Smooth duality in natural characteristic

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Abstract. We develop a duality theory for admissible smooth representations of p-adic Lie groups on vector spaces over fields of characteristic p. To this end we introduce certain higher smooth duality functors and relate our construction to the Auslander duality of completed group rings. We study the behavior of smooth duality under tensor products, inflation and induction, and discuss the dimension theory of smooth mod-p representations of a p-adic reductive group. Finally, we compute the higher smooth duals of the irreducible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ in characteristic p and relate our results to the contragredient operation of Colmez.

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0 Introduction

Let G denote a locally profinite topological group, let E denote a field, and let $\mathrm{Rep}_E^\infty(G)$ denote the category of E-linear smooth G-representations. The arithmetic interest in the category $\mathrm{Rep}_E^\infty(G)$ comes from the fact that if G is the group of rational points of a connected reductive group over a local field and if E is the field of complex numbers then the set of isomorphism classes of the irreducible objects of $\mathrm{Rep}_E^\infty(G)$ forms one side of the classical local Langlands correspondence.

Given an object V of $\mathrm{Rep}_E^\infty(G)$, the *smooth dual* $S^0(V)$ of V is the subspace of smooth vectors in the E-linear dual $\check{V} = \mathrm{Hom}_E(V, E)$ of V endowed with the contragredient action of G. Over coefficient fields of characteristic zero the functor S^0 restricts to an autoduality of the category $\mathrm{Rep}_E^\infty(G)^a$ of admissible smooth E-linear G-representations and is an elementary, yet fundamental tool in representation theory.

In view of the recent p-adic and mod-p variants of the local Langlands program there is a growing interest in smooth representations in *natural characteristic*. This means that E has positive characteristic p and that G is locally a pro-p group. If G is a p-adic Lie group it turns out that the functor S^0 is rather useless (cf. Corollary 3.9 and Remark 3.10). The aim of the present article is to overcome these deficiencies and to develop a useful duality theory for admissible smooth representations of p-adic Lie groups in natural characteristic.

In section 1 we study the case of a general locally profinite topological group G and an arbitrary field E. The Pontryagin duality functor $\check{(\cdot)} = \mathrm{Hom}_E(\cdot, E)$ sets up an anti-equivalence between the categories of discrete and pseudocompact E-vector spaces, respectively. Endowing a smooth E-linear G-representation with the discrete topology, the pseudocompact E-vector space $\check{V}$ turns out to be a module over a ring $\Lambda(G)$ generalizing the completed group ring for compact groups (cf. Theorem 1.5).

Borrowing terminology from [30], §3, we define the notion of a *coadmissible* $\Lambda(G)$-module and show that the category $\mathcal{C}_G$ of coadmissible $\Lambda(G)$-modules is anti-equivalent to the category $\mathrm{Rep}_E^\infty(G)^a$ of admissible smooth E-linear G-representations via Pontryagin duality (cf. Corollary 1.8). If G is a p-adic Lie group and if E has characteristic p then an abstract $\Lambda(G)$-module is coadmissible if and only if its underlying $\Lambda(G_0)$-module is finitely generated for any compact open subgroup G_0 of G (cf. Proposition 1.9).

In section 2 we let p be a prime number different from the characteristic of E and assume that G is locally a pro-p group. In this situation the functor S^0 restricts to an exact autoduality of the category $\mathrm{Rep}_E^\infty(G)^a$. Following ideas of Schneider and Teitelbaum from [31], §1, we explain how to describe the endofunctor of the category $\mathcal{C}_G$ induced by S^0 via Pontryagin duality. Up to twisting by the modulus character of G it is given by $\mathrm{Hom}_{\Lambda(G)}(\cdot, \Delta(G))$ where $\Delta(G)$ is the $(\Lambda(G), \Lambda(G))$-bimodule $C_c^\infty(G, E)\check{}$ dual to the space of compactly supported locally constant E-valued functions on G.

From section 3 on we assume that p is a prime number, that G is a finite dimensional Lie group of dimension $d = \dim(G)$ over the field $\mathbb{Q}_p$ of p-adic numbers and that the characteristic of the field E is equal to p. The previous results suggest to study the functors $E^i = E^i_G = \mathrm{Ext}^i_{\Lambda(G)}(\cdot, \Delta(G))$ on the category $\mathrm{Mod}_{\Lambda(G)}$ of left $\Lambda(G)$-modules. By fundamental results of Lazard and Venjakob the group G admits a compact open subgroup whose completed group ring over E is *Auslander regular* of global dimension d (cf.

[37], Theorem 3.30 (ii), as well as Theorem 3.1 below). Using a general duality theorem of Björk and closely following the strategy set forth by Schneider and Teitelbaum in [31] we define the *grade* $j(M)$ of a coadmissible $\Lambda(G)$-module M and use it to define a descending filtration

$$\mathcal{C}_G = \mathcal{C}_G^0 \supseteq \ldots \supseteq \mathcal{C}_G^d \supseteq \mathcal{C}_G^{d+1} = 0$$

of $\mathcal{C}_G$ by Serre subcategories such that the functor E^i induces an autoduality of the abelian quotient category $\mathcal{C}_G^i/\mathcal{C}_G^{i+1}$ for $0 \leq i \leq d$ (cf. Theorem 3.5).

If V is an object of $\mathrm{Rep}_E^\infty(G)^a$ we call $d(V) = d - j(\check{V})$ the *dimension* of V and introduce standard terminology such as purity, holonomicity and the property of being Cohen-Macaulay (cf. Definition 3.6). For Cohen-Macaulay objects the grade is related to the projective dimension over completed group rings (cf. Remark 3.7). Further, an object is holonomic if and only if its underlying E-vector space is finite dimensional (cf. Proposition 3.8).

The bimodule $\Delta(G)$ has the formal properties of a dualizing module in the sense that $E^i(E) = 0$ for $i \neq d$ and that $E^d(E) = \chi_G$ is given by an E-valued smooth character χ_G of the group G. If $i \geq 0$ we define the *i-th smooth duality functor* $S^i : \mathrm{Rep}_E^\infty(G) \to \mathrm{Rep}_E^\infty(G)$ by

$$S^i(V) = S_G^i(V) = \varinjlim_N \mathrm{Ext}^i_{\Lambda(N)}(E, \check{V}),$$

were N runs through the open subgroups of G. Following ideas of Venjakob from [37], §5, we prove that for $0 \leq i \leq d$ the diagram of functors

$$\begin{tikzcd}
\mathrm{Rep}_E^\infty(G) \arrow[r, "\check{(\cdot)}"] \arrow[d, "\chi_G \otimes_E S^{d-i}"'] & \mathrm{Mod}_{\Lambda(G)} \arrow[d, "E^i"] \\
\mathrm{Rep}_E^\infty(G) \arrow[r, "\check{(\cdot)}"] & \mathrm{Mod}_{\Lambda(G)}
\end{tikzcd}$$

commutes up to natural isomorphism. Thus, the δ-functor $(S^i)_{i\geq 0}$ gives rise to a d-step duality on the category $\mathrm{Rep}_E^\infty(G)^a$ of admissible smooth E-linear G-representations (cf. Theorem 3.14 and Corollary 3.15). Moreover, an object $V \in \mathrm{Rep}_E^\infty(G)^a$ is Cohen-Macaulay if and only if its higher smooth duals vanish in all degrees different from $d(V)$ (cf. Corollary 3.16). Further, our construction turns out to be compatible with Venjakob's approach using local cohomology groups (cf. Remark 3.11 (ii) and Remark 3.17).

In section 4 we study the behavior of the smooth duality functors under the change of the group G. We first show that they commute with inflation (cf. Theorem 4.1) and that on admissible representations the inflation functor preserves dimensions and the properties of being holonomic, pure and Cohen-Macaulay (cf. Corollary 4.2). For direct products of p-adic Lie groups and tensor products of admissible representations the higher smooth duals obey a formula of Künneth type (cf. Theorem 4.3). Therefore, the tensor product behaves additively with respect to dimensions and preserves the properties of being holonomic, pure and Cohen-Macaulay (cf. Corollary 4.4). Finally, if H is a closed subgroup of G such that $G = G_0H$ for some compact open subgroup G_0 of G then we describe how the smooth duality functors behave with respect to induction from H to G (cf. Theorem 4.7). As a consequence, we obtain that the induction functor preserves the grade and the properties of being pure and Cohen-Macaulay (cf. Corollary 4.8). We also show that there are natural restriction functors in the setting of smooth duality.

If G is the group of rational points of a connected reductive group $\mathbb{G}$ over $\mathbb{Q}_p$ then we use the previous results and classification techniques of Abe, Henniart, Herzig and Vignéras to give dimension bounds for the irreducible non-supercuspidal representations of G (cf. Theorem 4.9). It is a major open question of the theory if there are similar bounds for the supercuspidal representations of G as suggested by the examples in section 5 (cf. Remark 4.10).

In section 5 we explicitly compute the higher smooth duals in a number of examples. These include the irreducible smooth E-linear representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, as classified by Barthel, Livné and Breuil (cf. [3] and [10]).

In Theorem 5.1 we start by giving an explicit description of the duality character χ_G of a p-adic Lie group relying on results of Schneider and Teitelbaum from [31]. As a consequence, if G is open in the group of rational points of a connected reductive group then χ_G is trivial (cf. Corollary 5.2). For the group of rational points of a Borel subgroup the duality character can be computed according to the formula in Corollary 5.3.

In Proposition 5.4 we treat the principal series representations of a p-adic reductive group. They are Cohen-Macaulay and their unique non-zero higher smooth dual is a principal series representation again. A first result concerning the so-called *special* representations is contained in Proposition 5.5. However, we are currently not able to compute their higher smooth duals in general. Instead, we only treat the example of the Steinberg representation over E when $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and when $G = \mathrm{GL}_3(\mathbb{Q}_p)$ (cf. Proposition 5.6 and Proposition 5.7). These two examples show that irreducible representations need not be Cohen-Macaulay and that even on Cohen-Macaulay objects the smooth duality functors do not preserve irreducibility (cf. Remark 5.8 for a comparison with the case of characteristic zero).

Finally, we show that the *supersingular* representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over

E are Cohen-Macaulay of dimension one and that their first smooth dual is supersingular again (cf. Theorem 5.13). As a consequence of the classification results of Breuil, Barthel and Livné we thus obtain that every infinite dimensional irreducible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E is Cohen-Macaulay of dimension one. Further, in contrast to characteristic zero the supercuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ in characteristic p are not injective, as was already observed by Paskunas (cf. Remark 5.14). Finally, our explicit computations show that on the infinite dimensional irreducible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E the first smooth duality functor $S^1_{\mathrm{GL}_2(\mathbb{Q}_p)}$ coincides with the contragredient operation constructed by Colmez in his work on the p-adic local Langlands correspondence (cf. Remark 5.15).

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Conventions and notation. Let G denote a locally profinite topological group, and let E denote a field. We denote by $\mathrm{Rep}^\infty_E(G)$ the category of smooth E-linear representations of G, i.e. of E-vector spaces with an E-linear G-action such that the stabilizer of any vector is open in G.
If p is a prime number then by a p-adic Lie group we mean a finite dimensional Lie group over the field $\mathbb{Q}_p$ of p-adic numbers.
If R is a ring then we denote by Mod_R the category of left R-modules.

1 Pontryagin duality

The results from Pontryagin duality that we shall need are essentially contained in [17], §2. Certain statements need to be generalized to non-compact groups, however, so that we will give a brief summary.

Endowing the field E with the discrete topology, it is a pseudocompact ring in the sense of [19], IV.3. We say that an E-vector space is *discrete* if it is endowed with the discrete topology. A topological E-vector space is called *pseudocompact* if it is the topological projective limit of finite dimensional discrete E-vector spaces.

The category of pseudocompact E-vector spaces (with morphisms all continuous E-linear maps) is abelian (cf. [19], Théorème IV.3.3). In particular, every continuous E-linear map between pseudocompact E-vector spaces has a closed image and every continuous E-linear bijection between pseudocompact E-vector spaces is a topological isomorphism.

If V is a discrete E-vector space and if $(V_i)_{i\in I}$ is the family of its finite dimensional subspaces then we let $\check{V} = \mathrm{Hom}_E(V,E) \cong \varprojlim_{i\in I} \mathrm{Hom}_E(V_i,E)$ be its E-linear dual, viewed as a pseudocompact E-vector space. Conversely, if M is a pseudocompact E-vector space we let $\check{M} := \mathrm{Hom}_E^{\mathrm{cont}}(M,E)$ be the space of all continuous E-linear forms on M, endowed with the discrete topology. The following theorem is then immediate.

