z_0 := G_0\cap Z$ then we may assume exp to restrict to an isomorphism $\Lambda_\mathfrak{z}\to Z_0$ (note that exp is also an exponential map for Z_0). The K-linear topological isomorphism $exp_* : D(\Lambda_\mathfrak{g},K)\to D(G_0,K)$ therefore restricts to isomorphisms

$$\begin{array}{rll} exp_* : & D(\Lambda_\mathfrak{z},K) \longrightarrow D(Z_0,K) & \\ id : & S(\mathfrak{z},K) \longrightarrow U(\mathfrak{z},K) & \text{and} \\ exp_* : & S(\mathfrak{g},K) \longrightarrow U(\mathfrak{g},K). & \end{array}$$

Lemma 2.2.2. *If $\lambda\in D(\Lambda_\mathfrak{z},K)$ and $\delta\in D(\Lambda_\mathfrak{g},K)$ then $exp_*(\lambda\cdot\delta) = exp_*(\lambda)\cdot exp_*(\delta)$.*

Proof: Let $\mathfrak{y}\in\Lambda_\mathfrak{z}$ and $f\in C^{an}(G_0,K)$. Then

$$\begin{array}{rcl} exp_*(\delta_\mathfrak{y}\cdot\delta)(f) & = & (\delta_\mathfrak{y}\cdot\delta)(exp^* f) \\ & = & \delta(\mathfrak{x}\mapsto f(exp(\mathfrak{y}+\mathfrak{x}))) \\ & = & \delta(\mathfrak{x}\mapsto f(exp(\mathfrak{y})\cdot exp(\mathfrak{x}))) \\ & = & (exp_*(\delta_\mathfrak{y})\cdot exp_*(\delta))(f), \end{array}$$

since $\mathfrak{y}$ commutes with all $\mathfrak{x} \in \mathfrak{g}$. Since $K[\Lambda_\mathfrak{z}]$ is dense in $D(\Lambda_\mathfrak{z}, K)$, the assertion follows from the linearity and continuity of exp_*. $\square$

Together with Lemma 1.2.10 we obtain that exp_* restricts to an isomorphism $D(\Lambda_\mathfrak{g}, K)_{\Lambda_\mathfrak{z}} \to D(G_0, K)_{Z_0}$ and that the diagram

$$\begin{array}{ccc} D(\Lambda_\mathfrak{z}, K)\hat{\otimes}_{S(\mathfrak{z},K)} S(\mathfrak{g}, K) & \xrightarrow{\ \mu\ } & D(\Lambda_\mathfrak{g}, K)_{\Lambda_\mathfrak{z}} \\ {\scriptstyle exp_*\hat{\otimes} exp_*}\downarrow\wr & & \wr\downarrow{\scriptstyle exp_*} \\ D(Z_0, K)\hat{\otimes}_{U(\mathfrak{z},K)} U(\mathfrak{g}, K) & \xrightarrow{\ \mu\ } & D(G_0, K)_{Z_0} \end{array}$$

is commutative. G acts trivially on $D(\Lambda_\mathfrak{z}, K)$ and $D(Z_0, K)$. Moreover, G acts on $S(\mathfrak{g}, K)$ in such a way that $exp_* : S(\mathfrak{g}, K) \to U(\mathfrak{g}, K)$ is G-equivariant. Thus, there is an action of G on $D(\Lambda_\mathfrak{g}, K)_{\Lambda_\mathfrak{z}}$ such that $exp_* : D(\Lambda_\mathfrak{g}, K)_{\Lambda_\mathfrak{z}} \to D(G_0, K)_{Z_0}$ is G-equivariant and we may equally well show the above statements in the setting of $\Lambda_\mathfrak{g}$ and $\Lambda_\mathfrak{z}$.

Passing to an open subgroup of $\Lambda_\mathfrak{g}$, we may assume that $\Lambda_\mathfrak{g}$ and $\Lambda_\mathfrak{z}$ satisfy the compatibility conditions of Corollary 1.3.6. Hence for $r \in p^\mathbb{Q}$ with $1/p < r < 1$ the K-Banach algebra $D_r(\Lambda_\mathfrak{g}, K)_{\Lambda_\mathfrak{z}}$ admits a finite direct sum decomposition

$$D_r(\Lambda_\mathfrak{g}, K)_{\Lambda_\mathfrak{z}} = \bigoplus_{\alpha \in A'} \mathbf{b}^\alpha S_r(\mathfrak{g}, K)$$

with $\mathbf{b}^\alpha \in K[\Lambda_\mathfrak{z}]$ for all $\alpha \in A'$ (cf. Corollary 1.4.3).

