Homological vanishing theorems for locally analytic representations

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Abstract. Let G be the group of rational points of a split connected reductive group over a p-adic local field, and let Γ be a discrete and cocompact subgroup of G. Motivated by questions on the cohomology of p-adic symmetric spaces, we investigate the homology of Γ with coefficients in locally analytic principal series and related representations of G. The vanishing and finiteness results that we find partially rely on the compactness of certain Banach-Hecke operators. We also give a new construction of P. Schneider's reduced Hodge-de Rham spectral sequence and show that the induced filtration is the Hodge-de Rham filtration. In a previously unknown case, our vanishing theorems then also imply two other of P. Schneider's conjectures.

Contents

0 Introduction

Let p denote a prime number, let L denote a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, and let G denote the group of L-rational points of an L-split connected reductive group $\mathbb{G}$ over L.

If Γ is a discrete subgroup of G such that the quotient space $\Gamma\backslash G$ is compact, then the cohomology of representations of Γ has been an area of interest for many years. One of the most striking results in this direction is a vanishing theorem of Garland, Casselman, Prasad, Borel and Wallach whose proof relies on many deep results from the theory of smooth complex representations of G (cf. [5], Chapter XIII, Proposition 3.7). These methods also play a vital role in the computation of the de Rham cohomology of p-adic symmetric spaces by P. Schneider and U. Stuhler (cf. [28], §5).

Starting from the sheaf of differentials, the above computations were later extended to more general local systems on discrete cocompact quotients of Drinfeld's p-adic upper half space by P. Schneider (cf. [26]). One of the main open problems of [26] is the conjectured degeneration of the so-called *reduced Hodge-*

de Rham spectral sequence which converges to the de Rham cohomology of the local system. Its initial terms are given by the Γ-cohomology of so-called *p-adic holomorphic discrete series representations* of the group $G = \mathrm{SL}_{d+1}(L)$. By dualizing, the determination of these cohomology groups leads to the problem of computing the Γ-homology of the topologically dual representations. These are p-adic locally analytic representations of G in the sense of [30], section 3, whose structure was made explicit by Y. Morita, P. Schneider, J. Teitelbaum and S. Orlik (cf. [24] and the references therein).

The aim of our article is to study the Γ-homology of locally analytic G-representations in several important first cases and to apply our results to the above problems.

To start with, let T and P denote the groups of L-rational points of a maximal L-split torus $\mathbb{T}$ of $\mathbb{G}$ and of a Borel subgroup $\mathbb{P}$ of $\mathbb{G}$ containing $\mathbb{T}$. If K is a spherically complete non-archimedean valued field containing L then we let $\chi : T \to K^\times$ denote a K-valued locally analytic character of T and denote by $\mathrm{Ind}_P^G(\chi)$ the associated locally analytic principal series representation of G over K (cf. section 2). We first study the homology $\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi))$ of the abstract group Γ with coefficients in $\mathrm{Ind}_P^G(\chi)$.

Denoting by B a suitable Iwahori subgroup of G we consider a certain B-stable K-Banach subspace $\mathcal{A}$ of $\mathrm{Ind}_P^G(\chi)$ (cf. Lemma 2.1), a variant of which figures prominently in the theory of p-adic automorphic forms (cf. [13], section 4). The ring of G-endomorphisms of the compactly induced representation $\text{c-Ind}_B^G(\mathcal{A})$ contains a commuting family of Hecke operators U_t, parametrized by a certain subset T^- of T. As in the theory of smooth representations, we interprete $\mathrm{Ind}_P^G(\chi)$ as the specialization of $\text{c-Ind}_B^G(\mathcal{A})$ at a suitable Hecke character, assuming $\mathbb{G}$ to be semisimple and adjoint (cf. Proposition 2.4). We go much further, however. Namely, in this case the Hecke algebra in question is freely generated by a finite set $(U_{t_\alpha})_{\alpha\in\Delta}$ of Hecke operators parametrized by the positive simple roots Δ of $\mathbb{T}$ on the Lie algebra of $\mathbb{G}$ with respect to $\mathbb{P}$. We show the Koszul complex of $\text{c-Ind}_B^G(\mathcal{A})$ associated with the family $(U_{t_\alpha} - \chi(t_\alpha))_{\alpha\in\Delta}$ to be exact (cf. Theorem 2.5), thus obtaining an explicit resolution of $\mathrm{Ind}_P^G(\chi)$ by Γ-acyclic G-representations.

The Γ-homology of $\mathrm{Ind}_P^G(\chi)$ is the homology of the complex obtained by passing to the Γ-coinvariants of the above Koszul complex. It is the Koszul complex of $\text{c-Ind}_B^G(\mathcal{A})_\Gamma$ associated with the operators induced by the endomorphisms $U_{t_\alpha} - \chi(t_\alpha)$, $\alpha \in \Delta$. Our main observation is that $\text{c-Ind}_B^G(\mathcal{A})_\Gamma$ naturally is a K-Banach space and that the operators U_{t_α} are K-linear and norm decreasing. It follows that the endomorphism $U_{t_\alpha} - \chi(t_\alpha)$ is invertible if $\chi(t_\alpha) \in K$ has negative valuation, leading to our first vanishing theorem (cf. Theorem 3.2). We note that the arguments leading to its proof are similar to those leading to [28], §5 Theorem 6.

Now assume χ to be the restriction of a locally analytic K-valued character of the group $M_{\Delta\setminus\{\alpha\}}$ of L-rational points of a maximal proper Levi subgroup of $\mathbb{G}$ containing $\mathbb{T}$. In this case, the bijectivity of the operator $U_{t_\alpha} - \chi(t_\alpha)$ on c-$\mathrm{Ind}_B^G(\mathcal{A})_\Gamma$, which implies the vanishing of $\mathrm{H}_\bullet(\Gamma, \mathrm{Ind}_P^G(\chi))$, can be controlled much more precisely. Namely, U_{t_α} restricts to a *compact* K-linear endomorphism of a certain K-Banach subspace c-$\mathrm{Ind}_B^G(\mathcal{A}_\alpha)_\Gamma$ of c-$\mathrm{Ind}_B^G(\mathcal{A})_\Gamma$, and the endomorphism $U_{t_\alpha} - \chi(t_\alpha)$ is bijective on the latter once it is so on the former (cf. Proposition 3.4 and Lemma 3.6). Since the elements of the spectrum of U_{t_α} on c-$\mathrm{Ind}_B^G(\mathcal{A}_\alpha)_\Gamma$ form the set of zeros of an entire power series $\zeta_{\Gamma,t_\alpha,\chi}(T)$ over K, we obtain that $\mathrm{H}_\bullet(\Gamma, \mathrm{Ind}_P^G(\chi))$ vanishes in all degrees as along as $\chi(t_\alpha^{-1})$ avoids a *discrete* subset of K (cf. Theorem 3.7).

Going back to an arbitrary character χ, a different compactness argument leads to the very general fact that the Γ-homology of $\mathrm{Ind}_P^G(\chi)$ is always finite dimensional over K. As a consequence, it is naturally dual to the Γ-cohomology of the topological dual of $\mathrm{Ind}_P^G(\chi)$. Further, the Γ-Euler-Poincaré characteristic of $\mathrm{Ind}_P^G(\chi)$ is always trivial (cf. Theorem 3.9). We also obtain that the Γ-homology of $\mathrm{Ind}_P^G(\chi)$ vanishes in all degrees, once it does in degree 0 or d, where d is the semisimple rank of $\mathbb{G}$ (cf. Theorem 3.10).

For technical reasons, Theorem 3.2, Theorem 3.7, Theorem 3.9 and Theorem 3.10 are formulated for semisimple and adjoint groups, but one can deduce vanishing and finiteness criteria for general L-split connected reductive groups by using Proposition 3.11. We explicitly formulate one such vanishing result in the case that χ is an integral linear combination of the elements of Δ with a positive contribution from at least one positive simple root. In this situation we have $\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi)) = 0$ for all $q \geq 0$ (cf. Theorem 3.13).

We also show in which way such a vanishing result may fail once the algebraic character χ does not satisfy the above positivity condition. If $\chi = \mathbb{1}$ is the trivial character of T, for example, then we show that the inclusion $\mathrm{Ind}_P^G(\mathbb{1})^\infty \subset \mathrm{Ind}_P^G(\mathbb{1})$ of the smooth principal series representation induces an isomorphism between the respective Γ-homologies (cf. Theorem 3.14). The proof of this result uses one of our previous vanishing theorems, as well as a resolution of $\mathrm{Ind}_P^G(\mathbb{1})^\infty$ by locally analytic principal series representations which was constructed by M. Strauch and S. Orlik (cf. [25], section 4.9).

Making heavy use of arguments from [28], §5, Theorem 3.14 also shows that the dimension of the K-vector space $\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\mathbb{1}))$ can be expressed in terms of a constant $\mu(\Gamma)$ which also rules the dimension of the de Rham cohomology of the quotient of Drinfeld's upper half space by Γ if $G = \mathrm{PGL}_{d+1}(L)$ (cf. [28], §5, Theorem 5).

We finally broaden our point of view and study the Γ-homology of some of the locally analytic G-representations appearing in the images of the bifunctors $\mathcal{F}_{P_I}^G$ of S. Orlik and M. Strauch (cf. [25], section 4). Among these are the Jordan-Hölder constituents of G-representations which are topologically dual to the p-adic holomorphic discrete series representations of $\mathrm{SL}_{d+1}(L)$. The vanishing results that we find are recorded in Theorem 3.16 and Corollary 3.17.

In the last section of our article we take up some of P. Schneider's problems from [26]. We give an alternative construction of the reduced Hodge-de Rham complex which is based on the study of infinitesimal characters (cf. Proposition 4.5, Lemma 4.7 and Proposition 4.8). This approach has the advantage of showing directly that the Hodge-de Rham filtration and the reduced Hodge-de Rham filtration on the de Rham cohomology of the local system coincide (cf. Theorem 4.9). This was conjectured by P. Schneider in [26], page 648. We also use our vanishing theorems and results of the second named author on the structure of the p-adic holomorphic discrete series for $\mathrm{SL}_3(L)$ to prove the entirety of P. Schneider's conjectures in a previously unknown case (cf. Theorem 4.10).

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Conventions and notation. Let p be a prime number, and let L be a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers. We let $K|L$ be an extension of valued fields such that K is spherically complete. Let val denote the valuations of L and K, and let $\mathfrak{o}_L$ and $\mathfrak{o}_K$ denote the respective valuation rings. Choosing a uniformizer π of L, we shall assume $\mathrm{val}(\pi) = 1$.
If X is an affinoid variety over L then we denote by $\mathcal{O}(X)$ the K-affinoid algebra of rigid functions on the base extension of X from L to K.
If V is a locally convex K-vector space then we denote by V' the K-vector space of continuous K-linear forms on V, endowed with the strong topology.
If $\mathbb{H}$ is an algebraic group defined over L then we denote by $X^*(\mathbb{H})$ and $X_*(\mathbb{H})$ the group of L-rational characters and cocharacters of $\mathbb{H}$, respectively. We denote by $H := \mathbb{H}(L)$ the group of L-rational points of $\mathbb{H}$, endowed with its natural structure of a locally L-analytic group. We denote by $\mathfrak{h}$ the Lie algebra of H and by $U(\mathfrak{h})$ the universal enveloping algebra of $\mathfrak{h} \otimes_L K$.
If Γ is an abstract group, if M is an abelian group carrying an action of Γ, and if q is a non-negative integer, then we denote by $\mathrm{H}_q(\Gamma, M)$ the q-th homology group of Γ with coefficients in M.

1 The structure of p-adic reductive groups

Let $\mathbb{G}$ be an L-split connected reductive group defined over L, and let $\mathbb{T}$ be a maximal L-split torus of $\mathbb{G}$. The natural pairing

$$\langle \cdot, \cdot \rangle : X^*(\mathbb{T}) \times X_*(\mathbb{T}) \to \mathbb{Z}$$

is a perfect duality between finitely generated free abelian groups of the same rank. We fix a Borel subgroup $\mathbb{P}$ of $\mathbb{G}$ defined over L which contains $\mathbb{T}$. Let $\mathbb{N}$ denote the unipotent radical of $\mathbb{P}$, let $\Phi := \Phi(\mathbb{G}, \mathbb{T})$ denote the root system determined by the adjoint action of $\mathbb{T}$ on the Lie algebra of $\mathbb{G}$, and let Φ^+ (resp. Δ) denote the set of positive (resp. positive simple) roots in Φ with respect to $\mathbb{P}$. We let $W := \mathrm{N}_G(T)/T$ denote the Weyl group of Φ (cf. [3], §§13–14).

The group T determines an apartment A in the Bruhat-Tits building of G. We choose a chamber $C \subset A$ and a special vertex $x_0 \in C$ as in [10], section 3.5. The stabilizer G_0 of x_0 (resp. the stabilizer B of C) in G is a maximal compact open subgroup (resp. an Iwahori subgroup) of G. The subgroup $T_0 := T \cap G_0$ of T is its maximal compact open subgroup, and there is an isomorphism $T/T_0 \simeq X_*(\mathbb{T})$

which is characterized by the condition that

$$\langle \chi, t \rangle = \mathrm{val}(\chi(t)) \text{ for all } \chi \in X^*(\mathbb{T}) \text{ and all } t \in T/T_0.$$

For any root $\alpha \in \Phi$ we denote by $\check{\alpha} \in X_*(\mathbb{T})$ the corresponding coroot. We let

$$T^- := \{t \in T \mid \mathrm{val}(\alpha(t)) \leq 0 \text{ for all } \alpha \in \Phi^+\}$$

be the inverse image under $T \to T/T_0 \simeq X_*(\mathbb{T})$ of the monoid $X_*(\mathbb{T})^-$ of antidominant cocharacters of $\mathbb{T}$. If the group $\mathbb{G}$ is semisimple and adjoint, i.e. if $X^*(\mathbb{T})$ is generated by Φ, then the fundamental antidominant weights of the root system $\check{\Phi}$ are in fact cocharacters of $\mathbb{T}$. Thus, if $\mathbb{G}$ is semisimple and adjoint there are elements t_α, $\alpha \in \Delta$, such that

$$\mathrm{val}(\beta(t_\alpha)) = -\delta_{\alpha\beta} \text{ for all } \alpha, \beta \in \Delta.$$

We may and will even assume that $t_\alpha \in \bigcap_{\beta \in \Delta \setminus \{\alpha\}} \ker(\beta)$. Namely, since $\mathbb{G}$ is adjoint, the positive simple roots form a basis of the $\mathbb{Z}$-module $X^*(\mathbb{T})$. Therefore, the map $T \to \prod_{\beta \in \Delta} L^\times$, $t \mapsto (\beta(t))_{\beta \in \Delta}$, is an isomorphism (cf. the proof of [3], Proposition 8.5). If $\pi \in L$ is a uniformizer, then we may choose for t_α the inverse image of the family which has π^{-1} in component α and 1 everywhere else. We also note that in this case $X_*(\mathbb{T})^-$ is the free abelian monoid generated by the fundamental antidominant cocharacters. In particular, any element $t \in T^-$ can be written uniquely as $t = t_0 \prod_{\alpha \in \Delta} t_\alpha^{n_\alpha}$ with $t_0 \in T_0$ and suitable integers $n_\alpha \geq 0$.

If χ and $\tilde{\chi}$ are abstract weights of the root system Φ then we shall write $\chi \geq \tilde{\chi}$ if and only if the weight $\chi - \tilde{\chi}$ is a non-negative (rational) linear combination of the elements of Δ.

For any root $\alpha \in \Phi$ we denote by $\mathbb{N}_\alpha \subset \mathbb{G}$ the corresponding root subgroup and choose isomorphisms $\phi_\alpha : \mathbb{G}_a \to \mathbb{N}_\alpha$ over L such that

$$t\phi_\alpha(x)t^{-1} = \phi_\alpha(\alpha(t)x) \text{ for all } t \in \mathbb{T} \text{ and } x \in \mathbb{G}_a, \tag{1}$$

where $\mathbb{G}_a$ denotes the one dimensional additive group over L (cf. [19], Theorem 26.3).

Let I be a subset of Δ. We denote by $\mathbb{P}_I$ the corresponding standard parabolic subgroup of $\mathbb{G}$ and by $\mathbb{N}_I$ its unipotent radical (cf. [3], Proposition 14.18). In particular, we have $\mathbb{P}_\emptyset = \mathbb{P}$ and $\mathbb{N}_\emptyset = \mathbb{N}$. Denote by $w_0 \in W$ the longest element of the Weyl group with respect to the length function determined by Δ, and let $\overline{\mathbb{P}}_I$ be the parabolic subgroup of $\mathbb{G}$ containing $\mathbb{T}$ which is opposite to $\mathbb{P}_I$. Let $\overline{\mathbb{N}}_I$ denote the unipotent radical of $\overline{\mathbb{P}}_I$. Let $\Phi(I)^+$ (resp. $\Phi(I)^-$) denote the set of positive (resp. of negative) roots in Φ whose expressions as linear combinations of the elements in Δ have a non-zero contribution from at least one element in $\Delta \setminus I$. Any ordering of $\Phi(I)^\pm$ induces isomorphisms

$$\prod_{\alpha \in \Phi(I)^+} \mathbb{N}_\alpha \xrightarrow{\simeq} \mathbb{N}_I \quad \text{and} \quad \prod_{\alpha \in \Phi(I)^-} \mathbb{N}_\alpha \xrightarrow{\simeq} \overline{\mathbb{N}}_I \tag{2}$$

of algebraic varieties over L. There are induced isomorphisms of locally L-analytic manifolds

$$(3) \qquad \prod_{\alpha\in\Phi(I)^+}(B\cap N_\alpha) \xrightarrow{\simeq} (B\cap N_I) \quad \text{and} \quad \prod_{\alpha\in\Phi(I)^-}(B\cap N_\alpha) \xrightarrow{\simeq} (B\cap \overline{N}_I).$$

Further, for any $\alpha\in\Phi$ there is an integer n_α such that

$$(4) \qquad \phi_\alpha^{-1}(B\cap N_\alpha) = \{x\in L \mid \mathrm{val}(x)\geq n_\alpha\}.$$

If $\mathbb{M}_I$ denotes the Levi subgroup of $\mathbb{P}_I$ containing $\mathbb{T}$ then the Iwahori decomposition of B is the assertion that the multiplication map

$$(5) \qquad (B\cap\overline{N}_I)\times(B\cap M_I)\times(B\cap N_I) \xrightarrow{\simeq} B$$

is bijective and hence is an isomorphism of locally L-analytic manifolds. Further,

$$(6) \qquad t(B\cap\overline{N})t^{-1}\subseteq B\cap\overline{N} \quad \text{and} \quad t^{-1}(B\cap P)t\subseteq B\cap P$$

for all $t\in T^-$. We shall also need the fact that

$$B\cap P = G_0\cap P$$

(cf. [10], section 3.5, for the fact that $B\cap T = G_0\cap T = T_0$; using that $G_0\cap P = (G_0\cap T)\cdot(G_0\cap N)$, the missing equality $G_0\cap N = B\cap N$ can be deduced by constructing G_0 as the group of $\mathfrak{o}_L$-rational points of a reductive group scheme $\mathfrak{G}_{x_0}$ over $\mathfrak{o}_L$ and B as the inverse image in G_0 of a suitable Borel subgroup of the special fibre of $\mathfrak{G}_{x_0}$; cf. [34], sections 3.4 and 3.7).

According to [12], Proposition 1.4.4, or [14], Proposition 4.1.6, the group $B=B_0$ admits a family $(B_n)_{n\geq 0}$ of open normal subgroups which form a fundamental system of neighborhoods of the identity and which satisfy analogs of (3), (4), (5) and (6). This can be seen by constructing B_n as in the proof of [12], Proposition 1.4.4, i.e. as the kernel of the homomorphism $G_0\to\mathfrak{G}_{x_0}(\mathfrak{o}_L/\pi^n\mathfrak{o}_L)$ for $n\geq 1$. We note that if H denotes any of the groups G, $\overline{N}_I$, M_I or N_I then $B_n\cap H$ may also be viewed as the group of L-rational points of an affinoid group scheme over L. Therefore, it makes sense to talk about the affinoid algebra of rigid K-analytic functions on the corresponding base change from L to K. The inclusion $B_n\cap H\subseteq B_n$ (resp. $B_n\cap H\subseteq B\cap H$) is induced from a closed (resp. an open) immersion of rigid analytic spaces.