Theorem 1.1. *The functors $\check{(\cdot)}$ are mutually quasi-inverse equivalences between the categories of discrete and pseudocompact E-vector spaces.* $\square$

If G is compact then we denote by

$$\Lambda(G) = E[\![G]\!] = \varprojlim_{N \trianglelefteq G} E[G/N]$$

the *completed group ring* or *Iwasawa algebra* of G over E. Here N runs through the family of all normal open subgroups of G and $E[G/N]$ denotes the group ring of G/N over E. In this case $\Lambda(G)$ is naturally a pseudocompact E-algebra.

If G is an arbitrary locally profinite group and if G_0 is a fixed compact open subgroup of G then the $(E[G], \Lambda(G_0))$-bimodule $E[G] \otimes_{E[G_0]} \Lambda(G_0)$ admits a unique E-algebra structure making the maps

$$\begin{array}{rcl} (\delta \mapsto \delta \otimes 1) & : & E[G] \to E[G] \otimes_{E[G_0]} \Lambda(G_0) \quad \text{and} \\ (\delta' \mapsto 1 \otimes \delta') & : & \Lambda(G_0) \to E[G] \otimes_{E[G_0]} \Lambda(G_0) \end{array}$$

homomorphisms of E-algebras. To see this note that $\bigoplus_{g\in G/G_0} \Lambda(G_0) \to E[G] \otimes_{E[G_0]} \Lambda(G_0)$, $(\delta_g)_{g\in G/G_0} \mapsto \sum_{g\in G/G_0} g\delta_g$, is an isomorphism of right $\Lambda(G_0)$-modules. This implies that if N is an open normal subgroup of G_0 then the natural map $\rho_{G_0}^N : E[G] \otimes_{E[N]} \Lambda(N) \to E[G] \otimes_{E[G_0]} \Lambda(G_0)$ is an isomorphism of $(E[G], \Lambda(N))$-bimodules.

In order to construct the desired ring structure on $E[G] \otimes_{E[G_0]} \Lambda(G_0)$ it suffices to see that the right action of G_0 extends to an action of G by automorphisms of left $E[G]$-modules. Since

$$(\delta \otimes \delta' \mapsto \delta g \otimes g^{-1}\delta g) : E[G] \otimes_{E[G_0]} \Lambda(G_0) \to E[G] \otimes_{E[g^{-1}G_0 g]} \Lambda(g^{-1}G_0 g)$$

is an isomorphisms of left $E[G]$-modules we may let

$$(\delta \otimes \delta') \cdot g = [\rho_{G_0}^{G_0 \cap g^{-1}G_0 g} \circ (\rho_{g^{-1}G_0 g}^{G_0 \cap g^{-1}G_0 g})^{-1}](\delta g \otimes g^{-1}\delta' g).$$

The uniqueness assertion implies that up to isomorphism the E-algebra structure on $E[G] \otimes_{E[G_0]} \Lambda(G_0)$ does not depend on the choice of G_0. In particular, it coincides with the one considered previously if G is compact. We will therefore write $\Lambda(G)$ for the E-algebra $E[G] \otimes_{E[G_0]} \Lambda(G_0)$.

Remark 1.2. Let p be a prime number. If E is a complete valued field extension of $\mathbb{Q}_p$ and if G is a p-adic Lie group then $\Lambda(G)$ can be constructed as a quotient of the E-algebra $D(G,E)$ of locally analytic E-valued distributions on G (cf. [30], §6, and [31], §1, where $\Lambda(G)$ is denoted by $D^\infty(G,E)$). It is the E-algebra of locally constant E-valued distributions on G.

We denote by $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$ the category of pseudocompact E-vector spaces M carrying an E-linear action of G for which the structure map $G \times M \to M$ is *jointly continuous*, i.e. continuous for the product topology on the left.

The map $(g \mapsto g^{-1}) : G \to G$ extends to a canonical antiautomorphism $\Lambda(G) \to \Lambda(G)$ of E-algebras. In principle, this enables us to identify the categories of left and right $\Lambda(G)$-modules which we will often do. However, it will sometimes be necessary to distinguish between left and right $\Lambda(G)$-modules. By abuse of terminology, given a pseudocompact E-vector space with a jointly continuous action of G from the right, we will speak of a *right object* of $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$.

For the following assertions see [16], Lemma 3.1.1, and [17], Lemma 2.2.7.

Lemma 1.3. *Let M be a pseudocompact E-vector space carrying an action of G by continuous E-linear automorphisms. The following assertions are equivalent.*

(i) The structure map $G \times M \to M$ of the G-action is jointly continuous.

(ii) For any open subgroup H of G the structure map $H \times M \to M$ of the H-action is jointly continuous.

(iii) If G_0 is a compact open subgroup of G then the $E[G_0]$-module structure of M extends to a $\Lambda(G_0)$-module structure for which the map $\Lambda(G_0) \times M \to M$ is jointly continuous and such that M admits a basis of neighborhoods of 0 consisting of $\Lambda(G_0)$-submodules. $\square$

By Lemma 1.3 and by the construction of the E-algebra $\Lambda(G)$ we obtain the following assertion. A posteriori it justifies our notation $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$.

Corollary 1.4. *If M is an object of $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$ then the $E[G]$-module structure of M uniquely extends to a $\Lambda(G)$-module structure. Any morphism in $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$ is $\Lambda(G)$-linear. If G is compact then $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$ is the category of pseudocompact $\Lambda(G)$-modules in the sense of [11], page 443.* $\square$

In particular, if G_0 is a compact open subgroup of G and if M and N are right and left objects of $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$, respectively, then we can consider the complete tensor product $M \hat{\otimes}_{\Lambda(G_0)} N$ of M and N over $\Lambda(G_0)$ (cf. [11], page 446). Recall that by [11], Lemma 2.1,

$$M \hat{\otimes}_{\Lambda(G_0)} (\cdot) \quad \text{and} \quad (\cdot) \hat{\otimes}_{\Lambda(G_0)} N$$

are additive, covariant and right exact functors from the category $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$ to the category of pseudocompact E-vector spaces. If $G = \{1\}$ is the trivial group then $\Lambda(G) = \Lambda(G_0) = E$ and the complete tensor product is exact.

As in [16], Lemma 2.2.7, Theorem 1.1 and Lemma 1.3 imply the following result.

Theorem 1.5. *The functors* $\check{(\cdot)}$ *restrict to quasi-inverse equivalences of categories between* $\mathrm{Rep}^{\infty}_{E}(G)$ *and* $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$.

Proof. Let V be an object of $\mathrm{Rep}^{\infty}_{E}(G)$. By functoriality, G acts on $\check{V}$ by continuous E-linear automorphisms. Let G_0 be a compact open subgroup of G. If V_i is any finite dimensional subspace of V then there is an open normal subgroup N of G_0 which acts trivially on V_i. Thus, $\check{V}_i$ becomes a $\Lambda(G_0)$-module via $\Lambda(G_0) \to E[G_0/N]$. The resulting structure map $\Lambda(G_0) \times \check{V}_i \to \check{V}_i$ is jointly continuous. Passing to the projective limit, we see that the $E[G_0]$-structure of $\check{V} \cong \varprojlim_i \check{V}_i$ extends to a jointly continuous $\Lambda(G_0)$-structure. Therefore, Lemma 1.3 implies that $\check{V}$ is an object of $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$.

Conversely, if $M \cong \varprojlim_i M/M_i$ is an object of $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$ then Lemma 1.3 allows us to assume that all M_i are G_0-stable. Together with $G_0 \times M \to M$ also the maps $G_0 \times M/M_i \to M/M_i$ are continuous. Since M/M_i is discrete and finite dimensional, the action of G_0 is trivial upon restriction to a sufficiently small open subgroup. Therefore, the action of G_0 on $\check{M} \cong \varinjlim_i (M/M_i)\check{}$ and hence that of G is smooth. $\square$

Recall that an E-linear smooth representation V of G is called *admissible* if the space V^N of N-invariants is finite dimensional for any open subgroup N of G.

Dually, we borrow the terminology from [30], §3, and call an abstract $\Lambda(G)$-module M *coadmissible* if for any compact open subgroup G_0 of G and for any open normal subgroup N of G_0 the $E[G_0/N]$-module $E[G_0/N] \otimes_{\Lambda(G_0)} M$ is finitely generated (equivalently, finite dimensional over E) and if the natural homomorphism

$$M \longrightarrow \varprojlim_{N} E[G_0/N] \otimes_{\Lambda(G_0)} M \tag{1}$$

of $\Lambda(G_0)$-modules is bijective.

Remark 1.6. Let p be a prime number. If G possesses an open pro-p subgroup G_0 and if the characteristic of E is different from p then the E-algebra $\Lambda(G_0) = \varprojlim_N E[G_0/N]$ is the inverse limit of noetherian E-algebras with flat transition maps. Indeed, the group rings $E[G_0/N]$ are finite dimensional and semisimple. Thus, $\Lambda(G_0)$ satisfies the algebraic properties of a Fréchet-Stein algebra in the sense of [30], §3, where the notion of coadmissibility was introduced for the first time.

The isomorphism (1) allows us to endow a coadmissible $\Lambda(G)$-module M with the structure of a pseudocompact E-vector. Using that any continuous E-linear bijection between pseudocompact E-vector spaces is a topological isomorphism one can check that this topology does not depend on the choice of G_0 and that the action of G on M is by continuous E-linear automorphisms. It then follows from Lemma 1.3 that a coadmissible $\Lambda(G)$-module M is naturally an object of $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$.

If M is an object of $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$ and if H is any subgroup of G then we denote by M_H the largest Hausdorff quotient of M on which H acts trivially. It is the quotient of M by the closure $\overline{M(H)}$ of the subspace generated by all elements of the form $hm - m$ with $h \in H$ and $m \in M$.

Lemma 1.7. *Let M be an object of $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$. If G_0 is a compact open subgroup of G and if N is a normal open subgroup of G_0 then there is a natural isomorphism $E[G_0/N] \otimes_{\Lambda(G_0)} M \cong M_N$ of $\Lambda(G_0)$-modules. The pseudocompact $\Lambda(G)$-module M is coadmissible if and only if M_N is finite dimensional over E for any open subgroup N of G. Moreover, an abstract $\Lambda(G)$-module M is coadmissible if and only if the required conditions are satisfied for at least one compact open subgroup G_0 of G.*

Proof. We denote by I_N the kernel of the homomorphism $\Lambda(G_0) \to E[G_0/N]$. Since it is open in $\Lambda(G_0)$ and since $E[G_0] \subseteq \Lambda(G_0)$ is dense, the kernel of the homomorphism $E[G_0] \to E[G_0/N]$ is dense in I_N. The latter is the ideal generated by all elements of the form $n - 1$ with $n \in N$. This implies $I_N \cdot M \subseteq \overline{M(N)}$ because of the continuity of the $\Lambda(G_0)$-action on M (cf. Lemma 1.3). By definition $I_N \cdot M$ is a dense subspace of $\overline{M(N)}$.

On the other hand, the $\Lambda(G_0)$-bihomomorphism $I_N \times M \to M$ sending (δ, m) to $\delta \cdot m$ is jointly continuous. By the universal property of the complete tensor product it extends to a continuous E-linear map $I_N \hat{\otimes}_{\Lambda(G_0)} M \to M$ of pseudocompact E-vector spaces. Its image is closed and hence equal to $\overline{M(N)}$. Now the right exactness of the complete tensor product, together with [11], Lemma 2.1 (ii), implies that the sequence

$$I_N \hat{\otimes}_{\Lambda(G_0)} M \longrightarrow M \longrightarrow E[G_0/N] \otimes_{\Lambda(G_0)} M \longrightarrow 0$$

is exact. Therefore, $M_N \cong E[G_0/N] \otimes_{\Lambda(G_0)} M$, as claimed (cf. also [11], Lemma 4.2 (ii)). This also shows $\overline{M(N)} = I_N \cdot M$.

By Theorem 1.5 the E-linear G-representation $\check{M}$ is smooth. In particular, $\check{M} = \varinjlim_N \check{M}^N$. It is straightforward to check that $(\check{M}^N)\check{} \cong M_N$. Therefore, $M \cong \check{\check{M}} \cong \varprojlim_N M_N$, proving the first part of the lemma.

