

# The cohomology of locally analytic representations

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*à Jacqueline Weil*

**Abstract.** We develop a cohomology theory for locally analytic representations of  $p$ -adic Lie groups on nonarchimedean locally convex vector spaces. There are versions of Pontrjagin duality, Shapiro's lemma and a Hochschild-Serre spectral sequence. As an application, we give the definition of a supercuspidal locally analytic representation of a  $p$ -adic reductive group and study extensions between locally analytic principal series representations.

## Contents

0. Introduction	1
1. Preliminaries from functional analysis	5
2. Relative homological algebra	8
3. Pontrjagin duality	17
4. Finite resolutions	19
5. Shapiro's lemma	27
6. Spectral sequences	30
7. Supercuspidal representations	37
8. Extensions between principal series representations	39
References	51

## 0 Introduction

Let  $p$  be a prime number, let  $L$  be a finite extension of the field  $\mathbb{Q}_p$  of  $p$ -adic numbers, let  $K$  be a valued field extension of  $L$  such that  $K$  is spherically complete, and let  $H$  be a finite-dimensional locally  $L$ -analytic group.

The systematic study of locally analytic representations of  $H$  on locally convex topological  $K$ -vector spaces was initiated by P. Schneider and J. Teitelbaum (cf. [33], [35] and [36]). One of the most recent arithmetic applications of this theory is concerned with the  $p$ -adic local Langlands program, first conceived by C. Breuil, trying to establish a correspondence between  $p$ -adic Galois representations and certain continuous representations of  $p$ -adic reductive groups

on nonarchimedean topological vector spaces (see e.g. [9]). In contrast to the classical local Langlands program a  $p$ -adic correspondence would not only involve (topologically) irreducible representations of  $p$ -adic reductive groups but also certain extensions of such. In view of this phenomenon, and as a part of the general theory, it seemed necessary to develop a suitable cohomology theory for locally analytic representations of  $p$ -adic Lie groups. This is the aim of the present article.

With  $H$  as above we let  $D(H) = D(H, K)$  denote the  $K$ -algebra of locally analytic  $K$ -valued distributions on  $H$  (cf. section 2). A fundamental result of P. Schneider and J. Teitelbaum states that the category of locally analytic representations of  $H$  on locally convex  $K$ -vector spaces of compact type is anti-equivalent to the category  $\mathcal{M}_H^{nF}$  of separately continuous  $D(H)$ -modules on nuclear Fréchet spaces over  $K$  (cf. [35], Corollary 3.3).

More generally, let  $\mathcal{M}_H$  denote the category of complete Hausdorff locally convex  $K$ -vector spaces with the structure of a separately continuous  $D(H)$ -module, taking as morphisms all continuous  $D(H)$ -linear maps (cf. section 2). Neither of the categories  $\mathcal{M}_H^{nF}$  or  $\mathcal{M}_H$  is abelian, and the full abelian subcategory of coadmissible  $D(H)$ -modules of [36], section 6, does generally not have enough projective objects (confer the two paragraphs following Proposition 4.1). In the larger category of all abstract  $D(H)$ -modules the restriction and induction functors do not seem to preserve injective or projective objects. Therefore, one lacks certain necessary tools such as Shapiro's lemma and the spectral sequence of Hochschild-Serre.

In order to overcome these deficiencies, we carry over J.L. Taylor's general approach of a homology theory for topological algebras over the complex numbers (cf. [40]) to the nonarchimedean setting. It is based on the definition of projective and injective objects relative to the class of exact sequences in  $\mathcal{M}_H$  admitting a continuous  $K$ -linear section. Thus, the theory is closely related to that of the continuous cohomology of [5], chapter IX, and [13], §1.

Section 1 provides the necessary results from nonarchimedean functional analysis, whereas the specific homological algebra is developed in section 2. For each pair of objects  $V$  and  $W$  of  $\mathcal{M}_H$  we define relative extension groups  $\mathcal{E}xt_H^q(V, W)$ , torsion groups  $\mathcal{T}or_q^H(V, W)$ , locally analytic cohomology groups  $H_{an}^q(H, V)$  and locally analytic homology groups  $H_q^{an}(H, V)$  (cf. Definition 2.5 and Definition 2.7). Apart from being relatively universal  $\delta$ -functors these  $K$ -vector spaces can be endowed with natural locally convex topologies (cf. Remark 2.10) and can often be computed by means of a complex of locally analytic cochains (cf. Remark 2.17).

In section 3 we single out two full exact subcategories of  $\mathcal{M}_H$  which are anti-equivalent to each other under the duality functor (cf. Theorem 3.1, slightly generalizing the above mentioned result of P. Schneider and J. Teitelbaum). This leads to a version of Pontrjagin duality in Theorem 3.2 similar to the one between discrete and compact modules over profinite groups.

We begin section 4 by constructing  $K$ -linear comparison homomorphisms

$$\kappa^q : \mathcal{E}xt_H^q(V, W) \longrightarrow \text{Ext}_{D(H)}^q(V, W)$$

for any two objects  $V$  and  $W$  of  $\mathcal{M}_H$  where the groups on the right are computed in the category of all abstract  $D(H)$ -modules. We give an example in which  $\kappa^1$  is injective but not surjective (cf. Proposition 4.1 and Proposition 4.2). This example also shows that the space  $C^{an}(\mathbb{Z}_p, K)$  of  $K$ -valued locally analytic functions on  $\mathbb{Z}_p$  does not admit any continuous  $K$ -linear antiderivatives (cf. Corollary 4.3) – a piece of  $p$ -adic analysis which seems to be interesting of its own.

On the other hand, one can show the comparison homomorphisms  $\kappa^q$  to be bijective assuming the module  $V$  to admit a relatively projective resolution by objects of the form  $D(H) \widehat{\otimes}_{K, \iota} Y^q$  where all  $Y^q$  are  $K$ -vector spaces carrying their finest locally convex topology (cf. section 1 for the notion of the complete inductive tensor product). We closely examine the case of the trivial one dimensional  $D(H)$ -module  $V = \mathbb{1}$ , and define the group  $H$  to satisfy assumption (A) if  $\mathbb{1}$  admits a relatively projective resolution of the above type. Under this assumption, one can use results from [37], section 3, to relate the locally analytic cohomology of  $H$  to the cohomology of its Lie algebra and the cohomology over the  $K$ -algebra  $D^\infty(H)$  of locally constant  $K$ -valued distributions on  $H$  (cf. Theorem 4.9 and Theorem 4.10).

The first one to consider the cohomology of compact locally  $\mathbb{Q}_p$ -analytic groups, although in a different setting, was M. Lazard (cf. [25], Chapitre V, which was given a modern treatment in [39]). We crucially use his ideas involving a famous lemma of J-P. Serre's to prove that compact locally  $\mathbb{Q}_p$ -analytic groups satisfy assumption (A) (cf. Theorem 4.4). For finite extensions  $L$  of  $\mathbb{Q}_p$  and solvable groups we give a different argument relying on the  $p$ -adic Fourier theory of [34] (cf. Theorem 6.5).

Given a locally  $L$ -analytic group  $H_1$  and a closed locally  $L$ -analytic subgroup  $H_2$  we define induction and coinduction functors from  $\mathcal{M}_{H_2}$  to  $\mathcal{M}_{H_1}$  in section 5. They correspond to each other under Pontrjagin duality (cf. Proposition 5.3). Both functors, as well as the restriction functor, enjoy the necessary acyclicity properties to obtain a version of Shapiro's lemma, generalizing Frobenius reciprocity (cf. Theorem 5.5, Proposition 5.6 and Theorem 5.7).

It is one of our achievements to have built up a theory flexible enough and to have chosen assumption (A) sufficiently weak to be able to apply the results of section 4 to a vast class of non-compact groups. Namely, using a well-known technique due to C.T.C. Wall which we recall in Theorem 6.1, we show that the class of groups satisfying assumption (A) is closed under extensions and that it includes all solvable locally  $L$ -analytic groups (cf. Corollary 6.2 and Theorem 6.5). Moreover, we have all the necessary machinery at hand to follow an argument of W. Casselman and D. Wigner (cf. [13], section 3) showing that the group of  $\mathbb{Q}_p$ -rational points of a reductive group, and hence that of *any* linear algebraic group over  $\mathbb{Q}_p$ , satisfies assumption (A) (cf. Theorem 6.6 and Theorem 6.7). For such groups we are able to establish the existence of a (co)homological

spectral sequence in Theorem 6.8. Finally, using results of P. Schneider and J. Teitelbaum from [37] we establish a vanishing theorem giving an analog in our setting of M. Lazard's famous result that uniform pro- $p$  groups are Poincaré duality groups (cf. Theorem 6.9 and the remarks preceding it).

Sections 7 and 8 are concerned with applications to the theory of locally analytic representations.

In section 7 we assume  $N$  to be the group of  $L$ -rational points of a unipotent group defined over  $L$  with Lie algebra  $\mathfrak{n}$ . In this case the locally analytic  $N$ -homology is isomorphic to the  $N$ -coinvariants of the  $\mathfrak{n}$ -homology (cf. Theorem 7.1). In Definition 7.2 we give the notion of a supercuspidal locally analytic representation of a  $p$ -adic reductive group  $G$  and use the above result to relate our definition to the usual one in the smooth case. In fact, using a structure theorem of D. Prasad (cf. [28], Theorem 1) we can even characterize the supercuspidal locally algebraic representations of  $G$  (cf. Theorem 7.3). In Theorem 7.4 we prove a criterion allowing us to carry over the degeneracy result of Theorem 7.1 to  $N$ -cohomology and to directly relate the latter to the Lie algebra cohomology with respect to  $\mathfrak{n}$ . This gives a variant of a result of W. Casselman and D. Wigner for semisimple groups (cf. [13], Theorem 1).

In the final section 8 we let  $G$ ,  $P$ ,  $M$  and  $N$  be the groups of  $L$ -rational points of a reductive group over  $L$ , a minimal parabolic subgroup, its Levi quotient and its unipotent radical, respectively. We assume  $L = \mathbb{Q}_p$  or  $G$  to be  $L$ -split. If  $\chi_1, \chi_2 : M \rightarrow K^\times$  are two locally analytic characters we study the extension groups  $\mathcal{E}xt_G^q(I(\chi_1), I(\chi_2))$  where  $I(\chi_i) := \text{Ind}_P^G(\chi_i)$  are two locally analytic principal series representations. By Pontrjagin duality, Shapiro's lemma and the Hochschild-Serre spectral sequence this problem is split up into three steps. Letting  $M(\chi_i) := I(\chi_i)'_b$  denote the  $D(G)$ -module obtained by dualizing, one first has to compute the locally analytic cohomology groups  $H_{an}^q(N, M(\chi_1))$ . As in the smooth case the module  $M(\chi_1)$  is endowed with a  $P$ -invariant filtration and one tries to compute the locally analytic  $N$ -cohomology of the associated graded object. We carry out these computations for the highest and the lowest graded pieces, respectively (cf. Theorem 8.2 and Theorem 8.5). The cohomology of the latter turns out to be isomorphic to the  $\mathfrak{n}$ -cohomology of a certain Verma module, at least if  $G$  is  $L$ -split. The two cases we treat show the locally analytic theory to significantly deviate from the corresponding behavior in the smooth case (cf. Remark 8.6).

In a second step one has to determine the groups of extensions between two locally analytic characters of  $M$ . The most interesting situation turns out to be the one in which both characters are trivial, a case we treat in Theorem 8.9 and Corollary 8.10. Again, the result differs from the corresponding smooth one. On the other hand, it suggests that – had we introduced cup products – the ring structure of the total cohomology space of a  $p$ -adic reductive group would still be that of an exterior algebra (cf. Remark 8.11).

The third step concerns the analysis of the Hochschild-Serre spectral sequence which we carry out only in the explicit case of  $G = \text{GL}_2(L)$  (cf. Example 8.12). In this example our previous results are general enough to completely determine

the  $N$ -cohomology of the dual module of a principal series representation (cf. Theorem 8.13). This allows us to slightly generalize a theorem of P. Schneider and J. Teitelbaum on the space of  $G$ -homomorphisms between two such representations (cf. Theorem 8.14). More generally, we are able to determine the higher extension groups in a generic case and to show that the Euler-Poincaré characteristic of two principal series representations of  $G$  is always trivial (cf. Theorem 8.15, Theorem 8.16 and Theorem 8.17).

Coming back to the case of a general reductive group, we conclude our article by showing that for a pair of principal series representations the comparison homomorphisms  $\kappa^g$  for the dual modules are bijective in any degree (cf. Theorem 8.18). Thus, our approach is perfectly well-suited for the study of such representations.

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*Conventions and notation.* Let  $p$  be a prime number and let  $L$  be a finite extension of the field  $\mathbb{Q}_p$  of  $p$ -adic numbers. We let  $K|L$  be an extension of valued fields such that  $K$  is spherically complete. Let  $\mathfrak{o}_L$  and  $\mathfrak{o}_K$  denote the valuation rings of  $L$  and  $K$ , respectively. By  $H$  we denote a locally  $L$ -analytic group of finite dimension  $d$  with Lie algebra  $\mathfrak{h}$ . For technical reasons we will always assume  $H$  to be countable at infinity, i.e. the set of cosets  $H/H_0$  to be at most countable for any compact open subgroup  $H_0$  of  $H$ . We let  $\text{LCS}_K$  be the category of locally convex  $K$ -vector spaces. Its morphisms are all continuous  $K$ -linear maps. If  $R$  is a ring with unit then we denote by  $\text{Mod}_R$  the category of (unital left)  $R$ -modules.

## 1 Preliminaries from functional analysis

If  $V$  and  $W$  are two locally convex  $K$ -vector spaces we let  $\mathcal{L}_b(V, W)$  be the space of continuous  $K$ -linear maps from  $V$  to  $W$  endowed with the strong topology. The latter is defined by the family of lattices

$$(1) \quad \mathfrak{N}(B, U) := \{f \in \mathcal{L}(V, W) : f(B) \subseteq U\}$$

with  $B$  (resp.  $U$ ) running through all bounded subsets of  $V$  (resp. all open lattices of  $W$ ). In particular, we write  $V'_b := \mathcal{L}_b(V, K)$  for the strong dual of  $V$ . We write  $V \widehat{\otimes}_{K, \iota} W$  and  $V \widehat{\otimes}_{K, \pi} W$  for the Hausdorff completion of the inductive and the projective tensor product of  $V$  and  $W$ , respectively (cf. [32], §17). For the notion of barrelled and bornological locally convex  $K$ -vector spaces we refer to [32], §6.

Over the complex numbers, the following permanence property was first formulated by A. Grothendieck (cf. [17], I.3.1, page 78).

**Lemma 1.1.** *If  $V$  and  $W$  are barrelled then so is  $V \widehat{\otimes}_{K,\iota} W$ .*

Proof: As the locally convex inductive limit of a family of maps from  $V$  and  $W$  to  $V \otimes_K W$ , the space  $V \otimes_{K,\iota} W$  is barrelled whenever  $V$  and  $W$  are, and so is the associated Hausdorff space (cf. [32], page 35, Examples 3 and 4). It is true quite generally that the completion  $\widehat{U}$  of a Hausdorff barrelled locally convex  $K$ -vector space is barrelled again. Indeed, let  $M \subseteq \widehat{U}$  be a closed lattice. Then  $L := M \cap U$  is a closed lattice in  $U$ . Thus,  $L$  is open in  $U$  and  $M = \overline{L}$  is open in  $\widehat{U}$  (cf. [32], Remark 7.4 (ii)).  $\square$

Let  $V = \varinjlim_{i \in I} V_i$  be the locally convex inductive limit of a directed family  $(V_i)_{i \in I}$  of locally convex  $K$ -vector spaces  $V_i$ . If  $I = \mathbb{N}$  and if the maps  $V_i \rightarrow V_{i+1}$  are strict injective such that  $V = \cup_{i \geq 1} V_i$  then we speak of a strict inductive limit. We call *strict (LF)-space* any strict inductive limit of  $K$ -Fréchet spaces.

The first part of the following proposition can be proved as in [17], I.3.1 Proposition 14 (see also [15], Lemma 1.1.30).

**Proposition 1.2.** *The complete inductive tensor product commutes with locally convex direct sums. If  $V = \varinjlim_{i \in I} V_i$  and  $W = \varinjlim_{j \in J} W_j$  are locally convex inductive limits then there is a topological isomorphism*

$$(2) \quad V \otimes_{K,\iota} W \simeq \varinjlim_{(i,j) \in I \times J} (V_i \otimes_{K,\iota} W_j).$$

*If both  $V$  and  $W$  are strict (LF)-spaces then (2) extends to a topological isomorphism*

$$(3) \quad V \widehat{\otimes}_{K,\iota} W \simeq \varinjlim_{i \in \mathbb{N}} (V_i \widehat{\otimes}_{K,\iota} W_i).$$

*In particular,  $V \widehat{\otimes}_{K,\iota} W$  is again a strict (LF)-space and is barrelled and bornological.*

Proof: We prove only the two final assertions. Let us first remark that by a cofinality argument there is a topological isomorphism

$$\varinjlim_{i \in \mathbb{N}} (V_i \otimes_{K,\iota} W_i) \simeq \varinjlim_{i,j} (V_i \otimes_{K,\iota} W_j).$$

If  $V_i$  and  $W_i$  are Fréchet spaces then the spaces  $V_i \otimes_{K,\iota} W_i \simeq V_i \otimes_{K,\pi} W_i$  are metrizable (cf. [32], Proposition 17.6 and the ensuing remark) and the map  $V_i \otimes_{K,\iota} W_i \rightarrow V_{i+1} \otimes_{K,\iota} W_{i+1}$  is strict (cf. [32], Corollary 17.5). The induced map  $V_i \widehat{\otimes}_{K,\iota} W_i \rightarrow V_{i+1} \widehat{\otimes}_{K,\iota} W_{i+1}$  is strict injective, too. Thus,  $\varinjlim_{i \in \mathbb{N}} (V_i \widehat{\otimes}_{K,\iota} W_i)$  is complete (cf. [32], Lemma 7.9), whence (3). We see that  $V \widehat{\otimes}_{K,\iota} W$  is again a strict (LF)-space and as such is barrelled and bornological (cf. [32], page 33, Example 2, Proposition 6.14 and page 35, Examples 2 and 3).  $\square$

**Proposition 1.3.** *If  $V = \varinjlim_i V_i$  and  $W = \varinjlim_i W_i$  are strict (LF)-spaces and if all  $V_i$  are reflexive then any bounded  $\mathfrak{o}_K$ -submodule  $A$  of  $U := V \widehat{\otimes}_{K,\iota} W$  is contained in the closure of a set of the form  $B \otimes_{\mathfrak{o}_K} C$  where  $B$  and  $C$  are bounded  $\mathfrak{o}_K$ -submodules of  $V$  and  $W$ , respectively.*

Proof: By [32], Lemma 4.10, we may assume  $A$  to be closed. Let  $U_A$  be the vector subspace of  $U$  generated by  $A$  and let  $p_A$  be the gauge seminorm of  $U_A$  defined by  $A$  (cf. [32], page 8). According to [32], Lemma 7.17,  $(U_A, p_A)$  is a Banach space and the inclusion  $U_A \rightarrow U$  is continuous.

By Proposition 1.2 we have  $U = \varinjlim_i (V_i \widehat{\otimes}_{K, \iota} W_i)$  with Fréchet spaces  $V_i \widehat{\otimes}_{K, \iota} W_i$ . By [32], Corollary 8.9,  $U_A \rightarrow U$  factors through some  $V_i \widehat{\otimes}_{K, \iota} W_i \rightarrow U$ . Since  $V_i$  is reflexive it follows from [32], Proposition 20.11, that there are bounded  $\mathfrak{o}_K$ -submodules  $B$  and  $C$  of  $V_i$  and  $W_i$ , respectively, such that  $A$  is contained in the closure of  $B \otimes_{\mathfrak{o}_K} C$  in  $V_i \widehat{\otimes}_{K, \iota} W_i$ . A priori,  $B$  and  $C$  are bounded in  $V$  and  $W$ , and  $A$  is contained in the closure of  $B \otimes_{\mathfrak{o}_K} C$  in  $U$ .  $\square$

**Proposition 1.4.** *Let  $U, V$  and  $W$  be locally convex  $K$ -vector spaces. If  $U$  is barrelled and if  $W$  is Hausdorff and complete then there is a natural continuous  $K$ -linear bijection*

$$(4) \quad \alpha : \mathcal{L}_b(U \widehat{\otimes}_{K, \iota} V, W) \longrightarrow \mathcal{L}_b(U, \mathcal{L}_b(V, W)).$$

*If  $U$  and  $V$  are locally convex direct sums of strict (LF)-spaces such that all Fréchet spaces associated with  $U$  or  $V$  are reflexive, then (4) is even a topological isomorphism.*

Proof: For the first part of the proof see [15], Proposition 1.1.35. We prove only the second assertion.

Let  $A \subseteq U \widehat{\otimes}_{K, \iota} V$  be bounded. Since the complete inductive tensor product commutes with locally convex direct sums (cf. Proposition 1.2) we deduce from [8], III.1.4 Proposition 5 and our Proposition 1.3 that  $A \subseteq \overline{B \otimes_{\mathfrak{o}_K} C}$  for certain bounded  $\mathfrak{o}_K$ -submodules  $B$  and  $C$  of  $U$  and  $V$ , respectively. If  $L$  is an open lattice in  $W$  then  $\mathfrak{N}(B, \mathfrak{N}(C, L)) = \alpha(\mathfrak{N}(B \otimes_{\mathfrak{o}_K} C, L)) \subseteq \alpha(\mathfrak{N}(A, L))$ .  $\square$

**Proposition 1.5.** *Let  $V$  and  $W$  be locally convex  $K$ -vector spaces and assume that any bounded subset of  $W'_b$  is equicontinuous (e.g. that  $W$  is barrelled or bornological). Then the map  $\tau : \mathcal{L}_b(V, W) \rightarrow \mathcal{L}_b(W'_b, V'_b)$  induced by passing to the transpose is continuous.*

Proof: Note first that if  $f : V \rightarrow W$  is continuous and  $K$ -linear then so is its transpose  $f' : W'_b \rightarrow V'_b$  (cf. [32], Remark 16.1). Let  $A \subseteq V$  and  $B \subseteq W'_b$  be bounded subsets. By assumption on  $W$  there is an open lattice  $U$  in  $W$  such that  $\ell(U) \subseteq \mathfrak{o}_K$  for all  $\ell \in B$ . But then  $\mathfrak{N}(A, U) \subseteq \tau^{-1}(\mathfrak{N}(B, \mathfrak{N}(A, \mathfrak{o}_K)))$ .  $\square$

**Proposition 1.6.** *If  $(U_i)_{i \in \mathbb{N}}$  is a countable family of bornological Hausdorff locally convex  $K$ -vector spaces then the direct product  $U := \prod_{i \in \mathbb{N}} U_i$  is bornological.*

Proof: Denote by  $\mathcal{T}$  the product topology on  $U$  and by  $\mathcal{T}_{bor}$  the locally convex topology defined by all lattices  $L$  with the property that for all bounded subsets  $B$  of  $(U, \mathcal{T})$  there is an element  $a \in K$  such that  $B \subseteq aL$ . The topology  $\mathcal{T}_{bor}$  is finer than  $\mathcal{T}$  and both topologies have the same class of bounded subsets. By [32], Lemma 14.1, it suffices to show that  $\mathcal{T}$  is the Mackey topology on  $U$  and that the topology  $\mathcal{T}_{bor}$  is admissible in the sense of [32], §14.

As for the first claim, let  $B$  be a bounded and  $c$ -compact  $\mathfrak{o}_K$ -submodule of the weak dual space of  $U$ . By the arguments given in the proof of [32], Proposition

14.3, there is a finite set of indices  $I \subseteq \mathbb{N}$  such that  $B \subseteq \bigoplus_{i \in I} \pi_i(B)$  where  $\pi_i : U'_s \rightarrow (U_i)'_s$  is the canonical map  $\bigoplus_i (U_i)'_s \subseteq \prod_i (U_i)'_s \rightarrow (U_i)'_s$ . For the corresponding polars this implies

$$B^p \supseteq \prod_{i \notin I} U_i \times \bigoplus_{i \in I} \pi_i(B)^p.$$

The  $\mathfrak{o}_K$ -module  $\pi_i(B)$  is bounded and  $c$ -compact in the weak dual of  $U_i$  (cf. [32], Lemma 12.1) so that  $\pi_i(B)^p$  is an open lattice in the Mackey topology of  $U_i$ . Now  $U_i$  is bornological so that the Mackey topology coincides with the given topology (cf. [32], Proposition 14.4). Therefore,  $B^p$  is open in  $(U, \mathcal{T})$ .

As for the second claim, a  $K$ -linear functional on  $U$  is continuous with respect to  $\mathcal{T}_{bor}$  if and only if it respects bounded subsets. Let the functional  $\ell$  have this property. As in [21], §28.4, one shows that there is an index  $n_0 \in \mathbb{N}$  such that  $\ell(u) = 0$  for all elements  $u = (u_i)_{i \in \mathbb{N}}$  of  $U$  for which  $u_1 = \dots = u_{n_0} = 0$ . Letting  $\iota_i : U_i \rightarrow U$  be the natural inclusions, the functionals  $\ell \circ \iota_i : U_i \rightarrow K$  respect bounded subsets, hence are continuous because  $U_i$  is bornological (cf. [32], Proposition 6.13). Therefore, so is  $\ell = \sum_{i=1}^{n_0} \ell \circ \iota_i \circ \text{pr}_i$ .  $\square$

## 2 Relative homological algebra

Recall that a topological Hausdorff space  $M$  is called (strictly) paracompact if any open covering of  $M$  admits a locally finite refinement by (pairwise disjoint) open subsets. If  $M$  is a finite dimensional paracompact locally  $L$ -analytic manifold then  $M$  is strictly paracompact (cf. [31], Proposition 8.7). The  $K$ -vector space  $C^{an}(M, K)$  of locally analytic functions on  $M$  with values in  $K$  is the space of functions  $f : M \rightarrow K$  which are locally representable, via suitable charts, by convergent power series with coefficients in  $K$ . It carries a natural topology making it a Hausdorff barrelled, locally convex  $K$ -vector space (cf. [16], Satz 2.1.10). If  $M$  is compact then  $C^{an}(M, K)$  is a space of compact type, hence is complete, bornological and reflexive (cf. [35], Lemma 2.1 and Theorem 1.1). If  $(M_i)_{i \in I}$  is a covering of  $M$  by pairwise disjoint open subsets then there is a topological isomorphism

$$(5) \quad C^{an}(M, K) \simeq \prod_{i \in I} C^{an}(M_i, K)$$

(cf. [16], Korollar 2.2.4). Choosing the subsets  $M_i$  to be compact we see that  $C^{an}(M, K)$  is complete and reflexive (cf. [32], Proposition 9.10 and Proposition 9.11).

Let  $D(M, K) := C^{an}(M, K)'_b$  denote the space of so-called *locally analytic distributions* on  $M$  with values in  $K$ . Passing to the strong duals, we infer from (5) that

$$(6) \quad D(M, K) \simeq \bigoplus_{i \in I} D(M_i, K)$$

(cf. [32], Proposition 9.11). Since the strong dual of a space of compact type is a nuclear Fréchet space (cf. [35], Theorem 1.3), hence is complete, barrelled and bornological (cf. [32], Proposition 6.14 and page 35, Example 2), the same is

true of  $D(M, K)$  (cf. [32], Lemma 7.8, as well as page 33, Example 3 and page 35, Example 4).