Lemma 2.2.3. *The action of $\mathfrak{g}$ on $D(\Lambda_\mathfrak{g}, K)_{\Lambda_\mathfrak{z}}$ induced by that of G extends to a $\mathfrak{g}$-action on $D_r(\Lambda_\mathfrak{g}, K)_{\Lambda_\mathfrak{z}}$.*

Proof: It suffices to show that the action of $\mathfrak{g}$ on $S(\mathfrak{g}, K)$ is continuous with respect to the norm $||\cdot||_{\overline{r}}$. Note that by Corollary 1.4.5 there is a continuous K-linear surjection $\tau' : S(\mathfrak{g}_{\mathbb{Q}_p}, K) \to S(\mathfrak{g}, K)$ which is seen to be $\mathfrak{g}$-equivariant (use Proposition 2.1.1). As a direct consequence of Frommer's theorem $S(\mathfrak{g}_{\mathbb{Q}_p}, K)$ is a K-Fréchet-Stein algebra. Therefore, $S(\mathfrak{g}, K)$ and the kernel J of τ' are coadmissible modules over $S(\mathfrak{g}_{\mathbb{Q}_p}, K)$. According to Theorem B (cf. [26], section 3) the coherent sheaf corresponding to J is given by the kernels J_r of the surjections $S_r(\mathfrak{g}_{\mathbb{Q}_p}, K) \to S_r(\mathfrak{g}, K)$ (cf. (1.14)). Since J is $\mathfrak{g}$-invariant and dense in J_r (cf. Theorem A of [26], section 3), we may assume $L = \mathbb{Q}_p$ and hence $||\cdot||_{\overline{r}} = ||\cdot||_r$ to be multiplicative.

Recall from Frommer's theorem that there is a $\mathbb{Q}_p$-basis $\mathfrak{X} = (\mathfrak{x}_1, \ldots, \mathfrak{x}_d)$ of $\mathfrak{g}$ such that

$$S_r(\mathfrak{g}, K) = \left\{ \sum_\alpha d_\alpha \mathfrak{X}^\alpha \mid d_\alpha \in K,\ \lim_{|\alpha|\to\infty} |d_\alpha| ||\mathfrak{X}^\alpha||_r = 0 \right\}$$

with $||\sum_\alpha d_\alpha \mathfrak{X}^\alpha||_r = \sup_\alpha \{|d_\alpha| \prod_{i=1}^d ||\mathfrak{x}_i||_r^{\alpha_i}\}$. For $\mathfrak{x} \in \mathfrak{g}$ choose $\lambda \in \mathbb{Q}_p^*$ such that $||ad(\lambda\mathfrak{x})(\mathfrak{x}_i)||_r \le ||\mathfrak{x}_i||_r$ for all i. It follows that $||\mathfrak{x} * \delta||_r \le |\lambda^{-1}| \cdot ||\delta||_r$ for all $\delta \in S(\mathfrak{g}, K)$. $\square$

We obtain

$$D_r(\Lambda_{\mathfrak{g}}, K)^{\mathfrak{g}}_{\Lambda_{\mathfrak{z}}} = \bigoplus_{\alpha \in A'} \mathbf{b}^\alpha S_r(\mathfrak{g}, K)^{\mathfrak{g}}.$$

Since, as remarked in the proof of Corollary 1.4.3, $(\mathbf{b}^\alpha)_{\alpha \in A'}$ is also a basis for the free $S_r(\mathfrak{z}, K)$-module $D_r(\Lambda_{\mathfrak{z}}, K)$ we obtain a topological isomorphism

$$D_r(\Lambda_{\mathfrak{z}}, K) \otimes_{S_r(\mathfrak{z},K)} S_r(\mathfrak{g}, K)^{\mathfrak{g}} \longrightarrow D_r(\Lambda_{\mathfrak{g}}, K)^{\mathfrak{g}}_{\Lambda_{\mathfrak{z}}}.$$