As for the final assertion, if (1) is an isomorphism for some compact open subgroup G_0 of G then M is at least an object of $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G_0)}$. The first part of the proof then shows that M_N is finite dimensional for any open normal subgroup N of G_0 and that the natural map $M \to \varprojlim_N M_N$ is an isomorphism. However, for this description of the topology it is clear that G acts by continuous E-linear automorphisms. Therefore, M is an object of $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$. By the first part of the lemma, M is coadmissible. $\square$

By $\mathrm{Rep}^{\infty}_E(G)^a$ and $\mathcal{C}_G$ we denote the full subcategories of $\mathrm{Rep}^{\infty}_E(G)$ and $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$ consisting of admissible and coadmissible objects, respectively. As an immediate consequence of Theorem 1.5, Lemma 1.7 and the above relation $(\check{M}^N)\check{} \cong M_N$ one obtains the following result (for a particular situation see also [30], Theorem 6.6 and its proof).

Corollary 1.8. *The functors $\check{(\cdot)}$ restrict to quasi-inverse equivalences between the categories $\mathrm{Rep}^{\infty}_E(G)^a$ and $\mathcal{C}_G$.* $\square$

For the situation that we will be interested in most, the notion of coadmissibility has the following alternative characterization.

Proposition 1.9. *Let p be a prime number. Assume that G possesses an open subgroup which is a pro-p group and assume that the characteristic of E is equal to p.*

(i) An object $M \in \mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$ is coadmissible if and only if the underlying $\Lambda(G_0)$-module of M is finitely generated for some and hence for any compact open subgroup G_0 of G.

(ii) If G is a p-adic Lie group then an object $M \in \mathrm{Mod}_{\Lambda(G)}$ is coadmissible if and only if the underlying $\Lambda(G_0)$-module of M is finitely generated for some and hence for any compact open subgroup G_0 of G. In this case $\mathcal{C}_G$ is a full Serre subcategory of $\mathrm{Mod}_{\Lambda(G)}$.

Proof. By Lemma 1.7 the assertions do not depend on G_0. Therefore, we may assume G_0 to be an open pro-p subgroup of G. Since E is of characteristic p the ring $\Lambda(G_0)$ is then local with maximal ideal $I_{G_0} = \ker(\Lambda(G_0) \to E)$ (cf. [27], Proposition 5.2.16 (iii) and its proof).

If $M \in \mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$ is coadmissible then $I_{G_0}M$ is of finite codimension in M because $M/I_{G_0}M \cong E \otimes_{\Lambda(G_0)} M$. By the topological Nakayama lemma we obtain that the underlying $\Lambda(G_0)$-module of M is finitely generated (cf. [11], Corollary 1.5).

Conversely, if $M \in \mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$ is finitely generated over $\Lambda(G_0)$ then it is also finitely generated over $\Lambda(N)$ for any compact open subgroup N of G_0. The right exactness of the functor $(\cdot) \otimes_{\Lambda(N)} E$ implies that $M_N \cong E \otimes_{\Lambda(N)} M$ is finite dimensional for any N. Thus, M is coadmissible by Lemma 1.7.

As we shall recall in Theorem 3.1 below, if G is a p-adic Lie group then the ring $\Lambda(G_0)$ is noetherian. As a consequence, any finitely generated $\Lambda(G_0)$-module M is finitely presented. Since any $\Lambda(G_0)$-linear map between finitely generated free $\Lambda(G_0)$-modules is continuous with closed image, M is naturally a pseudocompact E-vector space. For this canonical topology the action of G is by continuous automorphisms so that M is coadmissible by (i). Further, any $\Lambda(G_0)$-linear map between finitely generated $\Lambda(G_0)$-modules is continuous for the canonical topology. As a consequence, $\mathcal{C}_G$ is a full subcategory of $\mathrm{Mod}_{\Lambda(G)}$. □

Remark 1.10. If G possesses an open pro-p subgroup G_0 and if the characteristic of E is zero then the $\Lambda(G_0)$-module $\check{V}$ associated with an admissible smooth E-linear representation V of G is not necessarily finitely generated. Rather, the representations for which this is true are characterized by a global multiplicity condition (cf. [32], Proposition 2.1 and its proof).

If M is an E-linear representation of G and if N runs through the set of open subgroups of G then we denote by

$$\Sigma^0(M) = \Sigma^0_G(M) := \varinjlim_N M^N \in \mathrm{Rep}^\infty_E(G)$$

the G-subrepresentation of M consisting of all smooth vectors, i.e. of vectors whose stabilizers in G are open. The endofunctor

$$S^0 = S^0_G = \Sigma^0_G \circ \check{(\cdot)} : \mathrm{Rep}^\infty_E(G) \longrightarrow \mathrm{Rep}^\infty_E(G)$$

of the category $\mathrm{Rep}^\infty_E(G)$ is called the 0-*th smooth dual.*

2 Smooth duality in non-natural characteristic

In this section we assume that p is a prime number, that G admits an open pro-p subgroup G_0 and that the characteristic of E is different from p. Following ideas of [31], §1, the endofunctor $\check{(\cdot)} \circ \Sigma^0$ of the category $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$ can be described as follows.

Denote by $C^\infty_c(G, E)$ the E-vector space of all compactly supported locally constant maps $G \to E$. It carries commuting smooth E-linear left and right

G-actions by left and right translation, respectively. According to Theorem 1.5 the pseudocompact E-vector space $\Delta(G) = C_c^\infty(G, E)\check{\ }$ admits commuting and jointly continuous E-linear actions of G from the left and from the right. In particular, Corollary 1.4 implies that $\Delta(G)$ is a $(\Lambda(G), \Lambda(G))$-bimodule.

If $f \in C_c^\infty(G, E)$ is right invariant under an open subgroup N of G_0 then we set

$$\mu_{G_0}(f) = (G_0 : N)^{-1} \sum_{g \in G/N} f(g) \in E,$$

using that the index $(G_0 : N)$ is a power of p and therefore invertible in E. Apparently, $\mu_{G_0}(f)$ is independent of the chosen subgroup N of G_0 and we obtain an element $\mu_{G_0} \in \Delta(G)$ satisfying $g \cdot \mu_{G_0} = \mu_{G_0}$ for all $g \in G$. Thus, μ_{G_0} is a left Haar measure on G and as in [12], Proposition 3.1, or [38], I.2, is seen to be unique up to scalars as a G-left invariant element of $\Delta(G)$.

We let $\delta_G : G \to E^\times$ be the locally constant modulus character of G defined by $\mu_{G_0} \cdot g = \delta_G(g)\mu_{G_0}$ for all $g \in G$. We have

$$\delta_G(g) = \frac{(gG_0g^{-1} : G_0 \cap gG_0g^{-1})}{(G_0 : G_0 \cap gG_0g^{-1})},$$

a value which is in fact independent of G_0 (cf. [38], I.2.7). We denote by $\Delta(G) \otimes_E \delta_G^{-1}$ the $(\Lambda(G), \Lambda(G))$-bimodule whose underlying G-action is defined by

$$g(\delta \otimes 1)g' := g\delta g' \otimes \delta_G(g')^{-1},$$

i.e. the left action of $\Lambda(G)$ is unchanged and the right action of $\Lambda(G)$ is twisted by δ_G^{-1}.

As in [31], Lemma 1.4, one shows that the outer square of the diagram

$$\begin{tikzcd}
\mathrm{Rep}_E^\infty(G) \arrow[r, "\check{(\cdot)}", "\cong"'] \arrow[d, "S^0"'] & \mathrm{Mod}_{\Lambda(G)}^{pc} \arrow[r, "\text{forget}"] \arrow[d, dotted] & \mathrm{Mod}_{\Lambda(G)} \arrow[d, "\mathrm{Hom}_{\Lambda(G)}(\cdot, \Delta(G) \otimes_E \delta_G^{-1})"] \\
\mathrm{Rep}_E^\infty(G) \arrow[r, "\check{(\cdot)}", "\cong"'] & \mathrm{Mod}_{\Lambda(G)}^{pc} \arrow[r, "\text{forget}"] & \mathrm{Mod}_{\Lambda(G)}
\end{tikzcd}$$

is commutative up to natural isomorphism. A priori, the right vertical functor takes values in the category of abstract left $\Lambda(G)$-modules by making use of the bimodule structure of $\Delta(G) \otimes_E \delta_G^{-1}$. A posteriori, Theorem 1.5 shows that $\mathrm{Hom}_{\Lambda(G)}(\cdot, \Delta(G) \otimes_E \delta_G^{-1})$ restricts to an endofunctor of the category $\mathrm{Mod}_{\Lambda(G)}^{\mathrm{pc}}$. This is the vertical arrow in the middle.

Under our assumptions on G and E, the category $\mathrm{Rep}_E^\infty(G)^a$ of admissible smooth E-linear G-representations is abelian and the 0-th smooth dual S^0 restricts to an exact anti-equivalence of $\mathrm{Rep}_E^\infty(G)^a$ (cf. [38], Proposition I.4.18).

It follows from Corollary 1.8 that the functor $\mathrm{Hom}_{\Lambda(G)}(\cdot, \Delta(G) \otimes_E \delta_G^{-1})$ respects coadmissibility and is exact. The latter property was given an algebraic explanation in [31], Proposition 2.4. Namely, the left $\Lambda(G)$-module $\Delta(G)$ is injective. This statement can be reduced to the selfinjectivity of the ring $\Lambda(G_0)$ which is in fact isomorphic to a direct product of finite dimensional simple E-algebras (cf. the proof of [31], Proposition 2.4).

3 Smooth duality in natural characteristic

In this section we will assume that p is a prime number, E is a field of characteristic p and G is a Lie group of finite dimension $d = \dim(G)$ over the field $\mathbb{Q}_p$ of p-adic numbers.

The $(\Lambda(G), \Lambda(G))$-bimodules $C_c^\infty(G, E)$ and $\Delta(G) = C_c^\infty(G, E)\check{}$ are defined as in the previous section. Given an object $M \in \mathrm{Mod}_{\Lambda(G)}$ we set

$$E^i(M) = E_G^i(M) = \mathrm{Ext}^i_{\Lambda(G)}(M, \Delta(G)).$$

Again the right $\Lambda(G)$-action on $\Delta(G)$ gives rise to a right $\Lambda(G)$-action on $E^i(M)$. We will view $E^i(M)$ as a left $\Lambda(G)$-module through the canonical antiautomorphism of $\Lambda(G)$ and hence view E^i as an endofunctor of the category $\mathrm{Mod}_{\Lambda(G)}$.

The results of the previous section suggest that in order to define a good notion of smooth duality in natural characteristic one has to study the functor $E^0 = \mathrm{Hom}_{\Lambda(G)}(\,\cdot\,, \Delta(G))$ and its derivatives E^i. Their behavior is governed by the Auslander duality for completed group rings with p-torsion coefficients. For compact groups this was worked out by Venjakob in [37]. We shall closely follow ideas of Schneider and Teitelbaum from [31] to extend these results to the general case.

For the notion of Auslander regularity we refer to [23], III.2.1. Recall that by [15], Theorem 8.32, the p-adic Lie group G admits a compact open subgroup G_0 which is a *uniform pro-p group* in the sense of [15], Definition 4.1. The first assertion of the following theorem is due to Lazard (cf. [25], Chapitre V, Proposition 2.2.4). In a more general form the second result is due to Venjakob (cf. [37], Theorem 3.30 (ii) and the remarks following its proof).

Theorem 3.1. *If G_0 is a compact open subgroup of G then the ring $\Lambda(G_0)$ is noetherian. If moreover G_0 is a uniform pro-p group then the ring $\Lambda(G_0)$ is Auslander regular of global dimension $d = \dim(G) = \dim(G_0)$.*

Proof. We may assume that G_0 is a uniform pro-p group. Let I_{G_0} denote the maximal ideal of the local ring $\Lambda(G_0)$ (cf. [27], Proposition 5.2.16 (iii) and its proof). The ring $\Lambda(G_0)$ is separated and complete for its I_{G_0}-adic topology and the graded ring associated with the I_{G_0}-adic filtration of $\Lambda(G_0)$ is isomorphic to a polynomial ring in d variables over E (cf. [15], Theorem 7.24). It follows from [23], II.2.2 Proposition 1, III.2.2 Theorem 5, III.2.3 Theorem 5 and III.2.4 Example 1, that $\Lambda(G_0)$ is a noetherian Auslander regular domain of finite global dimension d. $\square$

If G is compact then the following result can be found in [24], Lemma 2.3. By following the arguments of [31], Lemma 2.2, it can be proved for any locally profinite topological group G.