If the coefficient field  $K$  is clear from the context we omit it from the notation and write  $C^{an}(M) = C^{an}(M, K)$  and  $D(M) = D(M, K)$ .

If  $M = H$  is a locally  $L$ -analytic group the multiplication map  $H \times H \rightarrow H$  induces on  $D(H)$  the structure of a unital, separately continuous  $K$ -algebra (cf. [35], Proposition 2.3). The latter means that the multiplication map

$$(7) \quad D(H) \times D(H) \longrightarrow D(H), \quad (\delta_1, \delta_2) \mapsto \delta_1 \cdot \delta_2,$$

induces continuous  $K$ -linear endomorphisms of  $D(H)$  whenever  $\delta_1$  or  $\delta_2$  is fixed. Since the map (7) is  $K$ -bilinear and separately continuous and since  $D(H)$  is Hausdorff and complete, it induces a unique continuous  $K$ -linear map

$$(8) \quad D(H) \widehat{\otimes}_{K, \iota} D(H) \longrightarrow D(H).$$

We let  $\mathcal{M}_H$  be the category whose objects are complete Hausdorff locally convex  $K$ -vector spaces  $V$  with the structure of a separately continuous  $D(H)$ -module

$$(9) \quad D(H) \times V \longrightarrow V,$$

separately continuous meaning that for all  $v \in V$  and all  $\delta \in D(H)$  the maps  $(\delta' \mapsto \delta' \cdot v)$  and  $(v' \mapsto \delta \cdot v')$  are continuous. As in (8) the structure map (9) uniquely extends to a continuous  $K$ -linear map

$$(10) \quad D(H) \widehat{\otimes}_{K, \iota} V \longrightarrow V.$$

As morphisms in  $\mathcal{M}_H$  we choose all continuous  $D(H)$ -linear maps, writing

$$\mathcal{L}_H(V, W) := \text{Hom}_{D(H)}^{\text{cont}}(V, W)$$

for two objects  $V$  and  $W$  of  $\mathcal{M}_H$ . We endow  $\mathcal{L}_H(V, W)$  with the topology induced from  $\mathcal{L}_b(V, W)$ .

We call *hypomodule* any object  $V$  of  $\mathcal{M}_H$  whose structure map (9) is not only separately continuous but even hypocontinuous with respect to all bounded subsets of  $D(H)$  (cf. [8], III.5.3 Definition 2). This means that for any bounded subset  $B$  of  $D(H)$  and any open lattice  $L$  of  $V$  there exists another open lattice  $M$  of  $V$  such that the image of  $B \times M$  is contained in  $L$ . We note that if  $V$  is an object of  $\mathcal{M}_H$  whose underlying locally convex  $K$ -vector space is barrelled then  $V$  is automatically a hypomodule (cf. [8], III.5.3 Proposition 6).

Recall that a continuous  $K$ -linear map between locally convex  $K$ -vector spaces is called *strong* if it is strict with closed image and if both its kernel and its image admit complements by closed  $K$ -subspaces. We endow  $\text{LCS}_K$  and  $\mathcal{M}_H$  with the structure of exact categories by declaring a sequence

$$\dots \longrightarrow V_{q+1} \xrightarrow{\delta_{q+1}} V_q \xrightarrow{\delta_q} V_{q-1} \longrightarrow \dots$$

in the respective category to be *strongly exact* (or *s-exact*, for short) if it is exact as a sequence of abstract  $K$ -vector spaces and if all maps  $\delta_q$  are strong maps

between locally convex  $K$ -vector spaces. If the above sequence is a complex, i.e. if  $\delta_q \circ \delta_{q+1} = 0$  for all  $q$ , then its  $s$ -exactness is equivalent to the existence of a continuous  $K$ -linear contracting homotopy consisting of continuous  $K$ -linear maps  $\lambda_q : V_q \rightarrow V_{q+1}$  such that

$$(11) \quad \delta_{q+1} \circ \lambda_q + \lambda_{q-1} \circ \delta_q = id_{V_q} \text{ for all } q.$$

**Definition 2.1.** An object  $P$  of  $\mathcal{M}_H$  is called  $s$ -projective if the functor  $\mathcal{L}_H(P, \cdot)$  transforms all short  $s$ -exact sequences

$$(12) \quad 0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

in  $\mathcal{M}_H$  into exact sequences of abstract  $K$ -vector spaces.

We emphasize that although we view  $\mathcal{L}_H(P, V_i)$  as a locally convex  $K$ -vector space we do not require the functor  $\mathcal{L}_H(P, \cdot)$  to give rise to  $s$ -exact sequences in  $\text{LCS}_K$ .

If  $V$  is any complete Hausdorff locally convex  $K$ -vector space then  $D(H) \widehat{\otimes}_{K, \iota} V$  is naturally an object of  $\mathcal{M}_H$ . Indeed, being Hausdorff and complete by definition, it suffices to remark that by tensorizing the identity map on  $V$  with (8) we obtain a continuous  $K$ -linear map

$$D(H) \widehat{\otimes}_{K, \iota} (D(H) \widehat{\otimes}_{K, \iota} V) \simeq (D(H) \widehat{\otimes}_{K, \iota} D(H)) \widehat{\otimes}_{K, \iota} V \longrightarrow D(H) \widehat{\otimes}_{K, \iota} V.$$

The objects of  $\mathcal{M}_H$  of the form  $D(H) \widehat{\otimes}_{K, \iota} V$  are called  $s$ -free.

**Lemma 2.2.** For any Hausdorff complete locally convex  $K$ -vector space  $V$  and any object  $W$  of  $\mathcal{M}_H$  there is a natural continuous  $K$ -linear bijection

$$(13) \quad \alpha : \mathcal{L}_H(D(H) \widehat{\otimes}_{K, \iota} V, W) \longrightarrow \mathcal{L}_b(V, W)$$

given by composition with the map  $(v \mapsto 1 \otimes v) : V \rightarrow D(H) \widehat{\otimes}_{K, \iota} V$ . If  $V$  is the locally convex direct sum of strict (LF)-spaces and if  $W$  is a hypomodule then  $\alpha$  is a topological isomorphism.

Proof: For the bijectivity of  $\alpha$  see [40], Proposition 1.3. We show that  $\alpha$  is continuous. Let  $B \subseteq V$  be bounded and let  $L \subseteq W$  be an open lattice. Since the above map  $V \rightarrow D(H) \widehat{\otimes}_{K, \iota} V$  is continuous the image  $B'$  of  $B$  is bounded. One checks that  $\alpha(\mathfrak{N}(B', L) \cap \mathcal{L}_H(D(H) \widehat{\otimes}_{K, \iota} V, W)) \subseteq \mathfrak{N}(B, L)$ .

Under the additional assumptions on  $V$  and  $W$  let  $A \subseteq D(H) \widehat{\otimes}_{K, \iota} V$  be a bounded  $\mathfrak{o}_K$ -submodule and let  $L \subseteq W$  be an open lattice. We need to show that  $\alpha(\mathfrak{N}(A, L) \cap \mathcal{L}_H(D(H) \widehat{\otimes}_{K, \iota} V, W))$  is open in  $\mathcal{L}_b(V, W)$ , i.e. contains a set of the form  $\mathfrak{N}(C, M)$  where  $C \subseteq V$  is bounded and  $M \subseteq W$  is an open lattice.

As seen in the proof of Proposition 1.4 there are bounded  $\mathfrak{o}_K$ -submodules  $B$  and  $C$  of  $D(H)$  and  $V$ , respectively, such that  $A \subseteq \overline{B \otimes_{\mathfrak{o}_K} C}$ . Since the open lattice  $L$  is closed in  $W$  we have  $\mathfrak{N}(B \otimes_{\mathfrak{o}_K} C, L) = \mathfrak{N}(\overline{B \otimes_{\mathfrak{o}_K} C}, L) \subseteq \mathfrak{N}(A, L)$ . Since  $B$  is bounded and  $W$  is a hypomodule there is an open lattice  $M$  in  $W$  such that  $B \cdot M \subseteq L$ . Then  $\mathfrak{N}(C, M)$  is as required.  $\square$

**Proposition 2.3.** *An object  $P$  of  $\mathcal{M}_H$  is  $s$ -projective if and only if it is a direct summand (in  $\mathcal{M}_H$ ) of an  $s$ -free module. If  $V$  is the locally convex direct sum of strict (LF)-spaces and if  $P = D(H) \widehat{\otimes}_{K,\iota} V$  then the functor  $\mathcal{L}_H(P, \cdot)$  takes  $s$ -exact sequences of hypomodules to  $s$ -exact sequences of locally convex  $K$ -vector spaces.*

Proof: For a proof of the first part see [40], Proposition 1.4. The crucial observation is that by Lemma 2.2 the functors  $\mathcal{L}_H(D(H) \widehat{\otimes}_{K,\iota} V, \cdot)$  and  $\mathcal{L}_b(V, \cdot)$  are equivalent, the latter being exact on  $s$ -exact sequences in  $\text{LCS}_K$ . The second assertion of the proposition follows from the second assertion of Lemma 2.2.  $\square$

We point out that it is precisely the lack of exactness of the functors  $\mathcal{L}_b(V, \cdot)$  on general exact sequences of continuous maps between locally convex  $K$ -vector spaces that forces us to work with  $s$ -exact sequences.

If  $V$  is an object of  $\mathcal{M}_H$  then by an  $s$ -projective  $s$ -resolution of  $V$  we mean an  $s$ -exact sequence

$$\dots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} V \longrightarrow 0$$

in  $\mathcal{M}_H$  in which all objects  $X_i$  are  $s$ -projective.

For an object  $V$  of  $\mathcal{M}_H$  let  $B_{-1}(H, V) := V$ , and for  $q \geq 0$  let  $B_q(H, V) := D(H) \widehat{\otimes}_{K,\iota} B_{q-1}(H, V)$  with its structure of an  $s$ -free module. For  $q \geq 0$  define  $d_q : B_q(H, V) \rightarrow B_{q-1}(H, V)$  through

$$\begin{aligned} d_q(\delta_0 \otimes \dots \otimes \delta_q \otimes v) &:= \sum_{i=0}^{q-1} (-1)^i \delta_0 \otimes \dots \otimes \delta_i \delta_{i+1} \otimes \dots \otimes \delta_q \otimes v \\ &\quad + (-1)^q \delta_0 \otimes \dots \otimes \delta_{q-1} \otimes \delta_q v. \end{aligned}$$

As in [40], section 2, one proves the following statement.

**Proposition 2.4.** *For any object  $V$  of  $\mathcal{M}_H$  the sequence  $(B_q(H, V), d_q)_{q \geq 0}$  is an  $s$ -projective  $s$ -resolution of  $V$  in  $\mathcal{M}_H$ .  $\square$*

As usual, the complex  $(B_q(H, V), d_q)_{q \geq 0}$  is called the *unnormalized bar resolution* of  $V$ .

The trivial character  $\mathbf{1} : H \rightarrow K^\times$  is a locally analytic representation of  $H$  in the sense of [35], section 3. According to [35], Proposition 3.2, it extends to a continuous  $K$ -linear homomorphism

$$(14) \quad \mathbf{1} : D(H) \longrightarrow K$$

giving the one dimensional space  $K$  the structure of a separately continuous  $D(H)$ -module which we continue to denote by  $\mathbf{1}$ .

**Definition 2.5.** If  $V$  and  $W$  are objects of  $\mathcal{M}_H$  we define  $\mathcal{E}xt_H^q(V, W)$  to be the  $q$ -th cohomology group of the complex  $\mathcal{L}_H(B_\bullet(H, V), W)$ . In particular, we call  $H_{an}^q(H, V) := \mathcal{E}xt_H^q(\mathbf{1}, V)$  the  $q$ -th *locally analytic cohomology group* of  $H$  with coefficients in  $V$ .

The inversion map on the group  $H$  defines a topological anti-automorphism of the  $K$ -algebra  $D(H)$  allowing us to identify the categories of separately continuous left and right  $D(H)$ -modules. That is why we refrain from introducing left and right versions of the category  $\mathcal{M}_H$  and sometimes speak of left and right objects, instead.

If  $V$  and  $W$  are objects of  $\mathcal{M}_H$ ,  $V$  a right module, we define  $V \widetilde{\otimes}_{D(H), \iota} W$  to be the quotient of  $V \widehat{\otimes}_{K, \iota} W$  by the image of the natural map

$$V \widehat{\otimes}_{K, \iota} D(H) \widehat{\otimes}_{K, \iota} W \longrightarrow V \widehat{\otimes}_{K, \iota} W,$$

sending an element  $v \otimes \delta \otimes w$  to  $v\delta \otimes w - v \otimes \delta w$ , where  $v \in V$ ,  $w \in W$  and  $\delta \in D(H)$ . Endowed with the corresponding quotient topology the space  $V \widetilde{\otimes}_{D(H), \iota} W$  is generally neither Hausdorff nor complete. If one of the modules  $V$  or  $W$  is  $s$ -free, however, one has the following analog of Lemma 2.2.

**Lemma 2.6.** *For any Hausdorff complete locally convex  $K$ -vector space  $V$  and any object  $W$  of  $\mathcal{M}_H$  there is a natural  $K$ -linear topological isomorphism*

$$(V \widehat{\otimes}_{K, \iota} D(H)) \widetilde{\otimes}_{D(H), \iota} W \simeq V \widehat{\otimes}_{K, \iota} W.$$

*If the object  $P$  of  $\mathcal{M}_H$  is  $s$ -projective then the functor  $P \widetilde{\otimes}_{D(H), \iota} (\cdot)$  takes  $s$ -exact sequences in  $\mathcal{M}_H$  to exact sequences of  $K$ -vector spaces. If  $P$  is  $s$ -free this functor takes  $s$ -exact sequences in  $\mathcal{M}_H$  to  $s$ -exact sequences in  $LCS_K$ .*

Proof: The first part of the proposition is contained in [40], Proposition 1.5 and Proposition 1.6. The second part relies on the observation that the functor  $(\cdot) \widehat{\otimes}_{K, \iota} V$  preserves the  $s$ -exactness of sequences of locally convex  $K$ -vector spaces.  $\square$

**Definition 2.7.** If  $V$  and  $W$  are objects of  $\mathcal{M}_H$  we define  $\mathcal{T}or_q^H(V, W)$  to be the  $q$ -th homology group of the complex  $V \widetilde{\otimes}_{D(H), \iota} B_\bullet(H, W)$ . In particular, we call  $H_q^{an}(H, V) := \mathcal{T}or_q^H(V, \mathbb{1})$  the  $q$ -th *locally analytic homology group* of  $H$  with coefficients in  $V$ .

It is now a routine matter to prove the following two propositions (cf. [40], Proposition 2.1. and Proposition 2.2).

**Proposition 2.8.** *For all  $q \geq 0$  the assignment  $\mathcal{E}xt_H^q(\cdot, \cdot)$  is a bifunctor from  $\mathcal{M}_H \times \mathcal{M}_H$  to the category of  $K$ -vector spaces which is contravariant in the first and covariant in the second variable. We have  $\mathcal{E}xt_H^0(\cdot, \cdot) = \mathcal{L}_H(\cdot, \cdot)$ . If  $V$  is  $s$ -projective and if  $W$  is any object of  $\mathcal{M}_H$  then  $\mathcal{E}xt_H^q(V, W) = 0$  for all  $q > 0$ . If  $V$  is an object of  $\mathcal{M}_H$  and if*

$$(15) \quad 0 \longrightarrow W_1 \longrightarrow W_2 \longrightarrow W_3 \longrightarrow 0$$

*is an  $s$ -exact sequence in  $\mathcal{M}_H$  then there are natural  $K$ -linear maps*

$$\begin{aligned} \delta_q &: \mathcal{E}xt_H^q(W_1, V) \longrightarrow \mathcal{E}xt_H^{q+1}(W_3, V) \text{ and} \\ \delta_q &: \mathcal{E}xt_H^q(V, W_3) \longrightarrow \mathcal{E}xt_H^{q+1}(V, W_1) \end{aligned}$$

*for all  $q \geq 0$  such that the sequences*

$$(16) \quad \dots \xrightarrow{\delta_{q-1}} \mathcal{E}xt_H^q(W_3, V) \longrightarrow \mathcal{E}xt_H^q(W_2, V) \longrightarrow \mathcal{E}xt_H^q(W_1, V) \xrightarrow{\delta_q} \dots$$

$$(17) \quad \dots \xrightarrow{\delta_{q-1}} \mathcal{E}xt_H^q(V, W_1) \longrightarrow \mathcal{E}xt_H^q(V, W_2) \longrightarrow \mathcal{E}xt_H^q(V, W_3) \xrightarrow{\delta_q} \dots$$

*are exact.*

**Proposition 2.9.** For all  $q \geq 0$  the assignment  $\mathcal{T}or_q^H(\cdot, \cdot)$  is a bifunctor from  $\mathcal{M}_H \times \mathcal{M}_H$  to the category of  $K$ -vector spaces which is covariant in both variables. We have  $\mathcal{T}or_0^H(\cdot, \cdot) = (\cdot) \widetilde{\otimes}_{D(H), \iota} (\cdot)$ . If  $V$  is  $s$ -projective and if  $W$  is any object of  $\mathcal{M}_H$  then  $\mathcal{T}or_q^H(V, W) = 0$  for all  $q > 0$ . If  $W$  is an object of  $\mathcal{M}_H$  and if

$$(18) \quad 0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

is an  $s$ -exact sequence in  $\mathcal{M}_H$  then there are natural  $K$ -linear maps

$$\delta_q : \mathcal{T}or_{q+1}^H(V_3, W) \longrightarrow \mathcal{T}or_q^H(V_1, W)$$

for all  $q \geq 0$  such that the sequence

$$(19) \quad \dots \xrightarrow{\delta_q} \mathcal{T}or_q^H(V_1, W) \longrightarrow \mathcal{T}or_q^H(V_2, W) \longrightarrow \mathcal{T}or_q^H(V_3, W) \xrightarrow{\delta_{q-1}} \dots$$

is exact.

**Remark 2.10.** If  $V$  and  $W$  are objects of  $\mathcal{M}_H$  then the  $K$ -vector spaces  $\mathcal{L}_H(B_q(H, V), W)$  are locally convex with respect to the topology inherited from  $\mathcal{L}_b(B_q(H, V), W)$ . Therefore, the groups  $\mathcal{E}xt_H^q(V, W)$  may be viewed as locally convex  $K$ -vector spaces, as well. If  $X_\bullet \rightarrow V$  is another  $s$ -projective  $s$ -resolution of  $V$  in  $\mathcal{M}_H$  then the identity map on  $V$  induces continuous  $K$ -linear maps of complexes  $X_\bullet \rightleftarrows B_\bullet(H, V)$  inducing continuous  $K$ -linear bijections between the cohomology groups of the complexes  $\mathcal{L}_H(X_\bullet, W)$  and  $\mathcal{L}_H(B_\bullet(H, V), W)$ , respectively. This shows that not only can the  $K$ -vector spaces  $\mathcal{E}xt_H^q(V, W)$  be computed using any  $s$ -projective  $s$ -resolution of  $V$  in  $\mathcal{M}_H$  but that in addition they carry canonical topologies (cf. also [5], IX.3.3). For the groups  $\mathcal{T}or_q^H(V, W)$  one argues similarly. We stress that the locally convex topologies on  $\mathcal{E}xt_H^q(V, W)$  and  $\mathcal{T}or_q^H(V, W)$  thus obtained are generally neither Hausdorff nor complete.

It is straightforward to see that the  $K$ -linear maps in (16), (17) and (19) between extension (resp. torsion) groups of the same degree are continuous for the topology defined in Remark 2.10. For the connecting homomorphisms we have the following results.

**Proposition 2.11.** (i) If (15) is an  $s$ -exact sequence of locally convex direct sums of strict (LF)-spaces then the connecting homomorphisms  $\delta_q$  in (16) are continuous for any  $q \geq 0$  and any hypomodule  $V$  in  $\mathcal{M}_H$ .

(ii) If  $V$  is the locally convex direct sum of strict (LF)-spaces and if the sequence (15) consists of hypomodules then the connecting homomorphisms  $\delta_q$  in (17) are continuous for any  $q \geq 0$ .

(iii) The connecting homomorphisms  $\delta_q$  in (19) are continuous for any two objects  $V$  and  $W$  of  $\mathcal{M}_H$ .

Proof: We start with (iii), the connecting homomorphisms being constructed in the usual way via the snake lemma. Given the  $s$ -exact sequence (18) we know from Lemma 2.6 that setting  $B_\bullet := B_\bullet(H, W)$  the induced sequence of complexes

$$(20) \quad 0 \longrightarrow V_1 \widetilde{\otimes}_{D(H), \iota} B_\bullet \longrightarrow V_2 \widetilde{\otimes}_{D(H), \iota} B_\bullet \longrightarrow V_3 \widetilde{\otimes}_{D(H), \iota} B_\bullet \longrightarrow 0$$

is s-exact. For any  $q \geq 0$  we choose continuous  $K$ -linear sections

$$s_q : V_3 \widetilde{\otimes}_{D(H), \iota} B_q \rightarrow V_2 \widetilde{\otimes}_{D(H), \iota} B_q \quad \text{and} \quad t_q : V_2 \widetilde{\otimes}_{D(H), \iota} B_q \rightarrow V_1 \widetilde{\otimes}_{D(H), \iota} B_q$$

of the maps in (20). Given a class  $[c] \in \mathcal{T}or_{q+1}^H(V_3, W)$  we have  $\delta_q([c]) = [t_q \circ (id_{V_2} \otimes d_{q+1}) \circ s_{q+1}(c)]$ . The continuity of  $\delta_q$  is thus plain from the continuity of the maps  $t_q$ ,  $s_{q+1}$  and  $d_{q+1}$ .

Given Lemma 2.2 and Proposition 2.3, the proofs of (i) and (ii) are analogous. One needs to know s-exact sequences in  $\mathcal{M}_H$  to be transformed into s-exact sequences in  $\text{LCS}_K$ .  $\square$

Let  $I(H)$  be the kernel of the  $K$ -algebra homomorphism  $\mathbb{1} : D(H) \rightarrow K$  of (14). Similarly, let  $I$  be the augmentation ideal of the group ring  $K[H]$  of  $H$  over  $K$ . Since  $K[H]$  is a dense subalgebra of  $D(H)$  (cf. [35], Lemma 3.1) one obtains that  $I$  is dense in  $I(H)$ .

By Proposition 2.8 we have  $H_{an}^0(H, V) = \mathcal{L}_H(\mathbb{1}, V)$  which, via evaluation at  $1 \in K$  and since  $I$  is dense in  $I(H)$ , is precisely the space of  $H$ -invariants:

$$(21) \quad H_{an}^0(H, V) = V^H.$$

In particular,  $H_{an}^0(H, V)$  is always Hausdorff and complete. By Proposition 2.9 we have

$$(22) \quad H_0^{an}(H, V) = \mathbb{1} \widetilde{\otimes}_{D(H), \iota} V \simeq V / \text{im}(I(H) \widehat{\otimes}_{K, \iota} V \rightarrow V).$$

If we let  $V(H)$  be the  $K$ -vector subspace of  $V$  spanned by all elements of the form  $h \cdot v - v$  where  $h \in H$  and  $v \in V$  then  $V(H) = IV$ . By our above remark and the separate continuity of the  $D(H)$ -action on  $V$  there are inclusions

$$V(H) \subseteq I(H)V \subseteq \text{im}(I(H) \widehat{\otimes}_{K, \iota} V \rightarrow V) \subseteq \overline{V(H)}$$

where  $\overline{V(H)}$  denotes the topological closure of  $V(H)$  in  $V$ . In particular,  $H_0^{an}(H, V)$  is Hausdorff if and only if  $\text{im}(I(H) \widehat{\otimes}_{K, \iota} V \rightarrow V) = \overline{V(H)}$ .

As pointed out in [40], page 153, there is a lack of symmetry concerning s-projective and s-injective objects in  $\mathcal{M}_H$ . If our homological algebra is to be built upon relatively injective objects then one has to restrict to hypomodules.

**Definition 2.12.** A hypomodule  $I$  in  $\mathcal{M}_H$  is called *s-injective* if the functor  $\mathcal{L}_H(\cdot, I)$  transforms all short s-exact sequences (12) consisting of hypomodules into exact sequences of abstract  $K$ -vector spaces.

**Lemma 2.13.** *Let  $V$  and  $W$  be complete Hausdorff locally convex  $K$ -vector spaces such that  $V$  is barrelled and bornological. If  $V$  is a (right) object of  $\mathcal{M}_H$  then  $(\delta f)(v) := f(v\delta)$  makes  $\mathcal{L}_b(V, W)$  a (left) hypomodule over  $D(H)$ . If  $W$  is a (left) hypomodule then  $(\delta f)(v) := \delta f(v)$  makes  $\mathcal{L}_b(V, W)$  a (left) hypomodule over  $D(H)$ .*

A proof can be found in [40], Proposition 3.1, noting that by our general assumption the space  $\mathcal{L}_b(V, W)$  is Hausdorff and complete (cf. [32], Proposition

7.16).

Since  $D(H)$  is both barrelled and bornological it follows from Lemma 2.13 that for any complete Hausdorff locally convex  $K$ -vector space  $V$  the space  $\mathcal{L}_b(D(H), V)$  is a hypomodule over  $D(H)$ . We then have the following analog of Lemma 2.2 and Lemma 2.6.

**Lemma 2.14.** *If  $W$  is a complete Hausdorff locally convex  $K$ -vector space and if  $V$  is a hypomodule in  $\mathcal{M}_H$  then there is a natural continuous  $K$ -linear bijection*

$$\mathcal{L}_H(V, \mathcal{L}_b(D(H), W)) \longrightarrow \mathcal{L}_b(V, W)$$

given by composition with the map  $(f \mapsto f(1)) : \mathcal{L}_b(D(H), W) \rightarrow W$ . If  $V$  is bornological and barrelled then this is a topological isomorphism.

Proof: For the bijectivity of the map we refer to [40], Proposition 3.2, the continuity being clear. An inverse  $\beta$  is constructed as follows. Given  $f \in \mathcal{L}_b(V, W)$  define the  $D(H)$ -module homomorphism  $\beta(f) : V \rightarrow \mathcal{L}_b(D(H), W)$  by  $\beta(f)(v)(\delta) := f(\delta v)$ . We show that  $\beta$  is continuous if  $V$  is barrelled and bornological. As in Lemma 2.13 the space  $\mathcal{L}_b(V, W)$  is a (right) hypomodule via  $(f\delta)(v) = f(\delta v)$ . Thus, if  $B \subseteq D(H)$  is bounded, if  $L \subseteq W$  is an open lattice and if  $A \subseteq V$  is bounded, there is an open lattice of the form  $\mathfrak{N}(C, M)$  in  $\mathcal{L}_b(V, W)$  such that  $\mathfrak{N}(C, M) \cdot B \subseteq \mathfrak{N}(A, L)$ . But then  $\beta(\mathfrak{N}(C, M)) \subseteq \mathfrak{N}(A, \mathfrak{N}(B, L))$ .  $\square$

**Proposition 2.15.** *A hypomodule  $I$  in  $\mathcal{M}_H$  is  $s$ -injective if and only if it is a direct summand (in  $\mathcal{M}_H$ ) of an object of the form  $\mathcal{L}_b(D(H), W)$  for some complete Hausdorff locally convex  $K$ -vector space  $W$ . If  $I = \mathcal{L}_b(D(H), W)$  with  $W$  Hausdorff and complete then the functor  $\mathcal{L}_H(\cdot, I)$  takes  $s$ -exact sequences of bornological and barrelled objects of  $\mathcal{M}_H$  to  $s$ -exact sequences in  $LCS_K$ .*

Proof: The first part of the proposition is proved in [40], Proposition 3.3. The second assertion follows from the second part of Lemma 2.14.  $\square$

If  $W$  is a hypomodule in  $\mathcal{M}_H$  then by an  $s$ -injective  $s$ -resolution of  $W$  we mean an  $s$ -exact sequence

$$0 \longrightarrow W \longrightarrow Y^0 \xrightarrow{d^0} Y^1 \xrightarrow{d^1} \dots$$

in  $\mathcal{M}_H$  in which all objects  $Y^i$  are  $s$ -injective hypomodules.