Passing to the projective limit we obtain a topological isomorphism

$$D(\Lambda_{\mathfrak{g}}, K) \hat{\otimes}_{S(\mathfrak{z},K)} S(\mathfrak{g}, K)^{\mathfrak{g}} \longrightarrow D(\Lambda_{\mathfrak{g}}, K)^{\mathfrak{g}}_{\Lambda_{\mathfrak{z}}}$$

as in the proof of Proposition 1.2.12: To satisfy the Mittag-Leffler condition we need to know that $S(\mathfrak{g}, K)^{\mathfrak{g}}$ is dense in $S_r(\mathfrak{g}, K)^{\mathfrak{g}}$ for all r. This is true according to Remark 2.1.4 and Theorem 1.4.2 and is in fact the reason for our working with $\Lambda_{\mathfrak{g}}$ and $\Lambda_{\mathfrak{z}}$ instead of with G_0 and Z_0. By our assumption on G and Remark 2.1.3 $D(\Lambda_{\mathfrak{g}}, K)^{\mathfrak{g}}_{\Lambda_{\mathfrak{z}}} = D(\Lambda_{\mathfrak{g}}, K)^{G}_{\Lambda_{\mathfrak{z}}}$.

Since by Lemma 1.1.1 and Proposition 2.1.2 $K[Z_0]$ and $U(\mathfrak{g})^G \otimes_L K$ are dense in $D(Z_0, K)$ and $U(\mathfrak{g}, K)^G$, respectively, it follows from [22], Lemma 19.10 (i), that the space $K[Z_0] \otimes_K (U(\mathfrak{g})^G \otimes_L K)$ is dense in $D(Z_0, K) \hat{\otimes}_K U(\mathfrak{g}, K)^G$. Therefore, so is its image in the quotient space $D(Z_0, K) \hat{\otimes}_{U(\mathfrak{z},K)} U(\mathfrak{g}, K)^G$. Since the image of $K[Z_0] \otimes_K (U(\mathfrak{g})^G \otimes_L K)$ under μ is precisely $D^{pt}(G_0, K)^G_{Z_0}$, the proof of the theorem is complete. □

Let $\mathbb{G}$ be a connected, reductive, linear algebraic group defined over L. $\mathbb{G}$ is the almost direct product of its center and the finitely many minimal, closed, connected, normal L-subgroups $\mathbb{G}_i$ of positive dimension of its derived subgroup $\mathbb{D}$. Let us call $\mathbb{G}$ sufficiently L-isotropic if all $\mathbb{G}_i$ are L-isotropic. This is the case, for example, if $\mathbb{G}$ is L-split. For the following cf. [29], Theorem 2.4:

Theorem (Sit). *Assume G to be the group of L-rational points of a connected, reductive, sufficiently L-isotropic L-group $\mathbb{G}$. If the conjugacy class of an element $g \in G$ is relatively compact in G (endowed with the topology induced from L) then g is contained in the center of G.*

Corollary 2.2.4. *Assume G to be the group of L-rational points of a connected, reductive, sufficiently L-isotropic L-group $\mathbb{G}$. Then $D(G,K)^G = D(G,K)^G_Z$. Let $\mathbb{D}$ be the derived group of $\mathbb{G}$, D the group of L-rational points of $\mathbb{D}$ and $\mathfrak{d}$ the Lie algebra of D. If K is discretely valued then there is a topological isomorphism*

$$D(G, K)^G \simeq D(Z, K) \hat{\otimes}_{K,\iota} U(\mathfrak{d}, K)^{\mathfrak{d}} \tag{2.6}$$

of separately continuous K-algebras.

Proof: According to (2.1), (1.4) and Remark 1.2.3 any invariant distribution on G is supported on a union of relatively compact conjugacy classes. As a consequence of Sit's theorem we have $D(G,K)^G = D(G,K)^G_Z$.