Proposition 3.2. *Let G_0 be a compact open subgroup of G, and let M be a $\Lambda(G)$-module. If $\ell = \ell_{G,G_0} : \Delta(G) \cong \prod_{g\in G/G_0} g\Lambda(G_0) \longrightarrow \Lambda(G_0)$ denotes the projection onto the component corresponding to the trivial coset $gG_0 = G_0$ then the map $\mathrm{Hom}_{\Lambda(G)}(M, \Delta(G)) \to \mathrm{Hom}_{\Lambda(G_0)}(M, \Lambda(G_0))$ sending φ to $\ell \circ \varphi$ is an isomorphism of $\Lambda(G_0)$-modules. In particular, there are natural isomorphisms*

$$\mathrm{Ext}^i_{\Lambda(G)}(M, \Delta(G)) \xrightarrow{\sim} \mathrm{Ext}^i_{\Lambda(G_0)}(M, \Lambda(G_0))$$

of $\Lambda(G_0)$-modules for all $i \geq 0$. $\square$

Corollary 3.3. *For any integer $i \geq 0$ the functor $E^i : \mathrm{Mod}_{\Lambda(G)} \to \mathrm{Mod}_{\Lambda(G)}$ preserves coadmissibility, i.e. it restricts to a functor $E^i : \mathcal{C}_G \to \mathcal{C}_G$.*

Proof. Let G_0 be a compact open subgroup of G, and let $M \in \mathcal{C}_G$. According to Theorem 3.1 the ring $\Lambda(G_0)$ is noetherian. Since M is finitely generated over $\Lambda(G_0)$ (cf. Proposition 1.9 (ii)) this implies that M admits a free resolution $P^\bullet$ by finitely generated free $\Lambda(G_0)$-modules P^i. By Proposition 3.2 the underlying $\Lambda(G_0)$-module of $E^i(M)$ is isomorphic to the i-th cohomology group of the complex $\mathrm{Hom}_{\Lambda(G_0)}(P^\bullet, \Lambda(G_0))$, hence is finitely generated. By Proposition 1.9 (ii) the $\Lambda(G)$-module $E^i(M)$ is coadmissible. $\square$

For any $\Lambda(G)$-module M we denote by

$$j(M) = j_G(M) = \min\{i \geq 0 \mid E^i(M) \neq 0\}$$

the *grade* or *codimension* of M over $\Lambda(G)$. If $M = 0$ then it is infinite but otherwise is finite (cf. [23], Chapter III, §2, 2.1). If M is non-zero and coadmissible then Proposition 3.2 and Theorem 3.1 show that $j(M) \leq d =$

$\dim(G)$ and that $j(M) = j_{G_0}(M)$ is equal to the grade of M over $\Lambda(G_0)$ for any compact open subgroup G_0 of G. We call

$$d(M) = d_G(M) = d - j(M)$$

the *dimension* of M over $\Lambda(G)$.

Let G_0 be an open subgroup of G which is a uniform pro-p group. Since the ring $\Lambda(G_0)$ is Auslander regular (cf. Theorem 3.1), any finitely generated $\Lambda(G_0)$-module M admits a filtration

$$0 = F^{d+1}(M) \subseteq F^d(M) \subseteq \ldots \subseteq F^0(M) = M$$

by $\Lambda(G_0)$-submodules which is characterized by the property that a submodule $M' \subseteq M$ satisfies $M' \subseteq F^i(M)$ if and only if $j_{G_0}(M') \geq i$ (cf. [4], Chapter 2, Theorem 4.15). It is called the *dimension filtration* of M. If $M \neq 0$ then

$$j_{G_0}(M) = \max\{i \geq 0 \,|\, F^i(M) = M\} \tag{2}$$

by [23], III.2.1 Proposition 5.

If N is an open subgroup of G_0 then the dimension filtration of M over $\Lambda(N)$ coincides with that over $\Lambda(G_0)$. Indeed, denoting these filtrations by $F_N^\bullet(M)$ and $F_{G_0}^\bullet(M)$, respectively, we have $i \leq j_N(F_N^i(M)) = j_{G_0}(F_N^i(M))$ whence $F_N^i(M) \subseteq F_{G_0}^i(M)$ and similarly $F_{G_0}^i(M) \subseteq F_N^i(M)$. Using this invariance property of the dimension filtration, the proof of the following proposition is formally the same as that of [30], Proposition 8.11.

Proposition 3.4. *Let M be a coadmissible $\Lambda(G)$-module.*

(i) The dimension filtration $F^\bullet(M)$ of M with respect to any open uniform pro-p subgroup G_0 of M consists of coadmissible $\Lambda(G)$-submodules of M which are independent of the choice of G_0.

(ii) If M' is a $\Lambda(G)$-submodule of M then $j(M') \geq i$ if and only if $M' \subseteq F^i(M)$.

(iii) We have $j(M) = \max\{i \geq 0 \,|\, F^i(M) = M\}$.

(iv) All nonzero $\Lambda(G)$-submodules of $F^i(M)/F^{i+1}(M)$ have grade i. $\square$

Recall that $\mathcal{C}_G$ is a Serre subcategory of $\mathrm{Mod}_{\Lambda(G)}$ and hence is abelian (cf. Proposition 1.9 (ii)). If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of coadmissible $\Lambda(G)$-modules then

$$j(M) = \min\{j(M'), j(M'')\} \tag{3}$$

by [23], III.2.1 Corollary 6. For $0 \leq i \leq d+1$ we denote by $\mathcal{C}_G^i$ the full subcategory of $\mathcal{C}_G$ consisting of all objects M with $j(M) \geq i$. We have $\mathcal{C}_G = \mathcal{C}_G^0$, $\mathcal{C}_G^{d+1} = 0$, and (3) implies that $\mathcal{C}_G^{i+1}$ is a Serre subcategory of $\mathcal{C}_G^i$ for $0 \leq i \leq d$. Hence we may form the abelian quotient category $\mathcal{C}_G^i/\mathcal{C}_G^{i+1}$.

Let $i \geq 0$ and consider the functor $E^i : \mathcal{C}_G \to \mathcal{C}_G$ of Corollary 3.3. Let G_0 be an open subgroup of G which is a uniform pro-p group. By Proposition 3.2 and the Auslander regularity of $\Lambda(G_0)$ the functor E^i factors through the embedding $\mathcal{C}_G^i \subseteq \mathcal{C}_G$. Further, if $M \in \mathcal{C}_G^{i+1}$ then $E^i(M) = 0$ by definition of the grade. Therefore, the functor $\mathcal{C}_G^i \hookrightarrow \mathcal{C}_G \xrightarrow{E^i} \mathcal{C}_G^i \to \mathcal{C}_G^i/\mathcal{C}_G^{i+1}$ induces a functor $\mathcal{C}_G^i/\mathcal{C}_G^{i+1} \to \mathcal{C}_G^i/\mathcal{C}_G^{i+1}$ that will again be denoted by E^i.

Theorem 3.5. *If $0 \leq i \leq d$ then the functor $E^i : \mathcal{C}_G^i/\mathcal{C}_G^{i+1} \to \mathcal{C}_G^i/\mathcal{C}_G^{i+1}$ is an equivalence of abelian categories which is quasi-inverse to itself.*

Proof. If G is a uniform pro-p group then the ring $\Lambda(G)$ is Auslander regular (cf. Theorem 3.1). In this case, the statement in question is a more general d-step duality theorem for Auslander-Gorenstein rings (cf. [2], Theorem 1.2).

In the general case let $M \in \mathcal{C}_G$. Following the arguments of the proof of [31], Proposition 4.3, consider the isomorphism

$$\begin{aligned} & R\operatorname{Hom}_{\Lambda(G)}(R\operatorname{Hom}_{\Lambda(G)}(M, \Delta(G)), \Delta(G)) \\ \cong\ & R\operatorname{Hom}_{\Lambda(G_0)}(R\operatorname{Hom}_{\Lambda(G_0)}(M, \Delta(G_0)), \Delta(G_0)) \end{aligned}$$

of complexes of $\Lambda(G_0)$-modules coming from Proposition 3.2. The properties of $\Lambda(G_0)$ then show that the natural map

$$M \longrightarrow R\operatorname{Hom}_{\Lambda(G)}(R\operatorname{Hom}_{\Lambda(G)}(M, \Delta(G)), \Delta(G))$$

is a quasi-isomorphism of complexes of $\Lambda(G)$-modules. One can then argue as in the proof of [31], Proposition 5.2. In fact, the hypercohomology spectral sequence for the composition $R\operatorname{Hom}_{\Lambda(G)}(R\operatorname{Hom}_{\Lambda(G)}(M, \Delta(G)), \Delta(G))$ coincides with the spectral sequence (0-2) in [2] if M is viewed as an object of $\mathcal{C}_{G_0}$. Therefore, the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^i_{G_0}(M)/F^{i+1}_{G_0}(M) & \longrightarrow & E^i_{G_0}(E^i_{G_0}(M)) & \longrightarrow & Q_{i+2}(M) & \longrightarrow & 0 \\ & & \| & & \| & & & & \\ & & F^i(M)/F^{i+1}(M) & & E^i(E^i(M)) & & & & \end{array}$$

considered in [2], (0-3), is an exact sequence in $\mathcal{C}_G$. This can also be proved directly using the naturality of the spectral sequence and using the G-stability of the dimension filtration (cf. Proposition 3.4). Since $j(N) = j_{G_0}(N)$ for any $N \in \mathcal{C}_G$, the arguments in the proof of [2], Theorem 1.2, show that $M = F^i(M) \to F^i(M)/F^{i+1}(M) \to E^i(E^i(M))$ induces an isomorphism $M \cong E^i(E^i(M))$ in $\mathcal{C}_G^i/\mathcal{C}_G^{i+1}$ for any $M \in \mathcal{C}_G^i$. $\square$

If V is an object of $\mathrm{Rep}_E^\infty(G)$ then $\check{V} \in \mathrm{Mod}_{\Lambda(G)}$ and we call

$$j(V) = j(\check{V}) \quad \text{and} \quad d(V) = d(\check{V})$$

the *grade* and the *dimension* of V, respectively. For $0 \leq i \leq d+1$ we denote by $\mathrm{Rep}_E^\infty(G)_i^a$ the full subcategory of $\mathrm{Rep}_E^\infty(G)^a$ consisting of all objects V with $j(V) \geq i$. This gives rise to a descending filtration

$$\mathrm{Rep}_E^\infty(G)^a = \mathrm{Rep}_E^\infty(G)_0^a \supseteq \mathrm{Rep}_E^\infty(G)_1^a \supseteq \ldots \supseteq \mathrm{Rep}_E^\infty(G)_{d+1}^a = 0$$

of the category $\mathrm{Rep}_E^\infty(G)^a$ by Serre subcategories such that $\check{(\cdot)}$ restricts to an equivalence $\mathrm{Rep}_E^\infty(G)_i^a \cong \mathcal{C}_G^i$ (cf. Corollary 1.8).

Definition 3.6. Let $M \in \mathcal{C}_G$ and $V \in \mathrm{Rep}_E^\infty(G)^a$.

(i) We call M *holonomic* (resp. *pure*, resp. *Cohen-Macaulay*) if $j(M) \geq d$ (resp. if $F^{j(M)+1}(M) = 0$, resp. if $E^i(M) = 0$ for $i \neq j(M)$).

(ii) We call V *holonomic* (resp. *pure*, resp. *Cohen-Macaulay*) if the object $\check{V} \in \mathcal{C}_G$ is holonomic (resp. pure, resp. Cohen-Macaulay).

Remark 3.7. By definition any holonomic object is Cohen-Macaulay, and by [37], Proposition 3.5 (v) and Proposition 3.9, any Cohen-Macaulay object is pure. Further, since $j(M) = j_{G_0}(M)$ for any compact open subgroup G_0 of G, a non-zero coadmissible $\Lambda(G)$-module M is Cohen-Macaulay if and only if its grade is equal to its projective dimension over $\Lambda(G_0)$.

Proposition 3.8. *An object of $\mathcal{C}_G$ or $\mathrm{Rep}_E^\infty(G)^a$ is holonomic if and only if its underlying E-vector space is finite dimensional.*

Proof. Let $M \in \mathcal{C}_G$ be holonomic, and let G_0 be an open subgroup of G which is a uniform pro-p group. Since $j(M) = j_{G_0}(M)$ the underlying $\Lambda(G_0)$-module is holonomic, as well, hence is of finite length over $\Lambda(G_0)$ (cf. Theorem 3.1 and [2], Corollary 1.3, or [23], III.4.2 Theorem 3 (2)). Since $\Lambda(G_0)$ is a local ring with residue field E, the only simple $\Lambda(G_0)$-module is E with the trivial action of G_0. Thus, $\dim_E(M) < \infty$.