Given the unnormalized bar resolution of  $D(H)$ , consider the induced complex  $\mathcal{L}_H(B_\bullet(H, D(H)), W)$ . By Lemma 2.2 this complex may topologically be identified with the complex

$$(23) \quad 0 \longrightarrow \mathcal{L}_b(D(H), W) \xrightarrow{d^0} \mathcal{L}_b(D(H) \widehat{\otimes}_{K, \iota} D(H), W) \xrightarrow{d^1} \dots \\ \dots \xrightarrow{d^{q-1}} \mathcal{L}_b((\widehat{\otimes}_{K, \iota}^q D(H)) \widehat{\otimes}_{K, \iota} D(H), W) \xrightarrow{d^q} \dots$$

where

$$d^q(f)(\delta_0 \otimes \dots \otimes \delta_q \otimes \delta) = \delta_0 f(\delta_1 \otimes \dots \otimes \delta) + (-1)^{q+1} f(\delta_0 \otimes \dots \otimes \delta_{q-1} \otimes \delta_q \delta) \\ + \sum_{i=1}^q (-1)^i f(\delta_0 \otimes \dots \otimes \delta_{i-1} \delta_i \otimes \dots \otimes \delta)$$

(cf. [40], page 151). In particular,  $\ker(d^0) \simeq W$  via evaluation at  $1 \in D(H)$  so that we obtain the augmented complex  $(F^q(H, W), d^q)_{q \geq -1}$  defined by

$$F^{-1}(H, W) := W, \quad F^q(H, W) := \mathcal{L}_b(\widehat{\otimes}_{K,t}^{q+1} D(H), W) \text{ for } q \geq 0$$

together with the inclusion  $d^{-1} : W \rightarrow \mathcal{L}_b(D(H), W)$  and the maps  $d^q$  from (23). Further, by Proposition 1.4 there are  $K$ -linear topological isomorphisms

$$(24) \quad F^q(H, W) \simeq \mathcal{L}_b(D(H), \mathcal{L}_b(\widehat{\otimes}_{K,t}^q D(H), W))$$

for any  $q \geq 0$ . By Proposition 2.15 the spaces  $F^q(H, W)$ ,  $q \geq 0$ , may be viewed as  $s$ -injective hypomodules in  $\mathcal{M}_H$ .

**Proposition 2.16.** *If  $W$  is a hypomodule in  $\mathcal{M}_H$  then the augmented complex  $(F^q(H, W), d^q)_{q \geq -1}$  provides an  $s$ -injective  $s$ -resolution of  $W$  in  $\mathcal{M}_H$ .*

Proof: It is straightforward to check that for  $q \geq 0$  the maps  $d^q$  are continuous and  $D(H)$ -linear. It remains to prove that the augmentation  $W \rightarrow \mathcal{L}_b(D(H), W)$  is continuous and  $D(H)$ -linear and that the complex is  $s$ -exact. As for the first assertion, the augmentation is given by  $d^{-1}(w)(\delta) = \delta w$ , hence is  $D(H)$ -linear, and its continuity follows from the hypocontinuity of the structure map  $D(H) \times W \rightarrow W$ . A continuous section is given by evaluation at  $1 \in D(H)$ .

As for the second assertion, note that  $B_\bullet(H, D(H))$  is an  $s$ -exact sequence in  $\mathcal{M}_H$  (cf. Proposition 2.4). Since all occurring objects are  $s$ -projective by Proposition 2.3, it admits a continuous  $D(H)$ -linear contracting homotopy, inducing a continuous  $K$ -linear contracting homotopy of the complex  $F^\bullet(H, W)$ .  $\square$

**Remark 2.17.** By [37], Proposition A.3, there are  $K$ -linear topological isomorphisms  $\widehat{\otimes}_{K,t}^q D(H) \simeq D(H^q)$  for any  $q \geq 1$ . If the underlying locally convex  $K$ -vector space of a hypomodule  $W$  in  $\mathcal{M}_H$  is of compact type (cf. [35], section 1) then there are  $K$ -linear topological isomorphisms

$$(25) \quad F^q(H, W) = \mathcal{L}_b(\widehat{\otimes}_{K,t}^{q+1} D(H), W) \simeq C^{an}(H^{q+1}, W)$$

(cf. [32], Corollary 18.8, [15], Proposition 2.1.28, [16], Korollar 2.2.4 and [17], I.1.3 Proposition 6). In this case the  $s$ -injective  $s$ -resolution of  $W$  constructed in Proposition 2.16 may be interpreted as a resolution by locally analytic cochains analogous to [5], Chapter IX, §1.4, or [13], section 1.

**Proposition 2.18.** *If the object  $V$  of  $\mathcal{M}_H$  is barrelled and if  $W$  is an  $s$ -injective hypomodule in  $\mathcal{M}_H$  then  $\mathcal{E}xt_H^q(V, W) = 0$  for all  $q > 0$ . If  $W$  is a hypomodule then for any object  $V$  of  $\mathcal{M}_H$  there are  $K$ -linear bijections*

$$(26) \quad \mathcal{E}xt_H^q(V, W) \longrightarrow H^q \mathcal{L}_H(V, F^\bullet(H, W))$$

for any  $q \geq 0$ . If  $V$  is bornological and barrelled then these are continuous. If  $V$  is the locally convex direct sum of strict  $(LF)$ -spaces then they are even topological isomorphisms.

Proof: The first assertion is proved in [40], Proposition 3.4. One needs  $V$  to be barrelled in order for the  $s$ -projective  $s$ -resolution  $B_\bullet(H, V)$  of  $V$  to consist of

hypomodules. In fact, by Lemma 1.1, these spaces are even barrelled.

The second assertion follows from the construction as well as Lemma 2.2, Proposition 1.4 and Lemma 2.14. Namely, for all  $q \geq 0$  there are  $K$ -linear bijections

$$\begin{aligned}
(27) \quad \mathcal{L}_H(B_q(H, V), W) &\longrightarrow \mathcal{L}_H((\widehat{\otimes}_{K,\iota}^{q+1} D(H)) \widehat{\otimes}_{K,\iota} V, W) \\
&\longrightarrow \mathcal{L}_H(V, \mathcal{L}_b(\widehat{\otimes}_{K,\iota}^{q+1} D(H), W)) \\
&\longrightarrow \mathcal{L}_H(V, \mathcal{L}_b(D(H), \mathcal{L}_b(\widehat{\otimes}_{K,\iota}^q D(H), W))) \\
&= \mathcal{L}_H(V, F^q(H, W)).
\end{aligned}$$

Due to the naturality of these isomorphisms they induce  $K$ -linear isomorphisms of complexes  $\mathcal{L}_H(B_\bullet(H, V), W) \rightarrow \mathcal{L}_H(V, F^\bullet(H, W))$ , whence (26).

By Lemma 2.2 the first map in (27) is always continuous. If  $V$  is barrelled then so is the second (cf. Proposition 1.4). If  $V$  is barrelled and bornological then the third map is a topological isomorphism by Lemma 2.14. The last case follows directly from the contents of Lemma 2.2 and Proposition 1.4.  $\square$

**Remark 2.19.** We see that for hypomodules the groups of extensions as in Definition 2.5 can also be computed using  $s$ -injective  $s$ -resolutions. In many important cases, though not in all, the locally convex topology one obtains coincides with the one in Remark 2.10. If necessary we will explicitly say whether the groups  $\mathcal{E}xt_H^q(V, W)$  are topologized by using  $s$ -projective  $s$ -resolutions of  $V$  or (if  $W$  is a hypomodule)  $s$ -injective  $s$ -resolutions of  $W$ .

### 3 Pontrjagin duality

We let  $\mathcal{M}_H^{snF}$  and  $\mathcal{M}_H^{pct}$  be the full subcategories of  $\mathcal{M}_H$  consisting of all objects whose underlying locally convex  $K$ -vector space is the countable locally convex direct sum of nuclear Fréchet spaces and the countable direct product of spaces of compact type, respectively. For the notion of nuclearity see [32], §19, for that of spaces of compact type see [32], §16, or [35], section 1. We endow  $\mathcal{M}_H^{snF}$  and  $\mathcal{M}_H^{pct}$  with the induced structure of an exact category and prove the following slight generalization of a fundamental result in the theory of locally analytic representations obtained by P. Schneider and J. Teitelbaum (cf. [35], Proposition 3.2 and Corollary 3.3). Our proof uses their arguments.

**Theorem 3.1.** *All objects of  $\mathcal{M}_H^{snF}$  and  $\mathcal{M}_H^{pct}$  are barrelled (hence hypomodules), bornological and reflexive. Passing from  $V$  to  $V'_b$  induces an  $s$ -exact anti-equivalence of categories  $\mathcal{M}_H^{snF} \rightarrow \mathcal{M}_H^{pct}$ . More precisely, for any two objects  $V$  and  $W$  of  $\mathcal{M}_H^{snF}$ , passing to the transpose defines a  $K$ -linear topological isomorphism*

$$(28) \quad \mathcal{L}_H(V, W) \simeq \mathcal{L}_H(W'_b, V'_b),$$

and the composition  $(\cdot)'_b \circ (\cdot)'_b$  of the duality functor with itself is naturally equivalent to the identity functor.

Proof: Nuclear Fréchet spaces and spaces of compact type are barrelled, bornological and reflexive (cf. [32], Corollary 19.3, Proposition 6.14, page 35, Example

2, Proposition 16.10 and Proposition 15.5). Hence so are arbitrary locally convex direct sums and countable direct products of such (cf. [32], page 35, Example 4, page 33, Example 3, Proposition 9.10, Proposition 9.11 and Proposition 14.3, as well as our Proposition 1.6).

Let  $\text{LCS}_K^{snF}$  and  $\text{LCS}_K^{pct}$  be the full subcategories of  $\text{LCS}_K$  whose objects are countable locally convex direct sums of nuclear Fréchet spaces and countable direct products of spaces of compact type, respectively. By [35], Corollary 1.4 and [32], Proposition 9.10 and Proposition 9.11, the functor  $(V \mapsto V'_b) : \text{LCS}_K^{snF} \rightarrow \text{LCS}_K^{pct}$  is essentially surjective. Since all spaces in question are barrelled we know from Proposition 1.5 that passing to the transpose yields a continuous map

$$(29) \quad \mathcal{L}_b(V, W) \longrightarrow \mathcal{L}_b(W'_b, V'_b)$$

for any two objects  $V$  and  $W$  of  $\text{LCS}_K^{snF}$ . The same argument works for any two objects of  $\text{LCS}_K^{pct}$  whence by reflexivity the map (29) is a  $K$ -linear topological isomorphism and the duality functor is an anti-equivalence of categories. By reflexivity,  $(\cdot)'_b \circ (\cdot)'_b$  is equivalent to the identity functor.

It remains to show that an object  $V$  of  $\text{LCS}_K^{snF}$  is a separately continuous  $D(H)$ -module if and only if so is  $V'_b$ . But since all spaces are barrelled, the argument is the same as the one given in the proof of [35], Corollary 3.3.  $\square$

**Theorem 3.2.** *The exact category  $\mathcal{M}_H^{snF}$  (resp.  $\mathcal{M}_H^{pct}$ ) has enough  $s$ -projective (resp.  $s$ -injective) objects. If one topologizes the extension groups between objects of  $\mathcal{M}_H^{snF}$  (resp.  $\mathcal{M}_H^{pct}$ ) by using  $s$ -projective (resp.  $s$ -injective)  $s$ -resolutions (cf. Remark 2.10 and Remark 2.19) then there are  $K$ -linear topological isomorphisms*

$$(30) \quad \text{Ext}_H^q(V, W) \simeq \text{Ext}_H^q(W'_b, V'_b)$$

for any  $q \geq 0$  and any two objects  $V$  and  $W$  of  $\mathcal{M}_H^{snF}$ .

**Lemma 3.3.** *If an object  $V$  of  $\mathcal{M}_H$  is the locally convex direct sum of strict (LF)-spaces then  $V'_b$  is an object of  $\mathcal{M}_H$  and there are isomorphisms*

$$(31) \quad B_q(H, V)'_b \simeq F^q(H, V'_b)$$

in  $\mathcal{M}_H$  for any  $q \geq 0$ .

Proof: Together with any Fréchet space,  $V$  is barrelled and bornological. It follows from (6) and Proposition 1.2 that the spaces  $B_q(H, V)$  are again locally convex direct sums of strict (LF)-spaces, hence are complete Hausdorff barrelled and bornological. By Lemma 2.13 all spaces  $B_q(H, V)'_b$  are hypomodules in  $\mathcal{M}_H$ , and (31) is a special case of Proposition 1.4.  $\square$

Proof of Theorem 3.2: Since we assume  $H$  to be countable at infinity,  $D(H)$  is an object of  $\text{LCS}_K^{snF}$  by (6). By Proposition 1.2, as well as [32], Proposition 17.6, Proposition 20.7 and Corollary 20.14, the spaces  $B_q(H, V)$  are objects of  $\mathcal{M}_H^{snF}$  for all  $q \geq 0$  so that the unnormalized bar resolution of  $V$  (cf. Proposition 2.4) consists of objects of  $\mathcal{M}_H^{snF}$ . By Lemma 3.3 and Theorem 3.1 the standard  $s$ -injective  $s$ -resolution of  $V'_b$  (cf. Proposition 2.16) consists of objects of  $\mathcal{M}_H^{pct}$ .

Further, by (28) and Lemma 3.3 there are  $K$ -linear topological isomorphisms of complexes

$$(32) \quad \mathcal{L}_H(B_\bullet(H, V), W) \simeq \mathcal{L}_H(W'_b, F^\bullet(H, V'_b))$$

whence the theorem.  $\square$

**Remark 3.4.** The result [35], Corollary 3.3, of P. Schneider and J. Teitelbaum that we referred to earlier states that the category of locally analytic representations of  $H$  on locally convex  $K$ -vector spaces of compact type is anti-equivalent to the category of separately continuous  $D(H)$ -modules on nuclear Fréchet spaces via the duality functor. We extended the former category to countable direct products of spaces of compact type so as to have enough s-injective objects. This is not necessary if  $H$  is compact. In fact, it follows from (25) and [16], Satz 2.3.2, that in this case the standard s-injective s-resolution  $F^\bullet(H, V)$  of a locally analytic  $H$ -representation on a space  $V$  of compact type consists of spaces of compact type itself. Thus, if  $H$  is compact the category of locally analytic  $H$ -representations on locally convex  $K$ -vector spaces of compact type has enough s-injectives.

## 4 Finite resolutions

Recall that if  $V$  is an abstract  $D(H)$ -module then the (uncompleted) unnormalized bar resolution of  $V$  provides a resolution of  $V$  by free  $D(H)$ -modules. If  $V$  is an object of  $\mathcal{M}_H$  then the unnormalized bar resolution of Proposition 2.4 is obtained by passing to the completion. Due to the universal property of completions we obtain for any pair  $(V, W)$  of objects of  $\mathcal{M}_H$  a map of complexes

$$\begin{aligned} \mathcal{L}_H(B_\bullet(H, V), W) &= \mathcal{L}_H(\otimes_{K, \iota}^{\bullet+1} D(H) \otimes_{K, \iota} V, W) \\ &\longrightarrow \text{Hom}_{D(H)}(\otimes_K^{\bullet+1} D(H) \otimes_K V, W) \end{aligned}$$

giving rise to  $K$ -linear maps

$$(33) \quad \kappa^q : \mathcal{E}xt_H^q(V, W) \longrightarrow \text{Ext}_{D(H)}^q(V, W)$$

in cohomology. On the right,  $V$  and  $W$  are viewed as abstract  $D(H)$ -modules via the forgetful functor.

Similarly, the natural maps

$$V \otimes_{D(H)} ((\otimes_K^{q+1} D(H)) \otimes_K W) \longrightarrow V \widetilde{\otimes}_{D(H), \iota} B_q(H, W)$$

give rise to  $K$ -linear maps

$$(34) \quad \kappa_q : \text{Tor}_q^{D(H)}(V, W) \longrightarrow \mathcal{T}or_q^H(V, W).$$

**Proposition 4.1.** *For any two objects  $V$  and  $W$  of  $\mathcal{M}_H$  the maps  $\kappa^0$  and  $\kappa^1$  are injective.*

Proof: For  $\kappa^0$  this is simply the inclusion  $\mathcal{L}_H(V, W) \subseteq \text{Hom}_{D(H)}(V, W)$ . As for  $\kappa^1$  the relevant part of the complex computing  $\text{Ext}_{D(H)}^1(V, W)$  may be written as

$$\text{Hom}_{D(H)}(V, W) \xrightarrow{d_0} \text{Hom}_{D(H)}(D(H) \otimes_K V, W)$$

with  $d_0(f)(\delta \otimes v) = \delta f(v)$ . We need to show that if  $d_0(f) \in \mathcal{L}_H(D(H) \otimes_{K,\iota} V, W)$  then already  $f$  is continuous. This is clear by restricting  $d_0(f)$  to all vectors of the form  $1 \otimes v$ ,  $v \in V$ , and by noting that the map  $V \rightarrow D(H) \otimes_{K,\iota} V$  given by  $v \mapsto 1 \otimes v$  is a homeomorphism onto its image (cf. [32], Corollary 17.5).  $\square$

One of the major achievements in the theory of locally analytic representations was the definition of a large class of modules for which  $\kappa^0$  is an isomorphism. This was discovered by P. Schneider and J. Teitelbaum in [36] who defined the category of so-called *coadmissible*  $D(H)$ -modules, assuming  $K$  to be discretely valued. These modules have the property that any abstract  $D(H)$ -linear homomorphism is automatically continuous and strict (cf. [36], section 6).

For example, if  $H = \mathbb{Z}_p$  then, by a theorem of Y. Amice, the  $K$ -algebra  $D(\mathbb{Z}_p)$  is isomorphic to the ring  $\mathcal{O}(X)$  of global sections of the one dimensional rigid analytic open unit disc  $X$  over  $K$ . In this case, the category of coadmissible  $D(\mathbb{Z}_p)$ -modules is equivalent to the category of coherent module sheaves on  $X$  via the global section functor. It is known that any coadmissible projective  $\mathcal{O}(X)$ -module, i.e. the module of global sections of a vector bundle of finite rank on  $X$ , is finitely generated (cf. [19], Chapitre V, Théorème 1). On the other hand, it is a significant feature of the non-noetherian ring  $\mathcal{O}(X)$  that not every coadmissible  $\mathcal{O}(X)$ -module is finitely generated. Thus, the abelian category of coadmissible  $D(H)$ -modules of a locally  $L$ -analytic group  $H$  does generally not have enough projective objects.

We give an example of a pair of coadmissible  $D(H)$ -modules for which  $\kappa^1$  is not surjective. As usual,  $\mathcal{E}xt_H^1(V, W)$  can be interpreted as the set of equivalence classes of s-exact sequences

$$(35) \quad 0 \longrightarrow W \longrightarrow E \longrightarrow V \longrightarrow 0$$

with objects  $E$  of  $\mathcal{M}_H$ . It suffices to construct an exact sequence (35) which does not have a continuous  $K$ -linear section. To this end let  $\mathfrak{h}$  be the Lie algebra of  $H$  and let  $I(\mathfrak{h}) := D(H) \cdot \mathfrak{h}$  be the left ideal of  $D(H)$  generated by  $\mathfrak{h}$ . According to [37], Remark 1.1,  $I(\mathfrak{h})$  is a closed two sided ideal. By [33], section 2, the quotient  $D^\infty(H) = D(H)/I(\mathfrak{h})$  can be identified topologically with the separately continuous  $K$ -algebra of locally constant distributions on  $H$ , i.e.  $D^\infty(H) \simeq C^\infty(H)'_b$ . Here the space of locally constant  $K$ -valued functions  $C^\infty(H)$  on  $H$  carries the topology inherited from  $C^{an}(H)$ . Note that if  $H$  is compact and if  $K$  is discretely valued then all of the modules  $D(H)$ ,  $I(\mathfrak{h})$  and  $D^\infty(H)$  are coadmissible (cf. [36], Lemma 3.6).

**Proposition 4.2.** *Let  $H$  be an arbitrary locally  $L$ -analytic group. The exact sequence*

$$(36) \quad 0 \longrightarrow I(\mathfrak{h}) \longrightarrow D(H) \longrightarrow D^\infty(H) \longrightarrow 0$$

*of strict continuous  $K$ -linear maps is not s-exact unless  $H$  is discrete.*

Proof: If  $H$  is discrete then  $\mathfrak{h} = 0$  and the assertion is clear. Conversely, assume the sequence (36) to admit a continuous  $K$ -linear section. Since

$$(37) \quad D(H, K) \widehat{\otimes}_{K,\iota} \Omega \simeq D(H, \Omega)$$

for any complete valued field extension  $\Omega$  of  $K$  we may assume  $K$  to contain the completion of an algebraic closure  $\mathbb{C}_p$  of  $\mathbb{Q}_p$ . Let  $H_0$  be a compact open subgroup of  $H$  and consider the topological isomorphism  $C^{an}(H) \simeq \prod_{h \in H/H_0} C^{an}(h \cdot H_0)$ , inducing the topological isomorphism  $C^\infty(H) \simeq \prod_{h \in H/H_0} C^\infty(h \cdot H_0)$ . For any element  $h \in H$  the subspace  $C^\infty(h \cdot H_0) = C^{an}(h \cdot H_0)^\flat$  is closed, hence is of compact type and reflexive (cf. [35], Theorem 1.1 and Proposition 1.2). Therefore, both  $C^{an}(H)$  and  $C^\infty(H)$  are reflexive. Since  $D^\infty(H)$  is assumed to be a complemented subspace of  $D(H)$ , this implies that  $C^\infty(H)$  is a complemented subspace of  $C^{an}(H)$ . In other words, there is a strict surjection  $\varphi : C^{an}(H) \rightarrow C^\infty(H)$ .

We claim that if  $H$  is not discrete then  $C^{an}(H)$  contains a total bounded  $\mathfrak{o}_K$ -submodule  $B$ , i.e. a bounded  $\mathfrak{o}_K$ -submodule generating a dense subspace of  $C^{an}(H)$ . Indeed, by (5) we may assume  $H$  to be compact. Since the topology on  $C^{an}(H)$  does not depend on the group  $H$  but just on the underlying locally  $L$ -analytic manifold we may assume  $H = \mathfrak{o}_L^d$ . By [37], Lemma A.1 and Proposition A.2, as well as [32], Lemma 19.10, we may further assume  $d = 1$ . By the generalized Mahler expansion of locally analytic functions on  $\mathfrak{o}_L$  (cf. [34], Proposition 4.5) the space  $C^{an}(\mathfrak{o}_L)$  is the inductive limit of a sequence of Banach spaces whose transition maps have dense image. Therefore, the unit ball in any of these Banach spaces is as required.

Assuming the existence of  $\varphi$ , it follows that also  $C^\infty(H)$  and any of its quotients  $C^\infty(H_0)$  with a compact open subgroup  $H_0$  of  $H$  contains a bounded total  $\mathfrak{o}_K$ -submodule  $B$ . However, the space  $C^\infty(H_0) = \varinjlim_{H'_0 \triangleleft H_0} K[H_0/H'_0]$  is the strict inductive limit of finite dimensional  $K$ -vector spaces with their natural topology. Since these are necessarily closed in each other the set  $B$  has to be contained in one of them (cf. [32], Proposition 5.6). But then  $B$  cannot be total unless  $C^\infty(H_0)$  is finite dimensional and hence  $H$  is discrete.  $\square$

We remark that the arguments in the last part of the proof were already used by A. Grothendieck in order to construct non-split exact sequences of locally convex vector spaces (cf. [17], II.4.1 Lemme 10).

**Corollary 4.3.** *If the space  $C^{an}(\mathbb{Z}_p)$  of locally analytic  $K$ -valued functions on the locally  $\mathbb{Q}_p$ -analytic manifold  $\mathbb{Z}_p$  is endowed with its natural topology of a compact inductive limit (cf. [32], §16, page 97) then the surjective continuous  $K$ -linear endomorphism  $(f \mapsto df/dx)$  of  $C^{an}(\mathbb{Z}_p)$  does not have a continuous  $K$ -linear right inverse. In other words, there are no continuous  $K$ -linear antiderivatives on  $C^{an}(\mathbb{Z}_p)$ .*

Proof: The kernel of  $d/dx$  is  $C^\infty(\mathbb{Z}_p)$ .  $\square$

The case where  $H$  is a compact locally  $\mathbb{Q}_p$ -analytic group and  $V = \mathbb{1}$  is the trivial  $D(H)$ -module is one particularly important situation in which the comparison homomorphisms  $\kappa^q$  and  $\kappa_q$  can be shown to be bijective for all  $q \geq 0$ . This is due to the following result which, in a different setting, was first proved by M. Lazard and which relies on a lemma by J-P. Serre (cf. [25], Chapitre V, Théorème 3.2.7). We follow M. Lazard's strategy in [25], Chapitre V, (2.1) and (2.2). The technical difficulty of having to deal with all norm filtrations on  $D(H)$  simultaneously is overcome by directly proving a certain Koszul complex of  $D(\mathbb{Z}_p^d)$  to be s-exact.

**Theorem 4.4** (Lazard-Serre). *If  $H$  is a compact locally  $\mathbb{Q}_p$ -analytic group then the trivial module  $\mathbb{1}$  admits an  $s$ -projective  $s$ -resolution in  $\mathcal{M}_H$  consisting of finitely generated free  $D(H)$ -modules.*

Proof: Since the module  $\mathbb{1}$  is defined over any complete subfield of  $K$  we may, in view of (37), assume  $K$  to be discretely valued. Let us first assume  $H$  to be isomorphic, as a locally  $\mathbb{Q}_p$ -analytic group, to the additive group  $\mathbb{Z}_p^d$ . By a several variable version of Y. Amice's Fourier transform, the algebra  $D(\mathbb{Z}_p^d)$  is topologically isomorphic to the ring of holomorphic functions on the  $d$ -dimensional rigid  $K$ -analytic open unit disc  $X$  (cf. [30], Lecture 2), i.e. there is a topological isomorphism  $\mathcal{F} : D(\mathbb{Z}_p^d) \rightarrow \mathcal{O}(X)$  where

$$\mathcal{O}(X) = \left\{ \sum_{\alpha \in \mathbb{N}^d} d_\alpha t^\alpha \mid d_\alpha \in K, \forall r < 1 : \lim_{|\alpha| \rightarrow \infty} |d_\alpha| r^{|\alpha|} = 0 \right\}.$$

Here we used the notation  $t = (t_1, \dots, t_d)$  for a family of  $d$  pairwise commuting free variables,  $t^\alpha := t_1^{\alpha_1} \cdots t_d^{\alpha_d}$  and  $|\alpha| := \alpha_1 + \dots + \alpha_d$ . The Fréchet topology on  $\mathcal{O}(X)$  can be defined explicitly through the family of norms

$$\| \sum_{\alpha} d_\alpha t^\alpha \|_r := \sup_{\alpha} |d_\alpha| r^{|\alpha|}$$

with  $0 < r < 1$ . We also note that if  $\varepsilon_i \in \mathbb{Z}_p^d$  denotes the  $i$ -th unit vector and if  $\delta_{\varepsilon_i}$  denotes the Dirac distribution in  $\varepsilon_i$  then the element  $\delta_{\varepsilon_i} - 1$  of  $D(\mathbb{Z}_p^d)$  is mapped to  $t_i$  under  $\mathcal{F}$ . It follows that the augmentation ideal  $I(\mathbb{Z}_p^d)$  of  $D(\mathbb{Z}_p^d)$  corresponds to the ideal  $(t_1, \dots, t_d)$ .