Since $G = D \cdot Z$ with finite intersection $D \cap Z$ it follows from Remark 1.2.14 that there is a topological isomorphism

$$D(G, K)_Z \longrightarrow D(Z, K) \hat{\otimes}_{K,\iota} U(\mathfrak{d}, K)$$

of $D(Z,K)$-$U(\mathfrak{d},K)^{op}$-bimodules. The image of $D^{pt}(G,K)^G$ under this isomorphism is $D^{pt}(Z,K) \otimes_K (U(\mathfrak{d})^{\mathfrak{d}} \otimes_L K)$ (cf. Remark 2.1.3). Since $D^{pt}(G,K)^G$, $D^{pt}(Z,K)$ and $U(\mathfrak{d})^{\mathfrak{d}} \otimes_L K$ are dense in $D(G,K)^G$, $D(Z,K)$ and $U(\mathfrak{d},K)^{\mathfrak{d}}$, respectively, (cf. Theorem 2.2.1, Lemma 1.1.1 and Proposition 2.1.2) the above isomorphism restricts to an isomorphism $D(G,K)^G \simeq D(Z,K)\hat{\otimes}_{K,\iota} U(\mathfrak{d},K)^{\mathfrak{d}}$. The arguments given at the beginning of the proof of Theorem 2.2.1 show that it may naturally be viewed as a homomorphism of K-algebras. □

2.3 The Fourier transform

Let k be a field which is complete with respect to a non-trivial and non-archimedean absolute value. Recall that a rigid analytic k-variety X is called quasi-Stein if there is a countable, admissible affinoid covering $(X_i)_{i\in\mathbb{N}}$ of X such that $X_i \subseteq X_{i+1}$ and the image of the map $\mathcal{O}(X_{i+1}) \to \mathcal{O}(X_i)$ is dense for all $i \in \mathbb{N}$ (cf. [19], Definition 2.3). It is easy to see that if X and Y are quasi-Stein then so is their fibred product $X \times_k Y$. Also, if X' is a rigid analytic k-variety admitting a finite morphism to a quasi-Stein k-variety X then X' is quasi-Stein itself. If k' is a complete, valued field extension of k then any rigid analytic, quasi-Stein k-variety X admits a base extension to k' and the resulting rigid analytic k'-variety $X_{k'}$ is quasi-Stein.

Remark 2.3.1. If X is quasi-Stein over k and k' is a complete valued field extension of k then the algebra of global sections of $X_{k'}$ is a k'-Fréchet-Stein algebra: If $(X_i)_{i\in\mathbb{N}}$ is a covering of X as a quasi-Stein space then

$$\mathcal{O}_{X_{k'}}(X_{k'}) = \varprojlim_i \mathcal{O}_{X_{k'}}((X_i)_{k'}).$$

For each $i \in \mathbb{N}$ the algebra $\mathcal{O}_{X_{k'}}((X_i)_{k'})$ is a noetherian k'-Banach algebra for which the map $\mathcal{O}_{X_{k'}}((X_{i+1})_{k'}) \to \mathcal{O}_{X_{k'}}((X_i)_{k'})$ is flat (cf. [2], 7.3.2 Corollary 6). Moreover, the natural map $\mathcal{O}_{X_{k'}}(X_{k'}) \to \mathcal{O}_{X_{k'}}((X_i)_{k'})$ has dense image because this is true for all transition maps.

Recall that if Z is a commutative locally L-analytic group and X is a rigid analytic L-variety then the group $\hat{Z}(X)$ of locally analytic characters of Z with values in X consists of the homomorphisms $Z \to \mathcal{O}_X(X)^*$ of groups such that for any admissible open affinoid subset $X_0 = \mathrm{Sp}(A)$ of X the induced homomorphism $Z \to A^*$ is an element of $C^{an}(Z,A)$ (cf. [12], Definition 6.4.2). It is shown in [loc.cit.], Corollary 6.4.4, that $\hat{Z}$ is a functor on the category of all rigid analytic L-varieties.