Conversely, if $M \in \mathcal{C}_G$ is of finite dimension over E then it is of finite length over $\Lambda(G_0)$. Thus, it is a finite successive extension of the trivial $\Lambda(G_0)$-module E. Using (3) it suffices to show that E is holonomic over $\Lambda(G_0)$. This in turn follows from Lazard's result that G_0 is a Poincaré duality group at p (cf. [35], Theorem 5.1.5). This statement can be formulated as

$$\mathrm{H}^i_{\mathrm{cont}}(G_0, \mathbb{Z}_p[\![G_0]\!]) \cong \begin{cases} 0\ , & 0 \leq i < d \\ \mathbb{Z}_p, & i = d. \end{cases} \tag{4}$$

Note that there are isomorphisms $\mathrm{H}^i_{\mathrm{cont}}(G_0, \mathbb{Z}_p[\![G_0]\!]) \cong \mathrm{Ext}^i_{\mathbb{Z}_p[\![G_0]\!]}(\mathbb{Z}_p, \mathbb{Z}_p[\![G_0]\!])$ for all $i \geq 0$ by [25], Chapitre V, Théorème 3.2.7. Instead of deriving the analogous statement over the field E from (4) we simply refer to [23], III.2.5 Theorem 2, stating that $j_{G_0}(E)$ is equal to the grade of the $\mathrm{gr}(\Lambda(G_0))$-module E where $\mathrm{gr}(\Lambda(G_0))$ denotes the graded ring associated with the maximal adic filtration on $\Lambda(G_0)$. Since $\mathrm{gr}(\Lambda(G_0)) \cong E[X_1, \ldots, X_d]$ is isomorphic to a polynomial ring in d variables $X_1, \ldots, X_d$ over E (cf. [15], Theorem 7.24), we need to see that the $E[X_1, \ldots, X_d]$-module E with $X_i \cdot E = 0$ for $1 \leq i \leq d$ is holonomic. This is a well-known result (cf. [7], X.1.4 Corollaire 1) that can also be deduced from a theorem of Roos (cf. [23], III.4.1 Theorem 7). Since $\mathrm{Ext}^d_{\Lambda(G_0)}(E, \Lambda(G_0))$ is an E-linear subquotient of $\mathrm{Ext}^d_{\mathrm{gr}(\Lambda(G_0))}(E, \mathrm{gr}(\Lambda(G_0)))$ (cf. [23], III.2.2 Proposition 4) we actually obtain

$$\mathrm{Ext}^i_{\Lambda(G_0)}(E, \Lambda(G_0)) \cong \begin{cases} 0\,, & 0 \leq i < d \\ E, & i = d. \end{cases} \tag{5}$$

The statements involving V follow from the fact that V is finite dimensional if and only if $\check{V}$ is. $\square$

As a consequence we have the following result. It shows that the 0-th smooth duality functor does not yield much information about the category $\mathrm{Rep}^\infty_E(G)^a$ in natural characteristic.

Proposition 3.9. *If $V \in \mathrm{Rep}^\infty_E(G)^a$ then $S^0(V) = F^d(\check{V})$ is finite dimensional over E. In particular, if V is irreducible and not finite dimensional over E then $S^0(V) = 0$. The largest full subcategory of $\mathrm{Rep}^\infty_E(G)^a$ on which the 0-th smooth dual S^0 restricts to an anti-equivalence of categories is the category of finite dimensional smooth representations of G over E. Via Pontryagin duality it is anti-equivalent to the category $\mathcal{C}^d_G$.*

Proof. Let G_0 be a compact open subgroup of G. If $\check{v} \in S^0(V)$ then $\Lambda(G_0) \cdot \check{v}$ is a $\Lambda(G_0)$-submodule of $S^0(V)$ which is of finite dimension over E. By Proposition 3.8 it is holonomic, hence is contained in $F^d(\check{V})$ (cf. Proposition 3.4). This shows $S^0(V) \subseteq F^d(\check{V})$.

Conversely, $F^d(\check{V})$ is holonomic, hence is of finite dimension over E by Proposition 3.8. On the other hand, the $\Lambda(G)$-module $F^d(\check{V})$ is coadmissible, hence is a pseudocompact E-vector space. Alternatively, it is easy to see that any finite dimensional E-subspace of a pseudocompact E-vector space is automatically closed and pseudocompact. In particular, this applies to $F^d(\check{V})$ inside $\check{V}$. At any rate, the subspace topology of $F^d(\check{V})$ induced by $\check{V}$ is discrete, and by Lemma 1.1 the action of G_0 on $F^d(\check{V})$ is smooth. As a consequence, $F^d(\check{V}) \subseteq S^0(V)$.

If V is irreducible then $\check{V}$ is a simple $\Lambda(G)$-module (cf. Corollary 1.8 and Proposition 1.9 (ii)). If V is not finite dimensional then neither is $\check{V}$ and we must have $F^d(\check{V}) = 0$ because $F^d(\check{V})$ is a holonomic $\Lambda(G)$-submodule of $\check{V}$

and hence is finite dimensional over E (cf. Proposition 3.4 and Proposition 3.8). Thus, also $S^0(V) = 0$. $\square$

Remark 3.10. Apparently, $S^0(V)$ is always a $\Lambda(G)$-submodule of $\check{V}$, and the vanishing result of Proposition 3.9 can be proved without referring to Proposition 3.8. Namely, we have $S^0(V) = \varinjlim_N \check{V}^N$ where N runs through the open normal subgroups of some compact open subgroup G_0 of G. Since the ring $\Lambda(G_0)$ is noetherian (cf. Theorem 3.1) and since the $\Lambda(G_0)$-module $\check{V}$ is finitely generated (cf. Proposition 1.9) so is $S^0(V)$. Therefore, there is an open normal subgroup N of G_0 such that $S^0(V) = \check{V}^N$. However, this implies that $S^0(V)$ is finitely generated over $E[G_0/N]$, hence is finite dimensional over E. If $V \in \mathrm{Rep}_E^\infty(G)^a$ is irreducible and infinite dimensional then the $\Lambda(G)$-submodule $S^0(V)$ of the simple $\Lambda(G)$-module $\check{V}$ must be zero.

We shall now explain how the d-step duality of Theorem 3.5 can be interpreted in terms of S^0 and suitable higher smooth duality functors. In the case of a compact group G this was shown by Venjakob in [37], §5, using local cohomology groups. We shall slightly reformulate his results and extend them to the case of possibly non-compact groups.

If N_1 and N_2 are open subgroups of G with $N_1 \subseteq N_2$ then the restriction of scalars $\mathrm{Mod}_{\Lambda(N_2)} \to \mathrm{Mod}_{\Lambda(N_1)}$ is exact and therefore induces a morphism $(\mathrm{Ext}^i_{\Lambda(N_2)}(E, \cdot))_{i\geq 0} \to (\mathrm{Ext}^i_{\Lambda(N_1)}(E, \cdot))_{i\geq 0}$ of δ-functors $\mathrm{Mod}_{\Lambda(N_2)} \to \mathrm{Mod}_{\Lambda(N_1)}$, where E carries the trivial action of N_1 and N_2. Since $\Lambda(N_2)$ is a finitely generated free $\Lambda(N_1)$-module, the restriction of scalars $\mathrm{Mod}_{\Lambda(N_2)} \to \mathrm{Mod}_{\Lambda(N_1)}$ preserves projective objects and both δ-functors can be computed from a projective resolution of E over $\Lambda(N_2)$. This will be of importance later.

If $M \in \mathrm{Mod}_{\Lambda(G)}$ and if $i \geq 0$ we set

$$\Sigma^i(M) = \Sigma^i_G(M) = \varinjlim_N \mathrm{Ext}^i_{\Lambda(N)}(E, M),$$

where the limit is taken over the directed family of all open subgroups N of G and takes values in the category of E-vector spaces.

Given an open subgroup N of G and an element $g \in G$, conjugation by g induces an isomorphism $c(g) : \Lambda(N) \to \Lambda(gNg^{-1})$ of E-subalgebras of $\Lambda(G)$. Restriction of scalars $c(g)_* : \mathrm{Mod}_{\Lambda(N)} \to \mathrm{Mod}_{\Lambda(gNg^{-1})}$ along $c(g)$ is an equivalence of abelian categories. If $M \in \mathrm{Mod}_{\Lambda(G)}$ then $(m \mapsto gm) : c(g)_*(M) \to M$ is an isomorphism of $\Lambda(gNg^{-1})$-modules. Since $c(g)_*(E) = E$ over $\Lambda(gNg^{-1})$ we obtain E-linear isomorphisms

$$\mathrm{Ext}^i_{\Lambda(N)}(E, M) \cong \mathrm{Ext}^i_{\Lambda(gNg^{-1})}(c(g)_*(E), c(g)_*(M)) \cong \mathrm{Ext}^i_{\Lambda(gNg^{-1})}(E, M),$$

which are trivial if $g \in N$. Indeed, for $i = 0$ and $g \in N$ this is the restriction of the map $g : M \to M$ to $M^N \to M^{gNg^{-1}} = M^N$. In general, one uses that $\mathrm{Ext}^i_{\Lambda(N)}(E, M)$ is the i-th cohomology group of $(I^\bullet)^N$ where $M \to I^\bullet$ is an injective resolution of M in $\mathrm{Mod}_{\Lambda(N)}$.

In this way we obtain a smooth E-linear action of G on $\Sigma^i(M)$ and regard Σ^i as a functor $\mathrm{Mod}_{\Lambda(G)} \to \mathrm{Rep}^\infty_E(G)$. Note that for $i = 0$ evaluation at $1 \in E$ induces a G-equivariant isomorphism $\Sigma^0(M) = \varinjlim_N M^N$ so that our notation is consistent with that introduced at the end of section 1.

Remark 3.11. (i) In a different setting the functors Σ^i were considered by Lazard and called the *stable cohomology groups of* G (cf. [25], Chapitre V, Théorème 2.4.10 and Théorème 3.2.7).

(ii) If G_0 is an open subgroup of G which is a uniform pro-p group then one can also consider the i-th *local cohomology groups* $\mathrm{H}^i_{\mathfrak{m}}(M)$ of a $\Lambda(G_0)$-module M in the sense of [37], section 5. It will become clear later that the $\Lambda(G_0)$-modules $\Sigma^i(M)$ and $\mathrm{H}^i_{\mathfrak{m}}(M)$ are isomorphic whenever M is coadmissible (cf. Remark 3.17).

(iii) The exactness of the inductive limit functor shows that the family $(\Sigma^i)_{i\geq 0}$ is a δ-functor $\mathrm{Mod}_{\Lambda(G)} \to \mathrm{Rep}^\infty_E(G)$.

Definition 3.12. For $i \geq 0$ the functor $S^i = S^i_G = \Sigma^i \circ \check{(\cdot)} : \mathrm{Rep}^\infty_E(G) \to \mathrm{Rep}^\infty_E(G)$ is called the *i-th smooth duality functor*.

By Proposition 3.2 and (5) we have

$$\dim_E E^i(E) = \begin{cases} 0, & 0 \leq i < d \\ 1, & i = d. \end{cases}$$

We denote by $\chi_G : G \to E^\times$ the smooth character of G affording the left $\Lambda(G)$-module structure of $E^d(E)$ and call χ_G the *duality character* of G. We have $\chi_G|_{G_0} = \chi_{G_0}$ for any open subgroup G_0 of G (cf. Proposition 3.2). In particular, χ_G is trivial upon restriction to any open pro-p subgroup of G.

Recall that $C^\infty_c(G, E)$ may be seen as a $(\Lambda(G), \Lambda(G))$-bimodule through the smooth actions of G by left and right translation. Twisting the left action by χ_G and leaving the right action unchanged one obtains a $(\Lambda(G), \Lambda(G))$-bimodule that we denote by $\chi_G \otimes_E C^\infty_c(G, E)$.

For any $i \geq 0$ the left $\Lambda(G)$-module $\Sigma^i(\Lambda(G)) \in \mathrm{Rep}^\infty_E(G) \subseteq \mathrm{Mod}_{\Lambda(G)}$ admits a commuting right $\Lambda(G)$-action induced functorially from the right $\Lambda(G)$-action on $\Lambda(G)$. In this way, we may view $\Sigma^i(\Lambda(G))$ as a $(\Lambda(G), \Lambda(G))$-bimodule, as well. In a different guise, the following results are contained in [37], Lemma 5.3 and Lemma 5.5.

Proposition 3.13. *The functors $\Sigma^i : \mathrm{Mod}_{\Lambda(G)} \to \mathrm{Rep}_E^\infty(G)$ commute with direct sums and direct limits. As $(\Lambda(G), \Lambda(G))$-bimodules we have*

$$\Sigma^i(\Lambda(G)) \cong \begin{cases} 0 & , \quad i \neq d \\ \chi_G \otimes_E C_c^\infty(G, E), & \quad i = d. \end{cases}$$

Proof. If N is a compact open subgroup of G then $\Lambda(N)$ is a noetherian ring (cf. Theorem 3.1). Therefore, E admits a resolution by finitely generated free $\Lambda(N)$-modules. As a consequence, the functors $\mathrm{Ext}^i_{\Lambda(N)}(E, \cdot)$ and hence the functors Σ^i commute with direct sums and direct limits.