Example 1.1. Let $\mathbb{G} := \mathbb{PGL}_{d+1}$ for some integer $d\geq 1$. It is the quotient of $\tilde{\mathbb{G}} := \mathbb{GL}_{d+1}$ by its connected center, hence is semisimple. Let $\mathbb{P}$ and $\mathbb{T}$ be the images in $\mathbb{G}$ of the subgroups of $\tilde{\mathbb{G}}$ consisting of all upper triangular and all diagonal matrices, respectively. Then $\mathbb{N}$ is the image in $\mathbb{G}$ of the subgroup of all upper triangular unipotent matrices in $\tilde{\mathbb{G}}$. The root system $\Phi=\Phi(\mathbb{G},\mathbb{T})$ is isomorphic to the root system A_d, and the positive simple roots $\Delta=\{\alpha_1,\ldots,\alpha_d\}$ of $\mathbb{T}$ with respect to $\mathbb{P}$ are given by

$$\alpha_i(\mathrm{diag}(t_1,\ldots,t_{d+1})) = \frac{t_i}{t_{i+1}}.$$

It is easy to see that Δ spans the character group $X^*(\mathbb{T})$ and hence that $\mathbb{G}$ is adjoint. For each index $1\leq i\leq d$ we might choose for $t_{\alpha_i}\in T^-$ the image in G

of the element $\mathrm{diag}(1,\ldots,1,\pi,\ldots,\pi)\in\tilde{G}$ with the leftmost π in the $(i+1)$-th place. For G_0 we may choose the image in G of $\mathrm{GL}_{d+1}(\mathfrak{o}_L)$. For B we may choose the image in G of the subgroup of $\mathrm{GL}_{d+1}(\mathfrak{o}_L)$ consisting of all matrices which are upper triangular modulo $\pi\mathfrak{o}_L$. If $n\geq 1$ is an integer then we may choose for B_n the image in G of the subgroup of $\mathrm{GL}_{d+1}(\mathfrak{o}_L)$ consisting of all matrices reducing to the identity modulo $\pi^n\mathfrak{o}_L$.

In the remainder of this section we are going to list analogs of several technical results of [28], §4. Using the structure theory above, the proofs for $\mathbb{G}=\mathbb{GL}_{d+1}$, given in [28], §4, all carry over to our more general situation by replacing $\mathrm{GL}_n(\mathfrak{o}_L)$ by the subgroup G_0 of G. We would like to point out that similar generalizations have already been considered by Y. Aït-Amrane (cf. [2], section 3).

Proposition 1.2. *If $b,\tilde{b}\in G_0$ and $t\in T^-$ are elements with $btBP\cap\tilde{b}tBP\neq\emptyset$ then $btBP=\tilde{b}tBP$, $btB=\tilde{b}tB$ and $bB=\tilde{b}B$.*

Proof: Assuming $\tilde{b}=1$ one deduces as in [28], §4 Proposition 7, that there is an element $b'\in t(B\cap\overline{N})t^{-1}$ such that $b^{-1}b'\in B\cap P$ and $btBP=b'tBP$. This implies
$$btBP=b't(B\cap\overline{N})t^{-1}P=t(B\cap\overline{N})t^{-1}P=tBP.$$
Further, $t^{-1}(b')^{-1}bt\in B\cap P$ and $t^{-1}b't\in B\cap\overline{N}$, so that
$$btB=t(t^{-1}b't)(t^{-1}(b')^{-1}bt)B=tB.$$
That $b\in B$ (and hence that $bB=B$) is shown in [loc.cit.]. □

Lemma 1.3. *If $t,\tilde{t}\in T^-$ then $BtB\cdot B\tilde{t}B=Bt\tilde{t}B$. More precisely, if x (resp. $\tilde{x}$) runs through a system of coset representatives of $(B\cap\overline{N})/t(B\cap\overline{N})t^{-1}$ (resp. of $(B\cap\overline{N})/\tilde{t}(B\cap\overline{N})\tilde{t}^{-1}$) then $Bt\tilde{t}B$ is the disjoint union of the sets $xt\tilde{x}\tilde{t}B$.*

Proof: This can be shown as in [28], §4 Lemma 10. □

Lemma 1.4. *The group G is the disjoint union of the double cosets G_0tB with $t\in T/T_0$. For any $g\in G$ there is an element $t\in T^-$ such that $gBtB\subseteq G_0T^-B$.*

Proof: The first assertion is a direct consequence of the Bruhat-Tits decomposition of G (cf. [10], section 3.5). Concerning the second assertion, the proof of [28], §4 Lemma 12, was generalized in [18], Lemma 2.20. □

2 The Koszul complex

The theory of locally analytic representations of p-adic Lie groups on locally convex K-vector spaces was systematically developed by P. Schneider and J. Teitelbaum (cf. [30] and [31], for example).

Let $\chi:T\to K^\times$ be a locally analytic K-valued character of T, viewed as a character of P via $P\to P/N\simeq T$, and let
$$\mathrm{Ind}_P^G(\chi):=\{f\in\mathcal{C}^{an}(G,K)\mid\forall g\in G\;\forall p\in P:f(gp)=\chi(p)^{-1}f(g)\}$$
be the *locally analytic principal series representation of G associated with χ*. Here $\mathcal{C}^{an}(G,K)$ denotes the locally convex K-vector space of all locally analytic

K-valued functions of G. The group G acts on $\mathrm{Ind}_P^G(\chi)$ by left translation, i.e. we have $(gf)(g') = f(g^{-1}g')$ for all $g, g' \in G$ and all $f \in \mathrm{Ind}_P^G(\chi)$.

If C is an open P-right invariant subset of G, we let $\mathrm{Ind}_P^G(\chi)(C)$ denote the subspace of $\mathrm{Ind}_P^G(\chi)$ consisting of all functions whose support is contained in C. Since the product map $(B \cap \overline{N}) \times P \to BP$ is an isomorphism of locally L-analytic manifolds onto an open and closed subset of G, restriction of functions to B and further to $B \cap \overline{N}$ induces K-linear isomorphisms

$$\mathrm{Ind}_P^G(\chi)(BP) \xrightarrow{\simeq} \mathrm{Ind}_{B\cap P}^B(\chi) \xrightarrow{\simeq} \mathcal{C}^{an}(B \cap \overline{N}, K), \tag{7}$$

the first of which is B-equivariant. We choose a sufficiently large non-negative integer n such that the restriction of χ to any of the cosets in $(B \cap P)/(B_n \cap P)$ is rigid analytic. We denote by $\mathcal{A} \subseteq \mathrm{Ind}_P^G(\chi)$ the inverse image under (7) of the subspace of $\mathcal{C}^{an}(B \cap \overline{N}, K)$ consisting of all functions which are rigid analytic on any coset in $(B \cap \overline{N})/(B_n \cap \overline{N})$.

Lemma 2.1. *The space $\mathcal{A}$ coincides with the subspace of $\mathrm{Ind}_{B\cap P}^B(\chi)$ consisting of all functions which are rigid analytic on any coset in B/B_n. In particular, $\mathcal{A}$ is a B-stable subspace of $\mathrm{Ind}_P^G(\chi)$.*

Proof: It is clear that any function in $\mathrm{Ind}_{B\cap P}^B(\chi)$ which is rigid analytic on any coset in B/B_n is contained in $\mathcal{A}$, i.e. that its restriction to $B \cap \overline{N}$ is rigid analytic on any coset in $(B \cap \overline{N})/(B_n \cap \overline{N})$.

Conversely, given $F \in \mathcal{A}$ and $b \in B$ we need to see that the function F is rigid analytic on bB_n. Writing $b = \overline{n}p$ with $\overline{n} \in B \cap \overline{N}$ and $p \in B \cap P$, we have $bB_n = \overline{n}(B_n \cap \overline{N})(B_n \cap P)p$ by the Iwahori decomposition of B_n and since B_n is normal in B. Now conjugation by p is a rigid analytic automorphism of B_n, and the projections $B_n \to B_n \cap \overline{N}$ and $B_n \to B_n \cap P$ are rigid analytic, as well. Since $F(\overline{n}\tilde{n}\tilde{p}p) = \chi(p^{-1}\tilde{p}^{-1})F(\overline{n}\tilde{n})$ for all $\tilde{n} \in B_n \cap \overline{N}$ and all $\tilde{p} \in B_n \cap P$, the claim follows from the properties of F and χ.

The final assertion follows from the fact that, given $b \in B$, left multiplication by b^{-1} is a rigid analytic automorphism of the variety B and permutes the cosets in B/B_n. $\square$

We denote by c-$\mathrm{Ind}_B^G(\mathcal{A})$ the K-vector space of all compactly supported functions $f : G \to \mathcal{A}$ satisfying $f(gb) = b^{-1}f(g)$ for all $g \in G$ and $b \in B$. Given $g \in G$ and $F \in \mathcal{A}$, we denote by $[g, F]$ the element of c-$\mathrm{Ind}_B^G(\mathcal{A})$ which is uniquely determined by the conditions

$$\mathrm{supp}([g, F]) = gB \quad \text{and} \quad [g, F](g) = F.$$

Note that G acts on c-$\mathrm{Ind}_B^G(\mathcal{A})$ through left translation, and that the map $\mathcal{A} \to$ c-$\mathrm{Ind}_B^G(\mathcal{A})$, $F \mapsto [1, F]$, is a B-equivariant K-linear injection. By Frobenius reciprocity it induces a K-linear isomorphism

$$\mathrm{Hom}_G(\text{c-}\mathrm{Ind}_B^G(\mathcal{A}), W) \simeq \mathrm{Hom}_B(\mathcal{A}, W)$$

for any K-linear G-representation W (the proof of [9], Proposition 2.5, carries over without any changes). Specializing to the case $W =$ c-$\mathrm{Ind}_B^G(\mathcal{A})$, the ring of K-linear G-endomorphisms of c-$\mathrm{Ind}_B^G(\mathcal{A})$ can be identified with the K-vector space of all B-biequivariant functions $\Psi : G \to \mathrm{End}_K(\mathcal{A})$ such that for any element $F \in \mathcal{A}$ the function $(g \mapsto \Psi(g)(F)) : G \to \mathcal{A}$ is compactly supported. The condition of B-biequivariance means that

$$\Psi(b_1 g b_2) = b_1 \circ \Psi(g) \circ b_2 \text{ in } \mathrm{End}_K(\mathcal{A})$$

for all $g \in G$ and all $b_1, b_2 \in B$. To such a function Ψ one associates the element $(F \mapsto \sum_{g \in G/B} [g, \Psi(g^{-1})(F)])$ of $\mathrm{Hom}_B(\mathcal{A}, \text{c-}\mathrm{Ind}_B^G(\mathcal{A}))$. An inverse of this construction is given by associating with $\Phi \in \mathrm{Hom}_B(\mathcal{A}, \text{c-}\mathrm{Ind}_B^G(\mathcal{A}))$ the function Ψ satisfying $\Psi(g)(F) := \Phi(F)(g^{-1})$.

Given $t \in T^-$, consider the K-linear endomorphism ψ_t of $\mathrm{Ind}_{B \cap P}^B(\chi)$ determined by

$$\psi_t(F)(b) = F(tbt^{-1}) \quad \text{for} \quad b \in B \cap \overline{N}$$

(cf. (7)). According to (1), (3) and (4), the subset $t(B_n \cap \overline{N})t^{-1}$ of $B_n \cap \overline{N}$ can be viewed as the set of L-rational points of an affinoid subdomain. If F is rigid analytic on any coset modulo $(B_n \cap \overline{N})$ and if $b \in B \cap \overline{N}$ then F is also rigid analytic on the affinoid subdomain

$$tb(B_n \cap \overline{N})t^{-1} = tbt^{-1} \cdot t(B_n \cap \overline{N})t^{-1} \subseteq tbt^{-1}(B_n \cap \overline{N}).$$

One deduces that ψ_t restricts to an element of $\mathrm{End}_K(\mathcal{A})$, denoted by ψ_t again.

Lemma 2.2. *There is a unique B-biequivariant function $\Psi_t : G \to \mathrm{End}_K(\mathcal{A})$ with $\mathrm{supp}(\Psi_t) = Bt^{-1}B$ and $\Psi_t(t^{-1}) = \psi_t$.*

Proof: The uniqueness assertion is clear. As for the existence of Ψ_t, we have to show that if $b \in B \cap t^{-1}Bt$ then $b \circ \psi_t = \psi_t \circ tbt^{-1}$ in $\mathrm{End}_K(\mathcal{A})$. We are going to show the analogous assertion in $\mathrm{End}_K(\mathrm{Ind}_{B\cap P}^B(\chi))$. Note that under the first B-equivariant isomorphism in (7), ψ_t is given by $\psi_t(F)(g) = F(tgt^{-1})$ for all $F \in \mathrm{Ind}_P^G(\chi)(BP)$ and all $g \in BP$ because $\chi(tpt^{-1}) = \chi(p)$ for all $p \in P$. Thus, we can compute

$$\begin{aligned}(b \circ \psi_t)(F)(g) &= \psi_t(F)(b^{-1}g) = F(tb^{-1}gt^{-1}) \quad \text{and} \\ (\psi_t \circ tbt^{-1})(F)(g) &= (tbt^{-1}F)(tgt^{-1}) = F(tb^{-1}t^{-1}tgt^{-1}) = F(tb^{-1}gt^{-1})\end{aligned}$$

for all $g \in BP$. $\square$

Given $t \in T^-$ we denote by U_t the unique K-linear and G-equivariant endomorphism of c-$\mathrm{Ind}_B^G(\mathcal{A})$ corresponding to the function Ψ_t of Lemma 2.2. It follows from (5) and (6) that the natural maps

$$(B \cap \overline{N})/t(B \cap \overline{N})t^{-1} \longrightarrow B/(B \cap tBt^{-1}) \longrightarrow BtB/B$$

are bijective. Therefore, if $F \in \mathcal{A}$ and $g \in G$, then U_t is given by

$$\begin{aligned}U_t([g,F]) &= \sum_{x \in (B\cap\overline{N})/t(B\cap\overline{N})t^{-1}} [gxt, \psi_t(x^{-1}F)] \\ &= \sum_{BtB = \coprod_x xtB} [gxt, \psi_t(x^{-1}F)].\end{aligned} \tag{8}$$

Lemma 2.3. *If $t, \tilde{t} \in T^-$ then $U_t U_{\tilde{t}} = U_{\tilde{t}t}$. In particular, the G-endomorphisms U_t and $U_{\tilde{t}}$ of c-$\mathrm{Ind}_B^G(\mathcal{A})$ commute with each other.*

Proof: Letting x and $\tilde{x}$ run through systems of representatives of the coset spaces $(B \cap \overline{N})/t(B \cap \overline{N})t^{-1}$ and $(B \cap \overline{N})/\tilde{t}(B \cap \overline{N})\tilde{t}^{-1}$, respectively, we have

$$\begin{aligned}(U_t U_{\tilde{t}})([g, F]) &= \sum_{x,\tilde{x}} [g\tilde{x}\tilde{t}xt, \psi_t(x^{-1}\psi_{\tilde{t}}(\tilde{x}^{-1}F))] \quad \text{and} \\ U_{\tilde{t}t}([g, F]) &= \sum_{x,\tilde{x}} [g\tilde{x}(\tilde{t}x\tilde{t}^{-1})\tilde{t}t, \psi_{\tilde{t}t}(\tilde{t}x^{-1}\tilde{t}^{-1}\tilde{x}^{-1}F)]\end{aligned}$$

for all $g \in G$ and $F \in \mathcal{A}$ (cf. Lemma 1.3 and (8)). Therefore, it suffices to check that $\psi_t(x^{-1}\psi_{\tilde{t}}(\tilde{x}^{-1}F)) = \psi_{\tilde{t}t}(\tilde{t}x^{-1}\tilde{t}^{-1}\tilde{x}^{-1}F)$ for all $x, \tilde{x}$. Given $b \in B \cap \overline{N}$ we have

$$\begin{aligned}\psi_t(x^{-1}\psi_{\tilde{t}}(\tilde{x}^{-1}F))(b) &= (x^{-1}\psi_{\tilde{t}}(\tilde{x}^{-1}F))(tbt^{-1}) = \psi_{\tilde{t}}(\tilde{x}^{-1}F)(xtbt^{-1}) \\ &= (\tilde{x}^{-1}F)(\tilde{t}xtbt^{-1}\tilde{t}^{-1}) = F(\tilde{x}\tilde{t}xtbt^{-1}\tilde{t}^{-1}) \quad \text{and} \\ \psi_{\tilde{t}t}(\tilde{t}x^{-1}\tilde{t}^{-1}\tilde{x}^{-1}F)(b) &= (\tilde{t}x^{-1}\tilde{t}^{-1}\tilde{x}^{-1}F)(\tilde{t}tbt^{-1}\tilde{t}^{-1}) = F(\tilde{x}\tilde{t}x\tilde{t}^{-1}\tilde{t}tbt^{-1}\tilde{t}^{-1}) \\ &= F(\tilde{x}\tilde{t}xtbt^{-1}\tilde{t}^{-1}). \qquad \square\end{aligned}$$

In smooth representation theory the interpretation of a parabolically induced representation as the specialization at a Hecke character of a representation which is induced from an Iwahori subgroup, is a well-established technique (cf. [28], §4, or [18], section 3). As we shall show, it admits an analog in the locally analytic setting. Note that by Frobenius reciprocity there is a unique G-equivariant K-linear map

$$\varphi : \text{c-}\mathrm{Ind}_B^G(\mathcal{A}) \longrightarrow \mathrm{Ind}_P^G(\chi) \tag{9}$$

such that $\varphi([g, F]) = gF$ for all $g \in G$ and $F \in \mathcal{A}$. Assuming the group $\mathbb{G}$ to be semisimple and adjoint, we fix once and for all representatives $t_\alpha \in T^-$ of the fundamental antidominant cocharacters in $X_*(\mathbb{T})$ satisfying $\beta(t_\alpha) = 1$ for all $\alpha \in \Delta$ and all $\beta \in \Delta \setminus \{\alpha\}$ (cf. section 1). For $\alpha \in \Delta$ we set

$$y_\alpha := U_{t_\alpha} - \chi(t_\alpha) \in \mathrm{End}_G(\text{c-}\mathrm{Ind}_B^G(\mathcal{A})). \tag{10}$$

Proposition 2.4. *Assume $\mathbb{G}$ to be semisimple and adjoint. The G-equivariant homomorphism φ of (9) is surjective with kernel $\sum_{\alpha \in \Delta} \mathrm{im}(y_\alpha)$.*

Proof: We are going to closely follow the arguments of the proof of [28], §4 Proposition 11. For the surjectivity of φ it suffices to see that the image of φ contains $\mathrm{Ind}_P^G(\chi)(BP)$ because $G = \coprod_{w \in W} BwBP$ and thus

$$\mathrm{Ind}_P^G(\chi) = \sum_{w \in W} Bw \cdot \mathrm{Ind}_P^G(\chi)(BP).$$

Setting $t := \prod_{\alpha \in \Delta} t_\alpha$ the sets $t^m(B_n \cap \overline{N})P$, $m \geq 0$, form a fundamental system of open neighborhoods of P in G/P (cf. (1) and (4), and use the fact that $\mathrm{val}(\beta(t)) > 0$ for all negative roots β). Given $f \in \mathrm{Ind}_P^G(\chi)(BP) \simeq \mathcal{C}^{an}(B \cap \overline{N}, K)$ there is an integer $m \geq 0$ such that the restriction of f to any of the finitely many sets $bt^m(B_n \cap \overline{N})t^{-m}$ with $b \in (B \cap \overline{N})/t^m(B_n \cap \overline{N})t^{-m}$ is rigid analytic. Thus, it suffices to see that the image of φ contains the subspace $\mathcal{O}(t^m(B_n \cap \overline{N})t^{-m})$ of $\mathcal{C}^{an}(B \cap \overline{N}, K)$ for any $m \geq 0$. However, given $G \in \mathcal{O}(t^m(B_n \cap \overline{N})t^{-m})$ we have $F := \chi(t)^m t^{-m} G \in \mathcal{O}(B_n \cap \overline{N}) \subseteq \mathcal{A} \subset \operatorname{im}(\varphi)$ and therefore $G = \chi(t)^{-m} t^m F \in \operatorname{im}(\varphi)$.