Theorem (Emerton-Schneider-Teitelbaum). *If Z is a commutative, locally L-analytic, topologically finitely generated group then the functor $\hat{Z}$ is representable by a strictly σ-affinoid rigid analytic space over L.*

Recall that according to [loc.cit.], Definition 2.1.17, a rigid analytic L-variety X is called strictly σ-affinoid if X has an admissible covering $(X_i)_{i\in\mathbb{N}}$ by affinoid subdomains X_i such that for every $i \in \mathbb{N}$ X_i is relatively compact in X_{i+1} in the sense of [2], 9.6.2. As a corollary to the construction of $\hat{Z}$ we obtain:

Corollary 2.3.2. *$\hat{Z}$ is quasi-Stein.*

Proof: By [12], Proposition 6.4.1, there is an isomorphism $Z \to \Lambda \times Z_0$ of locally L-analytic groups where Λ is a free abelian group of finite rank, say r, and

Z_0 is a compact open subgroup of Z. Consequently, there is an isomorphism $\hat{Z} \to \hat{\Lambda} \times \hat{Z}_0$. $\hat{\Lambda}$ is represented by the r-fold direct product of the rigid analytification $\mathbb{G}_{m,L}^{an}$ of the multiplicative group $\mathbb{G}_{m,L}$ over L which is quasi-Stein. Further, $\hat{Z}_0$ admits a finite morphism to a finite direct product of copies of $\widehat{\mathfrak{o}_L}$ which is quasi-Stein by [24], p. 456. □

The ring of global sections of the structure sheaf of $\hat{Z}_K$ is denoted by $\mathcal{O}(\hat{Z}_K)$. Since $\hat{Z}_K$ is quasi-Stein and strictly σ-affinoid it follows from Remark 2.3.1 and [12], Proposition 2.1.16, that $\mathcal{O}(\hat{Z}_K)$ is a nuclear K-Fréchet-Stein algebra.

Theorem (Emerton-Schneider-Teitelbaum)**.** *If Z is a commutative, locally L-analytic, topologically finitely generated group then there is a natural continuous injection $D(Z,K) \to \mathcal{O}(\hat{Z}_K)$ of K-algebras with dense image.*

We briefly recall the construction of this map: As above we choose an isomorphism $Z \to \Lambda \times Z_0$. According to [28], Proposition A.3, there is a topological isomorphism

$$D(Z,K) \simeq D(\Lambda,K)\hat{\otimes}_{K,\iota} D(Z_0,K).$$

Λ being discrete, $D(\Lambda,K) = K[\Lambda]$ is the topological direct sum of one dimensional K-vector spaces. Hence $D(\Lambda,K) \otimes_{K,\iota} D(Z_0,K)$ is complete (cf. Lemma 1.2.13 and [22], Lemma 7.8) so that

$$D(Z,K) \simeq K[\Lambda] \otimes_{K,\iota} D(Z_0,K).$$

On the other hand, the Fourier transform of [24], Theorem 2.3, extends to an isomorphism $D(Z_0,K) \simeq \mathcal{O}((\widehat{Z_0})_K)$ of K-Fréchet algebras. Further, $D(\Lambda,K) = K[\Lambda]$ can be interpreted as the algebra of regular functions on the algebraic Cartier dual $D(\Lambda) = \mathbb{G}_{m,K}^r$ of Λ. It admits an embedding into $\mathcal{O}((\mathbb{G}_{m,K}^r)^{an}) = \mathcal{O}(\hat{\Lambda}_K)$ with dense image. Since

$$\mathcal{O}(\hat{Z}_K) \simeq \mathcal{O}(\hat{\Lambda}_K)\hat{\otimes}_K \mathcal{O}((\hat{Z}_0)_K) \simeq \mathcal{O}(\hat{\Lambda}_K)\hat{\otimes}_{K,\iota} \mathcal{O}((\hat{Z}_0)_K)$$

the claim follows.

Corollary 2.3.3. *Let G be a locally L-analytic group and assume that either*

i) G is commutative and topologically finitely generated or

ii) G is the group of L-rational points of a connected, split reductive L-group $\mathbb{G}$.

If K is discretely valued then there is a quasi-Stein rigid analytic L-variety X and an injective, continuous homomorphism $D(G,K)^G \to \mathcal{O}(X_K)$ of K-algebras with dense image.