Since the left $\Lambda(N)$-module $\Lambda(G) \cong \oplus_{g \in N\backslash G} \Lambda(N)g$ is free, we get $\Sigma^i(\Lambda(G)) = 0$ for $i \neq d$ from Proposition 3.8. Further, the isomorphism

$$\mathrm{Ext}^d_{\Lambda(N)}(E, \Lambda(G)) \cong \bigoplus_{g \in N\backslash G} \mathrm{Ext}^d_{\Lambda(N)}(E, \Lambda(N))g$$

shows that the right G-representation $\mathrm{Ext}^d_{\Lambda(N)}(E, \Lambda(G))$ is isomorphic to the right G-representation $\mathrm{ind}_N^G(\chi_N^{-1})$ consisting of all compactly supported functions $f : G \to \mathrm{Ext}^d_{\Lambda(N)}(E, \Lambda(N))$ satisfying $f(ng) = f(g)n^{-1}$. An explicit isomorphism is given by sending f to $\sum_{g \in N\backslash G} f(g) \cdot g$. It is checked directly that the restriction map $\mathrm{Ext}^d_{\Lambda(N)}(E, \Lambda(G)) \to \mathrm{Ext}^d_{\Lambda(N')}(E, \Lambda(G))$ corresponds to the natural inclusion $\mathrm{ind}_N^G(\chi_N^{-1}) \to \mathrm{ind}_{N'}^G(\chi_{N'}^{-1})$, noting that $\chi_N|_{N'} = \chi_{N'}$ whenever N' is an open subgroup of N. Since χ_N is trivial for any open pro-p subgroup N of G, passage to the limit shows that $\Sigma^d(\Lambda(G)) \cong C_c^\infty(G, E)$ as a right $\Lambda(G)$-module.

It now remains to exhibit the smooth left action of G. In doing so we identify $\Sigma^d(\Lambda(G))$ and $C_c^\infty(G, E)$ as E-vector spaces. If $\gamma \in G$ then left multiplication by γ on $\Sigma^d(\Lambda(G))$ is induced by left multiplication with γ on $\Lambda(G)$. It maps the subspace $\mathrm{Ext}^d_{\Lambda(N)}(E, \Lambda(N))g$ bijectively onto the subspace $\mathrm{Ext}^d_{\Lambda(\gamma N \gamma^{-1})}(E, \Lambda(\gamma N \gamma^{-1}))\gamma g$. Taking $g = 1$ and composing with left translation by γ^{-1} we obtain an E-linear isomorphism

$$E^d_G(E) \cong \mathrm{Ext}^d_{\Lambda(N)}(E, \Lambda(N)) \longrightarrow \mathrm{Ext}^d_{\Lambda(\gamma N \gamma^{-1})}(E, \Lambda(\gamma N \gamma^{-1})) \cong E^d_G(E)$$

of $E^d_G(E) = \mathrm{Ext}^d_{\Lambda(G)}(E, \Delta(G))$. It can be written as the composition of two E-linear automorphisms the first of which is induced by left γ-multiplication on $\Delta(G)$ and by γ-conjugation on G. By the usual argument, it is trivial (cf. [27], Proposition 1.6.2). The second one is right multiplication by γ^{-1}, hence is multiplication with $\chi_G(\gamma)$.

Overall, we obtain that left multiplication with γ on $\Sigma^d(\Lambda(G)) \cong C_c^\infty(G, E)$ maps the characteristic function of Ng to the characteristic function of

$\gamma N\gamma^{-1}\gamma g = \gamma N g$ multiplied by $\chi_G(\gamma)$. This proves the claim concerning the left G-action on $\Sigma^d(\Lambda(G))$. $\square$

By $\chi_G^{-1} \otimes_E \Delta(G)$ we denote the $(\Lambda(G), \Lambda(G))$-bimodule obtained by twisting the left $\Lambda(G)$-action by χ_G^{-1} and by leaving the right $\Lambda(G)$-action unchanged. For the following central result see also [37], Theorem 5.6, whose proof is formally the same.

Theorem 3.14. *For any $\Lambda(G)$-module M and any integer $i \geq 0$ there are natural $\Lambda(G)$-linear isomorphisms*

$$\Sigma^i(M)\check{} \cong \mathrm{Ext}^{d-i}_{\Lambda(G)}(M, \chi_G^{-1} \otimes_E \Delta(G)).$$

Proof. Since $(\Sigma^i)_{i\geq 0}$ is a δ-functor (cf. Remark 3.11 (iii)) with $\Sigma^i = 0$ for $i > d$ and since the functor $\mathrm{Rep}_E^\infty(G) \xrightarrow{\check{(\cdot)}} \mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)} \xrightarrow{\text{forget}} \mathrm{Mod}_{\Lambda(G)}$ is exact, the functor $\check{(\cdot)} \circ \Sigma^d : \mathrm{Mod}_{\Lambda(G)} \to \mathrm{Mod}_{\Lambda(G)}$ is right exact. By Proposition 3.13 and the properties of $\check{(\cdot)} = \mathrm{Hom}_E(\cdot, E)$ it transforms arbitrary direct sums into direct products.

Let $M \in \mathrm{Mod}_{\Lambda(G)}$. If $m \in M$, $\xi \in \Sigma^d(M)\check{}$ and if $c_m : \Lambda(G) \to M$ denotes the $\Lambda(G)$-linear map defined by $c_m(\lambda) := \lambda \cdot m$ then one defines $\varphi_M(\xi) \in \mathrm{Hom}_{\Lambda(G)}(M, \Sigma^d(\Lambda(G))\check{}) \cong \mathrm{Hom}_{\Lambda(G)}(M, \chi_G^{-1} \otimes_E \Delta(G))$ (cf. Proposition 3.13) as $(\check{(\cdot)} \circ \Sigma^d)(c_m)(\xi)$. In this way one obtains a natural transformation

$$\varphi : \check{(\cdot)} \circ \Sigma^d \to \mathrm{Hom}_{\Lambda(G)}(\cdot, \chi_G^{-1} \otimes_E \Delta(G))$$

which is an isomorphism of functors. Indeed, since both functors transform direct sums into direct products and since $\varphi_{\Lambda(G)}$ is obviously bijective, φ_M is bijective whenever M is free. If M is arbitrary and if $F_1 \to F_0 \to M \to 0$ is a presentation of M by free $\Lambda(G)$-modules then the right exactness of the functors imply that φ_M is bijective, as well.

By the universal property of derived functors φ extends to a morphism $(\check{(\cdot)} \circ \Sigma^{d-i})_{i\geq 0} \to (\mathrm{Ext}^i_{\Lambda(G)}(\cdot, \chi_G^{-1} \otimes_E \Delta(G)))_{i\geq 0}$ of δ-functors. To show that it is an isomorphism it suffices to see that $(\check{(\cdot)} \circ \Sigma^{d-i})_{i\geq 0}$ is coeffaceable. This follows from Proposition 3.13. $\square$

Corollary 3.15. *For any $0 \leq i \leq d$ the diagram*

$$\begin{tikzcd}
\mathrm{Rep}_E^\infty(G) \arrow[r, "\check{(\cdot)}"] \arrow[d, "\chi_G \otimes_E S^{d-i}"'] & \mathrm{Mod}_{\Lambda(G)} \arrow[d, "E^i"] \\
\mathrm{Rep}_E^\infty(G) \arrow[r, "\check{(\cdot)}"] & \mathrm{Mod}_{\Lambda(G)}
\end{tikzcd}$$

is commutative up to natural isomorphism. In particular, the higher smooth duality functors $(S^i)_{i\geq 0}$ *respect admissibility. Moreover, the functor* S^{d-i} *induces an endofunctor of the category* $\mathrm{Rep}_E^\infty(G)_i^a/\mathrm{Rep}_E^\infty(G)_{i+1}^a$ *which is quasi-inverse to itself.*

Proof. For any $\Lambda(G)$-module M there is a natural isomorphism

$$\mathrm{Ext}^i_{\Lambda(G)}(M, \chi_G^{-1} \otimes_E \Delta(G)) \cong E^i(\chi_G \otimes_E M) \tag{6}$$

of left $\Lambda(G)$-modules. Further, one checks that there is a natural isomorphism $\chi_G \otimes_E S^i(V) \cong S^i(\chi_G^{-1} \otimes_E V)$ in $\mathrm{Rep}_E^\infty(G)$. Theorem 3.14 then shows that the functors $\chi_G \otimes_E S^{d-i}$ and E^i correspond to each other under Pontryagin duality. Since the endofunctor $\chi_G \otimes_E (\cdot)$ of the category $\mathrm{Rep}_E^\infty(G)$ preserves admissibility, it follows from Corollary 3.3 that so do the functors S^i. Moreover, twisting by a smooth character does not change the grade, so that the functors $\chi_G \otimes_E (\cdot)$ and $\chi_G^{-1} \otimes_E (\cdot)$ preserve the categories $\mathrm{Rep}_E^\infty(G)_i^a$ and $\mathcal{C}_G^i$, respectively. Since $(\chi_G \otimes_E S^{d-i}) \circ (\chi_G \otimes_E S^{d-i}) \cong S^{d-i} \circ S^{d-i}$, Theorem 3.5 implies that S^{d-i} induces an autoduality of the category $\mathrm{Rep}_E^\infty(G)_i^a/\mathrm{Rep}_E^\infty(G)_{i+1}^a$. □

Corollary 3.16. *If* $V \in \mathrm{Rep}_E^\infty(G)^a$ *then* $d(V) = \max\{i \geq 0 \mid S^i(V) \neq 0\}$*. In particular,* V *is holonomic (and hence finite dimensional over* E*) if and only if* $S^i(V) = 0$ *for all* $i > 0$*. More generally,* V *is Cohen-Macaulay if and only if* $S^i(V) = 0$ *for all* $i \neq d(V)$*.*

Proof. Since the dimension is stable under twisting by a smooth character, the assertions follow from Proposition 3.8, Theorem 3.14 and (6). □

Remark 3.17. Let G_0 be an open subgroup G_0 of G which is a uniform pro-p group, and let M be a coadmissible $\Lambda(G)$-module. Theorem 3.14, Proposition 3.2 and [37], Theorem 5.6, show that the underlying $\Lambda(G_0)$-module of $\Sigma^i(M)$ is isomorphic to the i-th local cohomology group of the finitely generated $\Lambda(G_0)$-module M considered in [37], section 5. Thus, our construction is compatible with the one given by Venjakob.

Remark 3.18. We note that in the theory of admissible locally analytic G-representations as developed by Schneider and Teitelbaum a natural analog of the functors S^i is not known to exist (cf. [31], end of §4).

4 Functoriality

In this section we continue to assume that p is a prime number, that E is a field of characteristic p and that G is a p-adic Lie group of dimension $d = \dim(G)$. We are going to study the behavior of the Auslander duality functors $(E^i)_{i\geq 0}$ and the higher smooth duality functors $(S^i)_{i\geq 0}$ under inflation, induction and tensor products.

Let H be a closed and normal subgroup of G. Restriction of scalars along the canonical ring homomorphism $\Lambda(G) \to \Lambda(G/H)$ gives rise to functors $\mathrm{Mod}_{\Lambda(G/H)} \to \mathrm{Mod}_{\Lambda(G)}$, $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G/H)} \to \mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$ and $\mathcal{C}_{G/H} \to \mathcal{C}_G$ which are called *inflation* and which will be denoted by inf. Likewise, the group homomorphism $G \to G/H$ gives rises to functors $\mathrm{Rep}^{\infty}_E(G/H) \to \mathrm{Rep}^{\infty}_E(G)$ and $\mathrm{Rep}^{\infty}_E(G/H)^a \to \mathrm{Rep}^{\infty}_E(G)^a$ which are called inflation, as well, and which will likewise be denoted by inf. Apparently, the diagram

$$
\begin{tikzcd}
\mathrm{Rep}^{\infty}_E(G/H) \arrow[r, "\check{(\cdot)}"] \arrow[d, "\mathrm{inf}"'] & \mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G/H)} \arrow[d, "\mathrm{inf}"] \\
\mathrm{Rep}^{\infty}_E(G) \arrow[r, "\check{(\cdot)}"] & \mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}
\end{tikzcd}
\tag{7}
$$

of functors is commutative up to natural isomorphism.