To see that $\sum_{\alpha \in \Delta} \operatorname{im}(y_\alpha) \subseteq \ker(\varphi)$ fix $\alpha \in \Delta$ and let x run through a system of coset representatives of $B/(B \cap t_\alpha B t_\alpha^{-1})$. According to (5), (6) and Proposition 1.2 we have

$$BP = Bt_\alpha BP = \coprod_x x t_\alpha BP.$$

Given $f \in \mathcal{A}$ we have $\varphi(U_{t_\alpha}([1, F])) = \sum_x x t_\alpha \psi_{t_\alpha}(x^{-1}F)$, where $x t_\alpha \psi_{t_\alpha}(x^{-1}F)$ has support in $x t_\alpha BP$. Since further

$$(x t_\alpha \psi_{t_\alpha}(x^{-1}F))(x t_\alpha b) = F(x t_\alpha b t_\alpha^{-1}) = \chi(t_\alpha) F(x t_\alpha b)$$

for all $b \in B \cap \overline{N}$, we obtain $\varphi(U_{t_\alpha}([1, F])) = \chi(t_\alpha) F$. This proves the claim because the functions $[1, F]$ with $F \in \mathcal{A}$ generate $\text{c-Ind}_B^G(\mathcal{A})$ as a G-module.

Now set $V := \sum_{\alpha \in \Delta} \operatorname{im}(y_\alpha)$ and let $g \in G$ and $F \in \mathcal{A}$. If $\tilde{t} \in T^-$, $\alpha \in \Delta$ and $b \in B$, then

$$[b\tilde{t}, F] = (\chi(t_\alpha) - U_{t_\alpha})([b\tilde{t}, \chi(t_\alpha)^{-1}F]) + U_{t_\alpha}([b\tilde{t}, \chi(t_\alpha)^{-1}F])$$

is contained in $\text{c-Ind}_B^G(\mathcal{A})(B\tilde{t}t_\alpha B) + V$ (cf. Lemma 1.3 and (8)). Using that any element of T^- can be written as $t_0 \prod_{\alpha \in \Delta} t_\alpha^{n_\alpha}$ with $t_0 \in T_0$ and suitable integers $n_\alpha \geq 0$, we inductively obtain $[1, F] \in \text{c-Ind}_B^G(\mathcal{A})(B\tilde{t}B) + V$ for all $\tilde{t} \in T^-$. Lemma 1.4 then implies that

$$[g, F] \in \text{c-Ind}_B^G(\mathcal{A})(G_0 T^- B) + V$$

for all $g \in G$ and $F \in \mathcal{A}$. Given $t_1 \in T^-$ there is an element $t_2 \in T^-$ such that $t_1 t_2 = t^n$ for some integer $n \geq 0$. One shows as above that if $[g, F]$ is supported on $G_0 t_1 B$ then

$$[g, F] \in \text{c-Ind}_B^G(\mathcal{A})(G_0 t_1 t_2 B) + V = \text{c-Ind}_B^G(\mathcal{A})(G_0 t^n B) + V. \tag{11}$$

Now let $f \in \ker(\varphi)$. If $f = \sum_i [g_i, F_i]$, the above arguments show that there is an integer $n \geq 0$ such that $[g_i, F_i] \in \text{c-Ind}_B^G(\mathcal{A})(G_0 t^n B) + V$ for each i. As a consequence, there is an element $\tilde{f} \in \text{c-Ind}_B^G(\mathcal{A})(G_0 t^n B)$ such that $f - \tilde{f} \in V$. Since $V \subseteq \ker(\varphi)$, we have $\tilde{f} \in \ker(\varphi)$, and it suffices to see that $\tilde{f} = 0$. Writing $\operatorname{supp}(\tilde{f}) = \coprod_i g_i t^n B$ with $g_i \in G_0$, there are elements $F_i \in \mathcal{A}$ such that $\tilde{f} = \sum_i [g_i t^n, F_i]$. Now $0 = \varphi(\tilde{f}) = \sum_i g_i t^n F_i$. Since $\operatorname{supp}(\varphi([g_i t^n, F_i])) \subseteq g_i t^n BP$ and since the subsets $(g_i t^n BP)_i$ of G are pairwise disjoint (cf. Proposition 1.2) we must have $F_i = 0$ for all i. Thus, $\tilde{f} = 0$. $\square$

We continue to assume $\mathbb{G}$ to be semisimple and adjoint. The G-representation $\text{c-Ind}_B^G(\mathcal{A})$ is a module over the polynomial ring

$$R := K[X_\alpha \mid \alpha \in \Delta]$$

by letting X_α act via the endomorphism U_{t_α} (cf. Lemma 2.3). Choose a basis $(e_\alpha)_{\alpha \in \Delta}$ of the K-vector space K^Δ and denote by $\bigwedge^\bullet K^\Delta$ the exterior algebra of K^Δ. Recall [7], X.9.4, that the Koszul complex of c-$\mathrm{Ind}_B^G(\mathcal{A})$ defined by the endomorphisms $(y_\alpha)_{\alpha\in\Delta}$ of (10) is the complex $((\bigwedge^\bullet K^\Delta)\otimes_K \text{c-}\mathrm{Ind}_B^G(\mathcal{A}), d_\bullet)$ with

$$(12)\quad d_q(e_{\alpha_1}\wedge\ldots\wedge e_{\alpha_q}\otimes f)=\sum_{i=1}^q(-1)^{i+1}e_{\alpha_1}\wedge\ldots\wedge\widehat{e_{\alpha_i}}\wedge\ldots\wedge e_{\alpha_q}\otimes y_{\alpha_i}(f).$$

It is concentrated in degrees $0\le q\le|\Delta|$, and the boundary maps d_q are all K-linear and G-equivariant if we let G act diagonally on $(\bigwedge^\bullet K^\Delta)\otimes_K \text{c-}\mathrm{Ind}_B^G(\mathcal{A})$ with its given action on c-$\mathrm{Ind}_B^G(\mathcal{A})$ and the trivial action on $\bigwedge^\bullet K^\Delta$.

Note that the 0-th homology of this complex is c-$\mathrm{Ind}_B^G(\mathcal{A})/\sum_{\alpha\in\Delta}\mathrm{im}(y_\alpha)$ which, by Proposition 2.4, is G-equivariantly isomorphic to $\mathrm{Ind}_P^G(\chi)$.

Theorem 2.5. *Assume $\mathbb{G}$ to be semisimple and adjoint. The augmented G-equivariant complex*

$$(\bigwedge^\bullet K^\Delta)\otimes_K \text{c-}\mathrm{Ind}_B^G(\mathcal{A})\longrightarrow \mathrm{Ind}_P^G(\chi)\longrightarrow 0,$$

with augmentation (9) and boundary maps (12), is exact.

Proof: The assignment $X_\alpha\mapsto X_\alpha-\chi(t_\alpha)$, $\alpha\in\Delta$, defines an automorphism of the ring R so that the sequence $(X_\alpha-\chi(t_\alpha))_{\alpha\in\Delta}$ is a regular sequence of R in any ordering of its elements. By the remark following [7], X.9.1, Corollaire 2, the homology of the above Koszul complex is $\mathrm{Tor}_\bullet^R(R/\mathfrak{m}, \text{c-}\mathrm{Ind}_B^G(\mathcal{A}))$, where $\mathfrak{m}$ denotes the ideal of R generated by the elements $X_\alpha-\chi(t_\alpha)$, $\alpha\in\Delta$. By Proposition 2.4 we need to see that these torsion groups vanish in all positive degrees.

Since $\chi(t_\alpha)\in K^\times$ for all $\alpha\in\Delta$, the reduction map $R\to R/\mathfrak{m}$ extends to a homomorphism $S:=K[X_\alpha^{\pm1}\mid\alpha\in\Delta]\to R/\mathfrak{m}$ of R-algebras, and there are natural isomorphisms

$$(13)\qquad \mathrm{Tor}_q^R(R/\mathfrak{m}, \text{c-}\mathrm{Ind}_B^G(\mathcal{A}))\simeq \mathrm{Tor}_q^S(R/\mathfrak{m}, S\otimes_R \text{c-}\mathrm{Ind}_B^G(\mathcal{A}))$$

for all $q\ge0$ because S is flat over R (cf. [7], X.6.6, Proposition 8). Set

$$G^-:=\bigcup_{t\in T^-}G_0tB\qquad\text{and}\qquad M:=\text{c-}\mathrm{Ind}_B^G(\mathcal{A})(G^-).$$

Note that by Lemma 1.3 and (8) the K-subspace M of c-$\mathrm{Ind}_B^G(\mathcal{A})$ is in fact an R-submodule. Consider the induced injection

$$S\otimes_R M\longrightarrow S\otimes_R \text{c-}\mathrm{Ind}_B^G(\mathcal{A}).$$

According to Lemma 1.3 and Lemma 1.4, given $F\in\mathcal{A}$ and $g\in G$, we can find a monomial $X:=\prod_{\alpha\in\Delta}X_\alpha^{n_\alpha}\in R$ such that $X\cdot[g,F]=(\prod_{\alpha\in\Delta}U_{t_\alpha}^{n_\alpha})([g,F])$ lies in M. It follows that the above inclusion is an isomorphism. By (13) and the corresponding isomorphism for M, the inclusion $M\subset \text{c-}\mathrm{Ind}_B^G(\mathcal{A})$ induces isomorphisms

$$\mathrm{Tor}_q^R(R/\mathfrak{m},M)\simeq\mathrm{Tor}_q^R(R/\mathfrak{m}, \text{c-}\mathrm{Ind}_B^G(\mathcal{A}))$$

for all $q \geq 0$. As above, the left hand side is the homology of the Koszul complex $(\bigwedge^\bullet K^\Delta) \otimes_K M$ defined by the endomorphisms y_α of M, $\alpha \in \Delta$. Therefore, the assertion of the theorem is a consequence of [7], X.9.6, Proposition 5, and Theorem 2.6 below. $\square$

We set $d := |\Delta|$ and fix an arbitrary ordering $\Delta = \{\alpha_1, \ldots, \alpha_d\}$ of Δ. We continue to assume $\mathbb{G}$ to be semisimple and adjoint and note that in this situation

$$G^- = \coprod_{k \in \mathbb{N}} G_k^- \qquad \text{where} \qquad G_k^- := \coprod_{\substack{m \in \mathbb{N}^d \\ |m| = k}} G_0 t_{\alpha_1}^{m_1} \cdot \ldots \cdot t_{\alpha_d}^{m_d} B.$$

This leads to a non-negative grading $M = \text{c-Ind}_B^G(\mathcal{A})(G^-) = \bigoplus_{k \in \mathbb{N}} M_k$ of M where $M_k := \text{c-Ind}_B^G(\mathcal{A})(G_k^-)$ is the subspace of M consisting of all functions whose support is contained in the closed and open subset G_k of G^-. We also endow $R = K[X_{\alpha_1}, \ldots, X_{\alpha_d}]$ with its natural grading in which each X_{α_j} is of degree 1. We note that by Lemma 1.3 and (8) these definitions make M a graded R-module. Given an element $f \in M$ we write

$$f = \sum_{k \geq 0} f_k \text{ with } f_k \in M_k \text{ for any integer } k \geq 0$$

for its decomposition into homogeneous components, almost all of which are zero. In order to ease our notation let us set $U_j := U_{t_{\alpha_j}}$ for all $1 \leq j \leq d$.

Theorem 2.6. *Setting $y_j := U_j - \chi(t_{\alpha_j})$, the sequence $(y_1, \ldots, y_d)$ is regular for $M := \text{c-Ind}_B^G(\mathcal{A})(G^-)$, i.e. if $1 \leq j \leq d$ then the endomorphism of $M / \sum_{i=1}^{j-1} \text{im}(y_i)$, induced by y_j, is injective.*

Proof: Assume $f \in M$ to satisfy $y_j(f) = 0$ for some index j and write $f = \sum_{k \geq 0} f_k$ as the sum of its homogeneous components. As the endomorphisms U_j and $\chi(t_{\alpha_j}) \in K^\times$ of M are graded of degree 1 and 0, respectively, we obtain $f_0 = 0$ and then $f = 0$ by induction. Therefore, y_j is injective.

Now assume $f, f^1, \ldots, f^{j-1}$ to be elements of M such that $y_j(f) = \sum_{i=1}^{j-1} y_i(f^i)$. We need to show that $f \in \sum_{i=1}^{j-1} \text{im}(y_i)$. Consider the K-vector space

$$\hat{M} := \text{Ind}_B^G(\mathcal{A})(G^-) := \{f : G^- \to \mathcal{A} \mid \forall g \in G^- \, \forall b \in B : f(gb) = b^{-1} f(g)\}.$$

We have $M = \oplus_{k \geq 0} M_k \subseteq \hat{M} = \prod_{k \geq 0} M_k$, and the $K[X_\alpha \mid \alpha \in \Delta]$-module structure on M naturally extends to a $K[[X_\alpha \mid \alpha \in \Delta]]$-module structure on $\hat{M}$. Namely, given a power series $\phi = \sum_{n \in \mathbb{N}^d} \lambda_n X_{\alpha_1}^{n_1} \cdots X_{\alpha_d}^{n_d}$ and an element $f = \sum_{k=0}^\infty f_k \in \hat{M}$ with $f_k \in M_k$ for all $k \geq 0$, we set $\phi \cdot f := \sum_{k=0}^\infty g_k$ with

$$g_k := \sum_{i=0}^{k} \sum_{\substack{n \in \mathbb{N}^d \\ n_1 + \ldots + n_d = k - i}} \lambda_n (U_1^{n_1} \circ \ldots \circ U_d^{n_d})(f_i),$$

using that the endomorphisms U_i of M are all of degree 1.

For any index $1 \leq i \leq j-1$ consider the element

$$g^i := -\sum_{k=0}^{\infty} \chi(t_{\alpha_j})^{-(k+1)} U_j^k(f^i) \in \hat{M}.$$

A formal computation shows that $f = \sum_{i=1}^{j-1} y_i(g^i)$ in $\hat{M}$. For any integer $r \geq 0$ let V_r be the closed and open subset of G^- defined by

$$V_r := \coprod_{\substack{m \in \mathbb{N}^d \\ m_j = r}} G_0 t_{\alpha_1}^{m_1} \cdot \ldots \cdot t_{\alpha_d}^{m_d} B,$$

so that $G^- = \coprod_{r \geq 0} V_r$. Any element $g \in \hat{M}$ can be uniquely written as $g = \sum_{r=0}^{\infty} g^{(r)}$ with $g^{(r)} \in \hat{M}$ and $\mathrm{supp}(g^{(r)}) \subseteq V_r$. Note that if $g \in \hat{M}$ is supported in V_r (i.e. if $g = g^{(r)}$) then so is $y_i(g)$ for any $i \neq j$ (cf. Lemma 1.3 and (8)). Consequently, if $i \neq j$ and if $g \in \hat{M}$ is an arbitrary element, then $y_i(g)^{(r)} = y_i(g^{(r)})$.

Since f is compactly supported there is an integer $s \geq 0$ such that $f^{(r)} = 0$ for all $r \geq s$. It follows that

$$\begin{aligned} 0 = \sum_{r \geq s} f^{(r)} &= \sum_{r \geq s} \Big(\sum_{i=1}^{j-1} y_i(g^i)\Big)^{(r)} = \sum_{r \geq s} \sum_{i=1}^{j-1} y_i(g^i)^{(r)} \\ &= \sum_{r \geq s} \sum_{i=1}^{j-1} y_i(g^{i,(r)}) = \sum_{i=1}^{j-1} y_i\Big(\sum_{r \geq s} g^{i,(r)}\Big), \end{aligned}$$

where we used the obvious relation $(g_1 + g_2)^{(r)} = g_1^{(r)} + g_2^{(r)}$ for any two elements $g_1, g_2 \in \hat{M}$. Thus, $f = \sum_{i=1}^{j-1} y_i(\sum_{r<s} g^{i,(r)})$, and it suffices to show that the elements $\sum_{r<s} g^{i,(r)}$ are compactly supported and therefore lie in M. Now for any integer $r \geq 0$ we have

$$\begin{aligned} g^{i,(r)} &= \Big(-\sum_{k=0}^{\infty} \chi(t_{\alpha_j})^{-(k+1)} U_j^k(f^i)\Big)^{(r)} \\ &= \Big(-\sum_{k \leq r} \chi(t_{\alpha_j})^{-(k+1)} U_j^k(f^i)\Big)^{(r)} + \Big(-\sum_{k > r} \chi(t_{\alpha_j})^{-(k+1)} U_j^k(f^i)\Big)^{(r)}, \end{aligned}$$

where $\mathrm{supp}(-\sum_{k>r} \chi(t_{\alpha_j})^{-(k+1)} U_j^k(f^i)) \subseteq \coprod_{k>r} V_k$ by Lemma 1.3 and (8). Therefore, the right summand above is zero. On the other hand, the function $-\sum_{k \leq r} \chi(t_{\alpha_j})^{-(k+1)} U_j^k(f^i)$ is compactly supported, and so is its (r)-component. Indeed, the latter is obtained by restricting to V_r and by extending by zero to all of G^-. $\square$

Remark 2.7. The K-Banach space $\mathcal{A}$ is a B-subrepresentation of the locally analytic B-representation $\mathrm{Ind}_{B \cap P}^{B}(\chi)$, and the K-linear endomorphism ψ_t of $\mathcal{A}$ is constructed as the restriction of a K-linear endomorphism of $\mathrm{Ind}_{B \cap P}^{B}(\chi)$. The proof of Lemma 2.2 shows that likewise the G-endomorphism U_t of c-$\mathrm{Ind}_B^G(\mathcal{A})$ is in fact the restriction of a G-endomorphism of c-$\mathrm{Ind}_B^G(\mathrm{Ind}_{B \cap P}^{B}(\chi))$. As above, one can therefore construct a G-equivariant complex

$$(14) \qquad \Big(\bigwedge^{\bullet} K^{\Delta}\Big) \otimes_K \text{c-}\mathrm{Ind}_B^G(\mathrm{Ind}_{B \cap P}^{B}(\chi)) \longrightarrow \mathrm{Ind}_P^G(\chi) \longrightarrow 0,$$

which is independent of the choice of n and hence of $\mathcal{A}$. The proofs of Proposition 2.4, Theorem 2.5 and Theorem 2.6 can all be copied word by word to show that the augmented complex (14) is exact, as well. As we shall see, both (12) and (14) provide resolutions of the G-representation $\mathrm{Ind}_P^G(\chi)$ which are acyclic for any discrete cocompact subgroup Γ of G. Working with the K-Banach space $\mathcal{A}$ has the great advantage, however, of leading to rather simple criteria concerning the Γ-acyclicity of $\mathrm{Ind}_P^G(\chi)$. This is why we have chosen to present the less canonical construction involving $\mathcal{A}$.

Although we shall not need it in the sequel, we would like to record the following result.

Lemma 2.8. *Assume $\mathbb{G}$ to be semisimple and adjoint.*

(i) The family $(U_{t_\alpha})_{\alpha\in\Delta}$ of G-endomorphisms of $\text{c-Ind}_B^G(\mathcal{A})$ is algebraically independent over K, i.e. the ring homomorphism $K[X_\alpha \mid \alpha \in \Delta] \to \mathrm{End}_G(\text{c-Ind}_B^G(\mathcal{A}))$, sending the formal variable X_α to U_{t_α}, is injective. Its image is the K-subalgebra generated by the family U_t, $t \in T^-$.

(ii) Assume the restriction of χ to $B \cap P$ to be rigid analytic, and let $\alpha \in \Delta$. Choosing $n = 0$ in the definition of $\mathcal{A}$, the endomorphism U_{t_α} of $M := \text{c-Ind}_B^G(\mathcal{A})(G^-)$ is injective.

Proof: Choose a non-zero element $F \in K \subseteq \mathcal{O}(B\cap\overline{N}) \subseteq \mathcal{A}$. It has the property that $\psi_t(x^{-1}F) \neq 0$ for all $t \in T^-$ and all $x \in B \cap \overline{N}$. Given $m, n \in \mathbb{N}^d$ with $m \neq n$, we have $Bt_{\alpha_1}^{m_1} \cdot \ldots \cdot t_{\alpha_d}^{m_d} B \cap B t_{\alpha_1}^{n_1} \cdot \ldots \cdot t_{\alpha_d}^{n_d} B = \emptyset$. Thus, if f is a polynomial in the variables X_α such that $f(U_{t_{\alpha_1}}, \ldots, U_{t_{\alpha_d}}) = 0$ then evaluation at $[1, F]$ implies $f = 0$ by Lemma 2.3 and (8). This proves the first assertion of (i). The second assertion follows from Lemma 2.3 and the fact that T^- is generated by the elements t_α, $\alpha \in \Delta$, and $t \in T_0$ (note that $U_t = \chi(t)$ if $t \in T_0$).