Consider the augmented Koszul complex  $(Y_q, d_q)_{q \geq 0}$  given by  $Y_q := \mathcal{O}(X) \otimes_K \bigwedge^q K^d$  and

$$d_q(f \otimes \alpha_{i_1} \wedge \dots \wedge \alpha_{i_q}) := \sum_{j=1}^q (-1)^{j+1} t_{i_j} f \otimes \alpha_{i_1} \wedge \dots \wedge \widehat{\alpha_{i_j}} \wedge \dots \wedge \alpha_{i_q}$$

for  $q \geq 1$  and where the augmentation is the natural projection  $d_0 : \mathcal{O}(X) \rightarrow \mathcal{O}(X)/(t_1, \dots, t_d) = K$ .

If  $(e_i)_{1 \leq i \leq d}$  denotes the standard basis of  $K^d$  and if  $I$  denotes a general family of indices  $I = (i_1, \dots, i_q)$  where  $1 \leq i_1 < \dots < i_q \leq d$  then  $Y_q$  is topologized through the family of norms (again denoted by  $\| \cdot \|_r$ ) defined by

$$\| \sum_I \delta_I \otimes e_{i_1} \wedge \dots \wedge e_{i_q} \|_r := r^q \cdot \sup_I \| \delta_I \|_r.$$

With these conventions, the sequence  $(Y_q, d_q)_{q \geq 0}$  is an  $s$ -exact  $s$ -resolution of  $\mathbb{1}$  admitting a contracting homotopy  $(s_q)_{q \geq -1}$  which is norm decreasing with respect to all norms  $\| \cdot \|_r$ . Indeed, for  $d = 1$  it is just the exact sequence

$$(38) \quad 0 \longrightarrow \mathcal{O}(X) \otimes_K K \cdot e_1 \longrightarrow \mathcal{O}(X) \longrightarrow K \longrightarrow 0,$$

sending  $f \otimes e_1$  to  $f \cdot t_1$  and  $g \in \mathcal{O}(X)$  to  $g(0)$ . Here we may let  $s_{-1}$  be the inclusion of  $K$  into  $\mathcal{O}(X)$  and define  $s_0$  through  $s_0(g) := \frac{g-g(0)}{t_1} \otimes e_1$ . In general,

the augmented Koszul complex is the complete tensor product of complexes of the form (38) so that our claim follows from general principles (cf. [25], Chapitre V, (1.3.2)). Note that the boundary maps  $d_q$  are norm decreasing with respect to all norms  $\|\cdot\|_r$ , as well.

Now assume  $H$  to be a uniform pro- $p$  group in the sense of [14], Definition 4.1. The fundamental properties of  $D(H)$  were worked out by P. Schneider and J. Teitelbaum leading to the notion of a Fréchet-Stein algebra (cf. [36], sections 3 and 4). We briefly recall some of their constructions. It is known (cf. [14], Theorem 4.9 and Theorem 8.18 with its proof) that if  $h_1, \dots, h_d$  are topological generators of  $H$  with  $d = \dim(H)$  then the map  $\mathbb{Z}_p^d \rightarrow H$  given by  $(\alpha_1, \dots, \alpha_d) \mapsto h_1^{\alpha_1} \cdots h_d^{\alpha_d}$  is a homeomorphism and a global chart for  $H$  as a locally  $\mathbb{Q}_p$ -analytic manifold. It follows from the Mahler expansion of locally analytic functions on  $\mathbb{Z}_p^d$  that by setting  $b_i := h_i - 1 \in D(H)$  and  $b^\alpha := b_1^{\alpha_1} \cdots b_d^{\alpha_d}$  for  $\alpha \in \mathbb{N}^d$  the map

$$(39) \quad \begin{aligned} f : D(\mathbb{Z}_p^d) \simeq \mathcal{O}(X) &\longrightarrow D(H) \\ \sum_{\alpha \in \mathbb{N}^d} d_\alpha t^\alpha &\longmapsto \sum_{\alpha \in \mathbb{N}^d} d_\alpha b^\alpha \end{aligned}$$

is a well-defined topological isomorphism of Fréchet spaces. By transport of structure the Fréchet topology of  $D(H)$  can be defined by the family of norms

$$\left\| \sum_{\alpha} d_\alpha b^\alpha \right\|_r := \sup_{\alpha} |d_\alpha| r^{|\alpha|}$$

with  $0 < r < 1$ . It is crucial to the theory that although  $f$  is generally not multiplicative we still have

$$(40) \quad \|f(\delta_1 \delta_2) - f(\delta_1) f(\delta_2)\|_r < \|\delta_1 \delta_2\|_r$$

for all  $\delta_1, \delta_2 \in D(\mathbb{Z}_p^d)$  and all  $1/p < r < 1$ . This follows from [36], Lemma 4.4 and the remark on page 163. It can be expressed by saying that if the two sides of (39) are endowed with the filtrations induced by  $\|\cdot\|_r$  then the map induced by  $f$  on the associated graded objects is an isomorphism of *rings*.

For  $q \geq 0$  let  $X_q := D(H) \otimes_K \bigwedge^q K^d$  and  $f_q := f \otimes id : Y_q \rightarrow X_q$ . Starting from the augmented complex  $(Y_q, d_q)_{q \geq 0}$ , the continuous  $K$ -linear contracting homotopy  $(s_q)_{q \geq -1}$  and the  $K$ -linear topological isomorphisms  $f_q$ , we are going to construct continuous  $D(H)$ -linear maps  $\tilde{d}_q : X_q \rightarrow X_{q-1}$ ,  $q \geq 0$ , making  $X_\bullet$  a complex, and a continuous  $K$ -linear contracting homotopy  $(\tilde{s}_q)_{q \geq -1}$ , thus an  $s$ -resolution of  $\mathbb{1}$  as desired. More precisely, the operators  $\tilde{d}_q$  and  $\tilde{s}_q$  will be constructed inductively such that they are norm decreasing and such that the operators  $\tilde{d}_q - f_{q-1} d_q f_q^{-1}$  and  $\tilde{s}_q - f_{q+1} s_q f_q^{-1}$  are even *strictly* norm decreasing with respect to all norms  $\|\cdot\|_r$ .

We let  $\tilde{d}_0 = \mathbb{1} : X_0 \simeq D(H) \rightarrow K$  be the augmentation and define  $\tilde{s}_{-1}$  to be the map  $\alpha \mapsto \alpha \cdot 1 \in D(H)$ . If as above  $(e_1, \dots, e_d)$  denotes the standard basis of  $K^d$  we set  $\tilde{d}_1(e_i) := b_i$  and define  $\tilde{d}_1 : X_1 \rightarrow X_0$  by  $D(H)$ -linear continuation. Evidently,  $\tilde{d}_0$  and  $\tilde{d}_1$  are norm decreasing with respect to  $\|\cdot\|_r$  and

$$(41) \quad \tilde{d}_0 \circ \tilde{s}_{-1} = id_K \text{ and } \tilde{d}_0 \circ \tilde{d}_1 = 0.$$

Set  $\tilde{s}_0^{(0)} := f_1 \circ s_0 \circ f_0^{-1}$  and  $\tilde{s}_0^{(n)} := \tilde{s}_0^{(n-1)} + \tilde{s}_0^{(0)}(id_{X_0} - \tilde{s}_{-1}\tilde{d}_0 - \tilde{d}_1\tilde{s}_0^{(0)})^n$  for  $n \geq 1$ . Using (41) we have  $\tilde{d}_0(id_{X_0} - \tilde{s}_{-1}\tilde{d}_0 - \tilde{d}_1\tilde{s}_0^{(0)}) = 0$  and one inductively obtains

$$(42) \quad id_{X_0} - \tilde{s}_{-1}\tilde{d}_0 - \tilde{d}_1\tilde{s}_0^{(n)} = (id_{X_0} - \tilde{s}_{-1}\tilde{d}_0 - \tilde{d}_1\tilde{s}_0^{(0)})^{n+1}$$

(cf. [25], Chapitre V, Lemme 1.3.5). We claim the sequence  $(\tilde{s}_0^{(n)})_{n \geq 0}$  to have a limit in  $\mathcal{L}_b(X_0, X_1)$ . By the Banach-Steinhaus theorem (cf. [8], III.4.2 Corollary 2) we need only show  $(id_{X_0} - \tilde{s}_{-1}\tilde{d}_0 - \tilde{d}_1\tilde{s}_0^{(0)})^{n+1}$  to tend to zero pointwise. Checking directly that  $\tilde{d}_0 \circ f_0 = d_0$  and  $f_0^{-1} \circ \tilde{s}_{-1} = s_{-1}$  we obtain that

$$id_{X_0} - \tilde{s}_{-1}\tilde{d}_0 - \tilde{d}_1\tilde{s}_0^{(0)} = (f_0 d_1 f_1^{-1} - \tilde{d}_1)\tilde{s}_0^{(0)}.$$

If  $\delta \cdot e_i \in X_0$  and if  $1/p < r < 1$  then by (40)

$$\begin{aligned} \|(f_0 d_1 f_1^{-1} - \tilde{d}_1)(\delta \cdot e_i)\|_r &= \|\delta \cdot b_i - f(f^{-1}(\delta)t_i)\|_r \\ &< \|f^{-1}(\delta)t_i\|_r = \|\delta \cdot e_i\|_r. \end{aligned}$$

Since  $K$  is assumed to be discretely valued there is a constant  $c < 1$  such that the operator norm of  $f_0 d_1 f_1^{-1} - \tilde{d}_1$  with respect to  $\|\cdot\|_r$  is bounded above by  $c$ . Since  $\tilde{s}_0^{(0)}$  is norm decreasing with respect to all norms  $\|\cdot\|_r$  the claim follows and the sequence  $(\tilde{s}_0^{(n)})_{n \geq 0}$  has a limit  $\tilde{s}_0$  in  $\mathcal{L}_b(X_0, X_1)$ . We infer from (42) that  $id_{X_0} - \tilde{s}_{-1}\tilde{d}_0 - \tilde{d}_1\tilde{s}_0 = 0$ . Further, the operators  $\tilde{s}_0 - s_0^{(0)}$  and  $\tilde{d}_1 - f_0 d_1 f_1^{-1}$  are both strictly norm decreasing with respect to  $\|\cdot\|_r$ .

The construction proceeds now by induction, setting

$$\tilde{d}_{q+1}(e_{i_1} \wedge \dots \wedge e_{i_{q+1}}) := (id_{X_q} - \tilde{s}_{q-1}\tilde{d}_q)f_q d_{q+1} f_{q+1}^{-1}(e_{i_1} \wedge \dots \wedge e_{i_{q+1}})$$

for  $q \geq 1$ , and by defining  $\tilde{d}_{q+1}$  through  $D(H)$ -linear continuation. Since  $\tilde{d}_q$  and  $\tilde{s}_{q-1}$  are norm decreasing by hypothesis, one deduces that so is  $\tilde{d}_{q+1}$ . Using in addition that  $\tilde{d}_q - f_{q-1}d_q f_q^{-1}$  is even strictly norm decreasing, the above argument involving (40) shows that so is  $\tilde{d}_{q+1} - f_q d_{q+1} f_{q+1}^{-1}$ .

Set  $\tilde{s}_q^{(0)} := f_{q+1} \circ s_q \circ f_q^{-1}$  and  $\tilde{s}_q^{(n)} := \tilde{s}_q^{(n-1)} + \tilde{s}_q^{(0)}(id_{X_q} - \tilde{d}_{q+1}\tilde{s}_q^{(0)} - \tilde{s}_{q-1}\tilde{d}_q)^n$  for  $n \geq 1$ . The induction hypotheses and the same arguments as before show that  $\tilde{d}_q \tilde{d}_{q+1} = 0$  and that the sequence  $(\tilde{s}_q^{(n)})_{n \geq 0}$  has a limit  $\tilde{s}_q$  in  $\mathcal{L}_b(X_q, X_{q+1})$  which is norm decreasing with respect to  $\|\cdot\|_r$ , satisfies  $\tilde{d}_{q+1}\tilde{s}_q + \tilde{s}_{q-1}\tilde{d}_q = id_{X_q}$ , and which has the property that  $\tilde{s}_q - f_{q+1}s_q f_q^{-1}$  is strictly norm decreasing with respect to  $\|\cdot\|_r$ . This completes the proof for a uniform pro- $p$  group.

The case of an arbitrary compact locally  $\mathbb{Q}_p$ -analytic group follows from a principle due to C.T.C. Wall which we will come back to below (cf. Remark 6.4).  $\square$

As a corollary to the proof of Theorem 4.4 we obtain:

**Corollary 4.5.** *If  $H$  is a uniform pro- $p$  group then the  $D(H)$ -module  $\mathbf{1}$  admits an  $s$ -projective  $s$ -resolution of finite length  $d = \dim(H)$  consisting of finitely generated free  $D(H)$ -modules  $X_q$  of respective ranks  $\binom{d}{q}$ .  $\square$*

**Corollary 4.6.** *If  $H$  is a compact locally  $L$ -analytic group then the augmentation ideals of  $D(H)$  and  $D^\infty(H)$  are generated by finitely many elements of the form  $\delta_h - 1$ ,  $h \in H$ , where  $\delta_h$  is the Dirac distribution in  $h$ .*

Proof: Consider the sequence of surjective homomorphisms of  $K$ -algebras

$$(43) \quad D(\mathbb{R}^{L|\mathbb{Q}_p} H) \longrightarrow D(H) \longrightarrow D^\infty(H),$$

where  $\mathbb{R}^{L|\mathbb{Q}_p} H$  is the locally  $\mathbb{Q}_p$ -analytic group underlying  $H$ . The maps in (43) send augmentation ideals onto augmentation ideals so that it suffices to consider the case  $L = \mathbb{Q}_p$ . If  $H_0$  is an open subgroup of  $H$  then the augmentation ideal of  $D(H)$  is generated by that of  $D(H_0)$  and the finitely many elements  $h - 1$  with  $h$  running through a set of representatives of  $H/H_0$ . By [14], Theorem 8.32, we may therefore assume  $H$  to be uniform pro- $p$ . In this case the assertion follows directly from the construction in Theorem 4.4.  $\square$

**Remark 4.7.** It is currently an open question if Theorem 4.4 generalizes to compact locally analytic groups over finite extensions  $L$  of  $\mathbb{Q}_p$ . The main technical obstacle is that the ring  $D(\mathfrak{o}_L)$  is generally not a ring of convergent power series unless  $L = \mathbb{Q}_p$ , and that we lack a sufficiently explicit resolution of  $\mathbb{1}$ , replacing the Koszul complex. In fact, P. Schneider and J. Teitelbaum showed that if  $L \neq \mathbb{Q}_p$  then  $D(\mathfrak{o}_L)$  is isomorphic to the ring of global sections of a form of the rigid  $L$ -analytic open unit disc which does not become trivial over any discretely valued extension of  $L$  (cf. [34], Corollary 3.7 and Lemma 3.9). However, the properties of the ring  $D(\mathfrak{o}_L)$  are sufficiently nice to treat solvable groups (cf. Theorem 6.5). This will be sufficient for our applications in sections 7 and 8.

Most of our results will rely on the following assumption on the group  $H$ .

(A) The trivial  $D(H)$ -module  $\mathbb{1}$  admits an  $s$ -projective  $s$ -resolution consisting of  $s$ -free objects of the form  $D(H) \widehat{\otimes}_{K,\iota} Y$  where  $Y$  is a  $K$ -vector space carrying its finest locally convex topology.

**Theorem 4.8.** *If  $H$  satisfies (A) the comparison homomorphisms*

$$\begin{aligned} \kappa^q &: H_{an}^q(H, V) \longrightarrow Ext_{D(H)}^q(\mathbb{1}, V) \quad \text{and} \\ \kappa_q &: Tor_q^{D(H)}(V, \mathbb{1}) \longrightarrow H_q^{an}(H, V) \end{aligned}$$

of (33) and (34) are bijective for any  $q \geq 0$  and any object  $V$  of  $\mathcal{M}_H$ .

Proof: If  $Y$  is a locally convex  $K$ -vector space carrying its finest locally convex topology then

$$D(H) \widehat{\otimes}_{K,\iota} Y = D(H) \otimes_K Y$$

is both  $s$ -free and free over  $D(H)$ . Further, by Lemma 2.2 and Lemma 2.6 there are equivalences of functors

$$\begin{aligned} \mathcal{L}_H(D(H) \widehat{\otimes}_{K,\iota} Y, \cdot) &\simeq \mathcal{L}_b(Y, \cdot) \simeq \text{Hom}_K(Y, \cdot) \\ &\simeq \text{Hom}_{D(H)}(D(H) \otimes_K Y, \cdot) \end{aligned}$$

and

$$\begin{aligned} (\cdot) \widetilde{\otimes}_{D(H),\iota} (D(H) \widehat{\otimes}_{K,\iota} Y) &\simeq (\cdot) \widehat{\otimes}_{K,\iota} Y \simeq (\cdot) \otimes_K Y \\ &\simeq (\cdot) \otimes_{D(H)} (D(H) \otimes_K Y). \quad \square \end{aligned}$$

**Theorem 4.9.** *If  $H$  satisfies (A) then there are spectral sequences*

$$(44) \quad \text{Ext}_{D^\infty(H)}^p(\mathbb{1}, \text{Ext}_{D(H)}^q(D^\infty(H), V)) \implies H_{an}^{p+q}(H, V) \quad \text{and}$$

$$(45) \quad \text{Tor}_p^{D^\infty(H)}(\text{Tor}_q^{D(H)}(V, D^\infty(H)), \mathbb{1}) \implies H_{p+q}^{an}(H, V)$$

for all objects  $V$  of  $\mathcal{M}_H$ .

Proof: By Theorem 4.8 we have  $H_{an}^q(H, V) \simeq \text{Ext}_{D(H)}^q(\mathbb{1}, V)$  naturally in  $V$ . Let  $I^\bullet$  be an injective resolution of  $V$  as an abstract  $D(H)$ -module. Then there is an isomorphism of complexes

$$\text{Hom}_{D(H)}(\mathbb{1}, I^\bullet) \simeq \text{Hom}_{D^\infty(H)}(\mathbb{1}, \text{Hom}_{D(H)}(D^\infty(H), I^\bullet)).$$

If we can show the  $D^\infty(H)$ -modules  $\text{Hom}_{D(H)}(D^\infty(H), I^q)$  to be injective we can take for (44) the usual Grothendieck spectral sequence.

However, as in Proposition 2.16, one can take for  $V$  an injective resolution in which all objects are of the form  $\text{Hom}_K(D(H), J)$  with certain  $K$ -vector spaces  $J$ . But then  $\text{Hom}_{D(H)}(D^\infty(H), \text{Hom}_K(D(H), J)) \simeq \text{Hom}_K(D^\infty(H), J)$  is injective over  $D^\infty(H)$ . The arguments leading to (45) are analogous.  $\square$

We note that by [37], page 306, and a similar reasoning for torsion groups we have

$$(46) \quad \text{Ext}_{D(H)}^q(D^\infty(H), V) \simeq H^q(\mathfrak{h}, V) \quad \text{and}$$

$$(47) \quad \text{Tor}_q^{D(H)}(V, D^\infty(H)) \simeq H_q(\mathfrak{h}, V)$$

for all  $q \geq 0$  and all  $D(H)$ -modules  $V$ . Here  $H^\bullet(\mathfrak{h}, \cdot)$  and  $H_\bullet(\mathfrak{h}, \cdot)$  denotes the Lie algebra cohomology and the Lie algebra homology of  $\mathfrak{h}$ , respectively. Thus, restricting to the case of a compact group  $H$  we obtain the following analog of [25], Chapitre V, Théorème 2.4.10 (cf. also [39], Theorem 5.2.4). As usual, if  $H$  is any group acting on an abelian group  $X$  we denote by  $X^H$  and  $X_H$  the subgroup of  $H$ -invariants and the quotient group of  $H$ -coinvariants, respectively.

**Theorem 4.10.** *Assuming the compact locally  $L$ -analytic group  $H$  to satisfy assumption (A) there are natural  $K$ -linear isomorphisms*

$$(48) \quad H_{an}^q(H, V) \simeq H^q(\mathfrak{h}, V)^H \quad \text{and}$$

$$(49) \quad H_q^{an}(H, V) \simeq H_q(\mathfrak{h}, V)_H$$

for all  $q \geq 0$  and any object  $V$  of  $\mathcal{M}_H$ . In particular,  $H_{an}^q(H, V) = H_q^{an}(H, V) = 0$  if  $q > \dim H$ .

Proof: Note first that if  $H$  is any compact locally  $L$ -analytic group then the trivial  $D^\infty(H)$ -module  $\mathbb{1}$  is projective. In fact, the augmentation  $D^\infty(H) \rightarrow \mathbb{1}$  has a  $D^\infty(H)$ -linear section. Equivalently, there is a functional  $C^\infty(H) \rightarrow K$  which is invariant under right translation. This is the usual integration map available for any field of characteristic zero.

Therefore, the spectral sequences of Theorem 4.9 degenerate. By (46) and (47) it remains to remark that  $\text{Hom}_{D^\infty(H)}(\mathbb{1}, V) \simeq V^H$  and  $W \otimes_{D^\infty(H)} \mathbb{1} \simeq W_H$  for any  $D^\infty(H)$ -module  $W$ , as follows from Corollary 4.6.  $\square$

## 5 Shapiro's lemma

Let  $H_1$  be a locally  $L$ -analytic group and let  $H_2$  be a closed locally  $L$ -analytic subgroup. We note that by [22], Proposition 1.1.2,  $D(H_2)$  is a closed subalgebra of  $D(H_1)$  so that any object of  $\mathcal{M}_{H_1}$  becomes an object of  $\mathcal{M}_{H_2}$  via restriction.

For an object  $V$  of  $\mathcal{M}_{H_2}$  we set

$$(50) \quad \text{ind}_{H_2}^{H_1}(V) := D(H_1) \widehat{\otimes}_{D(H_2), \iota} V \quad \text{and}$$

$$(51) \quad \text{coind}_{H_2}^{H_1}(V) := \mathcal{L}_{H_2}(D(H_1), V).$$

**Proposition 5.1.** *The assignment  $(V \mapsto \text{ind}_{H_2}^{H_1}(V))$  is a functor  $\mathcal{M}_{H_2} \rightarrow \mathcal{M}_{H_1}$  taking  $s$ -exact sequences to  $s$ -exact sequences. The assignment  $(V \mapsto \text{coind}_{H_2}^{H_1}(V))$  is a functor  $\mathcal{M}_{H_2} \rightarrow \mathcal{M}_{H_1}$  taking  $s$ -exact sequences of hypomodules in  $\mathcal{M}_{H_2}$  to  $s$ -exact sequences of hypomodules in  $\mathcal{M}_{H_1}$ .*

**Lemma 5.2.** *The (right)  $D(H_2)$ -module  $D(H_1)$  is  $s$ -free.*

Proof: By [16], Satz 4.1.1, the projection  $H_1 \rightarrow H_1/H_2$  has a locally analytic section inducing an isomorphism  $H_1 \simeq H_1/H_2 \times H_2$  of locally  $L$ -analytic manifolds and right  $H_2$ -spaces, the action of  $H_2$  on  $H_1/H_2$  being trivial. According to [37], Proposition A.3, we obtain an isomorphism

$$(52) \quad D(H_1) \simeq D(H_1/H_2) \widehat{\otimes}_{K, \iota} D(H_2)$$

of right  $D(H_2)$ -modules.  $\square$

Proof of Proposition 5.1: By (52) and Lemma 2.6 there is a natural  $K$ -linear topological isomorphism

$$(53) \quad \text{ind}_{H_2}^{H_1}(V) \simeq D(H_1/H_2) \widehat{\otimes}_{K, \iota} V$$

whence  $\text{ind}_{H_2}^{H_1}(V)$  is Hausdorff and complete. Its structure of a separately continuous  $D(H_1)$ -module is the one induced from the  $s$ -free module  $D(H_1) \widehat{\otimes}_{K, \iota} V$ . It is clear that the formation of  $\text{ind}_{H_2}^{H_1}(V)$  is functorial in  $V$ . The final assertion follows from Lemma 5.2 and Lemma 2.6.

By Lemma 2.13 the space  $\mathcal{L}_b(D(H_1), V)$  is an object of  $\mathcal{M}_{H_1}$  via  $(\delta f)(\delta') = f(\delta'\delta)$ . This structure extends to the closed, hence complete and Hausdorff subspace  $\mathcal{L}_{H_2}(D(H_1), V)$ . Therefore,  $\text{coind}_{H_2}^{H_1}(V)$  is an object of  $\mathcal{M}_{H_1}$ . The final assertion follows from Lemma 5.2 and Lemma 2.2, implying that if  $V$  is a hypomodule then there is a natural  $K$ -linear topological isomorphism

$$(54) \quad \text{coind}_{H_2}^{H_1}(V) \simeq \mathcal{L}_b(D(H_1/H_2), V). \quad \square$$

**Proposition 5.3.** *The functors  $\text{ind}_{H_2}^{H_1}$  and  $\text{coind}_{H_2}^{H_1}$  restrict to functors  $\mathcal{M}_{H_2}^{snF} \rightarrow \mathcal{M}_{H_1}^{snF}$  and  $\mathcal{M}_{H_2}^{pct} \rightarrow \mathcal{M}_{H_1}^{pct}$ , respectively. The diagram of  $s$ -exact functors*

$$(55) \quad \begin{array}{ccc} \mathcal{M}_{H_1}^{snF} & \xrightarrow{(\cdot)'_b} & \mathcal{M}_{H_1}^{pct} \\ \text{ind}_{H_1}^{H_2} \uparrow & & \uparrow \text{coind}_{H_2}^{H_1} \\ \mathcal{M}_{H_2}^{snF} & \xrightarrow{(\cdot)'_b} & \mathcal{M}_{H_2}^{pct} \end{array}$$

commutes up to natural equivalence.

