Homological vanishing theorems for locally analytic representations

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(joint work with Benjamin Schraen)

Let p be a prime number, let L be a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, let $\mathbb{G}$ be a connected, reductive group over L, let $G := \mathbb{G}(L)$ be its group of L-rational points, and let $\Gamma \subseteq G$ be a discrete and cocompact subgroup of G. The (co)homology of Γ-representations has been an area of research for a long time. One of the most striking results in this direction is the following vanishing theorem due to Garland, Casselman, Prasad, Borel and Wallach.

Theorem (Garland et al.). *If* Γ *is irreducible, if the* L*-rank* r *of* $\mathbb{G}$ *is at least* 2*, and if* V *is a finite dimensional representation of* Γ *over a field of characteristic zero, then* $\mathrm{H}^i(\Gamma, V) = 0$*, unless* $i \in \{0, r\}$.

The proof of this theorem uses the full force of the theory of smooth complex representations of G. If $X \subseteq \mathbb{P}^d_L$ denotes Drinfeld's p-adic symmetric space, if $\Gamma \subseteq \mathrm{PGL}_{d+1}(L)$ acts without fixed points on X, and if $X_\Gamma := \Gamma\backslash X$ denotes the quotient of X by Γ, then the same methods were used by Schneider and Stuhler to compute the de Rham cohomology $\mathrm{H}_{\mathrm{dR}}(X_\Gamma)$ of X_Γ from that of X (cf. [3]). The interest in the rigid varieties X_Γ stems from the fact that they uniformize certain Shimura varieties.
The case of trivial coefficients was extended by Schneider who considered finite dimensional algebraic representations M of $\mathrm{SL}_{d+1}(L)$ over L and the induced locally constant sheaf $\mathcal{M}_\Gamma$ on X_Γ (cf. [2]). He formulated several conjectures on the structure of the de Rham cohomology $\mathrm{H}_{\mathrm{dR}}(X_\Gamma, \mathcal{M}_\Gamma)$ which are related to two spectral sequences

$$E_2^{p,q} = \mathrm{H}^p(\Gamma, \mathrm{H}_{\mathrm{dR}}(X) \otimes_L M) \implies \mathrm{H}^{p+q}(X_\Gamma, \mathcal{M}_\Gamma)$$
$$E_1^{p,q} = \mathrm{H}^q(\Gamma, \Omega^p_X(X) \otimes_L M) \implies \mathrm{H}^{p+q}(X_\Gamma, \mathcal{M}_\Gamma).$$

Whereas $\mathrm{H}_{\mathrm{dR}}(X)$ is the dual of a smooth representation, the global differential forms $\Omega^p_X(X)$ are Fréchet spaces over L which carry a *locally analytic* action of $\mathrm{PGL}_{d+1}(L)$ in the sense of Schneider-Teitelbaum. Representations of this type were intensively studied by Morita, Schneider-Teitelbaum and Orlik. The main motivation for our work [1] was to study the (co)homology of Γ with coefficients in locally analytic representations of p-adic reductive groups, and to apply our results to the conjectures of Schneider.

Let K be a spherically complete valued field containing L, denote by $\mathfrak{o}_L$ the valuation ring of L, and let π be a uniformizer of L. For simplicity we shall only consider the group $G := \mathrm{PGL}_{d+1}(L)$. Let $P = N \cdot T$ be the standard Levi decomposition of the subgroup of upper triangular matrices of G, let $G_0 := \mathrm{PGL}_{d+1}(\mathfrak{o}_L)$, and let B denote the subgroup of G_0 consisting of all matrices whose reduction modulo π is upper triangular. For any positive integer n let $B_n := \ker(G_0 \to \mathrm{PGL}_{d+1}(\mathfrak{o}_L/\pi^n\mathfrak{o}_L))$. We let $T^- := \{\mathrm{diag}(\lambda_1, \ldots, \lambda_{d+1}) \in T \mid |\lambda_1| \geq \ldots \geq |\lambda_{d+1}|\}$

and $t_i \in T^-$ for $1 \leq i \leq d$ be representatives of the fundamental antidominant cocharacters of the root system of (G, T) with respect to P.

Given a locally analytic character $\chi : T \to K^\times$ and a discrete and cocompact subgroup Γ of G, our first goal is to study the homology $\mathrm{H}_*(\Gamma, \mathrm{Ind}_P^G(\chi))$ of Γ with coefficients in the *locally analytic principal series representation*

$$\mathrm{Ind}_P^G(\chi) := \{f \in \mathcal{C}^{\mathrm{an}}(G, K) \mid \forall g \in G\ \forall p \in P : f(gp) = \chi(p)^{-1} f(g)\}.$$