Proof: Case (i) is just the previous theorem because $D(G,K)^G = D(G,K)$. In case (ii) let Z be the center of G and n be the dimension of the derived group of $\mathbb{G}$. Since Z is topologically finitely generated we may define $X := \hat{Z} \times_L (\mathbb{A}_L^n)^{an}$. Writing $Z = \Lambda \times Z_0$ we have $\mathcal{O}(X_K) \simeq \mathcal{O}(\hat{\Lambda}_K)\hat{\otimes}_K \mathcal{O}((\widehat{Z_0})_K)\hat{\otimes}_K \mathcal{O}((\mathbb{A}_K^n)^{an})$. Further, Corollary 2.2.4 yields

$$D(G,K)^G \simeq K[\Lambda] \otimes_{K,\iota} D(Z_0,K)\hat{\otimes}_{K,\iota} U(\mathfrak{d},K)^{\mathfrak{d}}, \tag{2.7}$$

where $\mathfrak{d}$ denotes the Lie algebra of the derived group of $\mathbb{G}$. It follows from our assumptions on G that $\mathfrak{d}$ is semisimple and L-split whence by Theorem 2.1.6 there is a topological isomorphism $U(\mathfrak{d},K)^{\mathfrak{d}} \simeq \mathcal{O}((\mathbb{A}_K^n)^{an})$ of K-Fréchet algebras. Tensoring the embedding $K[\Lambda] \subseteq \mathcal{O}(\hat{\Lambda}_K)$ with

$$D(Z_0,K)\hat{\otimes}_{K,\iota}U(\mathfrak{d},K)^{\mathfrak{d}} \simeq \mathcal{O}((\widehat{Z_0})_K)\hat{\otimes}_{K,\iota}\mathcal{O}((\mathbb{A}_K^n)^{an})$$

gives a continuous K-linear injection $D(G,K)^G \to \mathcal{O}(X_K)$. Since $K[\Lambda]$ is dense in $\mathcal{O}(\hat{\Lambda}_K)$ it has dense image (cf. [22], Lemma 19.10) and, by construction, is a homomorphism of K-algebras. □

Remark 2.3.4. The isomorphism (2.7) makes it possible to explicitly compute the center of $D(G,K)$ if $\mathbb{G}$ is L-split. The structure of $U(\mathfrak{d},K)^{\mathfrak{d}}$ has been determined in Theorem 2.1.6: if n is the rank of $\mathfrak{d}$ then $U(\mathfrak{d},K)^{\mathfrak{d}} \simeq \mathcal{O}((\mathbb{A}_K^n)^{an})$ is the K-algebra of all power series in n variables with infinite radius of convergence. Moreover, if r is the dimension of Z then Z contains an open subgroup isomorphic to $\mathfrak{o}_L^r$. Thus, $Z \simeq A \times \mathfrak{o}_L^r$ as locally L-analytic groups with a discrete, finitely generated abelian group A. Consequently,

$$D(Z,K) \simeq K[A] \otimes_{K,\iota} \underbrace{D(\mathfrak{o}_L,K)\hat{\otimes}_K \cdots \hat{\otimes}_K D(\mathfrak{o}_L,K)}_{r\text{-times}}$$

(cf. [28], Proposition A.3). The structure of $D(\mathfrak{o}_L,K)$ has been investigated in [24]. It is the K-algebra of holomorphic functions on a twisted form of the open unit disk.

Corollary 2.3.5. *Under the assumptions of Corollary 2.3.3 any maximal ideal of $D(G,K)^G$ which is closed with respect to the topology induced by $\mathcal{O}(X_K)$ is of finite codimension.*

Proof: Let $\mathfrak{m}$ be a maximal ideal of $A := D(G,K)^G$ which is closed with respect to the metric topology induced by $\hat{A} := \mathcal{O}(X_K)$ and let $\hat{\mathfrak{m}}$ be the closure of $\mathfrak{m}$ in $\hat{A}$. $\hat{A}/\hat{\mathfrak{m}} = \widehat{A/\mathfrak{m}}$ gives rise to a non-zero, coherent module $\mathcal{F}$ on X_K (cf. [26], Lemma 3.6). There is a point $x \in X_K$ such that $\mathcal{F}_x \neq 0$. By Nakayama's lemma also $\mathcal{F}_x/\mathfrak{m}_x \neq 0$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X_K,x}$. However, $\dim_K \mathcal{F}_x/\mathfrak{m}_x < \infty$, and $\mathcal{F}_x/\mathfrak{m}_x$ is also a module over $A/\mathfrak{m}$. □