Proof: The space  $D(H_1/H_2)$  is the countable locally convex direct sum of nuclear Fréchet spaces. If  $V$  is an object of  $\mathcal{M}_{H_2}^{snF}$  then so is  $\text{ind}_{H_2}^{H_1}(V)$  by Proposition 5.1, (53) and the arguments given in the proof of Theorem 3.2. Similarly, we infer from (54) and [32], Corollary 18.8, that as a locally convex  $K$ -vector space

$$(56) \quad \text{coind}_{H_2}^{H_1}(V) \simeq C^{an}(H_1/H_2) \widehat{\otimes}_{K,\pi} V$$

is the countable direct product of spaces of compact type (cf. [17], I.1.3 Proposition 6 and [15], Proposition 2.1.28). It remains to show that if  $V$  is an object of  $\mathcal{M}_{H_2}^{snF}$  then  $\text{coind}_{H_2}^{H_1}(V'_b) \simeq \text{ind}_{H_2}^{H_1}(V)_b'$  in  $\mathcal{M}_{H_1}^{pct}$  where  $V'_b$  is an object of  $\mathcal{M}_{H_2}^{pct}$  by Theorem 3.1. This is the natural map

$$\mathcal{L}_{H_2}(D(H_1), V'_b) \xrightarrow{\alpha} \mathcal{L}_b(D(H_1) \widetilde{\otimes}_{D(H_2),\iota} V, K)$$

given by  $\alpha(f)(\delta \otimes v) := f(\delta)(v)$  which is a topological isomorphism by (53), (54) and Proposition 1.4.  $\square$

**Remark 5.4.** If the underlying locally convex  $K$ -vector space of an object  $V$  of  $\mathcal{M}_{H_2}$  is of compact type then  $\text{coind}_{H_2}^{H_1}(V)$  can be realized as the space

$$\text{Ind}_{H_2}^{H_1}(V) := \{f \in C^{an}(H_1, V) \mid \forall h_i \in H_i : f(h_1 h_2) = h_2^{-1} f(h_1)\}.$$

This follows from (56) since for spaces of compact type there is a natural topological isomorphism  $C^{an}(H_1, V) \simeq C^{an}(H_1) \widehat{\otimes}_{K,\pi} V$  (cf. Remark 2.17). If  $H_1/H_2$  is compact then this is again a space of compact type.

**Theorem 5.5** (Frobenius reciprocity). *If  $V$  and  $W$  are objects of  $\mathcal{M}_{H_2}$  and  $\mathcal{M}_{H_1}$ , respectively, then there is a natural continuous  $K$ -linear bijection*

$$(57) \quad \mathcal{L}_{H_1}(\text{ind}_{H_2}^{H_1}(V), W) \longrightarrow \mathcal{L}_{H_2}(V, W).$$

*If  $V$  is the locally convex direct sum of strict (LF)-spaces and if  $W$  is a hypomodule then this is a topological isomorphism.*

*If  $V$  is an object of  $\mathcal{M}_{H_2}$  and if  $W$  is a hypomodule in  $\mathcal{M}_{H_1}$  then there is a natural continuous  $K$ -linear bijection*

$$(58) \quad \mathcal{L}_{H_1}(W, \text{coind}_{H_2}^{H_1}(V)) \longrightarrow \mathcal{L}_{H_2}(W, V)$$

*given by composition with the map  $\text{coind}_{H_2}^{H_1}(V) \rightarrow V$ ,  $f \mapsto f(1)$ . If  $W$  is bornological and barrelled then this is a topological isomorphism.*

Proof: The map  $(v \mapsto 1 \otimes v)$  makes  $V$  a  $D(H_2)$ -submodule of  $\text{ind}_{H_2}^{H_1}(V)$ . According to Lemma 2.2 restriction to  $V$  induces a continuous  $K$ -linear bijection

$$\alpha : \mathcal{L}_{H_1}(D(H_1) \widehat{\otimes}_{K,\iota} V, W) \longrightarrow \mathcal{L}_b(V, W)$$

which, under the additional assumptions on  $V$  and  $W$ , is a topological isomorphism. It follows directly from the definitions that a continuous  $D(H_1)$ -linear map  $f$  from the left factors through  $\text{ind}_{H_2}^{H_1}(V)$  if and only if  $\alpha(f)$  is  $D(H_2)$ -linear.

It remains to remark that the quotient map  $D(H_1) \widehat{\otimes}_{K,t} V \rightarrow \text{ind}_{H_2}^{H_1}(V)$  is strong as follows from (52) and (53) so that the strong topology of  $\mathcal{L}_{H_1}(\text{ind}_{H_2}^{H_1}(V), W)$  is the subspace topology of  $\mathcal{L}_{H_1}(D(H_1) \widehat{\otimes}_{K,t} V, W)$ .

Using Lemma 2.14 the second part of the theorem is proved similarly.  $\square$

One of the reasons for working with s-exact sequences are the following acyclicity properties of the restriction and induction functors which seem to fail in a purely algebraic setting.

**Proposition 5.6.** (i) *The functor  $\text{ind}_{H_2}^{H_1} : \mathcal{M}_{H_2} \rightarrow \mathcal{M}_{H_1}$  takes s-projective objects to s-projective objects.*

(ii) *The functor  $\text{coind}_{H_2}^{H_1} : \mathcal{M}_{H_2} \rightarrow \mathcal{M}_{H_1}$  takes s-injective hypomodules to s-injective hypomodules.*

(iii) *The restriction functor  $\mathcal{M}_{H_1} \rightarrow \mathcal{M}_{H_2}$  preserves both s-projective objects and s-injective hypomodules.*

Proof: Since  $\text{ind}_{H_2}^{H_1}$  respects direct sums it suffices to show that it takes s-free objects to s-free objects (cf. Proposition 2.3). The claim then follows from Lemma 2.6.

The functor  $\text{coind}_{H_2}^{H_1}$  respects (finite) direct sums, too, so that by Proposition 2.15 we are reduced to the case of a module of the form  $\mathcal{L}_b(D(H_2), W)$  for a complete Hausdorff locally convex  $K$ -vector space  $W$ . However, by Lemma 2.14 there is a  $K$ -linear topological isomorphism  $\text{coind}_{H_2}^{H_1}(\mathcal{L}_b(D(H_2), W)) \simeq \mathcal{L}_b(D(H_1), W)$ .

Finally, the fact that the restriction functor preserves s-projectives follows from Proposition 2.3 and Lemma 5.2. Also, by Proposition 1.4 and (52) there is a  $K$ -linear topological isomorphism  $\mathcal{L}_b(D(H_1), V) \simeq \mathcal{L}_b(D(H_2), \mathcal{L}_b(D(H_1/H_2), V))$  for any complete Hausdorff locally convex  $K$ -vector space  $V$ . By naturality, it is a homomorphism of  $D(H_2)$ -modules so that due to Proposition 2.15 the restriction functor takes s-injective hypomodules to s-injective hypomodules.  $\square$

**Theorem 5.7** (Shapiro's lemma). *If  $V$  and  $W$  are objects of  $\mathcal{M}_{H_2}$  and  $\mathcal{M}_{H_1}$ , respectively, then there are  $K$ -linear bijections*

$$(59) \quad \mathcal{E}xt_{H_1}^q(\text{ind}_{H_2}^{H_1}(V), W) \longrightarrow \mathcal{E}xt_{H_2}^q(V, W)$$

for all  $q \geq 0$ . If the spaces in (59) are topologized by using s-projective s-resolutions (cf. Remark 2.10) then all maps are continuous. If in addition  $V$  is the locally convex direct sum of strict (LF)-spaces and if  $W$  is a hypomodule then the maps in (59) are topological isomorphisms.

If  $V$  and  $W$  are hypomodules in  $\mathcal{M}_{H_2}$  and  $\mathcal{M}_{H_1}$ , respectively, then there are  $K$ -linear bijections

$$(60) \quad \mathcal{E}xt_{H_1}^q(W, \text{coind}_{H_2}^{H_1}(V)) \longrightarrow \mathcal{E}xt_{H_2}^q(W, V)$$

for all  $q \geq 0$ . If the spaces in (60) are topologized by using s-injective s-resolutions (cf. Remark 2.19) then all maps are continuous. If in addition  $W$  is bornological and barrelled then all maps in (60) are topological isomorphisms.

Proof: Choose an s-projective s-resolution  $X_\bullet \rightarrow V$  of  $V$  in  $\mathcal{M}_{H_2}$ . By Proposition 5.1 and Proposition 5.6 the complex  $\text{ind}_{H_2}^{H_1}(X_\bullet) \rightarrow \text{ind}_{H_2}^{H_1}(V)$  is an s-projective s-resolution of  $\text{ind}_{H_2}^{H_1}(V)$  in  $\mathcal{M}_{H_1}$ . Further, by Theorem 5.5 there is a continuous  $K$ -linear bijection of complexes

$$\mathcal{L}_{H_1}(\text{ind}_{H_2}^{H_1}(X_\bullet), W) \longrightarrow \mathcal{L}_{H_2}(X_\bullet, W)$$

inducing the continuous  $K$ -linear bijections in (59). One might choose for  $X_\bullet$  the unnormalized bar resolution  $B_\bullet(H_2, V)$  of Proposition 2.4 which consists of locally convex direct sums of strict (LF)-spaces if  $V$  does (cf. the arguments given in the proof of Lemma 3.3). Therefore, the final assertion follows from the one in Theorem 5.5. The second part of the theorem is proved similarly.  $\square$

The same line of arguments leading to Theorem 5.7 also gives the following analog for torsion groups for which one uses Lemma 2.6.

**Theorem 5.8.** *If  $V$  and  $W$  are objects of  $\mathcal{M}_{H_2}$  and  $\mathcal{M}_{H_1}$ , respectively, then there are topological isomorphisms*

$$(61) \quad \mathcal{T}or_q^{H_1}(\text{ind}_{H_2}^{H_1}(V), W) \simeq \mathcal{T}or_q^{H_2}(V, W) \quad \text{and}$$

$$(62) \quad \mathcal{T}or_q^{H_1}(W, \text{ind}_{H_2}^{H_1}(V)) \simeq \mathcal{T}or_q^{H_2}(W, V)$$

for all  $q \geq 0$ .  $\square$

## 6 Spectral sequences

In order to complete the proof of Theorem 4.4 and to broaden the class of groups satisfying assumption (A) we need the following technical construction due to C.T.C. Wall, generalizing the notion of a double complex. We again follow M. Lazard's exposition in [25], Chapitre V, (3.1) and (3.2).

Recall that a Wall complex in  $\mathcal{M}_H$  is a family  $(X_{ij}, d_{ij}^{(k)})_{i,j,k \geq 0}$  of objects  $X_{ij}$  in  $\mathcal{M}_H$  and morphisms  $d_{ij}^{(k)} : X_{ij} \rightarrow X_{i-k, j+k-1}$  such that if  $X_n := \bigoplus_{i+j=n} X_{ij}$  and  $\Delta_{n+1} := \sum_{i=0}^{n+1} \sum_{k=0}^i d_{i, n+1-i}^{(k)} : X_{n+1} \rightarrow X_n$  then  $\Delta_{n+1} \circ \Delta_{n+2} = 0$  for all  $n \geq 0$ . We call  $(X_n, \Delta_{n+1})_{n \geq 0}$  the total complex associated with the Wall complex  $(X_{ij}, d_{ij}^{(k)})_{i,j,k \geq 0}$ .

For  $r, n \geq 0$  we set  $F_r X_n := \bigoplus_{i+j=n, i \leq r} X_{ij}$  which defines a filtration on  $X_\bullet$  compatible with the differential  $\Delta_\bullet$ . This gives rise to a spectral sequence with initial terms  $E_{ij}^1 = H_j(X_{i, \bullet}, d_{i, \bullet}^{(0)})$ . If all complexes  $(X_{i, \bullet}, d_{i, \bullet}^{(0)})$  are exact then the spectral sequence degenerates and the homology of the total complex coincides with the homology of the complex  $Y_i := H_0(X_{i, \bullet})$ ,  $i \geq 0$ , with differential induced by  $d_{i,0}^{(1)}$ .

**Theorem 6.1** (Wall). *Let  $\dots \rightarrow Y_1 \xrightarrow{\delta_1} Y_0 \xrightarrow{\delta_0} V$  be an s-exact sequence in  $\mathcal{M}_H$ , and for each  $i \geq 0$  let  $(X_{ij}, d_{ij})_{j \geq 0}$  be an s-projective s-resolution of  $Y_i$ . There are morphisms  $d_{ij}^{(k)} : X_{ij} \rightarrow X_{i-k, j+k-1}$  in  $\mathcal{M}_H$  making  $(X_{ij}, d_{ij}^{(k)})_{i,j,k \geq 0}$  a Wall complex whose associated total complex is an s-projective s-resolution of  $V$ .*

Proof: For  $i, j \geq 0$  we set  $d_{ij}^{(0)} := d_{ij}$  and  $\Delta_0 := \delta_0 \circ d_{00}^{(0)} : X_0 \rightarrow V$ . For  $i \geq 0$  the module  $X_{i+1,0}$  is s-projective so that since  $d_{i,0}^{(0)}$  is a strong surjection there is a morphism  $d_{i+1,0}^{(1)}$  in  $\mathcal{M}_H$  making the following diagram commutative

$$\begin{array}{ccc} X_{i+1,0} & \xrightarrow{d_{i+1,0}^{(1)}} & X_{i,0} \\ d_{i+1,0}^{(0)} \downarrow & & \downarrow d_{i,0}^{(0)} \\ Y_{i+1} & \xrightarrow{\delta_{i+1}} & Y_i. \end{array}$$

The construction of the maps  $d_{ij}^{(k)}$  in the general case proceeds by induction and can be copied word by word from [25], Chapitre V, Théorème 3.1.3.

We are rather concerned about the augmented total complex  $(X_n, \Delta_n)_{n \geq 0}$  being an s-projective s-resolution of  $V$ . Since there is a misprint in formula (3.1.6.4) of [25], page 190, we shall give some of the details.

Let  $(\sigma_i)_{i \geq -1}$  be a continuous  $K$ -linear contracting homotopy of the complex  $Y_\bullet$ , and for each  $i \geq 0$  let  $(s_{ij})_{j \geq -1}$  be a continuous  $K$ -linear contracting homotopy of the complex  $(X_{ij}, d_{ij})_{j \geq 0}$ .

We set  $\Sigma_{-1} := s_{0,-1} \circ \sigma_{-1} : V \rightarrow X_0$  and have  $\Delta_0 \Sigma_{-1} = \delta_0 d_{00}^{(0)} s_{0,-1} \sigma_{-1} = id_V$ . Let  $\Sigma_{-1}^{(0)} := \Sigma_{-1}$ . For  $n \geq 0$  set

$$\Sigma_n^{(0)} := \sum_{i=0}^n s_{i,n-i} + s_{n+1,-1} \sigma_n d_{n,0}^{(0)} - s_{n,0} d_{n+1,0}^{(1)} s_{n+1,-1} \sigma_n d_{n,0}^{(0)}.$$

It is straightforward to check that for any  $n \geq 0$  and any  $0 \leq r \leq n$

$$(63) \quad (id_{X_n} - \Delta_{n+1} \Sigma_n^{(0)} - \Sigma_{n-1}^{(0)} \Delta_n)(F_r X_n) \subseteq F_{r-1} X_n$$

with the convention that  $F_{-1} X_n = 0$ . In particular,  $\Sigma_0 := \Sigma_0^{(0)}$  satisfies  $id_{X_0} - \Delta_1 \Sigma_0 - \Sigma_{-1} \Delta_0 = 0$ .

The construction of the maps  $\Sigma_n$  proceeds now by induction. For  $n \geq 0$  and  $m \geq 1$  we set  $\Sigma_n^{(m)} := \Sigma_n^{(m-1)} + \Sigma_n^{(0)} (id_{X_n} - \Delta_{n+1} \Sigma_n^{(0)} - \Sigma_{n-1}^{(0)} \Delta_n)^m$ . The sequence  $(\Sigma_n^{(m)})_{m \geq 0}$  of continuous  $K$ -linear maps becomes stationary for  $m \geq n$ . Indeed, we already saw this for  $n = 0$ . Since obviously

$$\Delta_n (id_{X_n} - \Delta_{n+1} \Sigma_n^{(0)} - \Sigma_{n-1}^{(0)} \Delta_n) = (id_{X_{n-1}} - \Delta_n \Sigma_{n-1}^{(0)} - \Sigma_{n-2}^{(0)} \Delta_{n-1}) \Delta_n$$

we may again apply [25], Chapitre V, Lemme 1.3.5, and obtain

$$id_{X_n} - \Delta_{n+1} \Sigma_n^{(m)} - \Sigma_{n-1}^{(m)} \Delta_n = (id_{X_n} - \Delta_{n+1} \Sigma_n^{(0)} - \Sigma_{n-1}^{(0)} \Delta_n)^{m+1}.$$

If  $m \geq n$  then by (63) the sequence becomes stationary so that  $\Sigma_n^{(m)} = \Sigma_n^{(n)} =: \Sigma_n$  satisfies  $id_{X_n} - \Delta_{n+1} \Sigma_n - \Sigma_{n-1}^{(n)} \Delta_n = 0$ . Note that  $\Sigma_{n-1}^{(n)} = \Sigma_{n-1}$  by hypothesis.  $\square$

Although Lazard does not state it in full generality, the following result is an analog of [25], Chapitre V, (3.2.1).

**Corollary 6.2.** *Let  $H_1$  be a locally  $L$ -analytic group and let  $H_2$  be a closed normal locally  $L$ -analytic subgroup of  $H_1$ . If both  $H_2$  and  $H_1/H_2$  satisfy assumption (A) then so does  $H_1$ .*

Proof: Let  $Y_\bullet \rightarrow \mathbb{1}$  be an s-projective s-resolution in  $\mathcal{M}_{H_1/H_2}$  with all  $Y_q$  s-free of the form  $Y_q = D(H_1/H_2) \widehat{\otimes}_{K,\iota} V_q$  where  $V_q$  is a  $K$ -vector space carrying its finest locally convex topology. Similarly, let  $Z_\bullet \rightarrow \mathbb{1}$  be an s-projective s-resolution in  $\mathcal{M}_{H_2}$  as in (A). By (52) and Lemma 2.6 we have  $D(H_1) \widetilde{\otimes}_{D(H_2),\iota} \mathbb{1} \simeq D(H_1/H_2)$  in  $\mathcal{M}_{H_1}$  where  $D(H_1/H_2)$  is an object of  $\mathcal{M}_{H_1}$  via the strict surjective homomorphism  $D(H_1) \rightarrow D(H_1/H_2)$  induced by the homomorphism  $H_1 \rightarrow H_1/H_2$  of locally  $L$ -analytic groups.

By Lemma 2.6 and Lemma 5.2 the complex  $D(H_1) \widetilde{\otimes}_{D(H_2),\iota} Z_\bullet$  is an s-resolution of  $D(H_1/H_2)$  in  $\mathcal{M}_{H_1}$  consisting of s-free objects of the required form. If we set  $X_{ij} := D(H_1) \widetilde{\otimes}_{D(H_2),\iota} Z_j \widehat{\otimes}_{K,\iota} V_i$  then for every  $i$  the family  $(X_{ij})_{j \geq 0}$  together with the maps induced from the complex  $Z_\bullet$  forms an s-projective s-resolution of  $Y_j$  in  $\mathcal{M}_{H_1}$  consisting of s-free modules of the required form. According to Theorem 6.1 one can associate with these data a Wall complex whose total complex is an s-projective s-resolution of  $\mathbb{1}$  in  $\mathcal{M}_{H_1}$ . Since all modules  $X_{ij}$  are s-free of the required type so are the members of the total complex.  $\square$

The proof of Corollary 6.2 gives the following more precise information.

**Corollary 6.3.** *Let  $H_1$  be a locally  $L$ -analytic group and let  $H_2$  be a closed normal locally  $L$ -analytic subgroup of  $H_1$ . If, as an object of  $\mathcal{M}_{H_2}$ , the module  $\mathbb{1}$  admits an s-projective s-resolution of finite length  $r$  (resp. an s-resolution consisting of finitely generated free objects) and if, as an object of  $\mathcal{M}_{H_1/H_2}$ , it admits an s-projective s-resolution of finite length  $s$  (resp. consisting of finitely generated free objects) then, as an object of  $\mathcal{M}_{H_1}$ , it admits an s-projective s-resolution of finite length  $r + s$  (resp. an s-resolution consisting of finitely generated free objects).  $\square$*

**Remark 6.4.** Suppose the locally  $\mathbb{Q}_p$ -analytic group  $H$  to be compact. By [14], Corollary 8.34,  $H$  contains an open normal subgroup  $H_0$  which is uniform pro- $p$ . Since the group  $H/H_0$  is discrete the space  $D(H/H_0) = K[H/H_0]$  carries its finest locally convex topology. Thus,  $H/H_0$  satisfies assumption (A) because the unnormalized bar resolution  $B_\bullet(H/H_0, \mathbb{1})$  (cf. Proposition 2.4) is as required. By the first part of the proof of Theorem 4.4 and by Corollary 6.2,  $H$  satisfies assumption (A), as well, thus completing the proof of Theorem 4.4.

The following three theorems show how Corollary 6.2 can be used to exhibit large classes of non-compact groups satisfying (A).

**Theorem 6.5.** *If  $H$  is a solvable locally  $L$ -analytic group then  $H$  satisfies (A).*

Proof: By Corollary 6.2 we may assume  $H$  to be commutative. Let  $H_0$  be a compact open subgroup of  $H$ . Our arguments in Remark 6.4 show that the discrete group  $H/H_0$  satisfies (A). Referring to Corollary 6.2 once again, it remains to show that so does  $H_0$ . If  $L = \mathbb{Q}_p$  then this is a special case of Theorem 4.4.

Otherwise, by the commutativity of  $H_0$  and by passing to a standard subgroup, we may assume  $H$  to be isomorphic to the additive group  $\mathfrak{o}_L^d$  (cf. [6], III.7.3 Theorem 4, III.7.6 Propositions 11 and 13). By Corollary 6.2 we may further assume  $d = 1$ . According to the remarks at the end of [34], section 3, the  $K$ -algebra  $D(\mathfrak{o}_L)$  is a coherent ring in which the classes of closed, finitely generated and invertible ideals coincide. In particular, any closed ideal is a projective  $D(\mathfrak{o}_L)$ -module. Moreover, any finitely generated submodule in a finitely generated free  $D(\mathfrak{o}_L)$ -module is closed.

Consider the s-exact sequence

$$0 \longrightarrow I(\mathfrak{o}_L) \longrightarrow D(\mathfrak{o}_L) \longrightarrow \mathbf{1} \longrightarrow 0.$$

By our above remarks,  $I(\mathfrak{o}_L)$  is finitely presented and projective. Thus, there is an exact sequence of  $D(\mathfrak{o}_L)$ -modules

$$(64) \quad 0 \longrightarrow Y_1 \longrightarrow D(\mathfrak{o}_L)^r \longrightarrow I(\mathfrak{o}_L) \longrightarrow 0$$

in which  $Y_1$  is finitely generated, hence closed, and projective. The sequence (64) is  $D(\mathfrak{o}_L)$ -split. We claim that any  $D(\mathfrak{o}_L)$ -linear section is automatically continuous. Indeed, by [34], Theorem 2.3 and the ensuing remark,  $D(\mathfrak{o}_L)$  is a  $K$ -Fréchet Stein algebra in the sense of [36], section 3. Therefore, the claim follows from the remark before Proposition 3.7 of [36].

Consider the composition  $D(\mathfrak{o}_L)^r \longrightarrow I(\mathfrak{o}_L) \longrightarrow D(\mathfrak{o}_L)$ . Its image is  $I(\mathfrak{o}_L)$  which admits a closed  $K$ -vector space complement. Its kernel is  $Y_1$  which is closed with a  $D(\mathfrak{o}_L)$ -module complement isomorphic to  $I(\mathfrak{o}_L)$ . Proceeding inductively, one obtains an s-resolution of  $\mathbf{1}$  over  $D(\mathfrak{o}_L)$  as required.  $\square$

For  $p$ -adic reductive groups the idea of using the Bruhat-Tits building to produce resolutions with good finiteness properties is, in this context, due to W. Casselman and D. Wigner (cf. [13], section 3, or [5], Chapter X.2). The techniques we have developed so far allow us to follow their arguments.

**Theorem 6.6.** *If  $G = \mathbb{G}(\mathbb{Q}_p)$  is the group of  $\mathbb{Q}_p$ -rational points of a connected reductive group  $\mathbb{G}$  defined over  $\mathbb{Q}_p$  then the object  $\mathbf{1}$  of  $\mathcal{M}_G$  admits an s-projective s-resolution consisting of finitely generated free  $D(G)$ -modules.*

Proof: By a standard procedure and Corollary 6.3 we may assume  $\mathbb{G}$  to be either a torus or a simply connected almost simple group.

In the first case, writing  $\mathbb{G}$  as the product of its maximal  $\mathbb{Q}_p$ -anisotropic and its maximal  $\mathbb{Q}_p$ -split subtorus, the same arguments allow us to assume  $\mathbb{G}$  to be either  $\mathbb{Q}_p$ -anisotropic or  $\mathbb{Q}_p$ -split. In the first case, the group  $G$  is compact (cf. [2], Corollaire 9.4) so that we may apply Theorem 4.4. In the second case, let  $G_0$  be the maximal compact open subgroup of  $G$ . We only need to consider the group  $G/G_0$ . Since this is a discrete, finitely generated free abelian group, the ring  $D(G/G_0)$  is Noetherian and carries its finest locally convex topology. Therefore, any finite free resolution of  $\mathbf{1}$  is as required.

In the second case again, if  $\mathbb{G}$  is  $\mathbb{Q}_p$ -anisotropic then  $G$  is compact and we may apply Theorem 4.4. Thus, we are reduced to the case where  $G = \mathbb{G}(\mathbb{Q}_p)$  with

an almost simple, simply connected  $\mathbb{Q}_p$ -isotropic group  $\mathbb{G}$ .

Let  $\mathcal{B}$  be the Bruhat-Tits building of  $G$  and let  $\ell$  be the  $\mathbb{Q}_p$ -rank of  $\mathbb{G}$ . Let  $K_1, \dots, K_{\ell+1}$  be the stabilizers in  $G$  of the vertices of a fixed chamber of  $\mathcal{B}$ , so that  $\bigcap_{i=1}^{\ell+1} K_i$  is an Iwahori subgroup of  $G$ . The parahoric subgroups containing it are in one-to-one correspondence with subsets  $\alpha \subseteq \{1, \dots, \ell+1\}$  via  $K_\alpha = \bigcap_{i \in \alpha} K_i$ . The set of simplices whose associated parahoric group is conjugate to  $K_\alpha$  is isomorphic to  $G/K_\alpha$  as a  $G$ -set (cf. [41], section 3). For any  $0 \leq i \leq \ell$  we set

$$F^i := \bigoplus_{|\alpha|=i+1} \operatorname{coind}_{K_\alpha}^G(\mathbb{1}).$$

If we let  $C(G/K_\alpha)$  denote the space of continuous  $K$ -linear functions on  $G/K_\alpha$  endowed with the compact-open topology then there are  $D(G)$ -linear topological isomorphisms

$$\operatorname{coind}_{K_\alpha}^G(\mathbb{1}) \simeq C^{an}(G/K_\alpha) \simeq C(G/K_\alpha)$$

by Remark 5.4 and since  $G/K_\alpha$  is a discrete topological space. We let

$$(65) \quad 0 \longrightarrow \mathbb{1} \longrightarrow F^0 \longrightarrow \dots \longrightarrow F^\ell \longrightarrow 0$$

be the usual chain complex on  $\mathcal{B}$  with coefficients in the module  $\mathbb{1}$ . Since  $\mathcal{B}$  is contractible (cf. [10], Proposition 7.4.20) the complex (65) is s-exact, a contracting homotopy of  $\mathcal{B}$  inducing a continuous  $K$ -linear contracting homotopy of (65). By Proposition 5.3 the strong dual

$$(66) \quad 0 \longrightarrow C_\ell \longrightarrow \dots \longrightarrow C_0 \longrightarrow \mathbb{1} \longrightarrow 0$$

of the complex (65) is s-exact in  $\mathcal{M}_G$  with  $C_i = \bigoplus_{|\alpha|=i+1} \operatorname{ind}_{K_\alpha}^G(\mathbb{1})$ .