Theorem 4.1. *If H is a closed normal subgroup of G and if $i \geq 0$ then there is a natural isomorphism*

$$\mathrm{inf} \circ S^i_{G/H} \cong S^i_G \circ \mathrm{inf}$$

of functors $\mathrm{Rep}^{\infty}_E(G/H) \to \mathrm{Rep}^{\infty}_E(G)$.

Proof. By Theorem 1.5 and the commutativity of (7) we may equivalently construct an isomorphism of functors $\mathrm{inf} \circ \Sigma^i_{G/H} \cong \Sigma^i_G \circ \mathrm{inf}$ from $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G/H)}$ to $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$. For the sake of brevity, we will omit the inflation functors from the notation. The rough idea of our proof is that in the direct limit suitable Hochschild-Serre spectral sequences degenerate.

Let us first assume that G is compact. Let $Q^\bullet \to E \to 0$ be a resolution of E by finitely generated free $\Lambda(G)$-modules, and let $P^\bullet \to E \to 0$ be a resolution of E by finitely generated free $\Lambda(G/H)$-modules (cf. Theorem 3.1). Let further $(G_k)_{k\geq 0}$ be a basis of neighborhoods of the identity of G consisting of open normal subgroups. Setting $H_k := H \cap G_k$, the family $(H_k)_{k\geq 0}$ is a basis of neighborhoods of the identity of H consisting of open subgroups of H which are normal in G.

Let $M \in \mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G/H)}$ and write $M = \varprojlim_{i\in I} M/M_i$ where $(M_i)_{i\in I}$ is a basis of neighborhoods of zero in M consisting of open $\Lambda(G/H)$-submodules. For any $s \geq 0$ and any $\ell \geq 0$ there is an E-linear isomorphism

$$\mathrm{Hom}^{\mathrm{cont}}_{\Lambda(H_\ell)}(Q^s, M) \cong \varprojlim_{i\in I} \mathrm{Hom}^{\mathrm{cont}}_{\Lambda(H_\ell)}(Q^s, M/M_i).$$

Since any continuous $\Lambda(H_\ell)$-linear map $Q^s \to M/M_i$ factors through a finite dimensional discrete quotient of Q^s, the G-action on $\mathrm{Hom}^{\mathrm{cont}}_{\Lambda(H_\ell)}(Q^s, M/M_i)$ is smooth and gives rise to a module structure over $\Lambda(G)$. Therefore, $\mathrm{Hom}^{\mathrm{cont}}_{\Lambda(H_\ell)}(Q^s, M)$ is naturally a $\Lambda(G)$-module. We may therefore consider the double complex

$$C^{\bullet,\bullet} = (C^{r,s})_{r,s\geq 0} = \varinjlim_{k\geq 0} \varinjlim_{\ell\geq 0} \mathrm{Hom}_{\Lambda(G_k)}(P^r, \mathrm{Hom}^{\mathrm{cont}}_{\Lambda(H_\ell)}(Q^s, M)).$$

There are two spectral sequences associated with $C^{\bullet,\bullet}$ which converge to the cohomology of the associated total complex. The initial terms are

$$E_2^{p,q} = \mathrm{H}^p((H^q C^{r,\bullet})_{r\geq 0}) \quad \text{and} \quad (E_2^{p,q})' = \mathrm{H}^q((H^p C^{\bullet,s})_{s\geq 0}),$$

respectively. Since the resolution $Q^\bullet \to E \to 0$ consists of finitely generated $\Lambda(G)$-modules, all its homomorphisms are automatically continuous homomorphisms of pseudocompact $\Lambda(H_\ell)$-modules. Further, the $\Lambda(H_\ell)$-modules Q^s are projective in $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(H_\ell)}$ (cf. [11], Lemma 4.5). By the usual argument, the cohomology of the complex $\mathrm{Hom}^{\mathrm{cont}}_{\Lambda(H_\ell)}(Q^\bullet, M)$ can be computed from any projective resolution of E in $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(H_\ell)}$. Since the ring $\Lambda(H_\ell)$ is noetherian, there is a resolution $R^\bullet \to E \to 0$ of E consisting of finitely generated free $\Lambda(H_\ell)$-modules. It is automatically a projective resolution of E in $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(H_\ell)}$ and we have $\mathrm{Hom}_{\Lambda(H_\ell)}(R^q, M) = \mathrm{Hom}^{\mathrm{cont}}_{\Lambda(H_\ell)}(R^q, M)$ for any $q \geq 0$. Thus, $\mathrm{H}^q \,\mathrm{Hom}^{\mathrm{cont}}_{\Lambda(H_\ell)}(Q^\bullet, M) \cong \mathrm{Ext}^q_{\Lambda(H_\ell)}(E, M)$ and therefore

$$\begin{aligned}
\mathrm{H}^q\, C^{r,\bullet} &\cong \varinjlim_{\ell\geq 0} \varinjlim_{k\geq \ell} \mathrm{H}^q\, \mathrm{Hom}_{\Lambda(G_k)}(P^r, \mathrm{Hom}^{\mathrm{cont}}_{\Lambda(H_\ell)}(Q^\bullet, M)) \\
&= \varinjlim_{\ell\geq 0} \varinjlim_{k\geq \ell} \mathrm{H}^q\, \mathrm{Hom}_{\Lambda(G_k/H_k)}(P^r, \mathrm{Hom}^{\mathrm{cont}}_{\Lambda(H_\ell)}(Q^\bullet, M)) \\
&\cong \varinjlim_{\ell\geq 0} \varinjlim_{k\geq \ell} \mathrm{Hom}_{\Lambda(G_k/H_k)}(P^r, \mathrm{Ext}^q_{\Lambda(H_\ell)}(E, M)) \\
&\cong \varinjlim_{\ell\geq 0} \varinjlim_{k\geq 0} \mathrm{Hom}_{\Lambda(G_k)}(P^r, \mathrm{Ext}^q_{\Lambda(H_\ell)}(E, M))
\end{aligned}$$

because for $k \geq \ell$ the action of H_k on $\mathrm{Hom}^{\mathrm{cont}}_{\Lambda(H_\ell)}(Q^\bullet, M)$ is trivial and because the $\Lambda(G_k/H_k)$-module P^r is projective.

Since the $\Lambda(G_k)$-module P^r is finitely generated, the natural homomorphism

$$\varinjlim_{\ell\geq 0} \mathrm{Hom}_{\Lambda(G_k)}(P^r, \mathrm{Ext}^q_{\Lambda(H_\ell)}(E, M)) \longrightarrow \mathrm{Hom}_{\Lambda(G_k)}(P^r, \Sigma^q_H(M)) \tag{8}$$

is injective. Moreover, the action of H on M is trivial so that the map

$$\begin{array}{rcl}
\mathrm{Hom}_{\Lambda(H_\ell)}(R^\bullet, E) \otimes_E M & \longrightarrow & \mathrm{Hom}_{\Lambda(H_\ell)}(R^\bullet, M) \\
\varphi \otimes m & \mapsto & (r \mapsto \varphi(r)m)
\end{array}$$

is a well-defined isomorphism of complexes. Thus, $\Sigma_H^q(M) \cong \Sigma_H^q(E) \otimes_E M$ is zero for $q > 0$ by Theorem 3.14 and (5). As a consequence of the injectivity of (8) the spectral sequence degenerates. Since $\mathrm{Ext}^0_{\Lambda(H_\ell)}(E, M) \cong M$ over $\Lambda(G_k)$ independently of ℓ its limit terms are

$$\begin{aligned} E_2^{p0} &= \varinjlim_{k\geq 0} \mathrm{H}^p \operatorname{Hom}_{\Lambda(G_k)}(P^\bullet, M) \\ &= \varinjlim_{k\geq 0} \mathrm{H}^p \operatorname{Hom}_{\Lambda(G_k/H_k)}(P^\bullet, M) = \Sigma^p_{G/H}(M). \end{aligned}$$

In order to compute the initial terms $(E_2^{pq})'$ of the second spectral sequence we fix $k, \ell, s \geq 0$ and claim that the functor $\operatorname{Hom}_{\Lambda(G_k)}(\cdot, \operatorname{Hom}^{\mathrm{cont}}_{\Lambda(H_\ell)}(Q^s, M))$ on the category of finitely generated $\Lambda(G/H)$-modules into the category of E-vector spaces is exact. To see this we may assume $Q^s = \Lambda(G)$ to be free of rank one over $\Lambda(G)$.

In this case, the projection $\Lambda(G) \to \Lambda(G/H_\ell)$ induces an isomorphism $\operatorname{Hom}^{\mathrm{cont}}_E(\Lambda(G/H_\ell), M) \to \operatorname{Hom}^{\mathrm{cont}}_{\Lambda(H_\ell)}(\Lambda(G), M)$ of $\Lambda(G)$-modules. Writing $M = \varprojlim_{i\in I} M/M_i$ as above, there is an isomorphism of functors

$$\begin{aligned} &\operatorname{Hom}_{\Lambda(G_k)}(\cdot, \operatorname{Hom}^{\mathrm{cont}}_E(\Lambda(G/H_\ell), M)) \cong \\ &\qquad\qquad \varprojlim_i \operatorname{Hom}_{\Lambda(G_k)}(\cdot, \operatorname{Hom}^{\mathrm{cont}}_E(\Lambda(G/H_\ell), M/M_i)). \end{aligned}$$

We endow any of the $\Lambda(G)$-modules $\operatorname{Hom}^{\mathrm{cont}}_E(\Lambda(G/H_\ell), M/M_i)$ with the discrete topology and note that the action of G is smooth because any continuous homomorphism factors through a finite dimensional discrete quotient.

If $f : P \to \operatorname{Hom}^{\mathrm{cont}}_E(\Lambda(G/H_\ell), M/M_i)$ is a homomorphism of abstract $\Lambda(G)$-modules where P is finitely generated then the image of f is G_m-invariant for some $m \geq k$. As in the proof of Lemma 1.7 let I_{G_m} be the kernel of the projection $\Lambda(G_k) \to E[G_k/G_m]$. By Theorem 3.1 it is generated by finitely many elements of the form $g - 1$ with $g \in G_m$. Hence the open submodule $I_{G_m}P$ is contained in the kernel of f. This shows that f is automatically continuous. By [11], Lemma 2.1 (ii) and Lemma 2.4, there are isomorphisms

$$\begin{aligned} & \operatorname{Hom}_{\Lambda(G_k)}(P, \operatorname{Hom}^{\mathrm{cont}}_E(\Lambda(G/H_\ell), M/M_i)) \\ = \;& \operatorname{Hom}^{\mathrm{cont}}_{\Lambda(G_k/(G_k\cap H_\ell))}(P, \operatorname{Hom}^{\mathrm{cont}}_E(\Lambda(G/H_\ell), M/M_i)) \\ \cong \;& \operatorname{Hom}^{\mathrm{cont}}_E(P \hat{\otimes}_{\Lambda(G_k/(G_k\cap H_\ell))} \Lambda(G/H_\ell), M/M_i) \\ \cong \;& \operatorname{Hom}^{\mathrm{cont}}_E(P \otimes_{\Lambda(G_k/(G_k\cap H_\ell))} \Lambda(G/H_\ell), M/M_i) \end{aligned}$$

of $\Lambda(G)$-modules which are natural in P. Passing to the projective limit over i we obtain an isomorphism of functors

$$\begin{aligned} &\operatorname{Hom}_{\Lambda(G_k)}(\cdot, \operatorname{Hom}^{\mathrm{cont}}_{\Lambda(H_\ell)}(\Lambda(G), M)) \cong \\ &\qquad\qquad \operatorname{Hom}^{\mathrm{cont}}_E((\cdot) \otimes_{\Lambda(G_k/(G_k\cap H_\ell))} \Lambda(G/H_\ell), M) \end{aligned}$$

from $\mathcal{C}_{G/H}$ to the category of E-vector spaces. Since $\Lambda(G/H_\ell)$ is a free module over $\Lambda(G_k/(G_k \cap H_\ell))$ our claim will be proved once we can show that the functor $\mathrm{Hom}_E^{\mathrm{cont}}(\cdot, M)$ on $\mathcal{C}_{G/H}$ is exact. However, $M \cong \mathrm{Hom}_E^{\mathrm{cont}}(\check{M}, E)$ is isomorphic in $\mathrm{Mod}_E^{\mathrm{pc}}$ to a direct product of copies of E. Therefore, as a functor from $\mathcal{C}_{G/H}$ to the category of E-vector spaces, $\mathrm{Hom}_E^{\mathrm{cont}}(\cdot, M)$ is isomorphic to a direct product of copies of the duality functor $\check{(\cdot)}$.