As for (ii), let $\alpha \in \Delta$ and set $U := U_{t_\alpha}$. If f is an element of M such that $U(f) = 0$ then $U(f_k) = 0$ for any k. Therefore, we may assume f to be homogeneous. By Lemma 1.3 and the first assertion of Lemma 1.4 we may even assume f to be supported on a single double coset G_0tB for some $t \in T^-$. Write $G_0tB = \coprod_y yBtB$ with $y \in G_0$ and choose elements $f^y \in \text{c-Ind}_B^G(\mathcal{A})(yBtB)$ such that $f = \sum_y f^y$. We have $\mathrm{supp}(U(f)) \subseteq \cup_y yBtt_\alpha B$, where the union is disjoint by Proposition 1.2. Since $\mathrm{supp}(U(f^y)) \subseteq yBtt_\alpha B$, we obtain $U(f^y) = 0$ for all y and may, without loss of generality, assume $\mathrm{supp}(f) \subseteq BtB$ for some $t \in T^-$. Let x run through a set of coset representatives of $(B\cap\overline{N})/t(B\cap\overline{N})t^{-1}$, so that $BtB = \coprod_x xtB$. Choosing elements $F_x \in \mathcal{A}$ with $f = \sum_x [xt, F_x]$, we have $\mathrm{supp}(U([xt, F_x])) \subseteq xtBt_\alpha B$, where $xtBt_\alpha B \cap \tilde{x}tBt_\alpha B = \emptyset$ for $x \neq \tilde{x}$ (cf. Lemma 1.3). This implies $U([xt, F_x]) = 0$ for all x, and we may assume $f = [1, F]$ for some $F \in \mathcal{A}$. However,

$$0 = U([1,F]) = \sum_{Bt_\alpha B = \coprod_x xt_\alpha B} [xt_\alpha, \psi_{t_\alpha}(x^{-1}F)]$$

implies $\psi_{t_\alpha}(x^{-1}F) = 0$ and thus $F = 0$ by the injectivity of $\psi_{t_\alpha} : \mathcal{O}(B\cap\overline{N}) \to \mathcal{O}(B\cap\overline{N})$. Note that it is only for the injectivity of ψ_{t_α} that we need $n = 0$. $\square$

3 Vanishing and finiteness theorems

Let $\mathbb{G}$ again be an arbitrary L-split connected reductive group defined over L. We endow $\mathcal{A}$ with the topology of a K-Banach space via the isomorphism

$$\mathcal{A} \simeq \prod_{b\in(B\cap\overline{N})/(B_n\cap\overline{N})} \mathcal{O}(b(B_n\cap\overline{N})),$$

where the reduced K-affinoid algebras $\mathcal{O}(b(B_n\cap\overline{N}))$ carry their spectral norms $||\cdot||$. We endow $\text{c-Ind}_B^G(\mathcal{A})$ with the topology defined by the corresponding maximum norm, denoted by $||\cdot||$ again, i.e. $||f|| := \sup_{g\in G}||f(g)||$. The action of G on $\text{c-Ind}_B^G(\mathcal{A})$ is then by K-linear isometries.

Let Γ be a discrete and cocompact subgroup of G (i.e. Γ is discrete and the quotient $\Gamma\backslash G$ is compact). We endow the K-vector space

$$\mathrm{H}_0(\Gamma, \text{c-Ind}_B^G(\mathcal{A})) = \text{c-Ind}_B^G(\mathcal{A})_\Gamma$$

of Γ-coinvariants of $\text{c-Ind}_B^G(\mathcal{A})$ with the quotient topology of $\text{c-Ind}_B^G(\mathcal{A})$. It is defined by the quotient seminorm coming from the maximum norm on $\text{c-Ind}_B^G(\mathcal{A})$. Note that if $t\in T^-$ then the G-endomorphism U_t of $\text{c-Ind}_B^G(\mathcal{A})$, defined in section 2, gives rise to a K-linear endomorphism of $\text{c-Ind}_B^G(\mathcal{A})_\Gamma$. We again denote it by U_t.

Proposition 3.1. *Let Γ be a discrete and cocompact subgroup of G. The quotient topology makes $\text{c-Ind}_B^G(\mathcal{A})_\Gamma$ a K-Banach space. If $t\in T^-$ then the K-linear endomorphism U_t of $\text{c-Ind}_B^G(\mathcal{A})$ is continuous. The operator norm of U_t on $\text{c-Ind}_B^G(\mathcal{A})_\Gamma$ with respect to the quotient norm is bounded above by* 1.

Proof: According to the arguments given in [5], page 237, Γ possesses a normal and torsion free subgroup Γ' of finite index. The group Γ' acts freely on G/B, and the set $\Gamma'\backslash G/B$ is finite. Choosing a set of double coset representatives $\{g_1,\ldots,g_r\}$ of the latter, the complete subspace $\text{c-Ind}_B^G(\mathcal{A})(\cup_i g_iB)$ of $\text{c-Ind}_B^G(\mathcal{A})$ admits the closed complement $\text{c-Ind}_B^G(\mathcal{A})(G\setminus\cup_i g_iB)$ and maps isomorphically onto $\text{c-Ind}_B^G(\mathcal{A})_{\Gamma'}$. Therefore, the quotient topology makes $\text{c-Ind}_B^G(\mathcal{A})_{\Gamma'}$ a K-Banach space on which the finite group Γ/Γ' acts by continuous K-linear endomorphisms. The closed subspace of Γ/Γ'-invariants is the image of the continuous K-linear endomorphism $\sum_{\gamma\in\Gamma/\Gamma'}\gamma$ and maps isomorphically onto the space $\text{c-Ind}_B^G(\mathcal{A})_\Gamma$. Therefore, the latter is a K-Banach space, as well.

Using (8) it is easy to see that the maximum norm on $\text{c-Ind}_B^G(\mathcal{A})$ makes U_t a continuous endomorphism whose operator norm is bounded above by 1. Note that the translation action of B on $\mathcal{A}$ is by isometries and that the endomorphism ψ_t of $\mathcal{A}$ is norm decreasing. The last statement of the theorem is then clear. $\square$

Theorem 3.2. *Let $\mathbb{G}$ be semisimple and adjoint, let Γ be a discrete and cocompact subgroup of G, and let χ be a locally analytic K-valued character of T. If $q>\dim(T)$ then $H_q(\Gamma, \mathrm{Ind}_P^G(\chi))=0$. If there is an element $\beta\in\Delta$ such that $\mathrm{val}(\chi(t_\beta))<0$ then $\mathrm{H}_q(\Gamma,\mathrm{Ind}_P^G(\chi))=0$ for all $q\geq 0$.*

Proof: Consider the augmented Koszul complex of Theorem 2.5 of length $|\Delta| = \dim(T)$ which is associated to the family of commuting endomorphisms $U_{t_\alpha} - \chi(t_\alpha)$, $\alpha \in \Delta$, of c-$\mathrm{Ind}_B^G(\mathcal{A})$. By [loc.cit.] and our assumption on Γ, it is an exact and Γ-equivariant resolution of $\mathrm{Ind}_P^G(\chi)$ by homologically acyclic Γ-modules. Indeed, Γ possesses a normal and torsion free subgroup Γ' of finite index (cf. [5], page 237), and the Γ'-module c-$\mathrm{Ind}_B^G(\mathcal{A})$ is even free over $K[\Gamma']$. Since we are working over a field of characteristic zero, we have $\mathrm{H}_q(\Gamma, \text{c-}\mathrm{Ind}_B^G(\mathcal{A})) \simeq \mathrm{H}_q(\Gamma', \text{c-}\mathrm{Ind}_B^G(\mathcal{A}))_{\Gamma/\Gamma'}$, and the claim follows.

In particular, $\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi))$ is the homology of the complex obtained by passing to the Γ-coinvariants of the Koszul complex in Theorem 2.5, thus proving the first statement. The latter complex in turn is the Koszul complex of c-$\mathrm{Ind}_B^G(\mathcal{A})_\Gamma$ associated with the family of commuting endomorphisms induced by the operators $U_{t_\alpha} - \chi(t_\alpha)$. It follows from Proposition 3.1 and [6], 1.2.4 Proposition 4, that the endomorphism $U_{t_\beta} - \chi(t_\beta)$ of c-$\mathrm{Ind}_B^G(\mathcal{A})_\Gamma$ is bijective if $\mathrm{val}(\chi(t_\beta)) < 0$. For trivial reasons, the sequence of endomorphisms $(U_{t_\beta} - \chi(t_\beta), (U_{t_\alpha} - \chi(t_\alpha))_{\alpha \in \Delta \setminus \{\beta\}})$ is then regular for c-$\mathrm{Ind}_B^G(\mathcal{A})_\Gamma$. Therefore, the statement of the theorem follows from [7], X.9.6, Proposition 5, and the fact that

$$\mathrm{Ind}_P^G(\chi)_\Gamma \simeq \text{c-}\mathrm{Ind}_B^G(\mathcal{A})_\Gamma / \sum_{\alpha \in \Delta} \mathrm{im}(U_{t_\alpha} - \chi(t_\alpha)) = 0$$

by the right exactness of $\mathrm{H}_0(\Gamma, \cdot)$. □

Remark 3.3. Since $\mathfrak{o}_K^\times$ contains all bounded subgroups of $K^\times$, the continuity of χ implies that $\chi(T_0) \subseteq \mathfrak{o}_K^\times$. Therefore, the map $(t \mapsto \mathrm{val}(\chi(t))) : T \to \mathrm{val}(K^\times)$ factors through T/T_0. It corresponds to an element $s(\chi)$ of $X^*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ which is called *the slope of* χ in [14], section 1.4. With this terminology, the condition on χ in Theorem 3.2 can be rephrased by requiring the slope of χ to have a positive contribution from at least one positive simple root when writing it as a real linear combination of the elements of Δ.

In a particular case we shall now significantly generalize the result of Theorem 3.2 and clarify the phenomena that lie behind it.

Given $\alpha \in \Delta$, we denote by $\mathbb{P}_{\Delta\setminus\{\alpha\}}$ the standard parabolic subgroup of $\mathbb{G}$ corresponding to the subset $\Delta \setminus \{\alpha\}$ of Δ and by $\mathbb{M}_{\Delta\setminus\{\alpha\}}$ its Levi subgroup containing $\mathbb{T}$ (cf. section 1). Let χ be a locally analytic K-valued character of $M_{\Delta\setminus\{\alpha\}}$, viewed also as a character of $P_{\Delta\setminus\{\alpha\}}$ and – via restriction – of P.

We consider the B-invariant subspace $\mathrm{Ind}_{B\cap P_{\Delta\setminus\{\alpha\}}}^B(\chi)$ of $\mathrm{Ind}_{B\cap P}^B(\chi)$. Choose $n \geq 0$ such that the restriction of χ to any coset in $(B \cap P_{\Delta\setminus\{\alpha\}})/(B_n \cap P_{\Delta\setminus\{\alpha\}})$ is rigid analytic. As in section 2, $\mathrm{Ind}_{B\cap P_{\Delta\setminus\{\alpha\}}}^B(\chi)$ contains the B-invariant subspace

$$\mathcal{A}_\alpha \simeq \prod_{b \in (B\cap \overline{N}_{\Delta\setminus\{\alpha\}})/(B_n \cap \overline{N}_{\Delta\setminus\{\alpha\}})} \mathcal{O}(b(B \cap \overline{N}_{\Delta\setminus\{\alpha\}}))$$

of all those functions whose restrictions to $B \cap \overline{N}_{\Delta\setminus\{\alpha\}}$ are rigid analytic on any coset modulo $(B_n \cap \overline{N}_{\Delta\setminus\{\alpha\}})$ (or equivalently, which are rigid analytic on any coset in B/B_n). We use the same integer n to define the subspace $\mathcal{A}$ of

$\mathrm{Ind}_{B\cap P}^{B}(\chi)$ so that $\mathcal{A}_\alpha = \mathcal{A} \cap \mathrm{Ind}_{B\cap P_{\Delta\setminus\{\alpha\}}}^{B}(\chi)$.

Note that the restriction of χ to the subgroup $\overline{N} \cap P_{\Delta\setminus\{\alpha\}}$ of $P_{\Delta\setminus\{\alpha\}}$ is trivial. Indeed, as in the proof of Proposition 2.4 there is an element $t \in T^- \subseteq M_{\Delta\setminus\{\alpha\}}$ such that the sequence $(t^m \overline{n} t^{-m})_{m\geq 0}$ of elements of $\overline{N}$ tends to 1 for any $\overline{n} \in \overline{N}$. Assuming $\overline{n} \in \overline{N} \cap P_{\Delta\setminus\{\alpha\}}$, we have $\chi(\overline{n}) = \chi(t^m \overline{n} t^{-m})$ for all integers $m \geq 0$, and the continuity of χ implies $\chi(\overline{n}) = 1$ as claimed.

Note further that the multiplication map $\overline{\mathbb{N}}_{\Delta\setminus\{\alpha\}} \times (\overline{\mathbb{N}} \cap \mathbb{P}_{\Delta\setminus\{\alpha\}}) \xrightarrow{\simeq} \overline{\mathbb{N}}$ is an isomorphism, as follows from [19], Theorem 30.1.b. By (3) it induces isomorphisms

$$(15)\quad (B_m \cap \overline{N}_{\Delta\setminus\{\alpha\}}) \times (B_m \cap \overline{N}_{(\alpha)}) \xrightarrow{\simeq} (B_m \cap \overline{N}) \text{ with } \overline{N}_{(\alpha)} := \overline{N} \cap P_{\Delta\setminus\{\alpha\}}$$

for any integer $m \geq 0$. We obtain that $\mathcal{A}_\alpha$ is the subspace of $(B \cap \overline{N}_{(\alpha)})$-right invariant functions of $\mathcal{A}$.

The group B_n is normal in B, and $\overline{N}_{\Delta\setminus\{\alpha\}}$ is normalized by $\overline{P}_{\Delta\setminus\{\alpha\}} \supset \overline{N}$. It follows that $B_n \cap \overline{N}_{\Delta\setminus\{\alpha\}}$ is normal in $B \cap \overline{N}$. Thus, given $b = b_1 b_2 \in B \cap \overline{N}$ with $b_1 \in B \cap \overline{N}_{\Delta\setminus\{\alpha\}}$ and $b_2 \in B \cap \overline{N}_{(\alpha)}$ we have

$$b(B_n \cap \overline{N}) = b_1(B_n \cap \overline{N}_{\Delta\setminus\{\alpha\}}) b_2 (B_n \cap \overline{N}_{(\alpha)}).$$

In other words, any coset in $(B \cap \overline{N})/(B_n \cap \overline{N})$ is a product of a coset in $(B \cap \overline{N}_{\Delta\setminus\{\alpha\}})/(B_n \cap \overline{N}_{\Delta\setminus\{\alpha\}})$ and of a coset in $(B \cap \overline{N}_{(\alpha)})/(B_n \cap \overline{N}_{(\alpha)})$. Letting b and $\tilde{b}$ run through representatives of the last two quotients, we conclude that the set of products $b\tilde{b}$ runs through a set of representatives of the first. Denoting by $\mathcal{B}_\alpha$ the K-Banach space of all functions in $\mathcal{C}^{an}(B \cap \overline{N}_{(\alpha)}, K)$ which are rigid analytic on any coset modulo $(B_n \cap \overline{N}_{(\alpha)})$, the above reasoning shows that there is a topological isomorphism

$$(16)\qquad \mathcal{A}_\alpha \widehat{\otimes}_K \mathcal{B}_\alpha \xrightarrow{\simeq} \mathcal{A}$$

with $(F \otimes G)(b\tilde{b}) = F(b)G(\tilde{b})$ for all $b \in B \cap \overline{N}_{\Delta\setminus\{\alpha\}}$ and $\tilde{b} \in B \cap \overline{N}_{(\alpha)}$. Under this isomorphism the inclusions $\mathcal{A}_\alpha \subseteq \mathcal{A}$ and $\mathcal{B}_\alpha \subseteq \mathcal{A}$ correspond to the maps $\mathcal{A}_\alpha \to \mathcal{A}_\alpha \widehat{\otimes}_K \mathcal{B}_\alpha$ and $\mathcal{B}_\alpha \to \mathcal{A}_\alpha \widehat{\otimes}_K \mathcal{B}_\alpha$ sending $F \in \mathcal{A}_\alpha$ to $F \otimes 1$ and $G \in \mathcal{B}_\alpha$ to $1 \otimes G$, respectively.

It follows from (1) and (4) that $B \cap \overline{N}_{(\alpha)}$ is stable under conjugation by any element $t \in T^-$. In particular, the endomorphism ψ_t of $\mathcal{A}$ restricts to an endomorphism of the B-invariant subspace $\mathcal{A}_\alpha$ of $\mathcal{A}$ consisting of $(B \cap \overline{N}_{(\alpha)})$-right invariant functions. It follows that the G-equivariant endomorphism U_t of c-$\mathrm{Ind}_B^G(\mathcal{A})$ restricts to a G-equivariant endomorphism of c-$\mathrm{Ind}_B^G(\mathcal{A}_\alpha)$, denoted by U_t again.

Let Γ continue to be a discrete and cocompact subgroup of G. As in Proposition 3.1, c-$\mathrm{Ind}_B^G(\mathcal{A}_\alpha)_\Gamma$ is naturally a K-Banach space and the operator induced by U_t is continuous. If $\mathbb{G}$ is semisimple and adjoint and if $t = t_\alpha$, however, we have the following more precise result.

Proposition 3.4. *Let $\mathbb{G}$ be semisimple and adjoint, and let Γ be a discrete and cocompact subgroup of G. Let $\alpha \in \Delta$ be a positive simple root, and let χ be a locally analytic K-valued character of $M_{\Delta\setminus\{\alpha\}}$. The continuous K-linear endomorphism U_{t_α} of the K-Banach space* $\text{c-Ind}_B^G(\mathcal{A}_\alpha)_\Gamma$ *is compact, i.e. it is the strong limit of continuous K-linear operators of finite rank.*

Proof: Using (8) and the fact that the K-subspace of compact endomorphisms of a K-Banach space forms an ideal inside the K-algebra of all continuous K-linear endomorphisms (cf. [27], Remark 16.7 (i)), it suffices to see that the K-linear continuous endomorphism ψ_{t_α} of the K-Banach space $\mathcal{A}_\alpha$ is compact. By the product structure of $\mathcal{A}_\alpha$ one is further reduced to showing that the restriction homomorphism $\mathcal{O}(t_\alpha(B_n \cap \overline{N}_{\Delta\setminus\{\alpha\}})t_\alpha^{-1}) \to \mathcal{O}(B_n \cap \overline{N}_{\Delta\setminus\{\alpha\}})$ is compact.

Any root β such that the root subgroup N_β appears in the product decomposition (3) of $B_n \cap \overline{N}_{\Delta\setminus\{\alpha\}}$ lies in $\Phi(\Delta \setminus \{\alpha\})^-$, i.e. has a negative contribution from α. Therefore, $\mathrm{val}(\beta(t_\alpha)) > 0$ for any such root, and the restriction of ψ_{t_α} to a continuous K-linear endomorphism of $\mathcal{O}(B_n \cap N_\beta)$ is compact (cf. (1), (4) and [27], proof of the claim on page 98). Since

$$\mathcal{O}(B_n \cap \overline{N}_{\Delta\setminus\{\alpha\}}) \simeq \widehat{\bigotimes}_{\beta\in\Phi(\Delta\setminus\{\alpha\})^-} \mathcal{O}(B_n \cap N_\beta),$$

the claim follows from [27], Lemma 18.12. □

Remark 3.5. The endomorphism of $\text{c-Ind}_B^G(\mathcal{A})_\Gamma$ induced by U_{t_α} is generally not compact. The reason is that the continuous K-linear map $\psi_{t_\alpha} : \mathcal{A} \to \mathcal{A}$ is generally not compact because $t_\alpha(B_n \cap \overline{N})t_\alpha^{-1}$ is generally not a relatively compact affinoid subdomain of $B_n \cap \overline{N}$. More precisely, by our choice of t_α, if $\beta \in \Phi^-$ has a trivial contribution from α then conjugation by t_α is the identity on N_β. It follows that ψ_{t_α} restricts to the identity on $\mathcal{B}_\alpha$. This fact will be used crucially in Lemma 3.6 below.

Let $\alpha \in \Delta$ and $\chi : M_{\Delta\setminus\{\alpha\}} \to K^\times$ be as in Proposition 3.4. We denote by

$$\zeta_{\Gamma,t_\alpha,\chi}(T) := \det(\mathrm{id} - T \cdot U_{t_\alpha} | \text{c-Ind}_B^G(\mathcal{A}_\alpha)_\Gamma)$$

the *characteristic power series* (or *Fredholm determinant*) of the compact operator U_{t_α} on the K-Banach space $\text{c-Ind}_B^G(\mathcal{A}_\alpha)_\Gamma$ (cf. [33], section 5), and point out that in spite of our notation it certainly depends on the K-Banach space $\mathcal{A}_\alpha$ and hence on the integer n. It is a power series with coefficients in K with an infinite radius of convergence and which has the property that $\zeta_{\Gamma,t_\alpha,\chi}(\lambda) \neq 0$ for $\lambda \in K$ if and only if $\mathrm{id} - \lambda U_{t_\alpha}$ is bijective on $\text{c-Ind}_B^G(\mathcal{A}_\alpha)_\Gamma$ (cf. [33], Proposition 7 and Proposition 11). We shall now see that in this case the operator $\mathrm{id} - \lambda U_{t_\alpha}$ is even bijective on the larger space $\text{c-Ind}_B^G(\mathcal{A})_\Gamma$, provided Γ is torsion free.