For any set  $\alpha$  we choose an s-projective s-resolution  $X_\bullet^{(\alpha)}$  of  $\mathbb{1}$  in  $\mathcal{M}_{K_\alpha}$  consisting of finitely generated free  $D(K_\alpha)$ -modules (cf. Theorem 4.4). Since  $D(G)$  is both free and s-free over  $D(K_\alpha)$  (cf. (6) and Lemma 5.2) we obtain from Lemma 2.6 that  $\operatorname{ind}_{K_\alpha}^G(X_\bullet^{(\alpha)})$  is an s-resolution of  $\operatorname{ind}_{K_\alpha}^G(\mathbb{1})$  consisting of finitely generated free  $D(G)$ -modules. Taking direct sums we may apply Theorem 6.1 to construct a Wall complex over (66) whose associated total complex is as desired.  $\square$

Combining Corollary 6.2, Theorem 6.5 and Theorem 6.6 we obtain the following result.

**Theorem 6.7.** *If  $H = \mathbb{H}(\mathbb{Q}_p)$  is the group of  $\mathbb{Q}_p$ -rational points of a linear algebraic group  $\mathbb{H}$  defined over  $\mathbb{Q}_p$  then  $H$  satisfies (A).*

Proof: The group  $\mathbb{H}$  is the extension of its identity component  $\mathbb{H}^\circ$  and the finite group scheme  $\mathbb{H}/\mathbb{H}^\circ$  whereas in characteristic zero the group  $\mathbb{H}^\circ$  is the semidirect product of its unipotent radical and a reductive group both of which are defined over  $\mathbb{Q}_p$  (cf. [2], section 0.8). The claim follows on passing to the groups of  $\mathbb{Q}_p$ -rational points.  $\square$

The functors  $H_{an}^0(H, \cdot)$  and  $H_0^{an}(H, \cdot)$ , although left (resp. right) exact in the algebraic sense, are not left (resp. right) exact in our relative situation. They generally do not take strong maps to strong maps. This leads to topological

problems when trying to establish a version of the Hochschild-Serre spectral sequence. These are vacuous, however, assuming the existence of s-resolutions with good finiteness properties such as the one in assumption (A).

**Theorem 6.8.** *Let  $H_1$  be a locally  $L$ -analytic group and let  $H_2$  be a closed normal locally  $L$ -analytic subgroup of  $H_1$ . If  $H_1/H_2$  satisfies (A) then there are spectral sequences*

$$(67) \quad E_2^{p,q} = H_{an}^p(H_1/H_2, H_{an}^q(H_2, V)) \implies H_{an}^{p+q}(H_1, V) \quad \text{and}$$

$$(68) \quad E_{p,q}^2 = H_p^{an}(H_1/H_2, H_q^{an}(H_2, V)) \implies H_{p+q}^{an}(H_1, V)$$

for all objects  $V$  of  $\mathcal{M}_{H_1}$ .

Proof: We establish (67), the case of locally analytic homology being similar. By Theorem 4.8 we have  $H_{an}^p(H_1/H_2, \cdot) \simeq \text{Ext}_{D(H_1/H_2)}^p(\mathbb{1}, \cdot)$ . As we shall see, the spaces  $H_{an}^q(H_2, V)$  are naturally modules over  $D(H_1/H_2)$  so that the terms on the left hand side of (67) have a well-defined meaning even if the spaces  $H_{an}^q(H_2, V)$  are not Hausdorff and complete (and hence not objects of  $\mathcal{M}_{H_1/H_2}$ ).

Let  $(Y_p, d_p)_{p \geq 0}$  be an s-projective s-resolution of  $\mathbb{1}$  in  $\mathcal{M}_{H_1/H_2}$  as in (A). Consider the double complex  $C_{pq} := \mathcal{L}_{H_1/H_2}(Y_p, \mathcal{L}_{H_2}(B_q(H_1, \mathbb{1}), V))$ ,  $p, q \geq 0$ . Associated with  $C_{\bullet, \bullet}$  are two spectral sequences both abutting to the cohomology of the associated total complex. For fixed  $q \geq 0$  there is a continuous  $K$ -linear bijection

$$(69) \quad \mathcal{L}_{H_2}(B_q(H_1, \mathbb{1}), V) \longrightarrow \mathcal{L}_b(D(H_1/H_2), \mathcal{L}_b(\widehat{\otimes}_{K, \iota}^q D(H_1), V))$$

(cf. Proposition 1.4, Lemma 2.2 and (52)) which is even a topological isomorphism if  $V$  is a hypomodule. Since  $\mathcal{L}_{H_1/H_2}(Y_p, \cdot) = \text{Hom}_{D(H_1/H_2)}(Y_p, \cdot)$  by assumption (A), the topology of  $\mathcal{L}_{H_2}(B_q(H_1, \mathbb{1}), V)$  is irrelevant when forming the complex  $C_{\bullet, q}$ . But then  $\mathcal{L}_{H_2}(B_q(H_1, \mathbb{1}), V)$  may be considered an s-injective hypomodule in  $\mathcal{M}_{H_1/H_2}$  (cf. (69) and Proposition 2.15) so that by Proposition 2.18

$$H^p C_{\bullet, q} = \begin{cases} \mathcal{L}_{H_1}(B_q(H_1, \mathbb{1}), V) & p = 0, \\ 0 & p > 0. \end{cases}$$

Thus, the spectral sequence degenerates and the cohomology of the total complex is that of the complex  $\mathcal{L}_{H_1}(B_{\bullet}(H_1, \mathbb{1}), V)$  which is  $H_{an}^{\bullet}(H_1, V)$ .

For the second spectral sequence associated with  $C_{\bullet, \bullet}$  we first fix  $p \geq 0$ . By our above remark  $\mathcal{L}_{H_1/H_2}(Y_p, \cdot)$  is an exact functor because the module  $Y_p$  is projective in the algebraic sense. Since  $B_{\bullet}(H_1, \mathbb{1})$  is also an s-projective s-resolution of  $\mathbb{1}$  in  $\mathcal{M}_{H_2}$  (cf. Proposition 5.6) we have

$$H^q C_{p, \bullet} = \mathcal{L}_{H_1/H_2}(Y_p, H_{an}^q(H_2, V)).$$

Taking cohomology in the  $p$ -direction gives the initial terms of the spectral sequence as required for (67).  $\square$

The following vanishing theorem heavily relies on the work [37] of P. Schneider and J. Teitelbaum. Together with Corollary 4.5 it implies that uniform pro- $p$  groups are Poincaré duality groups. In a different setting this was first proved

by M. Lazard (cf. [39], sections 4.4–4.5 and Theorem 5.1.5).

Let  $H$  be a locally  $L$ -analytic group of finite dimension  $d$  and let  $H_0$  be a compact open subgroup of  $H$ . Following [37], section 2, we let  $C_c^{an}(H)$  be the  $K$ -vector space of all compactly supported locally analytic  $K$ -valued functions on  $H$  endowed with the topology of a locally convex direct sum coming from the decomposition

$$C_c^{an}(H) = \bigoplus_{h \in H/H_0} C_c^{an}(h \cdot H_0).$$

This topology is independent of the choice of  $H_0$ . Let  $\mathcal{D}_K(H) := C_c^{an}(H)'_b$ . Since we assumed  $H$  to be countable at infinity both  $C_c^{an}(H)$  and  $\mathcal{D}_K(H)$  are objects of  $\mathcal{M}_H$  (cf. [37], Remark 2.1). Further, as in [37], section 3, we let  $\Delta_H$  be the one dimensional  $D(H)$ -module induced by the adjoint action of  $H$  on  $\bigwedge^d \mathfrak{h}$ . We note that if  $V$  is any object of  $\mathcal{M}_H$  and if  $\psi : H \rightarrow K^\times$  is any locally analytic character of  $H$  then the *twist*  $V \otimes_K \psi$  is again an object of  $\mathcal{M}_H$  (cf. [37], Appendix).

**Theorem 6.9.** *If  $H$  is a locally  $\mathbb{Q}_p$ -analytic group of dimension  $d$  and if  $\chi$  is any locally constant  $K$ -valued character of  $H$  then there are  $K$ -linear isomorphisms*

$$(70) \quad H_{an}^q(H, \mathcal{D}_K(H) \otimes_K \Delta_H \chi) = \begin{cases} K, & q = d \\ 0, & q \neq d \end{cases} \text{ and}$$

$$(71) \quad H_q^{an}(H, C_c^{an}(H) \otimes_K \Delta_H^{-1} \chi) = \begin{cases} K, & q = d \\ 0, & q \neq d. \end{cases}$$

Proof: Choose a compact open subgroup  $H_0$  of  $H$ . Using [37], Lemma 2.2, one can show that  $\mathcal{D}_K(H) \simeq \text{coind}_{H_0}^H(D(H_0))$  in  $\mathcal{M}_H$ .

If  $\psi$  is a locally analytic  $K$ -valued character of  $H$  then the functor  $V \mapsto V \otimes \psi$  on  $\mathcal{M}_H$  is s-exact and takes s-projective objects to s-projective objects. Together with the topological isomorphism  $\mathcal{L}_H(V \otimes_K \psi, W) \simeq \mathcal{L}_H(V, W \otimes_K \psi^{-1})$  for any two objects  $V$  and  $W$  of  $\mathcal{M}_H$  this implies

$$(72) \quad \mathcal{E}xt_H^q(V \otimes_K \psi, W) \simeq \mathcal{E}xt_H^q(V, W \otimes_K \psi^{-1})$$

for all  $q \geq 0$ . Combining this with Theorem 5.7 and Theorem 4.10 we obtain

$$H_{an}^q(H, \mathcal{D}_K(H) \otimes_K \Delta_H \chi) \simeq H^q(\mathfrak{h}, D(H_0) \otimes_K \Delta_{H_0})^{H_0}$$

for all  $q \geq 0$ , assuming  $\chi$  to be trivial on  $H_0$ . By what we saw in the proof of Theorem 4.10 we obtain from [37], Proposition 3.5, that

$$H^q(\mathfrak{h}, D(H_0) \otimes_K \Delta_{H_0}) \simeq \text{Ext}_{D(H_0)}^q(D^\infty(H_0), D(H_0) \otimes_K \Delta_{H_0})$$

is isomorphic, as a  $D^\infty(H_0)$ -module, to  $D^\infty(H_0)$  if  $q = d$  and is zero else. Applying [37], Lemma 1.4, we obtain (70).

For (71) one finds  $C_c^{an}(H) \simeq \text{ind}_{H_0}^H(C_c^{an}(H_0))$  and the proof is similar.  $\square$

**Remark 6.10.** If  $H$  is open in the group of  $L$ -rational points of a connected reductive or unipotent group then the character  $\Delta_H$  is trivial. Indeed, since the action of  $H$  on  $\bigwedge^d \mathfrak{h}$  is the restriction of an algebraic representation, for the character  $\Delta_H$  to be trivial it suffices to be trivial on  $\mathfrak{h}$ . This is well known.

## 7 Supercuspidal representations

For the group of  $L$ -rational points of unipotent algebraic group there is the following generalization of Theorem 4.10.

**Theorem 7.1.** *Let  $N = \mathbb{N}(L)$  be the group of  $L$ -rational points of a unipotent group  $\mathbb{N}$  defined over  $L$  and let  $\mathfrak{n}$  be its Lie algebra. The functor  $(\cdot)_N : \text{Mod}_{D^\infty(N)} \rightarrow \text{Mod}_K$  of  $N$ -coinvariants is exact. For any object  $V$  of  $\mathcal{M}_N$  and for all  $q \geq 0$  there are natural  $K$ -linear isomorphisms*

$$(73) \quad H_q^{an}(N, V) \simeq H_q(\mathfrak{n}, V)_N.$$

Proof: Since  $N$  is the union of its compact open subgroups  $N_i$  the functor  $(\cdot)_N : \text{Mod}_{D^\infty(N)} \rightarrow \text{Mod}_K$  is the direct limit of the functors  $(\cdot)_{N_i} : \text{Mod}_{D^\infty(N_i)} \rightarrow \text{Mod}_K$  which were shown to be exact in the proof of Theorem 4.10. Thus,  $(\cdot)_N$  is exact itself. By the same argument, any element of the augmentation ideal of  $D^\infty(N)$  is contained in the augmentation ideal of one of its subalgebras  $D^\infty(N_i)$  (note that by the analog of (6) for locally constant distributions on  $N$  any such is compactly supported). It follows from Corollary 4.6 that the augmentation ideal of  $D^\infty(N)$  is generated by all elements of the form  $\delta_n - 1$  with  $n \in N$ , where  $\delta_n$  denotes the Dirac distribution in  $n$ . As a consequence, there is an equivalence of functors  $(\cdot)_N \simeq (\cdot) \otimes_{D^\infty(N)} \mathbb{1}$  so that the  $D^\infty(N)$ -module  $\mathbb{1}$  is flat. By Theorem 6.5 we may consider the spectral sequence (45). It degenerates and, together with (47), gives (73).  $\square$

Let  $G = \mathbb{G}(L)$  be the group of  $L$ -rational points of a connected reductive group  $\mathbb{G}$  defined over  $L$ . Recall that if  $V$  is a  $K$ -vector space carrying an admissible irreducible smooth representation of  $G$  then  $V$  is called supercuspidal if for all proper parabolic  $L$ -subgroups  $\mathbb{P}$  of  $\mathbb{G}$  with unipotent radical  $\mathbb{N}$  we have  $V_N = 0$  where  $N := \mathbb{N}(L)$ .

The analog of the functor of  $N$ -coinvariants in our situation is the functor  $H_0^{an}(N, \cdot)$  which, in contrast to the smooth theory, is not exact. Therefore, we give the following definition of a supercuspidal locally analytic representation.

**Definition 7.2.** Let  $G$  be the group of  $L$ -rational points of a connected reductive group  $\mathbb{G}$  defined over  $L$ . A topologically irreducible locally analytic  $G$ -representation  $V$  is called supercuspidal if for all proper parabolic  $L$ -subgroups  $\mathbb{P}$  of  $\mathbb{G}$  and all  $q \geq 0$  we have  $H_q^{an}(N, V) = 0$  where  $N$  denotes the group of  $L$ -rational points of the unipotent radical of  $\mathbb{P}$ .

Using a structure theorem due to D. Prasad we can characterize the admissible supercuspidal locally algebraic  $G$ -representations, thereby clarifying the relation between Definition 7.2 and the usual definition in the smooth case. For the notion of a locally algebraic representation we refer the reader to [28]. Recall that by [28], Theorem 1, the irreducible locally algebraic representations  $V$  of  $G$  are of the form  $V = \pi \otimes_K \sigma$  with an irreducible smooth representation  $\pi$  and an irreducible finite dimensional algebraic representation  $\sigma$  of  $G$ .

**Theorem 7.3.** *Let  $G = \mathbb{G}(L)$  be the group of  $L$ -rational points of a connected reductive group  $\mathbb{G}$  defined over  $L$ . An admissible irreducible locally algebraic*

representation  $V = \pi \otimes_K \sigma$  of  $G$  is supercuspidal in the sense of Definition 7.2 if and only if  $\pi$  is a supercuspidal smooth representation.

Proof: Assume  $V = \pi \otimes_K \sigma$  with  $\pi$  supercuspidal smooth. Since  $\pi$  is trivial as an  $\mathfrak{n}$ -representation there are  $K$ -linear isomorphisms

$$(74) \quad H_q(\mathfrak{n}, V) \simeq \pi \otimes_K H_q(\mathfrak{n}, \sigma)$$

for all  $q \geq 0$  which are compatible with the action of  $N$ . Since  $H_q(\mathfrak{n}, \sigma)$  is algebraic and since the  $\mathfrak{n}$ -action is trivial, so is the action of  $N$ . Therefore,  $H_q(\mathfrak{n}, V)$  is  $N$ -isomorphic to a direct sum of copies of  $\pi$ . It follows from Theorem 7.1 that  $V$  is supercuspidal in the sense of Definition 7.2.

Conversely, assume  $\pi$  not to be supercuspidal. There is then a proper parabolic  $L$ -subgroup of  $\mathbb{G}$  with associated group  $N$  such that  $\pi_N$  is non-zero. By our above arguments and (74) for  $V$  to be supercuspidal it is necessary that  $H_q(\mathfrak{n}, \sigma)$  be zero in all degrees. This, however, is not true. Indeed, if  $q = 0$  or if  $q = \dim_L(\mathfrak{n})$  then the space  $H_q(\mathfrak{n}, \sigma)$  is of  $K$ -dimension one (cf. [20], pp. 296–297). This is an easy case of a more general theorem of B. Kostant (cf. [23], Theorem 5.14 and Corollary 5.14).  $\square$

Of course, one expects there to exist genuinely locally analytic examples of supercuspidal representations. First candidates of this kind were constructed in [18].

The degeneracy result of Theorem 7.1 does not hold for locally analytic  $N$ -cohomology, thus expressing a lack of locally analytic Poincaré duality for the group  $N$ . Still, we can prove the following criterion which will be needed in Theorem 8.5 below. We note that it admits a variant for semisimple groups which was inspired by a question of J-P. Serre's and which was proved by W. Casselman and D. Wigner (cf. [13], Theorem 1).

**Theorem 7.4.** *Let  $N = \mathbb{N}(L)$  be the group of  $L$ -rational points of a unipotent group  $\mathbb{N}$  defined over  $L$ . Assume the object  $V$  of  $\mathcal{M}_N$  to be a Fréchet space containing a dense  $N$ -stable subspace  $\tilde{V}$  which is the sum of its (finite dimensional) algebraic  $N$ -submodules. If the maps in the standard complex  $\mathrm{Hom}_L(\bigwedge^\bullet \mathfrak{n}, \tilde{V})$  computing the  $\mathfrak{n}$ -cohomology of  $\tilde{V}$  are strict with closed image then the natural maps  $H^q(\mathfrak{n}, \tilde{V}) \rightarrow H^q(\mathfrak{n}, V)$  are strict injective maps of Hausdorff spaces with dense image and there are  $K$ -linear isomorphisms  $H_{an}^q(N, V) \simeq H^q(\mathfrak{n}, V)$  for all  $q \geq 0$ .*

Proof: By assumption and since the completion functor is exact on strict exact sequences of Hausdorff metrizable topological groups (cf. [7], 3.1 Corollaire 1) the maps  $H^q(\mathfrak{n}, \tilde{V}) \rightarrow H^q(\mathfrak{n}, V)$  are strict injective maps of Hausdorff spaces with dense image. We claim the  $D^\infty(N)$ -action on  $H^q(\mathfrak{n}, V)$  to be trivial, i.e. to factor through  $\mathbf{1}$ . To see this, it suffices to remark that, by algebraicity, the  $N$ -action on the spaces  $H^q(\mathfrak{n}, \tilde{V})$  is trivial.

During the proof of Theorem 7.1 we saw that the augmentation ideal of  $D^\infty(N)$  is generated by all elements of the form  $\delta_n - 1$  with  $n \in N$ . Therefore, if  $W$  is any  $K$ -vector space, viewed as a trivial  $D^\infty(N)$ -module, there is an equivalence

of functors

$$\mathrm{Hom}_{D^\infty(N)}(\cdot, W) \simeq \mathrm{Hom}_K((\cdot)_N, W)$$

which is known to be exact by Theorem 7.1. Thus, all  $D^\infty(N)$ -modules  $H^q(\mathfrak{n}, V)$  are injective. By Theorem 6.5 we may consider the spectral sequence (44). By our above remarks and (46) it degenerates.  $\square$

## 8 Extensions between principal series representations

Let  $\mathbb{G}$  be a connected reductive group defined over  $L$  and let  $\mathbb{S}$  be a maximal  $L$ -split torus of  $\mathbb{G}$ . Let  $\mathbb{P}$  be a minimal parabolic subgroup of  $\mathbb{G}$  containing  $\mathbb{S}$ , let  $\mathbb{M}$  be the centralizer of  $\mathbb{S}$  in  $\mathbb{G}$  and let  $\mathbb{N}$  be the unipotent radical of  $\mathbb{P}$ . Let  $G$ ,  $P$ ,  $M$  and  $N$  be the group of  $L$ -rational points of  $\mathbb{G}$ ,  $\mathbb{P}$ ,  $\mathbb{M}$  and  $\mathbb{N}$ , respectively, and let  $\mathfrak{g}$ ,  $\mathfrak{p}$ ,  $\mathfrak{m}$  and  $\mathfrak{n}$  denote the corresponding Lie algebras.

If  $\mathbb{G}$  is  $L$ -split or if  $L = \mathbb{Q}_p$  then  $M$  satisfies (A) (cf. Theorem 6.5 and Theorem 6.6). For the rest of this section we will make this assumption.

Let  $\chi_1, \chi_2 : M \rightarrow K^\times$  be two locally analytic characters, viewed as locally analytic characters of  $P$  via the projection  $P \rightarrow P/N \simeq M$ , and set  $I(\chi_i) := \mathrm{coind}_P^G(\chi_i)$ . By Remark 5.4,  $I(\chi_i)$  can be realized as the space of locally analytic functions  $f : G \rightarrow K$  satisfying  $f(gp) = \chi_i^{-1}(p)f(g)$  for all  $g \in G$  and all  $p \in P$ . Together with the left regular representation this is a locally analytic  $G$ -representation on a locally convex  $K$ -vector space of compact type in the sense of [35], section 3, and the separate continuous  $D(G)$ -module structure on  $I(\chi_i)$  is the one coming from [35], Proposition 3.2.

The aim of this section is to analyze the spaces  $\mathcal{E}xt_G^q(I(\chi_1), I(\chi_2))$ . Letting  $M(\chi_i) := \mathrm{ind}_P^G(\chi_i^{-1})$  we use Proposition 5.3 and Pontrjagin duality (cf. Theorem 3.2) to obtain topological isomorphisms

$$(75) \quad \mathcal{E}xt_G^q(I(\chi_1), I(\chi_2)) \simeq \mathcal{E}xt_G^q(M(\chi_2), M(\chi_1)).$$

Further, by Shapiro's lemma (cf. Theorem 5.7) there are topological isomorphisms

$$(76) \quad \mathcal{E}xt_G^q(M(\chi_2), M(\chi_1)) \simeq \mathcal{E}xt_P^q(\chi_2^{-1}, M(\chi_1)).$$

By (72) and Theorem 6.8 there is a spectral sequence

$$(77) \quad \mathcal{E}xt_M^p(\chi_2^{-1}, H_{an}^q(N, M(\chi_1))) \implies \mathcal{E}xt_P^{p+q}(\chi_2^{-1}, M(\chi_1)).$$

We see that in a first step we have to compute the locally analytic cohomology groups  $H_{an}^q(N, M(\chi_1))$ . We omit the subscript from the notation and simply write  $I(\chi) := \mathrm{coind}_P^G(\chi)$  and  $M(\chi) := \mathrm{ind}_P^G(\chi^{-1})$  whenever  $\chi$  is a locally analytic  $K$ -valued character of  $M$ .

If  $C$  is a closed  $P$ -biinvariant subset of  $G$  we let  $I(\chi)_{G \setminus C}$  be the closed  $P$ -stable subspace of  $I(\chi)$  consisting of all functions  $f \in I(\chi)$  whose support is contained in  $G \setminus C$ . We let  $D(G)_C$  be the subspace of locally analytic distributions on  $G$  supported in  $C$  (cf. [22], section 1.2). The subspace  $D(G)_C$  is a closed left and right  $D(P)$ -submodule of  $D(G)$  (cf. [22], Lemma 1.2.4 and Lemma 1.2.5). Set  $M(\chi)_C := D(G)_C \tilde{\otimes}_{D(P), \iota} \chi^{-1}$ .

**Lemma 8.1.** *If  $C$  is a closed  $P$ -biinvariant subset of  $G$  then  $M(\chi)_C$  is an object of  $\mathcal{M}_P$  and the natural map  $M(\chi)_C \rightarrow M(\chi)$  is injective. There is a  $D(P)$ -linear topological isomorphism*

$$(78) \quad (I(\chi)/I(\chi)_{G \setminus C})'_b \simeq M(\chi)_C.$$

Proof: Choosing a locally analytic section  $\iota$  of the projection  $\pi : G \rightarrow G/P$  we obtain an isomorphism  $\kappa : G \rightarrow G/P \times P$  of locally  $L$ -analytic manifolds and right  $P$ -spaces via  $\kappa(g) := (\pi(g), (\iota \circ \pi)(g)^{-1} \cdot g)$  (cf. [16], Satz 4.1.1). We have  $\kappa(C) = \pi(C) \times P$  where  $\pi(C) = (G/P) \setminus \pi(G \setminus C)$  is a closed subset of  $G/P$ . Therefore, the topological isomorphism (52) restricts to a topological isomorphism  $D(G)_C \simeq D(G/P)_{\pi(C)} \widehat{\otimes}_{K, \iota} D(P)$ .

By Lemma 2.6 we have

$$(79) \quad M(\chi)_C \simeq D(G/P)_{\pi(C)} \widehat{\otimes}_{K, \iota} \chi^{-1}$$

as locally convex  $K$ -vector spaces. In particular,  $M(\chi)_C$  is Hausdorff and complete and thus is naturally an object of  $\mathcal{M}_P$ . Under the  $K$ -linear topological isomorphism (79) applied to the subsets  $C$  and  $G$  of  $G$ , the map  $M(\chi)_C \rightarrow M(\chi)$  is just the natural inclusion  $D(G/P)_{\pi(C)} \subseteq D(G/P)$ .

By [16], Satz 4.3.1,  $\iota^* : I(\chi) \rightarrow C^{an}(G/P)$  is a  $K$ -linear topological isomorphism. It is checked to restrict to a topological isomorphism  $I(\chi)_{G \setminus C} \simeq C^{an}(G/P)_{\pi(G \setminus C)}$  so that as in [22], Lemma 1.2.5, there is a  $K$ -linear topological isomorphism

$$(80) \quad (I(\chi)/I(\chi)_{G \setminus C})'_b \simeq D(G/P)_{\pi(C)}.$$

The  $K$ -linear topological isomorphism  $(C^{an}(G)/C^{an}(G)_{G \setminus C})'_b \simeq D(G)_C$  of [22], Lemma 1.2.5, induces a map  $D(G)_C \rightarrow (I(\chi)/I(\chi)_{G \setminus C})'_b$  in  $\mathcal{M}_P$  via restriction of functionals to the closed subspace  $I(\chi)/I(\chi)_{G \setminus C}$  of  $C^{an}(G)/C^{an}(G)_{G \setminus C}$ . It factors through  $M(\chi)_C$ . Using (79) and (80) one shows the resulting map  $M(\chi)_C \rightarrow (I(\chi)/I(\chi)_{G \setminus C})'_b$  to be a topological isomorphism as required.  $\square$

Let  $W := N_{\mathbb{G}}(\mathbb{S})/\mathbb{M}$  be the Weyl group of  $(\mathbb{G}, \mathbb{S})$  relative to  $L$  (cf. [1], §21.1). By the relative Bruhat decomposition (cf. [1], Theorem 21.15)  $G$  is the disjoint union of the double cosets  $PwP$ ,  $w \in W$ . If we let  $\Phi = \Phi(\mathbb{G}, \mathbb{S})$  be the roots of  $\mathbb{G}$  with respect to  $\mathbb{S}$  then  $\Phi$  is a root system with Weyl group  $W$  (cf. [1], Theorem 21.6 and its proof). Let  $\Delta$  be the basis of  $\Phi$  associated with  $\mathbb{P}$  (i.e. a basis of the reduced root system associated with  $\Phi$ ).