2.4 An extension of Harish-Chandra's isomorphism

Let $\mathbb{G}$ be a connected, split reductive, linear algebraic group defined over L with a maximal L-split torus $\mathbb{T}$. Let $\mathbb{D}$ and $\mathbb{Z}$ be the center and the derived group of $\mathbb{G}$, respectively. Then $\mathbb{D}$ is L-split and $\mathbb{T}' := (\mathbb{D}\cap\mathbb{T})^\circ$ is a maximal L-split torus of $\mathbb{D}$. Let G, Z, D, T and T' be the group of L-rational points of $\mathbb{G}, \mathbb{Z}, \mathbb{D}, \mathbb{T}$ and $\mathbb{T}'$, respectively, and $\mathfrak{g}, \mathfrak{z}, \mathfrak{d}, \mathfrak{t}$ and $\mathfrak{t}'$ be the respective Lie algebras. Note that $\mathfrak{d} = [\mathfrak{g},\mathfrak{g}]$ is a semisimple Lie algebra and that $\mathfrak{t}'$ is an L-split maximal toral subalgebra of $\mathfrak{d}$. Let finally $W = W(G,T) := N_G(T)/T$ be the Weyl group of G with respect to T. W acts on T by conjugation and hence on $D(T,K)$ such that the subalgebra $S(\mathfrak{t},K)$ of $D(T,K)$ is stable under W. W is also the Weyl group of D with respect to T', hence acts on T' and $D(T',K)$. The corresponding action on $S(\mathfrak{t}',K)$ is induced by the adjoint action of W on $\mathfrak{t}'$ (cf. the proof of Proposition 2.1.1). Recall that $S(\mathfrak{t}',K)$ is also acted on by the Weyl group $\mathfrak{W} = \mathfrak{W}(\mathfrak{d},\mathfrak{t}')$ of the pair $(\mathfrak{d},\mathfrak{t}')$ (cf. subsection 2.1). This action, too, is induced by viewing $\mathfrak{W}$ as a subgroup of $\mathrm{Aut}_L(\mathfrak{t}')$. The following fact is well known.

Lemma 2.4.1. *$Ad : W \to \mathfrak{W}$ is an isomorphism of groups. In particular, $S(\mathfrak{t}', K)^W = S(\mathfrak{t}', K)^{\mathfrak{W}}$.* □

Theorem 2.4.2. *Let G be the group of L-rational points of a connected, split reductive L-group $\mathbb{G}$ with T and W as above. If K is discretely valued then there is a topological isomorphism*

$$D(G, K)^G \simeq D(T, K)^W_Z$$

of separately continuous K-algebras.

Proof: According to Corollary 2.2.4 there is a topological isomorphism

$$\kappa : D(G, K)^G \longrightarrow D(Z, K)\hat{\otimes}_{K,\iota} U(\mathfrak{d}, K)^{\mathfrak{d}}$$

of separately continuous K-algebras.

Since $T = Z \cdot T'$ with finite intersection $Z \cap T'$ one proves in an analogous manner that there is a topological isomorphism of separately continuous K-algebras

$$\psi : D(Z, K)\hat{\otimes}_{K,\iota} S(\mathfrak{t}', K)^W \longrightarrow D(T, K)^W_Z .$$

According to Theorem 2.1.6 and Lemma 2.4.1 there is a topological isomorphism $\xi : U(\mathfrak{d}, K)^{\mathfrak{d}} \to S(\mathfrak{t}', K)^W$ of K-Fréchet algebras so that

$$\psi \circ (id\hat{\otimes}\xi) \circ \kappa : D(G, K)^G \to D(T, K)^W_Z$$

is as required. □

Remark 2.4.3. If $\mathbb{G}$ is semisimple then Z is finite and κ and ψ are the obvious isomorphisms

$$\begin{array}{lcl} K[Z] \otimes_K U(\mathfrak{g}, K)^G & \longrightarrow & D(G, K)^G_Z = D(G, K)^G \text{ and} \\ K[Z] \otimes_K S(\mathfrak{t}, K)^W & \longrightarrow & D(T, K)^W_Z . \end{array}$$

Since the isomorphism $\xi : U(\mathfrak{g}, K)^G \to S(\mathfrak{t}, K)^W$ was constructed without any restriction on K it follows that we have an isomorphism $D(G, K)^G \simeq D(T, K)^W_Z$ for any spherically complete coefficient field K.