Lemma 3.6. *Let $\Gamma \subset G$, $\alpha \in \Delta$ and $\chi : M_{\Delta\setminus\{\alpha\}} \to K^\times$ be as in Proposition 3.4, and assume Γ to be torsion free. If $\lambda \in K$ is not a zero of $\zeta_{\Gamma,t_\alpha,\chi}(T)$ then the endomorphism* $\mathrm{id} - \lambda U_{t_\alpha}$ *of* $\text{c-Ind}_B^G(\mathcal{A})_\Gamma$ *is bijective.*

Proof: As mentioned in Remark 3.5, our choice of t_α ensures that conjugation by t_α is the identity on $B \cap \overline{N}_{(\alpha)}$. First of all, this implies that the natural map

$$(B \cap \overline{N}_{\Delta\setminus\{\alpha\}})/t_\alpha(B \cap \overline{N}_{\Delta\setminus\{\alpha\}})t_\alpha^{-1} \xrightarrow{\simeq} (B \cap \overline{N})/t_\alpha(B \cap \overline{N})t_\alpha^{-1} \tag{17}$$

is bijective. Further, since Γ is torsion free, (16) gives rise to a K-linear topological isomorphism

$$\text{c-Ind}_B^G(\mathcal{A})_\Gamma \simeq \text{c-Ind}_B^G(\mathcal{A}_\alpha)_\Gamma \widehat{\otimes}_K \mathcal{B}_\alpha. \tag{18}$$

Indeed, as in the proof of Proposition 3.1 there are elements $g_1, \ldots, g_r \in G$ such that the projections $\text{c-Ind}_B^G(\mathcal{A}) \to \text{c-Ind}_B^G(\mathcal{A})_\Gamma$ and $\text{c-Ind}_B^G(\mathcal{A}_\alpha) \to \text{c-Ind}_B^G(\mathcal{A}_\alpha)_\Gamma$ restrict to isomorphisms

$$\text{c-Ind}_B^G(\mathcal{A})(\cup_i g_i B) \simeq \text{c-Ind}_B^G(\mathcal{A})_\Gamma \text{ and } \text{c-Ind}_B^G(\mathcal{A}_\alpha)(\cup_i g_i B) \simeq \text{c-Ind}_B^G(\mathcal{A}_\alpha)_\Gamma.$$

Since $\text{c-Ind}_B^G(\mathcal{A})(\cup_i g_i B) \simeq \text{c-Ind}_B^G(\mathcal{A}_\alpha)(\cup_i g_i B) \widehat{\otimes}_K \mathcal{B}_\alpha$ by (16), the claim follows.

Note that if $x \in B \cap \overline{N}_{\Delta\setminus\{\alpha\}}$ then under (16), the endomorphism $\psi_{t_\alpha} \circ x^{-1}$ of $\mathcal{A}$ is given by $\psi_{t_\alpha}(x^{-1}(F \otimes G)) = \psi_{t_\alpha}(x^{-1}F) \otimes \psi_{t_\alpha}(G) = \psi_{t_\alpha}(x^{-1}F) \otimes G$ for $F \in \mathcal{A}_\alpha$ and $G \in \mathcal{B}_\alpha$. It follows from (8) and the bijectivity of (17) that under (18) the endomorphism U_{t_α} of $\text{c-Ind}_B^G(\mathcal{A})_\Gamma$ is given by $(U_{t_\alpha} | \text{c-Ind}_B^G(\mathcal{A}_\alpha)_\Gamma) \widehat{\otimes} \mathrm{id}$. Therefore, the endomorphism $\mathrm{id} - \lambda U_{t_\alpha}$ of $\text{c-Ind}_B^G(\mathcal{A})$ is given by

$$(\mathrm{id} - \lambda U_{t_\alpha} | \text{c-Ind}_B^G(\mathcal{A}_\alpha)_\Gamma) \widehat{\otimes} \mathrm{id},$$

which is bijective if $\zeta_{\Gamma, t_\alpha, \chi}(\lambda) \neq 0$. □

Note that the set of zeros of $\zeta_{\Gamma, t_\alpha, \chi}(T)$ is a discrete subset of K because the power series in question is entire. Also, it is disjoint from $\{\lambda \in K \mid \mathrm{val}(\lambda) > 0\}$. Indeed, the arguments of the proof of Proposition 3.1 show that on $\text{c-Ind}_B^G(\mathcal{A}_\alpha)_\Gamma$, the norm of the operator U_{t_α} is bounded above by 1. Therefore, the claim follows from [6], 1.2.4 Proposition 4. These considerations show that if the character χ of T is the restriction of a locally analytic character of $M_{\Delta\setminus\{\alpha\}}$ for some positive simple root $\alpha \in \Delta$, then the following theorem gives a significant sharpening of the vanishing result of Theorem 3.2. Using Lemma 3.6, its proof proceeds as above.

Theorem 3.7. *Let $\mathbb{G}$ be semisimple and adjoint, and let Γ be a discrete, torsion free and cocompact subgroup of G. Let $\alpha \in \Delta$ be a positive simple root, and let χ be a locally analytic K-valued character of $M_{\Delta\setminus\{\alpha\}}$, viewed as a locally analytic character of T via restriction. If $\chi(t_\alpha)^{-1}$ is not a zero of $\zeta_{\Gamma, t_\alpha, \chi}(T)$ then $\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi)) = 0$ for all $q \geq 0$.* □

Remark 3.8. If Γ is a discrete and cocompact subgroup of G then Γ possesses a normal subgroup Γ' of finite index which is torsion free, as follows from the arguments in [5], page 237. Since we are working over a field of characteristic zero, there are natural equivalences $\mathrm{H}_\bullet(\Gamma, \cdot) \simeq \mathrm{H}_\bullet(\Gamma', \cdot)_{\Gamma/\Gamma'}$ of δ-functors, so that the homological triviality with respect to Γ' of any Γ-module implies its homological triviality with respect to Γ. Thus, Theorem 3.7 leads to a vanishing criterion for the Γ-homology of certain locally analytic principal series representations even if Γ is not necessarily torsion free.

A far reaching consequence of the above compactness arguments is given by the following two theorems.

Theorem 3.9. *Let $\mathbb{G}$ be semisimple and adjoint of rank d, and let Γ be a discrete and cocompact subgroup of G. If χ is a locally analytic K-valued character of T then the following assertions hold.*

(i) For each $q \geq 0$, the K-vector space $\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi))$ is finite dimensional.

(ii) Let $q \geq 0$. Endowing $\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi))$ with its natural topology of a finite dimensional K-vector space, there is a natural isomorphism

$$\mathrm{H}^q(\Gamma, \mathrm{Ind}_P^G(\chi)') \simeq \mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi))'.$$

(iii) We have $\sum_{q=0}^{d}(-1)^q \dim_K \mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi)) = 0$.

Proof: Setting $t = \prod_{\alpha\in\Delta} t_\alpha$, the operator U_t is a compact operator of the K-Banach space c-$\mathrm{Ind}_B^G(\mathcal{A})_\Gamma$. This can be shown by the same arguments as in the proof of Proposition 3.4. Namely, it suffices to see that the K-linear continuous endomorphism ψ_t of the K-Banach space $\mathcal{A}$ is compact. However, $\mathcal{A} \simeq \prod_{b\in(B\cap\overline{N})/(B_n\cap\overline{N})} \mathcal{O}(b(B_n\cap\overline{N}))$, and for any $b \in B$ the map ψ_t restricts to a map $\psi_t : \mathcal{O}(tbt^{-1}(B_n\cap\overline{N})) \to \mathcal{O}(b(B_n\cap\overline{N}))$. This map factors through the restriction map $\mathcal{O}(tbt^{-1}(B_n\cap\overline{N})) \to \mathcal{O}(tbt^{-1}\cdot t(B_n\cap\overline{N})t^{-1})$, induced by the open immersion $t(B_n\cap\overline{N})t^{-1} \subseteq (B_n\cap\overline{N})$ of rigid affinoid spaces. By construction and our choice of t we have $\mathrm{val}(\beta(t)) > 0$ for any negative root $\beta \in \Phi^-$. This implies that $t(B_n\cap N_\beta)t^{-1}$ is relatively compact in $(B_n\cap N_\beta)$ in the sense of [6], 9.6.2, and we may conclude as above.

If $y := U_t - \chi(t)$ then by [27], Proposition 22.8, there exists an integer $m \geq 0$ such that we have a decomposition

$$\text{c-}\mathrm{Ind}_B^G(\mathcal{A})_\Gamma = \ker(y^m) \oplus \mathrm{im}(y^m).$$

Moreover, y is a topological automorphism of $\mathrm{im}(y^m)$ and $\ker(y^m)$ is finite dimensional (cf. [27], Lemma 22.4). As the endomorphisms $y_\alpha := U_{t_\alpha} - \chi(t_\alpha)$ commute with y, both $\ker(y^m)$ and $\mathrm{im}(y^m)$ are stable under all y_α. Therefore, the Koszul complex of c-$\mathrm{Ind}_B^G(\mathcal{A})_\Gamma$ is the direct sum of the Koszul complexes of $\ker(y^m)$ and $\mathrm{im}(y^m)$. In particular,

$$\mathrm{Tor}_q^R(R/\mathfrak{m}, \text{c-}\mathrm{Ind}_B^G(\mathcal{A})_\Gamma) = \mathrm{Tor}_q^R(R/\mathfrak{m}, \ker(y^m)) \oplus \mathrm{Tor}_q^R(R/\mathfrak{m}, \mathrm{im}(y^m)),$$

where, as before, $R = K[X_\alpha,\, \alpha\in\Delta]$ and $\mathfrak{m}$ is the ideal generated by the elements $X_\alpha - \chi(t_\alpha)$ with $\alpha \in \Delta$.

As y is induced by an element of $\mathfrak{m}$, it acts trivially on $\mathrm{Tor}_q^R(R/\mathfrak{m}, \mathrm{im}(y^m))$. On the other hand, it is bijective on $\mathrm{im}(y^m)$ and hence on $\mathrm{Tor}_q^R(R/\mathfrak{m}, \mathrm{im}(y^m))$. Thus, $\mathrm{Tor}_q^R(R/\mathfrak{m}, \mathrm{im}(y^m)) = 0$ for any $q \geq 0$. Now $\ker(y^m)$ is finite dimensional and the Koszul complex of $\ker(y^m)$ is a complex of finite dimensional K-vector spaces. This implies that $\mathrm{Tor}_q^R(R/\mathfrak{m}, \ker(y^m))$ is finite dimensional for each q, thus proving (i).

Assertion (ii) is a consequence of (i) and the arguments given in the proof of

[32], Théorème 3.15. Finally,

$$\begin{aligned}
\sum_{q=0}^{d}(-1)^q \dim_K \mathrm{H}_q(\Gamma, \text{c-Ind}_B^G(\mathcal{A})_\Gamma) &= \sum_{q=0}^{d}(-1)^q \dim_K \mathrm{Tor}_q^R(R/\mathfrak{m}, \ker(y^m)) \\
&= \sum_{q=0}^{d}(-1)^q \dim_K \ker(y^m)^{\binom{d}{q}} \\
&= (\dim_K \ker(y^m)) \sum_{q=0}^{d}(-1)^q \binom{d}{q} \\
&= 0,
\end{aligned}$$

the second equality resulting from the dimension formula for K-linear maps between finite dimensional K-vector spaces. $\square$

The techniques of the above proof also lead to the following, *conditional vanishing criterion* for $\mathrm{H}_\bullet(\Gamma, \mathrm{Ind}_P^G(\chi))$.

Theorem 3.10. *Under the hypotheses of Theorem 3.9, the following assertions are equivalent.*

(i) For each $q \geq 0$ *we have* $\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi)) = 0$.

(ii) We have $\mathrm{H}_d(\Gamma, \mathrm{Ind}_P^G(\chi)) = 0$.

(iii) We have $\mathrm{H}_0(\Gamma, \mathrm{Ind}_P^G(\chi)) = 0$.

Proof: With the notations of the proof of Theorem 3.9, $H_q(\Gamma, \mathrm{Ind}_P^G(\chi))$ is the q-th homology space of the Koszul complex corresponding to the family $(y_\alpha)_{\alpha\in\Delta}$ acting on the finite dimensional K-vector space $\ker(y^m)$. It is clear that (i) implies both (ii) and (iii).

Let us denote by y_α the restriction of y_α to $\ker(y^m)$. The K-vector space $\ker(y^m)$ is finite dimensional so that for each $\alpha \in \Delta$ there is an integer $m_\alpha \geq 0$ such that $\ker(y^m)$ admits the y_α-stable decomposition

$$\ker(y^m) = \ker(y_\alpha^{m_\alpha}) \oplus \mathrm{im}(y_\alpha^{m_\alpha}).$$

Since the endomorphisms y_α of $\ker(y^m)$ pairwise commute and since the projections of $\ker(y^m)$ onto $\ker(y_\alpha^{m_\alpha})$ and $\mathrm{im}(y_\alpha^{m_\alpha})$ are polynomials in y_α, we obtain $\ker(y^m) = \bigoplus_{I\subset\Delta} \ker(y^m)_I$ with

$$\ker(y^m)_I = \bigcap_{\alpha\in I} \ker(y_\alpha^{m_\alpha}) \cap \bigcap_{\alpha\notin I} \mathrm{im}(y_\alpha^{m_\alpha}).$$

Each member of this direct sum is stable under y_α for all $\alpha \in \Delta$, so that the Koszul complex of $\ker(y^m)$ is the direct sum of the Koszul complexes of the spaces $\ker(y^m)_I$. Moreover, if $\alpha \in \Delta$ then y_α is bijective on $\mathrm{im}(y_\alpha^{m_\alpha})$. As $\ker(y^m)$ is finite dimensional, this implies that y_α is bijective on every subspace of $\mathrm{im}(y_\alpha^m)$ which is stable under y_α. In particular, y_α is invertible on $\ker(y^m)_I$ if $\alpha \notin I$. As a consequence, the Koszul complex of $\ker(y^m)_I$ is acyclic if $I \neq \Delta$, and the homology of the Koszul complex of $\ker(y^m)_\Delta$ is exactly $H_\bullet(\Gamma, \mathrm{Ind}_P^G(\chi))$.

The y_α are commuting nilpotent operators on $\ker(y^m)_\Delta$ so that their common kernel is trivial if and only if $\ker(y^m)_\Delta = 0$. By our above reasoning, the former is precisely $\mathrm{H}_d(\Gamma, \mathrm{Ind}_P^G(\chi))$, so that (ii) implies $\ker(y^m)_\Delta = 0$, which in turn implies (i).

Now suppose that $H_0(\Gamma, \mathrm{Ind}_P^G(\chi)) = 0$. This implies that if λ is a linear form on $\ker(y^m)_\Delta$ such that $\lambda \circ y_\alpha = 0$ for all $\alpha \in \Delta$, then $\lambda = 0$. In other words, the operators y_α^* on $\ker(y^m)_\Delta^*$ have no common zero. However, they commute with each other and are nilpotent, so that $\ker(y^m)_\Delta^* = 0$ and thus $\ker(y^m)_\Delta = 0$. This implies (i). $\square$

As we shall now explain, Theorem 3.2, Theorem 3.7, Theorem 3.9 and Theorem 3.10 lead to homological vanishing and finiteness theorems for principal series representations of arbitrary L-split connected reductive groups over L. Given such a group $\mathbb{G}$ let $\mathbb{G}^{ad}$ be its maximal semisimple adjoint quotient (cf. [3], section 24.1). According to [3], Proposition 11.14, given a maximal L-split torus $\mathbb{T}$ of $\mathbb{G}$ (resp. a Borel subgroup $\mathbb{P}$ of $\mathbb{G}$ containing $\mathbb{T}$) the image $\mathbb{T}^{ad}$ of $\mathbb{T}$ in $\mathbb{G}^{ad}$ (resp. the image $\mathbb{P}^{ad}$ of $\mathbb{P}$ in $\mathbb{G}^{ad}$) is a maximal L-split torus of $\mathbb{G}^{ad}$ (resp. a Borel subgroup of $\mathbb{G}^{ad}$ containing $\mathbb{T}^{ad}$). If Γ is a discrete and cocompact subgroup of G such that the image Γ^{ad} of Γ in G^{ad} is discrete then Γ^{ad} is a discrete and cocompact subgroup of G^{ad}. This follows from the fact that the image of the continuous homomorphism $G \to G^{ad}$ is open of finite index (cf. [4], Corollaire 3.20).

Proposition 3.11. *Let $\mathbb{G}$ be an L-split connected reductive group defined over L, and let $\mathbb{G}^{ad}$ be its maximal semisimple adjoint quotient. Let Γ be a discrete and cocompact subgroup of G whose image Γ^{ad} in G^{ad} is discrete (hence is discrete and cocompact). Let χ be a locally analytic K-valued character of T^{ad}, viewed as a character of T via the homomorphism $T \to T^{ad}$.*

(i) If the intersection of Γ with the center of G is finite then there are isomorphisms $\mathrm{H}_q(\Gamma^{ad}, \mathrm{Ind}_{P^{ad}}^{G^{ad}}(\chi)) \simeq \mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi))$ for all $q \geq 0$.

(ii) If $\mathrm{H}_q(\Gamma^{ad}, \mathrm{Ind}_{P^{ad}}^{G^{ad}}(\chi)) = 0$ for all $q \geq 0$ then $\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi)) = 0$ for all $q \geq 0$.

Proof: The kernels of the homomorphisms $\mathbb{G} \to \mathbb{G}^{ad}$ and $G \to G^{ad}$ are central by definition of a central isogeny and [3], Proposition 11.21. Therefore, the natural map $\mathbb{G}/\mathbb{P} \to \mathbb{G}^{ad}/\mathbb{P}^{ad}$ is an isomorphism of complete varieties and it follows from [3], Proposition 20.5, that the natural map

$$\mathrm{Ind}_{P^{ad}}^{G^{ad}}(\chi) \xrightarrow{\simeq} \mathrm{Ind}_P^G(\chi) \tag{19}$$

is bijective. Let Γ' be the kernel of the surjective homomorphism $\Gamma \to \Gamma^{ad}$. Since Γ' is central in G it is contained in the kernel of the homomorphism $T \to T^{ad}$. It follows that Γ' acts trivially on $\mathrm{Ind}_P^G(\chi)$ and that the action of Γ factors through Γ^{ad}. Therefore, the initial terms of the Hochschild-Serre spectral sequence

$$\mathrm{H}_p(\Gamma^{ad}, \mathrm{H}_q(\Gamma', \mathrm{Ind}_P^G(\chi))) \Longrightarrow \mathrm{H}_{p+q}(\Gamma, \mathrm{Ind}_P^G(\chi))$$

satisfy

$$\mathrm{H}_p(\Gamma^{ad}, \mathrm{H}_q(\Gamma', \mathrm{Ind}_P^G(\chi)) \simeq \mathrm{H}_q(\Gamma', K) \otimes_K \mathrm{H}_p(\Gamma^{ad}, \mathrm{Ind}_P^G(\chi)),$$

where $\mathrm{H}_p(\Gamma^{ad}, \mathrm{Ind}_P^G(\chi)) \simeq \mathrm{H}_p(\Gamma^{ad}, \mathrm{Ind}_{P^{ad}}^{G^{ad}}(\chi))$ by (19). This proves assertion (ii). Under assumption (i) the group Γ' is finite. Therefore, the spectral sequence degenerates and gives (i). $\square$

Remark 3.12. If $\mathbb{G}$ is semisimple and if $\mathbb{G}^{ad}$ is its maximal adjoint quotient then the image in G^{ad} of any discrete subgroup Γ of G is again discrete. This follows from the fact that in this case the kernel of the homomorphism $G \to G^{ad}$ is finite.

Combining Theorem 3.2 and Proposition 3.11 we obtain the following vanishing result for a large class of algebraic characters.