Recall the definition of the Bruhat ordering on  $W$ : If  $w, w' \in W$  then  $w' \geq w$  if and only if  $\mathbb{P}w\mathbb{P}$  is contained in the Zariski closure of  $\mathbb{P}w'\mathbb{P}$  in  $\mathbb{G}$ . By [3], Proposition 3.14, we have  $w' \geq w$  if and only if  $PwP$  is contained in the closure of  $Pw'P$  in  $G$  endowed with its topology of a locally  $L$ -analytic manifold.

Let  $\ell$  be the length function of  $W$  with respect to  $\Delta$ , let  $w_0$  be the longest element of  $W$  and set  $n := \ell(w_0)$ . For  $w \in W$  we let  $G_w := \cup_{w' \geq w} Pw'P$  which is the smallest open  $P$ -biinvariant subset of  $G$  containing  $PwP$ . For  $0 \leq r \leq n$  set  $G_r := \cup_{\ell(w) \geq r} PwP = \cup_{\ell(w)=r} G_w$ . Set  $F_w := G \setminus G_w$  and  $F_r := G \setminus G_r$  so

that  $F_r = \cup_{\ell(w) < r} PwP = \cap_{\ell(w) = r} F_w$ .

We endow  $I(\chi)$  with a descending filtration in  $\mathcal{M}_P$  by letting  $I(\chi)_r := I(\chi)_{G_r}$  for  $0 \leq r \leq n$ . We note that in contrast to the theory of smooth representations  $I(\chi)_r$  is not the space of functions vanishing on  $F_r$ , the latter being much bigger. In fact, a smooth function which vanishes on the closed set  $F_r$  automatically vanishes in an open neighborhood of  $F_r$ . Therefore, its support (i.e. the topological closure of the complement of its vanishing locus in  $G$ ) is contained in  $G \setminus F_r = G_r$ . For an arbitrary locally analytic or merely continuous function this is no longer true and greatly complicates the determination of the locally analytic  $N$ -homology of  $I(\chi)$ . As we shall see, it leads to a completely different behavior of extensions between principal series representations (cf. Remark 8.6 and the discussion in Example 8.12).

Since the topological space  $G_r/P$  is strictly paracompact and since the set  $PwP$  is closed in both  $G_{\ell(w)}$  and  $G_w$  one sees that the map

$$\bigoplus_{\ell(w)=r} I(\chi)_{G_w} \longrightarrow I(\chi)_r, \quad (f_w)_{\ell(w)=r} \mapsto \sum_{\ell(w)=r} f_w,$$

is surjective and that it induces a topological  $D(P)$ -linear isomorphism

$$(81) \quad I(\chi)_r/I(\chi)_{r+1} \simeq \bigoplus_{\ell(w)=r} (I(\chi)_{G_w}/I(\chi)_{G_w \setminus PwP}).$$

We set  $J_r := I(\chi)_r/I(\chi)_{r+1}$  and  $J_w := I(\chi)_{G_w}/I(\chi)_{G_w \setminus PwP}$ . Dually, we obtain an ascending filtration of  $M(\chi)$  in  $\mathcal{M}_P$  by letting  $M(\chi)_r := M(\chi)_{F_r}$  for  $1 \leq r \leq n+1$ . By Lemma 8.1 and (81) there are  $D(P)$ -linear topological isomorphisms

$$(82) \quad M(\chi)_{r+1}/M(\chi)_r \simeq (J_r)'_b \simeq \bigoplus_{\ell(w)=r} (J_w)'_b$$

and

$$(83) \quad M(\chi)_{F_w \cup PwP}/M(\chi)_{F_w} \simeq (J_w)'_b.$$

Although the functor  $H_{an}^0(N, \cdot)$  is not exact one may still infer a lot of information on  $H_{an}^\bullet(N, M(\chi))$  by computing  $H_{an}^\bullet(N, M(\chi)_{r+1}/M(\chi)_r)$ ,  $1 \leq r \leq n$ , as well as  $H_{an}^\bullet(N, M(\chi)_1)$ , and by analyzing the long exact sequences obtained from the exact sequences

$$0 \longrightarrow M(\chi)_r \longrightarrow M(\chi)_{r+1} \longrightarrow M(\chi)_{r+1}/M(\chi)_r \longrightarrow 0$$

in  $\mathcal{M}_N$  (cf. Theorem 4.8 and Theorem 6.5). We restrict ourselves to computing the locally analytic  $N$ -cohomology in the two extreme cases of  $M(\chi)_1$  and  $M(\chi)/M(\chi)_n$ . This will be sufficient for our calculations in Example 8.12 below.

Let  $\delta_P : P \rightarrow K^\times$  be the modulus character of  $P$  and let  $\mathfrak{d}_P := \delta_P \Delta_P$  where  $\Delta_P$  is the character which was introduced before Theorem 6.9. For any  $w \in W$  we let  $\chi^w$  be the character of  $M$  defined through  $\chi^w(m) := \chi(nmn^{-1})$  for a representative  $n$  of  $w$  in  $N_G(S)$ .

**Theorem 8.2.** *There are  $D(M)$ -linear isomorphisms*

$$(84) \quad H_{an}^q(N, M(\chi)/M(\chi)_n) \simeq \begin{cases} \mathfrak{d}_P^{-1}(\chi^{-1})^{w_0}, & q = \dim N \\ 0, & \text{else.} \end{cases}$$

Proof: We have  $I(\chi)_n = I(\chi)_{Pw_0P}$ . It follows from [1], Theorem 14.12, that the multiplication map  $N \times w_0P \rightarrow Pw_0P$  is an isomorphism of locally  $L$ -analytic manifolds. Quite generally, if  $U$  is a right  $P$ -invariant open subset of  $G$  then, since  $G/P$  is compact, the restriction to  $U$  defines a  $K$ -linear isomorphism from  $I(\chi)_U$  onto the space of all locally analytic functions  $f : U \rightarrow K$  whose support is compact modulo  $P$  and which satisfy  $f(up) = \chi^{-1}(p)f(u)$  for all  $u \in U$  and all  $p \in P$ . Together with our above remark we obtain that the map  $I(\chi)_n \rightarrow C_c^{an}(N)$  given by  $f \mapsto \tilde{f}$  with  $\tilde{f}(n) := f(nw_0)$  is a  $D(N)$ -linear topological isomorphism. The  $M$ -action on  $C_c^{an}(N)$  obtained by transport of structure is given by

$$(85) \quad (mf)(n) = \chi^{w_0}(m)\tilde{f}(m^{-1}nm).$$

By (83) there is a  $D(N)$ -linear topological isomorphism  $M(\chi)/M(\chi)_n \simeq \mathcal{D}_K(N)$ , so that by Theorem 6.9 and Remark 6.10 we have  $H_{an}^q(N, M(\chi)/M(\chi)_n) = 0$  if  $q$  is different from  $\dim N$  and that this space is one dimensional else. According to the proof of Theorem 6.9 we have

$$H_{an}^{\dim N}(N, M(\chi)/M(\chi)_n) \simeq H^{\dim N}(\mathfrak{n}, M(\chi)/M(\chi)_n)^N.$$

As in [20], Proposition VII.7.34, the latter space is  $M$ -equivariantly isomorphic to

$$H_0(\mathfrak{n}, M(\chi)/M(\chi)_n \otimes_K \Delta_N^{-1})^N \simeq H_0(\mathfrak{n}, M(\chi)/M(\chi)_n)^N \otimes_K \Delta_N^{-1},$$

where  $\Delta_N$  is seen as a character of  $M$  via the adjoint action of  $M$  on  $\bigwedge^{\dim N} \mathfrak{n}$ . Note that it coincides with the restriction to  $M$  of  $\Delta_P$  because  $\Delta_M$  is trivial (cf. Remark 6.10).

As seen in the proof of Theorem 6.9, the space  $H_0(\mathfrak{n}, M(\chi)/M(\chi)_n)^N$  is generated by a Haar measure  $\mu$  on  $C_c^\infty(N)$ . Using the unimodularity of  $M$  and (85) one finds  $m \cdot \mu = \delta_P(m)^{-1}(\chi^{-1})^{w_0}(m) \cdot \mu$ .  $\square$

**Remark 8.3.** By [6], III.3.16 Corollary to Proposition 55, one has  $\delta_P(m) = |\det \text{Ad}_{\mathfrak{p}}(m)|$ . If  $\mathbb{G}$  is  $L$ -split then this equals  $|\prod_{\alpha \in \Phi^+} \alpha(m)|$  by the weight space decomposition of  $\mathfrak{p}$ .

Now consider the  $D(P)$ -submodule  $M(\chi)_1 = D(G)_P \tilde{\otimes}_{D(P), L} \chi^{-1}$  of  $M(\chi)$ . If  $\mathfrak{h}$  is a Lie algebra over  $L$  we denote by  $U(\mathfrak{h})$  its universal enveloping algebra and set  $U_K(\mathfrak{h}) := U(\mathfrak{h}) \otimes_L K$ . If  $\mathfrak{h}$  is the Lie algebra of a locally  $L$ -analytic group  $H$  then there is an embedding  $U_K(\mathfrak{h}) \subseteq D(H)$  of  $K$ -algebras. It allows us to identify  $U_K(\mathfrak{h})$  with a dense subalgebra of  $D(H)_{\{1\}}$  (cf. [22], Lemma 1.2.5 and Proposition 1.2.8).

Consider the *generalized Verma module*  $\mathfrak{m}(\chi) := U_K(\mathfrak{g}) \otimes_{U_K(\mathfrak{p})} \chi^{-1}$ . This is a left  $\mathfrak{g}$ -module carrying an action of  $P$  via  $p \cdot (\mathfrak{x} \otimes 1) = \text{Ad}(p)(\mathfrak{x}) \otimes \chi^{-1}(p)$ . The inclusions  $U_K(\mathfrak{g}) \subseteq D(G)_P$  and  $U_K(\mathfrak{p}) \subseteq D(P)$  induce a homomorphism of

$P$ -modules  $\mathfrak{m}(\chi) \rightarrow M(\chi)_1$  which is injective. Indeed, letting  $\overline{P} = P^{w_0}$  be the group of  $L$ -rational points of the parabolic subgroup of  $\mathbb{G}$  containing  $\mathbb{M}$  which is opposite to  $\mathbb{P}$ ,  $\overline{N}$  the group of  $L$ -points of its unipotent radical with Lie algebra  $\overline{\mathfrak{n}}$ , we have  $\mathfrak{m}(\chi) \simeq U_K(\overline{\mathfrak{n}})$  as  $K$ -vector spaces by the Poincaré-Birkhoff-Witt theorem. Using (79) we have to show that the natural map  $U_K(\overline{\mathfrak{n}}) \rightarrow D(G/P)_{\pi(P)}$  is injective. Now  $P \cap \overline{N} = 1$  and  $\overline{N}$  maps homeomorphically onto an open neighborhood of  $\pi(P)$  in  $G/P$  whence  $D(G/P)_{\pi(P)} \simeq D(\overline{N})_{\{1\}}$  and the claim follows. This argument also shows that  $\mathfrak{m}(\chi)$  is dense in  $M(\chi)_1$ .

Choosing an  $L$ -basis of  $\mathfrak{g}$ , the Poincaré-Birkhoff-Witt theorem allows us to define an exhaustive filtration on  $\mathfrak{m}(\chi)$  by finite dimensional  $P$ -stable subspaces. Since the  $N$ -action on  $\chi$  is trivial and since the adjoint action of  $N$  on  $\mathfrak{g}$  is algebraic we deduce that  $\mathfrak{m}(\chi)$  is the sum of its finite dimensional algebraic  $N$ -subrepresentations. In order to apply Theorem 7.4 we need one more result.

**Lemma 8.4.** *Assume  $\mathbb{G}$  to be  $L$ -split. Endowing  $\mathfrak{m}(\chi)$  with the subspace topology of  $M(\chi)_1$  and the spaces  $\text{Hom}_L(\bigwedge^q \mathfrak{n}, \mathfrak{m}(\chi))$  with the corresponding direct sum topology, the complex  $\text{Hom}_L(\bigwedge^\bullet \mathfrak{n}, \mathfrak{m}(\chi))$  is strong.*

Proof: Put  $C^q := \text{Hom}_L(\bigwedge^q \mathfrak{n}, \mathfrak{m}(\chi))$ . Since  $M$  acts on  $\bigwedge^q \mathfrak{n}$  via the adjoint action of  $M$  on  $\mathfrak{n}$ , it also acts on  $C^q$  via  $(mf)(\mathfrak{x}) := m \cdot f(m^{-1}\mathfrak{x})$ . The differentials of the complex  $C^\bullet$  commute with the action of  $M$  (cf. [20], Proposition VI.6.11). Therefore, it suffices to show that any  $M$ -invariant subspace of  $C^q$  is closed and admits a closed (even  $M$ -invariant) vector space complement. Although we are dealing with metrizable instead of with normed vector spaces this is essentially the theory of diagonalizable modules as developed in [16], section 1.3.  $\square$

**Theorem 8.5.** *Assume  $\mathbb{G}$  to be  $L$ -split. There are  $D(M)$ -linear isomorphisms*

$$(86) \quad H_{an}^q(N, M(\chi)_1) \simeq H^q(\mathfrak{n}, \mathfrak{m}(\chi))$$

for all  $q \geq 0$ , and these are finite direct sums of one dimensional  $D(M)$ -modules.

Proof: According to Lemma 8.4 and the remarks preceding it we may apply Theorem 7.4 whence there are  $M$ -equivariant embeddings  $H^q(\mathfrak{n}, \mathfrak{m}(\chi)) \rightarrow H_{an}^q(N, M(\chi)_1)$  of Hausdorff locally convex  $K$ -vector spaces with dense image for all  $q \geq 0$ . The claim follows from the fact that the spaces  $H^q(\mathfrak{n}, \mathfrak{m}(\chi))$  are finite dimensional by the Casselman-Osborne theorem (cf. [20], Theorem VI.6.19 and the arguments given on pp. 318–319). The proof of Lemma 8.4 shows them to be the direct sums of their weight spaces.  $\square$

**Remark 8.6.** We point out two major differences between the smooth and the locally analytic theory, respectively. Firstly, Theorem 8.2 shows that the highest subquotient of  $M(\chi)$  obtained from the filtration  $(M(\chi)_r)_{1 \leq r \leq n+1}$  (and probably all intermediate ones) gives a trivial contribution to  $H_{an}^0(N, M(\chi))$ . Secondly, Theorem 8.5 shows that  $H_{an}^q(N, M(\chi)_1)$  and thus  $H_{an}^q(N, M(\chi))$  itself is a rather complicated object. Whereas the Jacquet module of a smooth principal series representation is easily determined (cf. [11], Theorem 3.5) in order to determine the dimensions of the  $K$ -vector spaces  $H^q(\mathfrak{n}, \mathfrak{m}(\chi))$  one has to refer to the Kazhdan-Lusztig conjecture (cf. [38], Introduction).

In view of our above results on  $H_{an}^\bullet(N, M(\chi))$  and the spectral sequence (77) we now have to compute the spaces  $\text{Ext}_M^q(\chi_1, \chi_2)$  where  $\chi_1, \chi_2 : M \rightarrow K^\times$  are

two locally analytic characters of  $M$ . The following lemma is proved as in [5], I.4.5.

**Lemma 8.7.** *Let  $H$  be a locally  $L$ -analytic group and let  $V$  and  $W$  be two objects of  $\mathcal{M}_H$  such that there exists an element  $\delta$  in the center of  $D(H)$  acting as multiplication by a non-zero scalar on  $V$  and as the zero map on  $W$ . We then have  $\text{Ext}_H^q(V, W) = 0$  for all  $q \geq 0$ .  $\square$*

For later reference we explicitly formulate the following special case.

**Corollary 8.8.** *Assume  $M$  to be the group of  $L$ -rational points of a connected reductive group defined over  $L$ . If  $\chi_1$  and  $\chi_2$  are two locally analytic characters of  $M$  whose restrictions to the center of  $M$  are distinct then  $\text{Ext}_M^q(\chi_1, \chi_2) = 0$  for all  $q \geq 0$ .*

Proof: There is an element  $s$  of the center of  $M$  such that  $\chi_1(s) \neq \chi_2(s)$ . One applies Lemma 8.7 with  $\delta := \chi_2^{-1}(s) \cdot \delta_s - 1$ .  $\square$

If  $M$  is as in Corollary 8.8 then two characters of  $M$  whose restrictions to the center of  $M$  are identical coincide on an open subgroup of finite index of  $M$  (cf. [4], Corollaire 3.20). In view of Corollary 8.8 the most important case to consider will therefore be the one in which  $\chi_1 = \chi_2$ . By (72) we may assume both characters to be trivial. Due to the appearance of Lie algebra cohomology the situation is more complicated than in the smooth setting for which we refer to [5], Proposition X.2.6. As usual, if  $\mathbb{H}$  is a linear algebraic group defined over  $L$  we denote by  $X_L(\mathbb{H})$  its group of  $L$ -rational characters. If  $\mathbb{H}$  is a torus then this is a finitely generated free abelian group of rank  $\text{rk}_L \mathbb{H}$ , the  $L$ -rank of  $\mathbb{H}$ .

**Theorem 8.9.** *Assume  $M$  to be the group of  $\mathbb{Q}_p$ -rational points of a connected reductive group  $\mathbb{M}$  defined over  $\mathbb{Q}_p$  with center  $\mathbb{S}$  and derived group  $\mathbb{D}$ . Setting  $S := \mathbb{S}(\mathbb{Q}_p)$  and  $D := \mathbb{D}(\mathbb{Q}_p)$  with Lie algebras  $\mathfrak{s}$  and  $\mathfrak{d}$ , respectively, there are  $K$ -linear isomorphisms*

$$(87) \quad H_{an}^n(M, \mathbf{1}) \simeq \bigoplus_{r+q=n} H_{an}^r(S, \mathbf{1}) \otimes_K H_{an}^q(D, \mathbf{1})$$

for all  $n \geq 0$  where

$$(88) \quad H_{an}^q(D, \mathbf{1}) \simeq H^q(\mathfrak{d}, \mathbf{1})$$

for all  $q \geq 0$  and

$$(89) \quad H_{an}^r(S, \mathbf{1}) \simeq \bigoplus_{t+s=r} \left( \bigwedge^t (X_{\mathbb{Q}_p}(\mathbb{S}) \otimes_{\mathbb{Z}} K) \otimes_K \bigwedge^s (\mathfrak{s} \otimes_{\mathbb{Q}_p} K) \right)$$

is a  $K$ -vector space of dimension  $\binom{\text{rk}_{\mathbb{Q}_p} \mathbb{S} + \dim S}{r}$ .

Proof: Since  $M/D$  is commutative there is a spectral sequence

$$(90) \quad H_{an}^r(M/D, H_{an}^q(D, \mathbf{1})) \implies H_{an}^{r+q}(M, \mathbf{1})$$

(cf. Theorem 6.5 and Theorem 6.8). The statement involving (88) is a theorem of W. Casselman and D. Wigner (cf. [13], Theorem 1) which can also be proved

using our theoretical setup. We leave the details to the reader.

The Lie algebra of  $M/D$  can be identified with  $\mathfrak{s}$ . Since the adjoint action of  $\mathfrak{s}$  on  $\mathfrak{d}$  is trivial it follows as above that the action of  $M/D$  on the spaces  $H^q(\mathfrak{d}, \mathbb{1})$  is trivial for all  $q \geq 0$ . Since the dimension of these spaces is finite over  $K$  we have

$$B_{\bullet}(M/D, H^q(\mathfrak{d}, \mathbb{1})) = B_{\bullet}(M/D, \mathbb{1}) \otimes_K H^q(\mathfrak{d}, \mathbb{1})$$

and

$$H_{an}^r(M/D, H_{an}^q(D, \mathbb{1})) \simeq H_{an}^r(M/D, \mathbb{1}) \otimes_K H_{an}^q(D, \mathbb{1}).$$

One sees that the double complex giving rise to the spectral sequence (90) is the tensor product of the complexes  $B_{\bullet}(M/D, \mathbb{1})$  and  $B_{\bullet}(D, \mathbb{1})$ . Thus, the cohomology of the associated total complex (which is  $H_{an}^n(M, \mathbb{1})$  by (90)) is the right hand side of (87) with  $S$  replaced by  $M/D$ .

The homomorphism  $S \rightarrow M/D$  has finite kernel and its image is open and normal of finite index in  $M/D$  (cf. [4], Corollaire 3.20). Since  $H_{an}^0(H, \cdot)$  is an  $s$ -exact functor whenever  $H$  is finite, we obtain  $H_{an}^r(S, \mathbb{1}) \simeq H_{an}^r(M/D, \mathbb{1})$  for all  $r \geq 0$ , and it remains to prove (89).

Let  $S_0$  be the largest compact open subgroup of  $S$ . By the arguments given in Remark 6.4, the discrete group  $S/S_0$  satisfies assumption (A) so that Theorem 6.8 gives a spectral sequence

$$H_{an}^t(S/S_0, H_{an}^s(S_0, \mathbb{1})) \implies H_{an}^{t+s}(S, \mathbb{1}).$$

As above, it leads to  $K$ -linear isomorphisms

$$H_{an}^r(S, \mathbb{1}) \simeq \bigoplus_{t+s=r} H_{an}^t(S/S_0, \mathbb{1}) \otimes_K H_{an}^s(S_0, \mathbb{1})$$

for all  $r \geq 0$  where  $H_{an}^s(S_0, \mathbb{1}) \simeq H^s(\mathfrak{s}, \mathbb{1})$ . This latter space is isomorphic to  $\bigwedge^s(\mathfrak{s} \otimes_{\mathbb{Q}_p} K)$ . In fact,  $H^{\bullet}(\mathfrak{s}, \mathbb{1})$  is the cohomology of the standard complex  $\text{Hom}_K(\bigwedge^{\bullet}(\mathfrak{s} \otimes_{\mathbb{Q}_p} K), \mathbb{1})$ , the boundary maps of which are all zero by the triviality of the  $\mathfrak{s}$ -action on  $\mathbb{1}$  and the commutativity of  $\mathfrak{s}$ . On the other hand,  $S/S_0$  is known to be a free abelian group of rank  $\text{rk}_{\mathbb{Q}_p} \mathbb{S} = \text{rk}_{\mathbb{Z}} X_{\mathbb{Q}_p}(\mathbb{S})$  such that  $X_{\mathbb{Q}_p}(\mathbb{S}) \otimes_{\mathbb{Z}} K \simeq \text{Hom}_{\mathbb{Z}}(S/S_0, \mathbb{Z}) \otimes_{\mathbb{Z}} K$  (cf. [5], section X.2.2). Therefore, the description of  $H_{an}^t(S/S_0, \mathbb{1})$  follows from [5], Lemma X.2.7.  $\square$

Using Theorem 6.5, the last part of the above proof generalizes to finite extensions  $L$  of  $\mathbb{Q}_p$  if  $\mathbb{M}$  is a torus.

**Corollary 8.10.** *Assume  $M$  to be the group of  $L$ -rational points of a torus  $\mathbb{M}$  defined over  $L$ . Letting  $\mathfrak{m}$  denote the Lie algebra of  $M$  there are  $K$ -linear isomorphisms*

$$H_{an}^r(M, \mathbb{1}) \simeq \bigoplus_{t+s=r} \left( \bigwedge^t (X_L(\mathbb{M}) \otimes_{\mathbb{Z}} K) \otimes_K \bigwedge^s (\mathfrak{m} \otimes_L K) \right)$$

for all  $r \geq 0$ . We have  $\dim_K H_{an}^r(M, \mathbb{1}) = \binom{\text{rk}_L \mathbb{M} + \dim M}{r}$ .  $\square$

**Remark 8.11.** With respect to the cup product the cohomology ring  $H(\mathfrak{d}) := \bigoplus_{q \geq 0} H^q(\mathfrak{d}, \mathbb{1})$  of a reductive Lie algebra  $\mathfrak{d}$  over a field of characteristic zero is known to be the exterior algebra over the subspace of primitive elements. This is a result of J-L. Koszul (cf. [24], Théorème 10.2). Although we have not introduced cup products between locally analytic cohomology groups, Koszul's result together with Theorem 8.9 strongly suggests that the algebra structure of the total cohomology space  $H(M) := \bigoplus_{q \geq 0} H_{an}^q(M, \mathbb{1})$  of a connected reductive  $\mathbb{Q}_p$ -group  $\mathbb{M}$  would be that of an exterior algebra, too.

We are going to illustrate our results by giving an explicit example.

**Example 8.12.** Let  $G := GL_2(L)$ , let  $T$  be the subgroup of diagonal matrices, let  $P$  be the subgroup of upper triangular matrices, and let  $N$  and  $\bar{N}$  be the subgroups of upper and lower triangular unipotent matrices in  $G$ , respectively. By abuse of notation we denote by  $\mathfrak{n}$ ,  $\bar{\mathfrak{n}}$  and  $\mathfrak{t}$  the following elements of the Lie algebras of  $N$ ,  $\bar{N}$  and  $T$ , respectively:

$$\mathfrak{n} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \bar{\mathfrak{n}} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathfrak{t} := [\mathfrak{n}, \bar{\mathfrak{n}}] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Further, we let  $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  be the longest element of the Weyl group of  $(G, T)$ .

Since any locally analytic character  $\chi : T \rightarrow K^\times$  extends to a continuous homomorphism  $\chi : D(T) \rightarrow K$  of  $K$ -algebras making  $\chi$  an object of  $\mathcal{M}_T$ , we may set  $c(\chi) := \chi(\mathfrak{t}) \in K$ . The Verma module  $\mathfrak{m}(\chi) = U_K(\mathfrak{g}) \otimes_{U_K(\mathfrak{p})} \chi^{-1}$  has the  $K$ -basis  $(\bar{\mathfrak{n}}^m)_{m \geq 0}$ . A direct calculation shows that

$$(91) \quad \mathfrak{n} \cdot \sum_m \lambda_m \bar{\mathfrak{n}}^m = \sum_m -\lambda_{m+1} (m+1)(m+c(\chi)) \bar{\mathfrak{n}}^m.$$

It follows that  $H^0(\mathfrak{n}, \mathfrak{m}(\chi))$  is one dimensional if  $c(\chi) \notin \mathbb{Z}_{\leq 0}$ , generated by  $\bar{\mathfrak{n}}^0$ , and that it is two dimensional if  $c(\chi) \in \mathbb{Z}_{\leq 0}$ , generated by  $\bar{\mathfrak{n}}^0$  and  $\bar{\mathfrak{n}}^{1-c(\chi)}$ . The action of  $T$  is determined to be via  $\chi^{-1}$  and  $\chi^{-1} \oplus \varepsilon^{c(\chi)-1} \chi^{-1}$ , respectively, where  $\varepsilon : T \rightarrow K^\times$  is the character given by

$$\varepsilon \left( \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \right) = t_1/t_2.$$

Further, by Poincaré duality, we have  $H^1(\mathfrak{n}, \mathfrak{m}(\chi)) \simeq H_0(\mathfrak{n}, \mathfrak{m}(\chi) \otimes \mathfrak{n}^*)$  as  $T$ -representations (cf. [20], Proposition VII.7.34). We deduce from (91) that this space is trivial if  $c(\chi) \notin \mathbb{Z}_{\leq 0}$  and that in the other cases it is one dimensional, generated by  $\bar{\mathfrak{n}}^{-c(\chi)}$  with  $T$ -action given by  $\varepsilon^{c(\chi)-1} \chi^{-1}$ .