This is done by constructing an explicit Γ-acyclic resolution in the following way. For any positive integer n we denote by $\mathcal{A}_n$ the subspace of $\mathrm{Ind}_P^G(\chi)$ consisting of all functions with support in $B \cdot P$ and whose restriction to $B \cap \overline{N}$ is rigid analytic on every coset modulo $B_n \cap \overline{N}$. Here $\overline{N}$ denotes the group of all lower triangular unipotent matrices. If n is sufficiently large then $\mathcal{A} := \mathcal{A}_n$ is a B-stable K-Banach space inside $\mathrm{Ind}_P^G(\chi)$. By Frobenius reciprocity there exists a unique G-equivariant map

$$\varphi : \text{c-}\mathrm{Ind}_B^G(\mathcal{A}) \to \mathrm{Ind}_P^G(\chi),$$

which will be the final term of the desired resolution. In fact, we show that φ is surjective and that there is a homomorphism $K[T^-] \to \mathrm{End}_G(\text{c-}\mathrm{Ind}_B^G(\mathcal{A}))$, $t \mapsto U_t$, of K-algebras such that $\ker(\varphi) = \sum_{i=1}^d \mathrm{im}(U_{t_i} - \chi(t_i))$ (cf. [1], Proposition 2.4). This suggests to consider the following Koszul complex whose exactness is the main technical result of our work (cf. [1], Theorem 2.5).

Theorem 1. *The augmented Koszul complex*

$$(\bigwedge^{\bullet} K^d) \otimes_K \text{c-}\mathrm{Ind}_B^G(\mathcal{A}) \longrightarrow \mathrm{Ind}_P^G(\chi) \xrightarrow{\varphi} 0$$

defined by the endomorphisms $(U_{t_i} - \chi(t_i))_{1 \leq i \leq d}$ *of* $\text{c-}\mathrm{Ind}_B^G(\mathcal{A})$ *is a* G*-equivariant exact resolution of* $\mathrm{Ind}_P^G(\chi)$ *by* Γ*-acyclic representations.*

As a corollary one immediately obtains the following result.

Corollary 2. *We have* $\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi)) \simeq \mathrm{H}_q((\bigwedge^\bullet K^d) \otimes_K \text{c-}\mathrm{Ind}_B^G(\mathcal{A})_\Gamma)$ *for any integer* $q \geq 0$*. In particular, if* $q > d$ *then* $\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi)) = 0$.

It is a crucial observation that $\text{c-}\mathrm{Ind}_B^G(\mathcal{A})_\Gamma$ is naturally a K-Banach space and that the operator induced by U_{t_i} is continuous with operator norm ≤ 1 for any i. This leads to the following vanishing theorem (cf. [1], Theorem 3.2).

Theorem 3. *If* $|\chi(t_i)| > 1$ *for some* $1 \leq i \leq d$ *then* $\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi)) = 0$ *for all* $q \geq 0$.

For the proof one simply refers to Corollary 2 and uses the fact that under the above hypothesis the endomorphism $U_{t_i} - \chi(t_i)$ of the K-Banach space $\text{c-}\mathrm{Ind}_B^G(\mathcal{A})_\Gamma$ is invertible.
A similarly far-reaching observation is that if $t := t_1 \cdot \ldots \cdot t_d$ then the K-linear endomorphism U_t of $\text{c-}\mathrm{Ind}_B^G(\mathcal{A})$ is not only continuous but even *compact*, i.e. it is the strong limit of continuous operators with finite rank. A Fredholm argument for

$U_t - \chi(t)$ then leads to the following very general finiteness result (cf. [1], Theorem 3.9).

Theorem 4. *For any integer* $q \geq 0$ *the* K*-vector space* $\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi))$ *is finite dimensional.*

We finally broaden our point of view and consider locally analytic G-representations V over K possessing a G-equivariant finite resolution

$$0 \longrightarrow V \longrightarrow M_0 \longrightarrow \cdots \longrightarrow M_n \longrightarrow 0,$$

in which all M_i are finite direct sums of locally analytic principal series representations $\mathrm{Ind}_P^G(\chi_{ij})$. Theorems 3 and 4 and a spectral sequence argument lead to vanishing and finiteness theorems for V. Examples to which this procedure applies include *locally algebraic* representations of the form $V = \mathrm{Ind}_P^G(\mathbf{1})^\infty \otimes_K M$, for which the necessary resolution is provided by the locally analytic BGG-resolution of Orlik-Strauch. Here $\mathrm{Ind}_P^G(\mathbf{1})^\infty$ denotes the smooth principal series representation associated with the trivial character $\mathbf{1}$ and M is a finite dimensional algebraic representation of G. Another example is given by certain subquotients of *p-adic holomorphic discrete series representations*, i.e. representations of the form $\Omega_X^p(X) \otimes_K M$. In fact, our vanishing theorems eventually allow us to prove Schneider's conjectures in several previously unknown cases (cf. [1], Theorem 4.10).

References

[1] J. Kohlhaase, B. Schraen, *Homological vanishing theorems for locally analytic representations*, Mathematische Annalen, to appear.

[2] P. Schneider, *The cohomology of local systems on p-adically uniformized varieties*, Mathematische Annalen **293** (1992), 623–650.

[3] P. Schneider, U. Stuhler, *The cohomology of p-adic symmetric spaces*, Inventiones Mathematicae **105** (1991), 47–122.