Theorem 3.13. *Let $\mathbb{G}$ be an L-split connected reductive group and let Γ be a discrete and cocompact subgroup of G whose image in the group of L-rational points of the maximal semisimple adjoint quotient of $\mathbb{G}$ is discrete. If the algebraic character $\chi \in X^*(\mathbb{T})$ is contained in the root lattice of $\Phi(\mathbb{G}, \mathbb{T})$ with a positive contribution from at least one positive simple root $\beta \in \Delta$ then*

$$\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi)) = 0$$

for all $q \geq 0$.

Proof: $\mathbb{G}$ and its maximal semisimple adjoint quotient have the same root systems so that the hypotheses of Proposition 3.11 are satisfied. We may thus assume $\mathbb{G}$ to be semisimple and adjoint. Writing $\chi = \sum_{\alpha \in \Delta} m_\alpha \alpha$, we have $m_\beta > 0$ by assumption. By our choice of $t_\beta \in T^-$ the element $\chi(t_\beta) = \beta(t_\beta)^{m_\beta}$ of K has negative valuation. Thus, the assertion follows from Theorem 3.2. $\square$

Unfortunately, the previous theorem does not cover the important case of algebraic characters which are contained in the subset $\sum_{\alpha \in \Delta} \mathbb{Z}_{\leq 0} \cdot \alpha$ of $X^*(\mathbb{T})$. For characters of the form $\chi = m_\alpha \alpha$ with an integer $m_\alpha \leq 0$, one might hope to apply the stronger vanishing result of Theorem 3.7. However, we have practically no information on the set of zeros of the power series $\zeta_{\Gamma, t_\alpha, \chi}(T)$ apart from its disjointness from the open unit ball in K. Using a technique of M. Strauch and S. Orlik we can show, however, that the above vanishing theorem has at least one possible exception.

Theorem 3.14. *Let $\mathbb{G}$ and Γ be as in Theorem 3.13. Let $\mathbb{1}$ be the trivial character of T, and let $\mathrm{Ind}_P^G(\mathbb{1})^\infty$ be the K-vector space of all locally constant K-valued P-right invariant functions on G, endowed with the action of G by left translation. The G-equivariant inclusion $\mathrm{Ind}_P^G(\mathbb{1})^\infty \to \mathrm{Ind}_P^G(\mathbb{1})$ induces isomorphisms*

$$\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\mathbb{1})^\infty) \simeq \mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\mathbb{1})) \tag{20}$$

for all $q \geq 0$. We have $\dim_K \mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\mathbb{1})) = \binom{\dim(T)}{q} \cdot \mu(\Gamma)$, where the constant $\mu(\Gamma)$ is as in [28], page 92 (see also (22) below). In particular, if $\mu(\Gamma) \neq 0$, if $\mathbb{G}$ is semisimple and adjoint, and if Γ is torsion free, then $\zeta_{\Gamma, t_\alpha, \mathbb{1}}(1) = 0$ for all $\alpha \in \Delta$.

Proof: Let ℓ denote the length function on W with respect to Δ and consider the weight $\delta := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ of the root system $\Phi = \Phi(\mathbb{G},\mathbb{T})$. The group W acts on $X^*(\mathbb{T})$ via $w(\chi)(t) := \chi(w^{-1}tw)$. We consider the so-called *affine action* of W on $X^*(\mathbb{T})$ given by $w*\chi := w(\chi+\delta)-\delta$, where the addition takes place in the weight lattice of the root system Φ. If $\chi \in X^*(\mathbb{T})$ is a dominant character, i.e. if $\langle\chi,\check{\alpha}\rangle \geq 0$ for all $\alpha\in\Delta$, and if $V(\chi)$ denotes the finite dimensional algebraic representation of $\mathbb{G}$ of highest weight χ, then there is a G-equivariant exact sequence

$$0 \longrightarrow V(\chi)' \otimes_K \mathrm{Ind}_P^G(\mathbb{1})^\infty \longrightarrow \bigoplus_{\substack{w\in W\\ \ell(w)=\bullet}} \mathrm{Ind}_P^G((w*\chi)^{-1}) \longrightarrow 0 \tag{21}$$

of length $\ell(w_0)+1$ (cf. [25], section 4.9). For $0 \leq q \leq \ell(w_0)$ the above direct sum is taken over all elements $w\in W$ of length q, so that for $q=0$ we simply obtain $\mathrm{Ind}_P^G(\chi^{-1})$. Associated with this exact resolution is a spectral sequence

$$E_2^{pq} := \mathrm{H}_p(\bigoplus_{\substack{w\in W\\ \ell(w)=\bullet}} \mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G((w*\chi)^{-1}))) \Longrightarrow \mathrm{H}_{p+q}(\Gamma, V(\chi)\otimes_K \mathrm{Ind}_P^G(\mathbb{1})^\infty).$$

If $w \in W\setminus\{1\}$ then the character $(w*\mathbb{1})^{-1} = \delta - w(\delta)$ is non-trivial and is a non-negative linear combination of positive simple roots (cf. [8], VI.1.6 Proposition 18). Thus, if $\chi = \mathbb{1}$ then Theorem 3.13 implies that the above spectral sequence degenerates. Since $V(\mathbb{1}) = \mathbb{1}$, this proves (20).

In order to compute the Γ-homology of $\mathrm{Ind}_P^G(\mathbb{1})^\infty$ we are going to closely follow the strategy of [28], §5, propositions 5 to 8, making strong use of results of Borel and Casselman concerning smooth representations of p-adic reductive groups.

Note first that as in the proof of [26], §1 Lemma 3, we have

$$\mathrm{Hom}_K(\mathrm{H}_\bullet(\Gamma, \mathrm{Ind}_P^G(\mathbb{1})^\infty), K) \simeq \mathrm{Ext}^\bullet_{K[\Gamma]}(\mathrm{Ind}_P^G(\mathbb{1})^\infty, \mathbb{1}).$$

The G-representation $\mathrm{Ind}_P^G(\mathbb{1})^\infty$ admits a resolution by finitely generated free $K[\Gamma]$-modules (cf. [28], §6 Proposition 16, for the case $\mathbb{G} = \mathbb{GL}_{d+1}$, and the generalized arguments of [29], Theorem II.3.1; one can also use (21) and Theorem 2.5). As in [26], §1 Lemma 4, this implies that we may replace K by the field $\mathbb{C}$ of complex numbers everywhere. If $\mathrm{Ext}^\bullet_H(\cdot,\cdot)$ denotes the bifunctor of extensions in the category of smooth complex representations of a locally compact totally disconnected group H then Shapiro's lemma implies

$$\mathrm{Ext}^\bullet_{K[\Gamma]}(\mathrm{Ind}_P^G(\mathbb{1})^\infty, \mathbb{1}) \simeq \mathrm{Ext}^\bullet_G(\mathrm{Ind}_P^G(\mathbb{1})^\infty, \mathrm{Ind}_\Gamma^G(\mathbb{1})^\infty)$$

(cf. [5], Proposition IX.2.3). By the arguments given in [28], page 88, the representation $\mathrm{Ind}_\Gamma^G(\mathbb{1})^\infty$ decomposes into a direct sum

$$\mathrm{Ind}_\Gamma^G(\mathbb{1})^\infty \simeq V_0 \oplus V_1 \oplus \ldots \oplus V_m$$

of admissible unitary G-representations V_j such that $V_0^B = 0$, and such that for $1 \leq j \leq m$ the G-representation V_j is irreducible with $V_j^B \neq 0$. By [11], Theorem 1.1, the irreducible composition factors of $\mathrm{Ind}_P^G(\mathbb{1})^\infty$ are the so-called *generalized Steinberg representations* $v_J = v^G_{P_J}$, parametrized by the subsets $J \subseteq \Delta$. As in [28], §5 Proposition 5, 6 and 7, we obtain

$$\mathrm{Ext}^\bullet_G(\mathrm{Ind}^G_P(\mathbb{1})^\infty, \mathrm{Ind}^G_\Gamma(\mathbb{1})^\infty) \simeq \mathrm{Ext}^\bullet_G(\mathrm{Ind}^G_P(\mathbb{1})^\infty, \mathbb{1}) \oplus [\mathrm{Hom}_G(v_\emptyset, \mathrm{Ind}^G_\Gamma(\mathbb{1})^\infty) \otimes_{\mathbb{C}} \mathrm{Ext}^\bullet_G(\mathrm{Ind}^G_P(\mathbb{1})^\infty, v_\emptyset)].$$

Denote by $\delta_P : P \to \mathbb{C}^\times$ the modulus character of P. Since the Steinberg representation $v_\emptyset$ is selfdual, [11], Proposition A.11, [12], Theorem 2.4.2, and Shapiro's lemma imply

$$\begin{aligned} \mathrm{Ext}^\bullet_G(\mathrm{Ind}^G_P(\mathbb{1})^\infty, v_\emptyset) &\simeq \mathrm{Ext}^\bullet_G(v_\emptyset, \mathrm{Ind}^G_P(\delta_P)^\infty) \\ &\simeq \mathrm{Ext}^\bullet_P(v_\emptyset, \delta_P) \\ &\simeq \mathrm{Ext}^\bullet_T((v_\emptyset)_N, \delta_P). \end{aligned}$$

According to [11], Corollary 1.3, we have $(v_\emptyset)_N \simeq \delta_P$ as a T-representation where $\dim_{\mathbb{C}}(\mathrm{Ext}^q_T(\delta_P, \delta_P)) = \binom{\dim(T)}{q}$ for all $q \geq 0$ (cf. the arguments of [5], Proposition X.2.6).

Similarly, $\mathrm{Ext}^\bullet_G(\mathrm{Ind}^G_P(\mathbb{1})^\infty, \mathbb{1}) \simeq \mathrm{Ext}^\bullet_T(\mathbb{1}, \delta_P) = 0$ by the same argument as in [5], Theorem I.4.1 (cf. the proof in [5], I.4.5). We obtain that

$$\dim_K \mathrm{H}_q(\Gamma, \mathrm{Ind}^G_P(\mathbb{1})) = \dim_K \mathrm{H}_q(\Gamma, \mathrm{Ind}^G_P(\mathbb{1})^\infty) = \binom{\dim(T)}{q} \cdot \mu(\Gamma)$$

for all $q \geq 0$, where

$$\mu(\Gamma) := \dim_{\mathbb{C}} \mathrm{Hom}_G(v_\emptyset, \mathrm{Ind}^G_\Gamma(\mathbb{1})^\infty) \tag{22}$$

is the multiplicity of the Steinberg representation $v_\emptyset$ in $\mathrm{Ind}^G_\Gamma(\mathbb{1})^\infty$. The final assertion of the theorem follows from Theorem 3.7. □

Remark 3.15. There is one easy case in which we can also treat the Γ-homology of principal series representations associated with antidominant weights. Namely, if $\mathbb{G}$ has semisimple rank 1 and if Γ is as in Theorem 3.14, then

$$\mathrm{H}_q(\Gamma, \mathrm{Ind}^G_P(\chi^{-1})) \simeq \mathrm{H}_q(\Gamma, V(\chi)' \otimes_K \mathrm{Ind}^G_P(\mathbb{1})^\infty)$$

for all $q \geq 0$ and all dominant characters $\chi \in X^*(\mathbb{T})$ which are contained in the root lattice of Φ. Indeed, in this case the resolution (21) is just a short exact sequence in which $\mathrm{Ind}^G_P((w_0 * \chi)^{-1})$ has vanishing Γ-homology in all degrees. As in the proof of Theorem 3.14 this follows from Theorem 3.13 and the fact the $(w_0 * \chi)^{-1}$ is a strictly dominant weight. As above, the vanishing of $\mathrm{H}_\bullet(\Gamma, \mathrm{Ind}^G_P(\chi^{-1}))$ can be related to the vanishing of the constant $\mu(\Gamma, V(\chi))$ of [26], §1. We leave the details to the reader.

We shall now show how the above vanishing results for locally analytic principal series representations lead to vanishing theorems for the much broader class of locally analytic representations considered by S. Orlik and M. Strauch in [25]. We note that according to [25], section 7, these representations figure prominently in the (topological dual of) global sections of equivariant vector bundles on Drinfeld's upper half space if $G = \mathrm{PGL}_{d+1}(L)$ for some integer $d \geq 1$ (see also our section 4 below).

In order to introduce these representations let $\mathbb{G}$ again be an arbitrary L-split connected reductive group. If $I \subseteq \Delta$ is a subset and if $\mathfrak{p}_I$ denotes the Lie algebra of the standard parabolic subgroup $\mathbb{P}_I$ of $\mathbb{G}$, then we denote by $\mathcal{O}_{alg}^{\mathfrak{p}_I}$ the category of $U(\mathfrak{g})$-modules introduced in [25], section 2.2. It is a subcategory of the category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand. If $I = \emptyset$, for example, then the category $\mathcal{O}_{alg}^{\mathfrak{p}}$ contains all *Verma modules*

$$\mathfrak{m}(\chi) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \chi \quad \text{for} \quad \chi \in X^*(\mathbb{T}),$$

as well as their simple quotients. If V is a K-vector space carrying an admissible smooth representation of the Levi subgroup M_I of P_I, and if $\mathfrak{m}$ is an object of $\mathcal{O}_{alg}^{\mathfrak{p}_I}$, then Orlik and Strauch construct a locally analytic representation $\mathcal{F}_{P_I}^G(\mathfrak{m}, V)$ of G over K which is cut out from a parabolically induced representation by certain differential equations (cf. [25], section 4). According to [25], Proposition 4.6, the assignment $(\mathfrak{m}, V) \mapsto \mathcal{F}_{P_I}^G(\mathfrak{m}, V)$ is a bifunctor which is exact in both arguments. If $\chi \in X^*(\mathbb{T})$ and if ψ is a smooth K-valued character of T, for example, then

$$\mathcal{F}_P^G(\mathfrak{m}(\chi), \psi) = \mathrm{Ind}_P^G(\chi^{-1}\psi). \tag{23}$$

Theorem 3.16. *Assume $\mathbb{G}$ and Γ to be as in Proposition 3.11. Let $\mathfrak{m}$ be a simple object of the category $\mathcal{O}_{alg}^{\mathfrak{p}}$ whose highest weight χ is contained in the root lattice of $\Phi(\mathbb{G}, \mathbb{T})$, and let ψ be a smooth K-valued character of T^{ad}. If the slope $s(\chi^{-1}\psi)$ of the character $\chi^{-1}\psi$ in the sense of Remark 3.3 has a positive contribution from at least one positive simple root $\beta \in \Delta$ then*

$$\mathrm{H}_q(\Gamma, \mathcal{F}_P^G(\mathfrak{m}, \psi)) = 0$$

for all $q \geq 0$.

Proof: According to [20], Proposition 1.15 (b) and 1.16 (3), the isomorphism class of $\mathfrak{m}$ is contained in the subgroup of the Grothendieck group of $\mathcal{O}_{alg}^{\mathfrak{p}}$ which is generated by the Verma modules $\mathfrak{m}(w * \chi)$ with $w \in W$ and $\chi \geq w * \chi$. The exactness of the functor $\mathcal{F}_P^G(\cdot, \psi)$ and the long exact homology sequence reduce us to showing that $\mathrm{H}_q(\Gamma, \mathcal{F}_P^G(\mathfrak{m}(w * \chi), \psi)) = 0$ for all $q \geq 0$ if $w \in W$ satisfies $\chi \geq w * \chi$. With the notation of Remark 3.3 we have

$$s((w * \chi)^{-1}\psi) = s(\psi) - w * \chi \geq s(\psi) - \chi = s(\chi^{-1}\psi).$$

Hence, (23), Proposition 3.11, Theorem 3.2 and Remark 3.3 allow us to conclude. $\square$

If I is a subset of Δ then we denote by W_I the corresponding subgroup of W and by w_I^0 the longest element of W_I. If $J \subseteq I$ then we denote by $v_{P_J}^{P_I}$ the generalized smooth Steinberg representation of M_I over K corresponding to J, i.e.

$$v_{P_J}^{P_I} := \mathrm{Ind}_{M_I \cap P_J}^{M_I}(\mathbb{1})^\infty / (\sum_{J \subsetneq H \subseteq I} \mathrm{Ind}_{M_I \cap P_H}^{M_I}(\mathbb{1})^\infty).$$

Corollary 3.17. *Assume $\mathbb{G}$ and Γ to be as in Proposition 3.11. Let I be a subset of Δ, and let $\mathfrak{m}$ be a simple object of the category $\mathcal{O}_{alg}^{\mathfrak{p}_I}$ whose highest weight χ is contained in the root lattice of $\Phi(\mathbb{G},\mathbb{T})$. If the character $\chi^{-1}(w_I^0 * \mathbb{1})^{[L:\mathbb{Q}_p]}$ has a positive contribution from at least one positive simple root $\beta \in \Delta$ then*

$$\mathrm{H}_q(\Gamma, \mathcal{F}_{P_I}^G(\mathfrak{m}, v_{P_J}^{P_I})) = 0$$

for all $q \geq 0$ and all subsets $J \subseteq I$.

Proof: As above, let $\delta_P : P \to K^\times$ be the modulus character of P. If $t \in T^-$ then

$$\begin{aligned}
\delta_P(t) &= \frac{(t(B\cap P)t^{-1} : t(B\cap P)t^{-1} \cap (B \cap P))}{(B\cap P : t(B\cap P)t^{-1}\cap(B\cap P))} = (t(N\cap B)t^{-1} : N \cap B) \\
&= \prod_{\alpha\in\Phi^+} (\alpha(t)\mathfrak{o}_L : \mathfrak{o}_L) = (\mathfrak{o}_L : \pi\mathfrak{o}_L)^{-\sum_{\alpha\in\Phi^+}\mathrm{val}(\alpha(t))}.
\end{aligned}$$

In the notation of Remark 3.3 this implies $s(\delta_P) = -2[L:\mathbb{Q}_p]\cdot\delta$ because the element $(\mathfrak{o}_L : \pi\mathfrak{o}_L) \in \mathbb{Z} \subset K$ has valuation $[L:\mathbb{Q}_p]$. According to [8], VI.1.10 Proposition 29, $\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ is an integral weight so that $w(\delta)-\delta$ is contained the root lattice of Φ for each $w \in W$. The character $\delta_P w(\delta_P^{-1})$ admits the square root $\delta_P^{1/2} w(\delta_P^{-1/2}) : P \to K^\times$ defined by

$$\delta_P^{1/2} w(\delta_P^{-1/2})(tn) = (\mathfrak{o}_L : \pi\mathfrak{o}_L)^{-\mathrm{val}((\delta - w(\delta))(t))}$$

for all $t \in T$ and all $n \in N$. We then have $[L:\mathbb{Q}_p]^{-1} s(\delta_P^{-1/2} w(\delta_P^{1/2})) = -w(\delta) + \delta = -w * \mathbb{1}$ for all $w \in W$. Given $w \in W_I$ we have $w \leq w_I^0$ for the Bruhat ordering and $w(\delta) \geq w_I^0(\delta)$ (cf. [20], §5.2), so that

$$\begin{aligned}
s(\chi^{-1}\delta_P^{1/2} w(\delta_P^{-1/2})) &= -s(\chi) + [L:\mathbb{Q}_p](-\delta + w(\delta)) \\
&\geq -s(\chi) + [L:\mathbb{Q}_p](-\delta + w_I^0(\delta)) = s(\chi^{-1}(w_I^0 * \mathbb{1})^{[L:\mathbb{Q}_p]}).
\end{aligned}$$

Since $\mathfrak{m}$ is also a simple object of the category $\mathcal{O}_{alg}^{\mathfrak{p}}$ with highest weight χ, we can apply Theorem 3.16 with $\psi := \delta_P^{1/2} w(\delta_P^{-1/2})$ and obtain

$$\mathrm{H}_q(\Gamma, \mathcal{F}_P^G(\mathfrak{m}, \delta_P^{1/2} w(\delta_P^{-1/2}))) = 0$$

for all $q \geq 0$. By [25], Proposition 4.6 (b), this implies

$$\mathrm{H}_q(\Gamma, \mathcal{F}_{P_I}^G(\mathfrak{m}, \mathrm{Ind}_P^{P_I}(\delta_P^{1/2} w(\delta_P^{-1/2}))^\infty)) = 0 \tag{24}$$

for all $q \geq 0$.

The restriction of δ_P from P to $M_I \cap P$ is the modulus character δ_{P_I} of $M_I \cap P$. Therefore, restriction from P_I to M_I induces a P_I-equivariant isomorphism

$$\mathrm{Ind}_P^{P_I}(\delta_P^{1/2} w(\delta_P^{-1/2}))^\infty \simeq \mathrm{Ind}_{M_I\cap P}^{M_I}(\delta_{P_I}^{1/2} w(\delta_{P_I}^{-1/2}))^\infty =: I_w$$

for any element w of the Weyl group W_I of the reductive group $\mathbb{M}_I$; here N_I acts trivially on I_w. It follows from [5], X.3.2, that each I_w admits a unique irreducible quotient J_w and that I_w admits a Jordan-Hölder series whose irreducible subquotients are among the representations $(J_{w'})_{w' \in W_I}$. For any element $w \in W_I$ let Q_w be the kernel of the surjection $I_w \to J_w$.