As above we let  $M(\chi) = I(\chi)'_b$  denote the  $D(G)$ -module obtained by dualizing the principal series representation  $I(\chi) := \text{coind}_P^G(\chi) = \text{Ind}_P^G(\chi)$ .

**Theorem 8.13.** (i) As  $D(T)$ -modules we have  $H_{an}^0(N, M(\chi)) = \chi^{-1}$  if  $c(\chi) \notin \mathbb{Z}_{\leq 0}$  whereas if  $c(\chi) \in \mathbb{Z}_{\leq 0}$  then  $H_{an}^0(N, M(\chi)) = \chi^{-1} \oplus \varepsilon^{c(\chi)-1} \chi^{-1}$ .

(ii) As  $D(T)$ -modules we have  $H_{an}^1(N, M(\chi)) = \varepsilon^{-1}|\varepsilon|^{-1}(\chi^{-1})^{w_0}$  if  $c(\chi) \notin \mathbb{Z}_{\leq 0}$  whereas if  $c(\chi) \in \mathbb{Z}_{\leq 0}$  then there is a short exact sequence

$$(92) \quad 0 \longrightarrow \varepsilon^{c(\chi)-1}\chi^{-1} \longrightarrow H_{an}^1(N, M(\chi)) \longrightarrow \varepsilon^{-1}|\varepsilon|^{-1}(\chi^{-1})^{w_0} \longrightarrow 0.$$

The sequence (92) admits a  $D(T)$ -linear section if and only if  $\varepsilon^{1-c(\chi)}\chi \neq \varepsilon|\varepsilon|\chi^{w_0}$ .

(iii) We have  $H_{an}^q(N, M(\chi)) = 0$  if  $q > 1$ .

Proof: Consider the exact sequence

$$0 \longrightarrow M(\chi)_1 \longrightarrow M(\chi) \longrightarrow M(\chi)/M(\chi)_1 \longrightarrow 0$$

with  $M(\chi)_1$  as above. By Theorem 4.8 and Theorem 6.5 it gives rise to a long exact sequence in the locally analytic cohomology with respect to  $N$ . Referring to Theorem 8.2, Remark 8.3, Theorem 8.5 and our above computations, everything follows except for the splitting assertion in (ii). Together with its proof, the latter was communicated to the author by an anonymous referee.

The sufficiency of the condition follows from Corollary 8.8. Therefore, assume  $\varepsilon^{1-c(\chi)}\chi = \varepsilon|\varepsilon|\chi^{w_0}$ . It suffices to show that  $H_{an}^1(N, M(\chi))$  is not semisimple as a module over  $D(T')$  with  $T' := T \cap \mathrm{SL}_2(L)$ . Let  $V(-c(\chi))$  be the algebraic representation of  $G' := \mathrm{SL}_2(L)$  of highest weight  $-c(\chi)$ , and let  $|\cdot| : T' \rightarrow K^\times$  be the character given by

$$\left| \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right| := |t|,$$

viewed as a character of  $P' := P \cap \mathrm{SL}_2(L)$  which is trivial on  $N$ . Let  $I^\infty(|\cdot|)$  be the  $K$ -vector space of all locally constant functions  $f : G' \rightarrow K$  satisfying  $f(gp) = |p|^{-1}f(g)$  for all elements  $g \in G'$  and  $p \in P'$ , and let  $G'$  act on  $I^\infty(|\cdot|)$  via the left regular representation. Endowed with its finest locally convex topology, the locally algebraic  $G'$ -representation  $V := V(-c(\chi)) \otimes_K I^\infty(|\cdot|)$  is an object of  $\mathcal{M}_{G'}$  and a reflexive locally convex  $K$ -vector space.

According to the proof of [26], Theorem 1, or the forthcoming [27], there is an exact sequence

$$0 \longrightarrow M(\varepsilon^{1-c(\chi)}\chi) \longrightarrow M(\chi) \longrightarrow V'_b \longrightarrow 0$$

of continuous  $D(G')$ -linear maps. Using Corollary 8.8 for  $q = 0$  and the results already proved, the associated long exact sequence in the locally analytic  $N$ -cohomology gives rise to a  $D(T')$ -linear isomorphism

$$(93) \quad H_{an}^1(N, M(\chi)) \simeq H_{an}^1(N, V'_b).$$

Using Theorem 6.5, consider the five-term exact sequence deduced from (44) for the object  $V'_b \simeq V(-c(\chi))'_b \otimes_K I^\infty(|\cdot|)'_b$  of  $\mathcal{M}_N$ . We have

$$\mathrm{Ext}_{D^\infty(N)}^1(\mathbb{1}, H^0(\mathfrak{n}, V'_b)) \simeq H^0(\mathfrak{n}, V(-c(\chi))'_b) \otimes_K \mathrm{Ext}_{D^\infty(N)}^1(\mathbb{1}, I^\infty(|\cdot|)'_b)$$

by arguments similar to those appearing in the proof of Theorem 7.3. The reflexivity of  $I^\infty(|\cdot|)_b'$  implies  $\text{Ext}_{D^\infty(N)}^1(\mathbb{1}, I^\infty(|\cdot|)_b') = 0$  because the  $D^\infty(N)$ -module  $\mathbb{1}$  is injective (cf. the proof of Theorem 7.4). Further,

$$\mathrm{H}^1(\mathfrak{n}, V_b')^N \simeq \mathrm{H}^1(\mathfrak{n}, V(-c(\chi)))' \otimes_K (I^\infty(|\cdot|)_N)_b'$$

as modules over  $D(T')$ , where the space  $\mathrm{H}^1(\mathfrak{n}, V(-c(\chi)))$  has  $K$ -dimension one and  $I^\infty(|\cdot|)_N$  is of  $K$ -dimension two (cf. [20], pp. 296–297, and [12], Proposition 9.3.1). Since  $\mathrm{H}_{an}^1(N, V_b')$  is of  $K$ -dimension two (cf. (93)), the five-term exact sequence and (93) yield a  $D(T')$ -equivariant isomorphism

$$\mathrm{H}_{an}^1(N, M(\chi)) \simeq \mathrm{H}^1(\mathfrak{n}, V(-c(\chi)))' \otimes_K (I^\infty(|\cdot|)_N)_b'.$$

However, the  $T'$ -representation  $I^\infty(|\cdot|)_N$  is not semisimple (cf. [12], Proposition 9.4.5).  $\square$

Using different methods and assuming  $L = \mathbb{Q}_p$ , the following result was first proved by P. Schneider and J. Teitelbaum (cf. [35], Proposition 6.2; note that the authors work with the opposite Borel subgroup containing  $T$ ).

**Theorem 8.14.** *Let  $\chi_1, \chi_2 : T \rightarrow K^\times$  be two locally analytic characters of  $T$ . The space  $\mathcal{L}_G(I(\chi_1), I(\chi_2)) = \mathcal{E}xt_G^0(I(\chi_1), I(\chi_2))$  is one dimensional if either  $\chi_1 = \chi_2$  or if  $c(\chi_1) \in \mathbb{Z}_{\leq 0}$  and  $\chi_2 = \chi_1 \varepsilon^{1-c(\chi_1)}$ . Otherwise it is zero.*

Proof: By (75) and (76) and since  $N$  acts trivially on  $\chi_2$  we have

$$(94) \quad \mathcal{L}_G(I(\chi_1), I(\chi_2)) \simeq \mathcal{L}_T(\chi_2^{-1}, \mathrm{H}_{an}^0(N, M(\chi_1))).$$

Using the description of  $\mathrm{H}_{an}^0(N, M(\chi_1))$  given in Theorem 8.13 together with Corollary 8.8 and Corollary 8.10, the claim follows.  $\square$

We now analyze the spectral sequence (77) which we abbreviate as  $E_2^{p,q} \Rightarrow E^{p+q}$ . By Theorem 8.13 we have  $E_2^{p,q} = 0$  for all  $q > 1$ . By construction of the edge morphisms we obtain a long exact sequence

$$(95) \quad \begin{array}{ccccccc} 0 & \longrightarrow & E_2^{1,0} & \longrightarrow & E^1 & \longrightarrow & E_2^{0,1} & \longrightarrow \\ & & \longrightarrow & E_2^{2,0} & \longrightarrow & E^2 & \longrightarrow & E_2^{1,1} & \longrightarrow \\ & & & \longrightarrow & E_2^{3,0} & \longrightarrow & E^3 & \longrightarrow & E_2^{2,1} & \longrightarrow & \dots \end{array}$$

If  $\chi : T \rightarrow K^\times$  is a character we let  $\chi', \chi'' : \mathbb{G}_m(L) \rightarrow K^\times$  be the characters given by

$$\chi'(\alpha) := \chi \left( \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \right) \text{ and } \chi''(\alpha) := \left( \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \right).$$

We let  $x : \mathbb{G}_m(L) \rightarrow K^\times$  denote the character given by the inclusion of  $L^\times$  into  $K^\times$ .

If  $\chi_2^{-1}$  does not simultaneously appear in the  $D(T)$ -semisimplifications of both  $\mathrm{H}_{an}^0(N, M(\chi_1))$  and  $\mathrm{H}_{an}^1(N, M(\chi_1))$  then Corollary 8.8 implies that  $E_2^{p,0} = 0$  or  $E_2^{p,1} = 0$  for all  $p \geq 1$ . In this case (72), (95) and Corollary 8.10 give the following result.

**Theorem 8.15.** *Let  $1 \leq q \leq 4$ .*

- (i) *Assume that  $c(\chi_1) \notin \mathbb{Z}_{\leq 0}$  and that  $\chi_1'(\chi_1'')^{-1} \neq x|\cdot|$  (so that we have  $\chi_1 \neq \varepsilon|\varepsilon|\chi_1^{w_0}$ ). If  $\chi_2 = \chi_1$  then  $\mathcal{E}xt_G^q(I(\chi_1), I(\chi_2))$  is of  $K$ -dimension  $\binom{4}{q}$ . If  $\chi_2 = \varepsilon|\varepsilon|\chi_1^{w_0}$  then  $\mathcal{E}xt_G^q(I(\chi_1), I(\chi_2))$  is of  $K$ -dimension  $\binom{4}{q-1}$ . Otherwise it is zero.*
- (ii) *Assume that  $c(\chi_1) \notin \mathbb{Z}_{\leq 0}$  and that  $\chi_1'(\chi_1'')^{-1} = x|\cdot|$  (so that we have  $\chi_1 = \varepsilon|\varepsilon|\chi_1^{w_0}$ ). If  $\chi_2 \neq \chi_1$  then  $\mathcal{E}xt_G^q(I(\chi_1), I(\chi_2)) = 0$ .*
- (iii) *Assume that  $c(\chi_1) \in \mathbb{Z}_{\leq 0}$ , that  $\chi_1'(\chi_1'')^{-1} \neq x|\cdot|$  and that  $\chi_1'(\chi_1'')^{-1} \neq |\cdot|x^{c(\chi_1)}$ . If  $\chi_2 = \chi_1$  then  $\mathcal{E}xt_G^q(I(\chi_1), I(\chi_2))$  is of  $K$ -dimension  $\binom{4}{q}$ . If  $\chi_2 = \varepsilon|\varepsilon|\chi_1^{w_0}$  then  $\mathcal{E}xt_G^q(I(\chi_1), I(\chi_2))$  is of  $K$ -dimension  $\binom{4}{q-1}$ .*
- (iv) *Assume that  $c(\chi_1) \in \mathbb{Z}_{\leq 0}$ . If  $\chi_2$  is different from  $\chi_1$ ,  $\varepsilon|\varepsilon|\chi_1^{w_0}$  and  $\chi_1\varepsilon^{1-c(\chi_1)}$  then  $\mathcal{E}xt_G^q(I(\chi_1), I(\chi_2)) = 0$ .  $\square$*

Our knowledge of the terms  $H_{an}^q(N, M(\chi))$  (cf. Theorem 8.13) allows us to conclude that by (72), Corollary 8.8 and Corollary 8.10 the terms  $E_2^{p,q}$  vanish if  $p > 4$  or  $q > 1$ . In view of the exact sequence (95) we conclude as follows.

- Theorem 8.16.** (i) *Assume that  $c(\chi_1) \notin \mathbb{Z}_{\leq 0}$ . The space  $\mathcal{E}xt_G^5(I(\chi_1), I(\chi_2))$  is of  $K$ -dimension one or zero according to whether or not  $\chi_2$  is equal to  $\varepsilon|\varepsilon|\chi_1^{w_0}$ .*
- (ii) *Assume that  $c(\chi_1) \in \mathbb{Z}_{\leq 0}$  and that  $\chi_1'(\chi_1'')^{-1} \neq |\cdot|x^{c(\chi_1)}$  (so that we have  $\varepsilon|\varepsilon|\chi_1^{w_0} \neq \varepsilon^{1-c(\chi_1)}\chi_1$ ). The space  $\mathcal{E}xt_G^5(I(\chi_1), I(\chi_2))$  is of  $K$ -dimension one or zero according to whether or not  $\chi_2$  is equal to one of the characters  $\varepsilon|\varepsilon|\chi_1^{w_0}$  or  $\varepsilon^{1-c(\chi_1)}\chi_1$ .*
- (iii) *We have  $\mathcal{E}xt_G^q(I(\chi_1), I(\chi_2)) = 0$  if  $q > 5$ .  $\square$*

In the cases which are not covered by Theorem 8.15 and Theorem 8.16 we do not know how to use (95) to obtain information on the individual terms  $E^q$ . At least, we are able to prove the following result on the Euler-Poincaré characteristic of two principal series representations.

**Theorem 8.17.** *If  $\chi_1, \chi_2 : T \rightarrow K^\times$  are two locally analytic characters then  $\sum_{q=0}^{\infty} (-1)^q \dim_K \mathcal{E}xt_G^q(I(\chi_1), I(\chi_2)) = 0$ .*

Proof: By (75), (76) and (94) we have to show that

$$\sum_{q=1}^{\infty} (-1)^q \dim_K E^q + \dim_K \mathcal{E}xt_T^0(\chi_2^{-1}, H_{an}^0(N, M(\chi_1))) = 0.$$

Setting  $E_2^{0,0} := \mathcal{E}xt_T^0(\chi_2^{-1}, H_{an}^0(N, M(\chi_1)))$  and using (95) this is equivalent to

$$\sum_{p=0}^{\infty} (-1)^p \dim_K E_2^{p,0} + \sum_{p=0}^{\infty} (-1)^p \dim_K E_2^{p,1} = 0.$$

According to Theorem 8.13 the space  $H_{an}^0(N, M(\chi_1))$  is the direct sum of one dimensional  $D(T)$ -modules. It follows from (72), Corollary 8.8 and Corollary 8.10 that  $\sum_{p=0}^{\infty} (-1)^p \dim_K E_2^{p,0} = 0$ . The same arguments work for  $H_{an}^1(N, M(\chi_1))$

unless the latter is a non-split extension of a character  $\eta$  with itself. In this case one uses (72), Corollary 6.5 and Theorem 4.8 to obtain the additivity formula

$$\sum_{p=0}^{\infty} (-1)^p \dim_K E_2^{p,1} = \sum_{p=0}^{\infty} (-1)^p \dim_K H_{an}^p(T, \eta\chi_2) + \sum_{p=0}^{\infty} (-1)^p \dim_K H_{an}^p(T, \eta\chi_2).$$

By the arguments already given, both alternating sums on the right vanish.  $\square$

This ends our elaboration of the example of  $GL_2$ .

We conclude our article with the following comparison result which, in view of our example in section 4, is certainly special to representations with good finiteness properties.

**Theorem 8.18.** *Let  $G$  be the group of  $L$ -rational points of a connected reductive group  $\mathbb{G}$  defined over  $L$  with  $P$ ,  $M$ ,  $N$  and  $\chi_1, \chi_2$  as above. Assuming  $L = \mathbb{Q}_p$  or  $\mathbb{G}$  to be  $L$ -split, the comparison homomorphisms*

$$\kappa^q : \mathcal{E}xt_G^q(M(\chi_2), M(\chi_1)) \longrightarrow \text{Ext}_{D(G)}^q(M(\chi_2), M(\chi_1))$$

of (33) are bijective for all  $q \geq 0$ .

Proof: By (76) we have  $\mathcal{E}xt_G^q(M(\chi_2), M(\chi_1)) \simeq \mathcal{E}xt_P^q(\chi_2^{-1}, M(\chi_1))$ . As noted earlier, together with  $\mathbb{1}$  (cf. Theorem 6.5 and Theorem 6.7) also the module  $\chi_2^{-1}$  admits an s-projective s-resolution in  $\mathcal{M}_P$  as in (A). The arguments of the proof of Theorem 4.8 show that the comparison homomorphisms

$$\kappa^q : \mathcal{E}xt_P^q(\chi_2^{-1}, M(\chi_1)) \longrightarrow \text{Ext}_{D(P)}^q(\chi_2^{-1}, M(\chi_1))$$

are bijective for any  $q \geq 0$ . After restriction to a compact open subgroup of  $P$ , the module  $\chi_2^{-1}$  even has an s-resolution by finitely generated free modules because this is true for the trivial module  $\mathbb{1}$  (cf. Theorem 4.4 if  $L = \mathbb{Q}_p$  and Corollary 6.3 as well as the proof of Theorem 6.5 if  $\mathbb{G}$  is  $L$ -split). Thus,  $\chi_2^{-1}$  satisfies assumption (FIN) of [37], section 6, and the arguments of [37], Lemma 6.3, show that there are isomorphisms

$$\text{Ext}_{D(P)}^q(\chi_2^{-1}, M(\chi_1)) \simeq \text{Ext}_{D(G)}^q(D(G) \otimes_{D(P)} \chi_2^{-1}, M(\chi_1))$$

where  $D(G) \otimes_{D(P)} \chi_2^{-1} = M(\chi_2)$  by Lemma 8.1. We note that if  $L$  is a proper extension of  $\mathbb{Q}_p$ , in order for this reasoning to work, one first has to generalize the flatness result of [37], Proposition 6.2, to this more general situation. This was achieved by T. Schmidt (cf. [29], Proposition 8.2).

It remains to remark that the resulting diagram

$$\begin{array}{ccc} \mathcal{E}xt_G^q(M(\chi_2), M(\chi_1)) & \xrightarrow{\sim} & \mathcal{E}xt_P^q(\chi_2^{-1}, M(\chi_1)) \\ \kappa^q \downarrow & & \downarrow \kappa^q \\ \text{Ext}_{D(G)}^q(M(\chi_2), M(\chi_1)) & \xrightarrow{\sim} & \text{Ext}_{D(P)}^q(\chi_2^{-1}, M(\chi_1)) \end{array}$$

is commutative for any  $q \geq 0$ .  $\square$

**Remark 8.19.** One might look for yet another type of comparison maps. Assuming  $H$  to be a locally  $L$ -analytic group we denote by  $\mathcal{C}_H$  the abelian category of all admissible locally analytic representations of  $H$  in the sense of [36], section 6. By  $\mathcal{U}_H$  we denote the abelian category of all continuous (unitary) admissible Banach space representations of the topological group  $H$ . Passing to the subspace of locally analytic vectors defines an additive functor  $(V \mapsto V^{an}) : \mathcal{U}_H \rightarrow \mathcal{C}_H$  (cf. [36], section 7). In view of the results of [36], Theorem 7.1, and of Lazard’s comparison isomorphism [25], Chapitre V, Théorème 2.3.10, one might wonder about the behavior of the induced homomorphisms

$$\mathrm{Ext}_{\mathcal{U}_H}^q(V_1, V_2) \longrightarrow \mathrm{Ext}_{\mathcal{C}_H}^q(V_1^{an}, V_2^{an})$$

at least if  $L = \mathbb{Q}_p$  and if the spaces  $V_i$  are (unitary) principal series representations. To our knowledge, there are no general results in that direction.

## References

- [1] A. BOREL: *Linear Algebraic Groups*, 2nd Edition, Graduate Texts in Mathematics **126**, Springer, 1991
- [2] A. BOREL, J. TITS: Groupes réductifs, *Publ. Math. IHÉS* **27**, 1965, pp. 55–151
- [3] A. BOREL, J. TITS: Compléments à l’article: Groupes réductifs, *Publ. Math. IHÉS* **41**, 1972, pp. 253–276
- [4] A. BOREL, J. TITS: Homomorphismes abstraits de groupes algébriques simples, *Ann. Math.* **97**, 1973, pp. 499–571
- [5] A. BOREL, N. WALLACH: *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*, 2nd Edition, Mathematical Surveys and Monographs **67**, AMS, 2000
- [6] N. BOURBAKI: *Lie Groups and Lie Algebras*, Chapters 1–3, Springer, 1989
- [7] N. BOURBAKI: *Topologie Générale*, Chapitre 9, Hermann, Paris, 1958
- [8] N. BOURBAKI: *Topological Vector Spaces*, Chapters 1–5, Springer, 2003
- [9] L. BERGER, C. BREUIL, P. COLMEZ (EDITORS): Représentations  $p$ -adiques de groupes  $p$ -adiques I–III, *Astérisque* **319**, **330**, **331**, Soc. Math. France, 2008/2010
- [10] F. BRUHAT, J. TITS: Groupes réductifs sur un corps local I, Données radicielles valuées, *Publ. Math. IHÉS* **41**, 1972, pp. 5–251
- [11] P. CARTIER: Representations of  $p$ -adic groups: A survey, *Proc. Symp. Pure Math.* **33**, AMS, Providence, 1979, pp. 111–156
- [12] W. CASSELMAN: Introduction to the theory of admissible representations of  $p$ -adic reductive groups, *preprint*, 1995

- [13] W. CASSELMAN, D. WIGNER: Continuous Cohomology and a Conjecture of Serre's, *Invent. Math.* **25**, 1974, pp. 199–211
- [14] J.D. DIXON, M.P.F. DU SAUTOY, A. MANN, D. SEGAL: *Analytic pro- $p$  groups*, 2nd Edition, Cambridge Studies in Advanced Mathematics **61**, Cambridge University Press, 2003
- [15] M. EMERTON: Locally analytic vectors in representations of locally  $p$ -adic analytic groups, to appear in *Memoirs of the AMS*
- [16] C.T. FÉAUX DE LACROIX: Einige Resultate über die topologischen Darstellungen  $p$ -adischer Liegruppen auf unendlich dimensionalen Vektorräumen über einem  $p$ -adischen Körper, *Schriftenreihe Math. Inst. Univ. Münster*, 3. Serie, Heft **23**, 1999
- [17] A. GROTHENDIECK: Produits tensoriels topologiques et espaces nucléaires, *Memoirs AMS* **16**, 1966
- [18] M. KISIN, M. STRAUCH: Locally analytic cuspidal representations for  $GL_2$  and related groups, *J. Inst. Math. Jussieu* **5**, 2006, pp. 373–421
- [19] L. GRUSON: Fibrés vectoriels sur un polydisque ultramétrique, *Ann. Scient. Éc. Norm. Sup.*, 4e série, t. 1, 1968, pp. 45–89
- [20] A. KNAPP: *Lie Groups, Lie Algebras, and Cohomology*, Princeton Univ. Press, 1988
- [21] G. KÖTHE: *Topological Vector Spaces I*, Grundlehren d. Math. Wiss. **159**, Springer, 1969
- [22] J. KOHLHAASE: Invariant distributions on  $p$ -adic analytic groups, *Duke Math. J.* **137**, 2007, pp. 19–62
- [23] B. KOSTANT: Lie algebra cohomology and the generalized Borel-Weil theorem, *Ann. Math* **74**, 1961, pp. 329–387
- [24] J-L. KOSZUL: Homologie et Cohomologie des Algèbres de Lie, *Bull. Soc. Math. France* **78**, 1950, pp. 65–127
- [25] M. LAZARD: Groupes analytiques  $p$ -adiques, *Publ. Math. IHÉS* **26**, 1965, pp. 5–219
- [26] Y. MORITA: Analytic Representations of  $SL_2$  over a  $p$ -adic Number Field, *Adv. Studies Pure Math.* **7**, 1985, pp. 185–222
- [27] S. ORLIK, M. STRAUCH: On Jordan-Hölder series of locally analytic representations, *preprint*, 2010
- [28] D. PRASAD: Locally algebraic representations of  $p$ -adic groups, appendix to [33]
- [29] T. SCHMIDT: Analytic vectors in continuous  $p$ -adic representations, *Compos. Math.* **145**, 2009, pp. 247–270

- [30] P. SCHNEIDER:  $p$ -adic representation theory, *The 1999 Britton Lectures at McMaster University*, available at <http://www.math.uni-muenster.de/math/u/schneider>
- [31] P. SCHNEIDER:  *$p$ -adic Analysis and Lie Groups*, Lecture Notes, Universität Münster, 2008, available at <http://www.math.uni-muenster.de/math/u/schneider>
- [32] P. SCHNEIDER: *Nonarchimedean Functional Analysis*, Springer Monographs in Mathematics, Springer, 2002
- [33] P. SCHNEIDER, J. TEITELBAUM:  $U(\mathfrak{g})$ -finite locally analytic representations, *Representation Theory* **5**, 2001, pp. 111–128
- [34] P. SCHNEIDER, J. TEITELBAUM:  $p$ -adic Fourier theory, *Documenta Math.* **6**, 2001, pp. 447–481
- [35] P. SCHNEIDER, J. TEITELBAUM: Locally analytic distributions and  $p$ -adic representation theory, with applications to  $GL_2$ , *Journ. AMS* **15**, 2002, pp. 443–468
- [36] P. SCHNEIDER, J. TEITELBAUM: Algebras of  $p$ -adic distributions and admissible representations, *Invent. Math.* **153**, 2003, pp. 145–196
- [37] P. SCHNEIDER, J. TEITELBAUM: Duality for admissible locally analytic representations, *Representation Theory* **9**, 2005, pp. 297–326
- [38] W. SOERGEL:  $\mathfrak{n}$ -Cohomology of simple highest weight modules on walls and purity, *Invent. Math.* **98**, 1989, pp. 565–580
- [39] P. SYMONDS, T. WEIGEL: Cohomology of  $p$ -adic analytic groups, *New Horizons in pro- $p$  Groups*, Progress in Mathematics **184**, Birkhäuser, 2000, pp. 349–410
- [40] J.L. TAYLOR: Homology and Cohomology for Topological Algebras, *Adv. Math.* **9**, 1972, pp. 137–182
- [41] J. TITS: Reductive groups over local fields, *Automorphic Forms, Representations, and  $L$ -Functions*, *Proc. Symp. Pure Math.* **33**, AMS, 1977, pp. 29–70

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