References

[1] M. Andler, A. Dvorsky, S. Siddhartha: Kontsevich quantization and invariant distributions on Lie groups, *Ann. Sci. École Norm. Sup.* **35**, 2002, p. 371–390

[2] S. Bosch, U. Güntzer, R. Remmert: *Non-Archimedean Analysis*, Grundlehren der mathematischen Wissenschaften **261**, Springer, 1984

[3] N. Bourbaki: *Commutative Algebra*, Hermann, Paris, 1980

[4] N. Bourbaki: *Groupes et algèbres de Lie*, Chapitres 2 et 3, Hermann, Paris, 1972

[5] C. Breuil, L. Berger: Towards a p-adic Langlands programme, available at `http://www.ihes.fr/~breuil`

[6] N. Bourbaki: *Topological Vector Spaces*, Chapters 1–5, Springer, 2003

[7] N. Bourbaki: *Variétés différentielles et analytiques*, Fascicule de résultats, Hermann, Paris, 1971

[8] J. Dixmier: *Algèbres enveloppantes*, Gauthiers-Villars, 1974

[9] J.D. Dixon, M.P.F. du Sautoy, A. Mann, D. Segal: *Analytic pro-p groups*, 2nd Edition, Cambridge studies in advanced mathematics **61**, Cambridge University Press, 2003

[10] M. Duflo: Caractères des groupes et algèbres de Lie résolubles, *Ann. Sci. École Norm. Sup.* **3**, 1970, p. 23–74

[11] M. Duflo: Opérateurs différentiels bi-invariants sur un groupe de Lie, *Ann. Sci. École Norm. Sup.* **10**, 1977, p. 265–288

[12] M. Emerton: Locally analytic vectors in representations of locally p-adic analytic groups, to appear in *Memoirs of the AMS*

[13] M. Emerton: On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms, to appear in *Invent. Math.*

[14] C.T. Féaux de Lacroix: Einige Resultate über die topologischen Darstellungen p-adischer Liegruppen auf unendlich dimensionalen Vektorräumen über einem p-adischen Körper, *Schriftenreihe Math. Inst. Univ. Münster*, 3. Serie, Heft **23**, 1999

[15] J. Fresnel, M. van der Put, *Rigid Analytic Geometry and its Applications*, Birkhäuser, 2004

[16] H. Frommer: The locally analytic principal series of split reductive groups, *preprint*, 2003, available at
`http://wwwmath1.uni-muenster.de/sfb/`

[17] A. Grothendieck, J. Dieudonné: Éléments de géométrie algébrique, Chapitre III, *Publ. IHES* **11**, 1961

[18] A. Grothendieck: Produits tensoriels topologiques et espaces nucléaires, Memoirs AMS **16**, 1966

[19] R. Kiehl: Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie, *Inv. Math.* **2**, 1967, p. 256 – 273

[20] U. Köpf: Über eigentliche Familien algebraischer Varietäten über affinoiden Räumen, *Schriftenreihe Math. Inst. Univ. Münster*, 2. Serie, Heft **7**, 1974

[21] A.M. Robert: *A course in p-adic analysis*, Graduate Texts in Mathematics **198**, Springer, 2000

[22] P. Schneider: *Nonarchimedean Functional Analysis*, Springer Monographs in Mathematics, Springer, 2002

[23] P. Schneider: *p-adische Analysis*, Lecture Notes, Universität Münster, 2000, available at
`http://www.math.uni-muenster.de/math/u/schneider`

[24] P. Schneider, J. Teitelbaum: p-adic Fourier theory, *Documenta Math.* **6**, 2001, p. 447–481

[25] P. Schneider, J. Teitelbaum: Locally analytic distributions and p-adic representation theory, with applications to GL_2, *Journ. AMS* **15**, 2002, p. 443–468

[26] P. Schneider, J. Teitelbaum: Algebras of p-adic distributions and admissible representations, *Invent. Math.* **153**, 2003, p. 145–196

[27] P. Schneider, J. Teitelbaum: p-adic boundary values, *Astérisque* **278**, 2002, p. 51–125, and **295**, p. 291–299, 2004

[28] P. Schneider, J. Teitelbaum: Duality for admissible locally analytic representations, *preprint*, 2004

[29] K.-Y. Sit: On bounded elements of linear algebraic groups, *Trans. AMS* **209**, 1975, p. 185–198