We are going to prove by induction on q that $\mathcal{F}^G_{P_I}(\mathfrak{m}, J_w)$ and $\mathcal{F}^G_{P_I}(\mathfrak{m}, Q_w)$ have vanishing Γ-homology in degree q. For $q = 0$ and any of the representations J_w this follows from (24) (with $q = 0$) and the right exactness of the functor $\mathrm{H}_0(\Gamma, \mathcal{F}^G_{P_I}(\mathfrak{m}, \cdot))$. But then it also follows for $q = 0$ and any of the representations Q_w because the irreducible constituents of Q_w are among those of I_w and hence among the representations $J_{w'}$ with $w' \in W_I$. Using (24), the induction step is achieved by considering the long exact homology sequence.

According to [11], Theorem 1.1, the representations $v^{P_I}_{P_J}$ are the Jordan-Hölder constituents of I_1. This completes the proof of the corollary. $\square$

4 Applications to p-adic symmetric spaces

In this final section of our article we take up some of the problems concerning the de Rham cohomology of local systems on p-adic symmetric spaces which were identified by P. Schneider in [26]. For this we assume $K = L$.

Let $d \geq 1$ be an integer, and let X be Drinfeld's p-adic upper half space of dimension d over K. Recall that X is the open rigid analytic subvariety of the projective space $\mathbb{P}^d_K$ obtained by removing all K-rational hyperplanes. It carries an action of the group $\mathrm{PGL}_{d+1}(K)$ and hence of $G := \mathrm{SL}_{d+1}(K)$.

We fix a discrete and cocompact subgroup Γ of G which acts without fixed points on X. Let M denote the underlying K-vector space of an irreducible algebraic representation of $\mathbb{G} := \mathbb{SL}_{d+1}$. It gives rise to a locally constant sheaf $\mathcal{M}_\Gamma$ on the étale site of the quotient variety $X_\Gamma := \Gamma\backslash X$. As in [26] we define the de Rham cohomology $\mathrm{H}^\bullet_{dR}(X_\Gamma, \mathcal{M}_\Gamma)$ of $\mathcal{M}_\Gamma$ to be the étale hypercohomology of the complex $\Omega^\bullet_{X_\Gamma} \otimes_K \mathcal{M}_\Gamma$. The dimensions of these K-vector spaces were computed in [26], §1 Corollary 6, by considering the *covering spectral sequence*

$$E_2^{pq} := \mathrm{H}^p(\Gamma, \mathrm{H}^q_{dR}(X) \otimes_K M) \Longrightarrow \mathrm{H}^{p+q}_{dR}(X_\Gamma, \mathcal{M}_\Gamma). \tag{25}$$

We let $F^\bullet_\Gamma$ be the filtration it induces on $\mathrm{H}^d_{dR}(X_\Gamma, \mathcal{M}_\Gamma)$. §1 Theorem 2 of [26] gives the K-dimensions of the graded pieces of the filtration $F^\bullet_\Gamma$ and asserts that the spectral sequence (25) degenerates.

According to [28], §1 Proposition 4, the variety X is a Stein space, so that one of the standard hypercohomology spectral sequences can be rewritten as

$$E_1^{pq} := H^q(\Gamma, \Omega^p_X(X) \otimes_K M) \Longrightarrow \mathrm{H}^{p+q}_{dR}(X_\Gamma, \mathcal{M}_\Gamma) \tag{26}$$

and is called the *Hodge-de Rham spectral sequence*. It does generally not degenerate in E_1 (cf. [26], page 649). We denote by $F^\bullet_{dR}$ the filtration of $\mathrm{H}^d_{dR}(X_\Gamma, \mathcal{M}_\Gamma)$ induced by the spectral sequence (26).

We let $\mathbb{P}$ denote the Borel subgroup of $\mathbb{G} = \mathbb{SL}_{d+1}$ of upper triangular matrices and choose for $\mathbb{T}$ the subgroup of diagonal matrices. In the notation of Example

1.1 we then have $\Delta = \{\alpha_1, \ldots, \alpha_d\}$. Set $I := \{\alpha_2, \ldots, \alpha_d\}$. If V is a finite dimensional algebraic representation of $\mathbb{P}_I$ over K then we denote by $\mathcal{F}_V$ the G-equivariant vector bundle (as well as its analytification) on $\mathbb{P}^d_K \simeq \mathbb{G}/\mathbb{P}_I$ whose fiber in $[\mathbb{P}_I]$ is V (cf. [22], I.5.8). We denote by

$$X^*(\mathbb{T})_I^- := \{\mu \in X^*(\mathbb{T}) \mid \langle \mu, \check{\alpha} \rangle \leq 0 \text{ for all } \alpha \in I\}$$

the subset of characters of $\mathbb{T}$ whose restriction to $\mathbb{M}_I$ is antidominant. Given $\mu \in X^*(\mathbb{T})_I^-$ we denote by V_μ the irreducible algebraic representation of $\mathbb{M}_I$ of lowest weight μ. We view V_μ as a representation of $\mathbb{P}_I$ via the projection $\mathbb{P}_I \to \mathbb{P}_I/\mathbb{N}_I \simeq \mathbb{M}_I$ and set $\mathcal{F}_\mu := \mathcal{F}_{V_\mu}$, as well as $D_\mu := \mathcal{F}_\mu(X)$.

Denoting by W_I the subgroup of W corresponding to I, the coset space $W_I \setminus W$ has cardinality $d+1$ and admits a unique set of representatives $\{c_j\}_{0 \leq j \leq d}$ such that c_j is of minimal length in $W_I c_j$. The representatives c_j are characterized by the condition that $\ell(s_\alpha c_j) > \ell(c_j)$ for any simple reflection $s_\alpha \in W$ with $\alpha \in I$. As before, we set $\delta := \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$. Given $w \in W$ and a weight λ of $\Phi(\mathbb{G}, \mathbb{T})$ we set $w \bar{*} \lambda := w(\lambda - \delta) + \delta$. Denoting by $\lambda \in X^*(\mathbb{T})$ the lowest weight of the $\mathbb{G}$-representation M we set $\lambda(j) := c_j \bar{*} \lambda$ for $0 \leq j \leq d$. For any j this is a character of $\mathbb{T}$ whose restriction to $\mathbb{M}_I$ is antidominant, i.e. $\lambda(j) \in X^*(\mathbb{T})_I^-$, as follows from the above characterization of the representatives c_j.

In [26], §3, P. Schneider constructs G-equivariant maps $d^j : D_{\lambda(j)} \to D_{\lambda(j+1)}$ making $(D_{\lambda(\bullet)}, d^\bullet)$ a complex which is quasi-isomorphic to the global de Rham complex $\Omega^\bullet_X(X) \otimes_K M$ (cf. [26], §3 Theorem 3). Note that Schneider works with the *highest* weight $w_0(\lambda)$ of the representation M, so that $D_{\lambda(j)}$ coincides with what is called $D_{w_0(\lambda)(j)}$ in [26] (cf. [26], §3 Lemma 2 and Remark 4). The resulting spectral sequence

$$E_1^{pq} := \mathrm{H}^q(\Gamma, D_{\lambda(p)}) \Longrightarrow \mathrm{H}^{p+q}_{dR}(X_\Gamma, \mathcal{M}_\Gamma) \tag{27}$$

is called the *reduced Hodge-de Rham spectral sequence*. We denote by $F^\bullet_{red}$ the filtration it induces on $\mathrm{H}^d_{dR}(X_\Gamma, \mathcal{M}_\Gamma)$. In [26], page 630 and page 648, P. Schneider formulates the following conjectures.

Conjecture 4.1 (Schneider)**.**

(i) For any integer j with $0 \leq j \leq d+1$ we have $\mathrm{H}^d_{dR}(X_\Gamma, \mathcal{M}_\Gamma) = F^j_\Gamma \oplus F^{d+1-j}_{dR}$.

(ii) The reduced Hodge-de Rham spectral sequence (27) degenerates in E_1.

(iii) The filtrations $F^\bullet_{dR}$ and $F^\bullet_{red}$ of $\mathrm{H}^d_{dR}(X_\Gamma, \mathcal{M}_\Gamma)$ coincide.

Remark 4.2. Apart from our result in Theorem 4.10 below, Conjecture 4.1 is known to be true in many cases. In [26], P. Schneider himself proved it unconditionally for $d = 1$, and for $d = 2$ when M is *integral*. Later, A. Iovita and M. Spieß gave a proof of Conjecture 4.1 for arbitrary d when M is the trivial representation (cf. [21]). Their proof uses certain integral structures of the de Rham complex of X. A second proof of this case was given by G. Alon and E. de Shalit in [1] and a third one by E. Große-Klönne in [15]. Generalizing the integral structures of Iovita and Spieß to arbitrary coefficients, E. Große-Klönne proved the conjecture for arbitrary d when M is the restriction of the regular representation of $\mathrm{GL}_{d+1}(L)$ or its dual (cf. [16]). Finally, in [17], E. Große-Klönne used global methods to prove part *(ii)* of Conjecture 4.1 under the assumption that Γ is of *arithmetic type* in the sense of [17], §4.

We are now going to give an alternative construction of the reduced Hodge-de Rham complex $(D_{\lambda(\bullet)}, d^\bullet)$ which has the advantage of directly leading to a proof of Conjecture 4.1 (iii) (cf. Theorem 4.9 below).

Given any G-equivariant vector bundle $\mathcal{F}$ on $\mathbb{P}^d_K$, the space $\mathcal{F}(X)$ of rigid analytic sections of $\mathcal{F}$ over X naturally is a K-Fréchet space carrying a locally analytic representation of G in the sense of [30], section 3 (cf. [24], section 1.1). According to [30], Proposition 3.2, the action of G on $\mathcal{F}(X)$ extends to an action of the K-algebra $D(G)$ of locally analytic K-valued distributions on G. The K-algebra $D(G)$ contains the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G as a subalgebra. We denote by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. The restriction of the projection

$$\mathrm{pr} : U(\mathfrak{g}) \simeq U(\mathfrak{t}) \oplus (\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\overline{\mathfrak{n}}) \longrightarrow U(\mathfrak{t})$$

from $U(\mathfrak{g})$ to $Z(\mathfrak{g})$ is a homomorphism of K-algebras. Given $\mu \in X^*(\mathbb{T})$ we denote by $\chi_\mu := (\mu + \delta) \circ \mathrm{pr} : Z(\mathfrak{g}) \to K$ the corresponding *Harish-Chandra character*. The following fact is proved in [32], Lemme 6.4 (note that the author works with $-\Delta$ instead of with Δ, which changes both the definition of δ and the notion of dominance).

Lemma 4.3. *If $\mu \in X^*(\mathbb{T})^-_I$ then $Z(\mathfrak{g})$ acts on $D_\mu = \mathcal{F}_\mu(X)$ through the character $\chi_{\mu-\delta}$.* □

If V is a $Z(\mathfrak{g})$-module, and if $\chi : Z(\mathfrak{g}) \to K$ is a K-valued character (i.e. a homomorphism of K-algebras), then we denote by V_χ the K-subspace of V consisting of all elements $v \in V$ for which there exists an integer $n \geq 1$ such that $(\mathfrak{x} - \chi(\mathfrak{x}))^n v = 0$ for all $\mathfrak{x} \in Z(\mathfrak{g})$. It is known that $Z(\mathfrak{g})$ is contained in the center of the ring $D(G)$ (cf. [30], Proposition 3.7). It follows that if the $Z(\mathfrak{g})$-action on V comes from an action of $D(G)$, then V_χ is even a $D(G)$-submodule of V.

Lemma 4.4. *Let $\mathcal{G}$ and $\mathcal{H}$ be G-equivariant vector bundles on $\mathbb{P}^d_K$.*

(i) Letting χ run through all K-valued characters of $Z(\mathfrak{g})$, the natural map $\oplus_\chi \mathcal{G}(X)_\chi \to \mathcal{G}(X)$ is bijective. We have $\mathcal{G}(X)_\chi = 0$ for almost all χ. For all characters χ and all elements $\mathfrak{x} \in Z(\mathfrak{g})$ the endomorphism $\mathfrak{x} - \chi(\mathfrak{x})$ of $\mathcal{G}(X)_\chi$ is nilpotent.

(ii) Let $d : \mathcal{G}(X) \to \mathcal{H}(X)$ be a homomorphism of $D(G)$-modules, let χ be a K-valued character of $Z(\mathfrak{g})$, and let $d_\chi : \mathcal{G}(X)_\chi \to \mathcal{H}(X)_\chi$ be the induced $D(G)$-linear map. In the category of (abstract) $D(G)$-modules we have $\ker(d_\chi) = \ker(d)_\chi$, $\mathrm{im}(d_\chi) = \mathrm{im}(d)_\chi$ and $\mathrm{coker}(d_\chi) = \mathrm{coker}(d)_\chi$. Further, $d = \oplus_\chi d_\chi$, $\ker(d) = \oplus_\chi \ker(d_\chi)$, $\mathrm{im}(d) = \oplus_\chi \mathrm{im}(d_\chi)$ and $\mathrm{coker}(d) = \oplus_\chi \mathrm{coker}(d_\chi)$.

Proof: Denoting by V the fiber of $\mathcal{G}$ in $[\mathbb{P}_I] \in \mathbb{G}/\mathbb{P}_I \simeq \mathbb{P}^d_K$, there is an isomorphism $\mathcal{G} \simeq \mathcal{F}_V$ of G-equivariant vector bundles on $\mathbb{P}^d_K$. The $\mathbb{P}_I$-representation V admits a decomposition series whose irreducible subquotients have the property that the action of $\mathbb{P}_I$ factors through $\mathbb{P}_I/\mathbb{N}_I \simeq \mathbb{M}_I$ (cf. [22], I.2.14 (8)). Since the functor $V \mapsto \mathcal{F}_V$ is exact (cf. [22], Proposition I.5.9), Theorem A for rigid analytic Stein spaces implies that $\mathcal{G}(X)$ admits a finite filtration by $D(G)$-stable $\mathcal{O}_X(X)$-submodules, the graded pieces of which are of the form D_μ for certain elements $\mu \in X^*(\mathbb{T})_I^-$.

We will prove (i) by induction on the length of such a filtration. Using Lemma 4.3 it suffices to show that if V is a $D(G)$-module containing a submodule of the form D_μ and such that V/D_μ has the properties in (i), then so does V. Given χ let $V(\chi)$ denote the inverse image of $(V/D_\mu)_\chi$ in V under the projection $V \to V/D_\mu$. By assumption, V is the sum of finitely many of its $D(G)$-submodules $V(\chi)$ each of which is an extension $0 \to D_\mu \to V(\chi) \to (V/D_\mu)_\chi \to 0$. Further, given $\mathfrak{x} \in Z(\mathfrak{g})$, the action of $\mathfrak{x} - \chi(\mathfrak{x})$ on $(V/D_\mu)_\chi$ is nilpotent. If $\chi = \chi_{\mu-\delta}$ then so is the action on $V(\chi)$ (cf. Lemma 4.3). If $\chi \neq \chi_{\mu-\delta}$ then the above exact sequence admits a $D(G)$-linear section. Indeed, there is an element $\mathfrak{x} \in Z(\mathfrak{g})$ such that $\chi(\mathfrak{x}) \neq \chi_{\mu-\delta}(\mathfrak{x})$, so that the endomorphism $\mathfrak{x} - \chi(\mathfrak{x})$ is nilpotent on $(V/D_\mu)_\chi$ whereas it is bijective on D_μ. Thus, one can argue as in [5], I.4.5.

It follows that the natural map $\oplus_\chi V_\chi \to V$ is surjective. Further, if $\chi \neq \chi_{\mu-\delta}$ is chosen such that $(V/D_\mu)_\chi = 0$ then $V_\chi = 0$ because V_χ maps to zero in V/D_μ and $V_\chi \cap D_\mu = 0$. Our above reasoning also shows that for any element $\mathfrak{x} \in Z(\mathfrak{g})$ the endomorphism $\mathfrak{x} - \chi(\mathfrak{x})$ of V_χ is nilpotent. In fact, this was pointed out for $V_{\chi_{\mu-\delta}} = V(\chi_{\mu-\delta})$ and otherwise follows from the fact that V_χ injects into $(V/D_\mu)_\chi$. Finally, if $(v_\chi)_\chi \in \oplus_\chi V_\chi$ is an element with $\sum_\chi v_\chi = 0$ in V, then $v_\chi \in D_\mu$ for all χ by assumption on V/D_μ. Since $V_\chi \cap D_\mu = 0$ unless $\chi = \chi_{\mu-\delta}$, this implies the map $\oplus_\chi V_\chi \to V$ to be injective.

Assertion (ii) follows from (i) and the fact that if χ and χ' are two distinct K-valued characters of $Z(\mathfrak{g})$ then $\mathrm{Hom}_{Z(\mathfrak{g})}(\mathcal{G}(X)_\chi, \mathcal{H}(X)_{\chi'}) = 0$. $\square$

Via restriction, the $\mathbb{G}$-representation M of lowest weight λ can be viewed as a representation of $\mathbb{P}_I$. Further, if j is an integer with $0 \le j \le d$ then $\bigwedge^j \mathfrak{n}_I^*$ is an algebraic $\mathbb{P}_I$-representation via the adjoint action of $\mathbb{P}_I$ on $\mathfrak{n}_I$. It is shown in [26], §3 Proposition 1 and Lemma 2, that

$$\Omega_X^j(X) \otimes_K M \simeq \mathcal{F}_{\bigwedge^j \mathfrak{n}_I^* \otimes_K M}(X). \tag{28}$$

Proposition 4.5. *If M is the irreducible $\mathbb{G}$-representation of lowest weight $\lambda \in X^*(\mathbb{T})$, and if j is an integer with $0 \le j \le d$, then there is a $D(G)$-linear isomorphism $(\Omega_X^j(X) \otimes_K M)_{\chi_{\lambda-\delta}} \simeq D_{\lambda(j)}$.*

Proof: Choose $\mu_1, \ldots, \mu_r \in X^*(\mathbb{T})_I^-$ such that the irreducible $\mathbb{M}_I$-representations $V_{\mu_1}, \ldots, V_{\mu_r}$ are the (not necessarily pairwise non-isomorphic) Jordan-Hölder constituents of $\bigwedge^j \mathfrak{n}_I^* \otimes_K M$. As can be seen from (28) and the proof of Lemma 4.4, the $D(G)$-module $(\Omega_X^j(X) \otimes_K M)_{\chi_{\lambda-\delta}}$ admits a finite filtration by $D(G)$-submodules whose associated graded object is naturally isomorphic to the direct sum over those modules $D_{\mu_i} = \mathcal{F}_{\mu_i}(X)$ for which $\chi_{\mu_i-\delta} = \chi_{\lambda-\delta}$. Note that $\chi_{\mu_i-\delta} = \chi_{\lambda-\delta}$ if and only if $\mu_i = w \bar{*} \lambda$ for some element $w \in W$ (cf. [20], Theorem 1.10; note that our character χ_μ is the character which is denoted by $\chi_{\mu-\delta}$ in [loc.cit.]).

Together with λ also $c_j(\lambda)$ is a weight of M. Since $c_j\overline{*}\mathbb{1}$ is a weight of $\bigwedge^j \mathfrak{n}_I^*$, it follows that $\lambda(j) = c_j\overline{*}\lambda = c_j(\lambda) + c_j\overline{*}\mathbb{1}$ is a weight of $\bigwedge^j \mathfrak{n}_I^* \otimes_K M$. Note that if $w \in W$ then $w\overline{*}\lambda \in X^*(\mathbb{T})_I^-$ if and only if $w = c_i$ for some i. Indeed, otherwise there is an element $\alpha \in I$ for which $\ell(s_\alpha w) < \ell(w)$ so that $w(\alpha)$ and α lie on opposite sides of the hyperplane determined by the simple reflection s_α (cf. [20], 0.3 (4) and [8], VI.1.6 Proposition 17).

Altogether, it suffices to prove that the weight $\lambda(j) = c_j\overline{*}\lambda$ appears with multiplicity one in $\bigwedge^j \mathfrak{n}_I^* \otimes_K M$, and that $c_i\overline{*}\lambda$ does not appear in $\bigwedge^j \mathfrak{n}_I^* \otimes_K M$ if $i \neq j$. This can be seen by considering the $\mathfrak{t}$-representation $\oplus_{i=0}^d(\bigwedge^i \mathfrak{n}^* \otimes_K M)$ in which each of the weights $w\overline{*}\lambda = w_0((w_0ww_0^{-1}) * w_0(\lambda))$ for $w \in W$ appears with multiplicity one (cf. [23], Lemma 5.12). $\square$

Remark 4.6. Our proof of Proposition 4.5 also shows the following. If V and W are $\mathbb{P}_I$-subrepresentations of $\bigwedge^j \mathfrak{n}_I^* \otimes_K M$ with $W \subset V$ and $V/W \simeq V_{\lambda(j)}$ then $(\Omega_X^j(X) \otimes_K M)_{\chi_{\lambda-\delta}}$ is contained in the $D(G)$-submodule $\mathcal{F}_V(X)$ of $\Omega_X^j(X) \otimes_K M$ and the surjective $D(G)$-linear homomorphism $\mathcal{F}_V(X) \to \mathcal{F}_{V/W}(X)$ restricts to an isomorphism $(\Omega_X^j(X) \otimes_K M)_{\chi_{\lambda-\delta}} \simeq \mathcal{F}_{V/W}(X) \simeq D_{\lambda(j)}$.

The boundary maps of the complex $\Omega_X^\bullet(X) \otimes_K M$ are defined by differentiating along local coordinates. Thus, one can verify that they are continuous for the K-Fréchet topology of the spaces $\Omega_X^j(X) \otimes_K M$. According to [30], Proposition 3.2, they are even $D(G)$-linear, so that we obtain the $D(G)$-stable subcomplex

$$C^\bullet := (\Omega_X^\bullet(X) \otimes_K M)_{\chi_{\lambda-\delta}} \subseteq \Omega_X^\bullet(X) \otimes_K M.$$

Lemma 4.7. *For any non-negative integer j the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts on $\mathrm{H}^j(\Omega_X^\bullet(X) \otimes_K M)$ through the character $\chi_{\lambda-\delta}$. As a consequence, the inclusion $C^\bullet \subseteq \Omega_X^\bullet(X) \otimes_K M$ is a quasi-isomorphism.*

Proof: Since $\mathrm{H}^j(\Omega_X^\bullet(X) \otimes_K M) \simeq \mathrm{H}_{dR}^j(X) \otimes_K M$, the computation of $\mathrm{H}_{dR}^j(X)$ in [28], §3 Theorem 1 and §4 Lemma 1, shows that $U(\mathfrak{g})$ acts trivially on $\mathrm{H}_{dR}^j(X)$. Therefore, $Z(\mathfrak{g})$ acts on $\mathrm{H}^j(\Omega_X^\bullet(X) \otimes_K M)$ through the central character of M, which is $\chi_{\lambda-\delta}$. Together with Lemma 4.4 this implies

$$\begin{aligned}
\mathrm{H}^j(\Omega_X^\bullet(X) \otimes_K M) &\simeq \oplus_\chi \mathrm{H}^j((\Omega_X^\bullet(X) \otimes_K M)_\chi) \\
&= \oplus_\chi(\mathrm{H}^j(\Omega_X^\bullet(X) \otimes_K M))_\chi = (\mathrm{H}^j(\Omega_X^\bullet(X) \otimes_K M))_{\chi_{\lambda-\delta}} \\
&= \mathrm{H}^j((\Omega_X^\bullet(X) \otimes_K M)_{\chi_{\lambda-\delta}}) = \mathrm{H}^j C^\bullet. \qquad \square
\end{aligned}$$

Proposition 4.5 and Lemma 4.7 suggest that the complex $C^\bullet$ is isomorphic to the reduced Hodge-de Rham complex. This is indeed the case, so that Lemma 4.7 reproves [26], §3 Theorem 3.

Proposition 4.8. *The complex $C^\bullet := (\Omega_X^\bullet(X) \otimes_K M)_{\chi_{\lambda-\delta}}$ is G-equivariantly isomorphic to the reduced Hodge-de Rham complex $D_{\lambda(\bullet)}$ of [26], §3.*

Proof: In [26], §3, P. Schneider defines a certain filtration $F^\bullet M$ of M by $\mathbb{P}_I$-subrepresentations. It leads to a corresponding filtration $\mathcal{F}^{\bullet,\bullet}$ of the complex $\Omega_X^\bullet(X) \otimes_K M$ by the $D(G)$-subcomplexes

$$\mathcal{F}^{r,s} := \mathcal{F}_{\bigwedge^s \mathfrak{n}_I^* \otimes F^{r-s}M}(X).$$

Write the highest weight $w_0(\lambda)$ of M as $w_0(\lambda) = (\lambda_1, \ldots, \lambda_d)$ with integers λ_j satisfying $0 \geq \lambda_1 \geq \ldots \geq \lambda_d$. Let $0 \leq j \leq d$ and consider the quotient complex $\mathcal{F}^{\lambda_{d-j}+j,\bullet}/\mathcal{F}^{\lambda_{d-j}+j+1,\bullet}$. According to the proof of [26], §3 Lemma 9, there are $\mathbb{P}_I$-subrepresentations V' and W' of $\bigwedge^j \mathfrak{n}_I^* \otimes_K \mathrm{gr}^{\lambda_{d-j}} M$ such that $W' \subset V'$, and such that $\mathcal{F}_{V'}(X)$ (resp. $\mathcal{F}_{W'}(X)$) can be identified with the kernel (resp. with the image) of the differential in degree j of the above quotient complex. Further, $V'/W' \simeq V_{\lambda(j)}$.

Let V and W be the inverse images of V' and W' in $\bigwedge^j \mathfrak{n}_I^* \otimes_K F^{\lambda_{d-j}} M$ under the projection $\bigwedge^j \mathfrak{n}_I^* \otimes_K F^{\lambda_{d-j}} M \to \bigwedge^j \mathfrak{n}_I^* \otimes_K \mathrm{gr}^{\lambda_{d-j}} M$, respectively. In the notation of the proof of [26], §3 Theorem 3, we then have

$$B^j = \mathcal{F}_W(X) \subseteq Z^j = \mathcal{F}_V(X) \subseteq \mathcal{F}^{\lambda_{d-j}+j,j}(\Omega^j_X(X) \otimes_K M) \subseteq \Omega^j_X(X) \otimes_K M.$$

According to Remark 4.6 we have $C^j \subset Z^j$, and the natural projection $Z^j \to Z^j/B^j$ restricts to an isomorphism $C^j \simeq Z^j/B^j \simeq D_{\lambda(j)}$. In particular, $C^\bullet$ is a subcomplex of the complex $Z^\bullet + dB^{\bullet-1}$ of [26], page 647, and the surjective homomorphism of complexes $Z^\bullet + dB^{\bullet-1} \to Z^\bullet/B^\bullet$ restricts to an isomorphism $C^\bullet \simeq Z^\bullet/B^\bullet$. Since the quotient $Z^\bullet/B^\bullet$ of the subcomplex $Z^\bullet + dB^{\bullet-1}$ of $\Omega^\bullet_X(X) \otimes_K M$ is the reduced Hodge-de Rham complex (cf. the proof of [26], §3 Theorem 3), we are done. □

For the rest of this article we are going to identify the two complexes $C^\bullet$ and $D_{\lambda(\bullet)}$.

Given any complex $(C^\bullet, d^\bullet)$ in non-negative degrees and any integer $i \geq 0$, we define the following truncations of $C^\bullet$:

$$\begin{aligned} t_{\leq i}C^\bullet &:= [C^0 \to \ldots \to C^{i-1} \to \ker(d^i) \to 0 \to \ldots] \\ C^\bullet_{\leq i} &:= [C^0 \to \ldots \to C^{i-1} \to C^i \to 0 \to \ldots] \\ t_{\geq i}C^\bullet &:= [0 \to \ldots \to 0 \to C^i/\mathrm{im}(d^{i-1}) \to C^{i+1} \to \ldots] \\ C^\bullet_{\geq i} &:= [0 \to \ldots \to 0 \to C^i \to C^{i+1} \to \ldots]. \end{aligned}$$

Theorem 4.9. *If j is an integer with $0 \leq j \leq d+1$ then $F^j_{dR} = F^j_{red}$ and*

$$\begin{aligned} F^j_\Gamma &= \mathrm{im}(\mathrm{H}^d(\Gamma, t_{\leq d-j} D_{\lambda(\bullet)}) \to \mathrm{H}^d(\Gamma, \Omega^\bullet_X(X) \otimes_K M)) \\ &= \ker(\mathrm{H}^d(\Gamma, \Omega^\bullet_X(X) \otimes_K M) \to \mathrm{H}^d(\Gamma, t_{\geq d+1-j} D_{\lambda(\bullet)})). \end{aligned}$$

Proof: By functoriality, $\mathrm{H}^d_{dR}(X_\Gamma, \mathcal{M}_\Gamma) \simeq \mathrm{H}^d(\Gamma, \Omega^\bullet_X(X) \otimes_K M)$ carries the structure of a $Z(\mathfrak{g})$-module. Further, the degeneration of the covering spectral sequence implies that it admits a filtration by $Z(\mathfrak{g})$-submodules such that the successive quotients are among the modules $\mathrm{H}^r(\Gamma, \mathrm{H}^s_{dR}(X) \otimes_K M)$ on which $Z(\mathfrak{g})$ acts through the character $\chi_{\lambda-\delta}$ (cf. the proof of Lemma 4.7). If χ is a character of $Z(\mathfrak{g})$ which is distinct from $\chi_{\lambda-\delta}$, and if V is any $Z(\mathfrak{g})$-module with $V = V_\chi$, then this implies $\mathrm{Hom}_{Z(\mathfrak{g})}(V, \mathrm{H}^d_{dR}(X_\Gamma, \mathcal{M}_\Gamma)) = \mathrm{Hom}_{Z(\mathfrak{g})}(\mathrm{H}^d_{dR}(X_\Gamma, \mathcal{M}_\Gamma), V) = 0$.

It follows from Lemma 4.4 that

$$\mathrm{H}^d(\Gamma, (\Omega^\bullet_X(X) \otimes_K M)_{\leq j-1}) = \oplus_\chi \mathrm{H}^d(\Gamma, ((\Omega^\bullet_X(X) \otimes_K M)_{\leq j-1})_\chi),$$

where $((\Omega^\bullet_X(X)\otimes_K M)_{\leq j-1})_\chi = ((\Omega^\bullet_X(X)\otimes_K M)_\chi)_{\leq j-1}$ for any K-valued character χ of $Z(\mathfrak{g})$. Referring to Proposition 4.5, we obtain that the natural map

$$\mathrm{H}^d(\Gamma,(D_{\lambda(\bullet)})_{\leq j-1}) \to \mathrm{H}^d(\Gamma,(\Omega^\bullet_X(X)\otimes_K M)_{\leq j-1}) \tag{29}$$

is injective. Further, our above reasoning implies that the natural $Z(\mathfrak{g})$-linear map $\mathrm{H}^d_{dR}(X_\Gamma,\mathcal{M}_\Gamma)\to \mathrm{H}^d(\Gamma,(\Omega^\bullet_X(X)\otimes_K M)_{\leq j-1})$ factors through the inclusion (29). Since F^j_{dR} (resp. F^j_{red}) is the kernel of the $Z(\mathfrak{g})$-equivariant homomorphism $\mathrm{H}^d_{dR}(X_\Gamma,\mathcal{M}_\Gamma)\to \mathrm{H}^d(\Gamma,(\Omega^\bullet_X(X)\otimes_K M)_{\leq j-1})$ (resp. of the $Z(\mathfrak{g})$-equivariant homomorphism $\mathrm{H}^d_{dR}(X_\Gamma,\mathcal{M}_\Gamma)\to \mathrm{H}^d(\Gamma,(D_{\lambda(\bullet)})_{\leq j-1})$), the first assertion follows.

Using Lemma 4.4, the formation of the truncations $t_{\leq d-j}$ and $t_{\geq d+1-j}$ commutes with $(\cdot)_\chi$, as well. Given the descriptions of $F^\bullet_\Gamma$ in [26], page 631, the proof of the second assertion is then similar. $\square$

As is customary, we identify a character $\lambda\in X^*(\mathbb{T})$ with the family $\lambda = (\lambda_1,\ldots,\lambda_{d+1})$ of integers λ_j determined by $\lambda(\mathrm{diag}(t_1,\ldots,t_{d+1})) = \prod_{j=1}^{d+1} t_j^{\lambda_j}$ for all elements $\mathrm{diag}(t_1,\ldots,t_{d+1})\in T$.

We conclude our article by proving the following special case of Conjecture 4.1, using the vanishing theorems of section 3.

Theorem 4.10. *Assume $d=2$. If the character $\lambda\in X^*(\mathbb{T})$ is of the form $\lambda=(a,0,-a)$ for some integer $a\leq 0$, then all assertions of Conjecture 4.1 are true.*

Proof: We have $\Delta=\{\alpha_1,\alpha_2\}$ as in Example 1.1 and denote by $s_1 := s_{\alpha_1}\in W$ and $s_2 := s_{\alpha_2}\in W$ the simple reflections corresponding to α_1 and α_2, respectively. We also set $P_1 := P_{\{\alpha_1\}}$ and $P_2 := P_{\{\alpha_2\}}$.

Given $\mu\in X^*(\mathbb{T})$ we denote by $L(\mu)$ the irreducible $U(\mathfrak{g})$-module of highest weight μ. Set $\lambda' := w_0(\lambda) = (-a,0,a)$, which is a dominant character. The $U(\mathfrak{g})$-modules $L(s_1*\lambda')$ and $L((s_1s_2)*\lambda')$ are objects of the category $\mathcal{O}^{\mathfrak{p}_2}_{alg}$. Setting $n := [L:\mathbb{Q}_p]$ we compute

$$\begin{aligned}
-s_1*\lambda' + n(s_2*\mathbb{1}) &= (1,a-1,-a)+n(0,-1,1)\\
&= (1,a-1-n,-a+n) \qquad \text{and}\\
-(s_1s_2)*\lambda' + n(s_2*\mathbb{1}) &= (2-a,a-1,-1)+n(0,-1,1)\\
&= (2-a,a-1-n,-1+n).
\end{aligned}$$

These weights both have a positive contribution from the root α_1, so that Corollary 3.17 applies and yields

$$\mathrm{H}_q(\Gamma,\mathcal{F}^G_{P_2}(L(s_1*\lambda'),v_P^{P_2})) = \mathrm{H}_q(\Gamma,\mathcal{F}^G_{P_2}(L((s_1s_2)*\lambda'),v_P^{P_2}))=0$$

for all $q\geq 0$. By a similar calculation Corollary 3.17 implies

$$\mathrm{H}_q(\Gamma,\mathcal{F}^G_{P_1}(L(s_2*\lambda'),\mathbb{1})) = \mathrm{H}_q(\Gamma,\mathcal{F}^G_{P_1}(L((s_2s_1)*\lambda'),\mathbb{1}))=0$$

for all $q\geq 0$.

Based on results from [24], the Jordan-Hölder constituents of the $D(G)$-modules $D_{\lambda(j)}$ were computed in [32], §6. More precisely, the strong dual $D'_{\lambda(0)}$ of $D_{\lambda(0)}$ possesses a finite exhaustive filtration by $D(G)$-submodules whose associated graded pieces are $\ker(d^0)' = \mathcal{F}^G_G(L(\lambda'), \mathbb{1}) = M'$, $\mathcal{F}^G_{P_1}(L(s_2 * \lambda'), \mathbb{1})$ and $\mathcal{F}^G_{P_2}(L((s_1 s_2) * \lambda'), v^{P_2}_P)$. As a consequence, we have isomorphisms

$$\mathrm{H}_q(\Gamma, D'_{\lambda(0)}) \simeq \mathrm{H}_q(\Gamma, M') \simeq \mathrm{H}_q(\Gamma, \ker(d^0)')$$

for all $q \geq 0$.

The $D(G)$-module $D'_{\lambda(1)}$ has a finite exhaustive filtration by $D(G)$-submodules whose associated graded pieces are $\mathcal{F}^G_{P_1}(L(s_2 * \lambda'), \mathbb{1})$, $\mathcal{F}^G_{P_2}(L((s_1 s_2) * \lambda'), v^{P_2}_P)$, $\mathrm{H}^1_{dR}(X)' \otimes_K M'$, $\mathcal{F}^G_{P_1}(L((s_2 s_1) * \lambda'), \mathbb{1})$ and $\mathcal{F}^G_{P_2}(L(s_1 * \lambda'), v^{P_2}_P)$. The graded pieces of the induced filtration on $\ker(d^1)'$ are given by $\mathcal{F}^G_{P_1}(L(s_2 * \lambda'), \mathbb{1})$, $\mathcal{F}^G_{P_2}(L((s_1 s_2) * \lambda'), v^{P_2}_P)$ and $\mathrm{H}^1_{dR}(X)' \otimes_K M'$. Further, the graded pieces of the induced filtration on $\mathrm{coker}(d^0)'$ are $\mathrm{H}^1_{dR}(X)' \otimes_K M'$, $\mathcal{F}^G_{P_1}(L((s_2 s_1) * \lambda'), \mathbb{1})$ and $\mathcal{F}^G_{P_2}(L(s_1 * \lambda'), v^{P_2}_P)$. As before we deduce isomorphisms

$$\begin{aligned} \mathrm{H}_q(\Gamma, D'_{\lambda(1)}) &\simeq \mathrm{H}_q(\Gamma, \mathrm{H}^1_{dR}(X)' \otimes_K M') \simeq \mathrm{H}_q(\Gamma, \ker(d^1)') \\ &\simeq \mathrm{H}_q(\Gamma, \mathrm{coker}(d^0)') \end{aligned}$$

for all $q \geq 0$.

Using [32], (6.37), one similarly obtains isomorphisms

$$\begin{aligned} \mathrm{H}_q(\Gamma, D'_{\lambda(2)}) &\simeq \mathrm{H}_q(\Gamma, \mathrm{H}^2_{dR}(X)' \otimes_K M') \simeq \mathrm{H}_q(\Gamma, \ker(d^2)') \\ &\simeq \mathrm{H}_q(\Gamma, \mathrm{coker}(d^1)') \end{aligned}$$

for all $q \geq 0$.

It follows from [26], §1 Proposition 2, that $\mathrm{H}^q(\Gamma, \mathrm{H}^p_{dR}(X) \otimes_K M)$ is a finite dimensional K-vector space for any $q \geq 0$. Arguing as in [32], Théorème 3.15, we can dualize the above results and conclude that

$$\begin{aligned} \mathrm{H}^q(\Gamma, D_{\lambda(p)}) &\simeq \mathrm{H}^q(\Gamma, \mathrm{H}^p_{dR}(X) \otimes_K M) \simeq \mathrm{H}^q(\Gamma, \ker(d^p)) \\ &\simeq \mathrm{H}^q(\Gamma, \mathrm{coker}(d^{p-1})) \end{aligned}$$

for all $q \geq 0$ and all $p \geq 0$.

Using Theorem 4.9 and reasoning as in the proof of [26], §2 Lemma 2 (iii), Conjecture 4.1 will be proved once we can show the natural maps $\mathrm{H}^d(\Gamma, t_{\leq j} D_{\lambda(\bullet)}) \to \mathrm{H}^d(\Gamma, (D_{\lambda(\bullet)})_{\leq j})$ and $\mathrm{H}^d(\Gamma, (D_{\lambda(\bullet)})_{\geq j}) \to \mathrm{H}^d(\Gamma, t_{\geq j} D_{\lambda(\bullet)})$ to be bijective for all j. To see this, consider the following diagram of complexes with exact rows

$$\begin{tikzcd} 0 \arrow[r] & (D_{\lambda(\bullet)})_{\leq j-1} \arrow[r] \arrow[d, equal] & t_{\leq j} D_{\lambda(\bullet)} \arrow[r] \arrow[d] & \ker(d^j)[-j] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (D_{\lambda(\bullet)})_{\leq j-1} \arrow[r] & (D_{\lambda(\bullet)})_{\leq j} \arrow[r] & D_{\lambda(j)}[-j] \arrow[r] & 0. \end{tikzcd}$$

We showed above that $\mathrm{H}^q(\Gamma, \ker(d^j)) \simeq \mathrm{H}^q(\Gamma, D_{\lambda(j)})$ for any $q \geq 0$. The long exact hypercohomology sequence therefore leads to the desired isomorphism

$$\mathrm{H}^d(\Gamma, t_{\leq j} D_{\lambda(\bullet)}) \simeq \mathrm{H}^d(\Gamma, (D_{\lambda(\bullet)})_{\leq j}).$$

The bijectivity of the second map is proved similarly. $\square$

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