Thus, also the second spectral sequence degenerates with limit terms

$$\begin{aligned}(E_2^{0q})' &= \varinjlim_{\ell \geq 0} \varinjlim_{k \geq 0} \mathrm{H}^q \mathrm{Hom}_{\Lambda(G_k)}(E, \mathrm{Hom}^{\mathrm{cont}}_{\Lambda(H_\ell)}(Q^\bullet, M)) \\ &\cong \varinjlim_{k \geq 0} \varinjlim_{\ell \geq 0} \mathrm{H}^q \mathrm{Hom}^{\mathrm{cont}}_{\Lambda(G_k H_\ell)}(Q^\bullet, M) \\ &\cong \varinjlim_{k \geq 0} \mathrm{Ext}^q_{\Lambda(G_k)}(E, M) = \Sigma^q_G(M).\end{aligned}$$

The construction of the above spectral sequence is functorial in M and so is the resulting isomorphism $\Sigma^i_G(M) \cong \Sigma^i_{G/H}(M)$. If G is no longer assumed to be compact note that $\Sigma^i_G(M) \cong \Sigma^i_{G_0}(M)$ and $\Sigma^i_{G/H}(M) \cong \Sigma^i_{G_0/H_0}(M)$ as G_0-representations for any compact open subgroup G_0 of G if $H_0 = H \cap G_0$. The above functoriality result can then be used to see that the G_0-equivariant isomorphism $\Sigma^i_G(M) \cong \Sigma^i_{G_0}(M) \cong \Sigma^i_{G_0/H_0}(M) \cong \Sigma^i_{G/H}(M)$ is actually G-equivariant. □

Corollary 4.2. *Let H be a closed normal subgroup of G and let V be an object of $\mathrm{Rep}_E^\infty(G/H)^a$. We have*

$$d_G(\inf(V)) = d_{G/H}(V).$$

Further, V is holonomic (resp. pure, resp. Cohen-Macaulay) over G/H if and only if $\inf(V)$ is holonomic (resp. pure, resp. Cohen-Macaulay) over G.

Proof. We may assume that G and H are pro-p groups, hence have trivial duality characters. Since $\dim(G) = \dim(G/H) + \dim(H)$, Corollary 1.8, Corollary 3.15 and Theorem 4.1 show that for any $i \geq 0$ there is an isomorphism of functors $E^i_G \circ \inf \cong \inf \circ E^{i-\dim(H)}_{G/H}$ from $\mathcal{C}_{G/H}$ to $\mathcal{C}_G$. Using the characterization of purity given in [23], III.4.2 Theorem 6, this implies all assertions of the corollary. □

Next we will discuss tensor products. Let G_1 and G_2 be p-adic Lie groups and let $V_i \in \mathrm{Rep}_E^\infty(G_i)$ for $i \in \{1, 2\}$. Via the diagonal action the tensor product $V_1 \otimes_E V_2$ becomes an object of $\mathrm{Rep}_E^\infty(G_1 \times G_2)$. The smooth E-linear $(G_1 \times G_2)$-representation $V_1 \otimes_E V_2$ is admissible if V_1 and V_2 are admissible smooth E-linear representations of G_1 and G_2, respectively. The converse holds if V_1 and V_2 are non-zero. There is a $(G_1 \times G_2)$-biequivariant isomorphism

$$C_c^\infty(G_1 \times G_2, E) \cong C_c^\infty(G_1, E) \otimes_E C_c^\infty(G_2, E).$$

Likewise, if $M_i \in \mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G_i)}$ for $i \in \{1,2\}$ then the diagonal action of $G_1 \times G_2$ on $M_1 \times M_2$ is jointly continuous for the product topology and extends to a jointly continuous action on the complete tensor product $M_1 \hat{\otimes}_E M_2$. In particular, $M_1 \hat{\otimes}_E M_2 \in \mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G_1 \times G_2)}$ is a module over $\Lambda(G_1 \times G_2)$.

The formation of tensor products is compatible with Pontryagin duality in the sense that there are natural isomorphisms

$$(V_1 \otimes_E V_2)\check{} \cong \check{V}_1 \hat{\otimes} \check{V}_2 \quad \text{and} \quad (M_1 \hat{\otimes}_E M_2)\check{} \cong \check{M}_1 \otimes_E \check{M}_2 \tag{9}$$

in $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G_1 \times G_2)}$ and $\mathrm{Rep}^\infty_E(G_1 \times G_2)$, respectively. For example, we have $\Delta(G_1 \times G_2) \cong \Delta(G_1) \hat{\otimes}_E \Delta(G_2)$.

If $M_i \in \mathcal{C}_{G_i}$ then the $\Lambda(G_1 \times G_2)$-module $M_1 \hat{\otimes}_E M_2$ is coadmissible, as follows from the exactness of the complete tensor product over E. Using (9) and Corollary 1.5 this also follows from the corresponding statement for smooth representations.

Theorem 4.3. *If M_1 and M_2 are coadmissible modules over $\Lambda(G_1)$ and $\Lambda(G_2)$, respectively, and if $i \geq 0$, then the coadmissible $\Lambda(G_1 \times G_2)$-module $E^i_{G_1 \times G_2}(M_1 \hat{\otimes}_E M_2)$ admits a finite filtration by $\Lambda(G_1 \times G_2)$-submodules whose associated graded module is isomorphic to $\bigoplus_{p+q=i} E^p_{G_1}(M_1) \hat{\otimes}_E E^q_{G_2}(M_2)$.*

Proof. Using Proposition 3.2 one can reduce to the case that both G_1 and G_2 are compact and hence that M_1 and M_2 are finitely generated over $\Lambda(G_1)$ and $\Lambda(G_2)$, respectively (cf. Proposition 1.9).

For $i \in \{1,2\}$ let $P_i^\bullet \to M_i \to 0$ be resolutions by finitely generated free modules over $\Lambda(G_i)$ (cf. Theorem 3.1). The total complex $T(P_1^\bullet \hat{\otimes} P_2^\bullet)$ is then a resolution of $M_1 \hat{\otimes}_E M_2$ by finitely generated free $\Lambda(G_1 \times G_2)$-modules. Recall once again that the complete tensor product over E is exact.

Let us consider the spectral sequence associated with the double complex $\mathrm{Hom}_{\Lambda(G_1 \times G_2)}(P_1^\bullet \hat{\otimes}_E P_2^\bullet, \Lambda(G_1 \times G_2))$. Since $\mathrm{Hom}_{\Lambda(G_1 \times G_2)}(\cdot, \Lambda(G_1 \times G_2))$ commutes with finite direct sums, the total complex of this double complex is given by $\mathrm{Hom}_{\Lambda(G_1 \times G_2)}(T(P_1^\bullet \hat{\otimes} P_2^\bullet), \Lambda(G_1 \times G_2))$. Therefore, the limit terms of the spectral sequence are isomorphic to $E^i_{G_1 \times G_2}(M_1 \hat{\otimes}_E M_2)$.

In order to compute the initial terms of the spectral sequence we note that there is an isomorphism

$$\mathrm{Hom}_{\Lambda(G_1\times G_2)}(P_1^\bullet\hat{\otimes}_E P_2^\bullet, \Lambda(G_1\times G_2)) \cong \\ \mathrm{Hom}_{\Lambda(G_1)}(P_1^\bullet, \Lambda(G_1))\hat{\otimes}_E \mathrm{Hom}_{\Lambda(G_2)}(P_2^\bullet, \Lambda(G_2))$$

of complexes of pseudocompact $\Lambda(G_1\times G_2)$-modules. It is given by sending $\varphi_1\hat{\otimes}\varphi_2$ to the map φ with $\varphi(p_1\hat{\otimes}p_2) := \varphi_1(p_1)\hat{\otimes}\varphi_2(p_2) \in \Lambda(G_1)\hat{\otimes}_E\Lambda(G_2) \cong \Lambda(G_1\times G_2)$.

The exactness of $(\cdot)\hat{\otimes}_E(\cdot)$ implies that the initial terms of the spectral sequence are given by $E_2^{pq} \cong E^p_{G_1}(M_1)\hat{\otimes}_E E^q_{G_2}(M_2)$. Further, the arguments leading to the usual Künneth formula show that the spectral sequence degenerates at E_2. Therefore, the filtration of $E^i_{G_1\times G_2}(M_1\hat{\otimes}_E M_2)$ arising from the spectral sequence is as required. $\square$

Corollary 4.4. *For $i \in \{1,2\}$ let $V_i \in \mathrm{Rep}_E^\infty(G_i)^a$ be non-zero. We have*

$$d_{G_1\times G_2}(V_1 \otimes_E V_2) = d_{G_1}(V_1) + d_{G_2}(V_2).$$

Further, $V_1 \otimes_E V_2$ is holonomic (resp. pure, resp. Cohen-Macaulay) over $G_1 \times G_2$ if and only if V_1 and V_2 are holonomic (resp. pure, resp. Cohen-Macaulay) over G_1 and G_2, respectively.

Proof. Since $\dim(G_1\times G_2) = \dim(G_1)+\dim(G_2)$ all assertions are direct consequences of (9), Theorem 4.3 and [23], III.4.2 Theorem 6. $\square$

Finally, we treat a special type of induction functors. Let H be a closed subgroup of G, and let $V \in \mathrm{Rep}_E^\infty(H)$. The compact induction $\mathrm{ind}_H^G(V) \in \mathrm{Rep}_E^\infty(G)$ is the E-vector space of all maps $f : G \to V$ whose support is compact modulo H and which satisfy $f(gh) = h^{-1}f(g)$ for all $h \in H$ and all $g \in G$. The action of G on $\mathrm{ind}_H^G(V)$ is given by left translation, i.e. by $(gf)(g') = f(g^{-1}g')$ for all $g, g' \in G$. In this way we obtain a functor $\mathrm{ind}_H^G : \mathrm{Rep}_E^\infty(H) \to \mathrm{Rep}_E^\infty(G)$. If the quotient space G/H is compact then it preserves admissibility (cf. [38], I.5.6). Further, the compact induction functors are transitive in the sense that if K is a closed subgroup of H then the functors ind_K^G and $\mathrm{ind}_H^G \circ \mathrm{ind}_H^K$ are naturally isomorphic (cf. [38], I.5.3).

In order to translate the induction functor to the dual side, we assume that G is compact and let $M \in \mathrm{Mod}^{\mathrm{pc}}_{\Lambda(H)}$. We define the pseudocompact E-vector space $\Lambda(G)\hat{\otimes}_{\Lambda(H)}M$ as the quotient of $\Lambda(G)\hat{\otimes}_E M$ by the closure U of the kernel of the natural map $\Lambda(G)\otimes_E M \to \Lambda(G)\otimes_{\Lambda(H)} M$.

The structure map $\Lambda(G)\times\Lambda(G) \to \Lambda(G)$ of the pseudocompact left $\Lambda(G)$-module $\Lambda(G)$ gives rise to a continuous E-linear map $\Lambda(G)\hat{\otimes}_E\Lambda(G) \to \Lambda(G)$.

Taking the complete tensor product with M over E we obtain a continuous E-linear map

$$\Lambda(G)\hat{\otimes}_E\Lambda(G)\hat{\otimes}_E M \longrightarrow \Lambda(G)\hat{\otimes}_E M$$

whose restriction to $\Lambda(G)\hat{\otimes}_E U$ takes values in U. Therefore, it induces a continuous E-linear map

$$\Lambda(G)\hat{\otimes}_E(\Lambda(G)\hat{\otimes}_{\Lambda(H)}M) \longrightarrow \Lambda(G)\hat{\otimes}_{\Lambda(H)}M.$$

According to Lemma 1.3 this makes $\Lambda(G)\hat{\otimes}_{\Lambda(H)}M$ an object of $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$. Apparently, $\Lambda(G)\hat{\otimes}_{\Lambda(H)}(\cdot) : \mathrm{Mod}^{\mathrm{pc}}_{\Lambda(H)} \to \mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$ is a functor.

Lemma 4.5. *Let G be compact, let H be a closed subgroup of G, and let $M \in \mathrm{Mod}^{\mathrm{pc}}_{\Lambda(H)}$.*

(i) The complete tensor product $\Lambda(G)\hat{\otimes}_{\Lambda(H)}M$ coincides with that of [11], p. 446, i.e. $\Lambda(G)\hat{\otimes}_{\Lambda(H)}M \cong \varprojlim_{X,Y}(\Lambda(G)/X) \otimes_{\Lambda(H)} (M/Y)$ where X and Y run through the open $\Lambda(H)$-submodules of $\Lambda(G)$ and M, respectively.

(ii) Up to natural isomorphism the diagram

$$\begin{tikzcd}
\mathrm{Rep}^{\infty}_E(H) \arrow[r, "\check{(\cdot)}"] \arrow[d, "\mathrm{ind}_H^G"'] & \mathrm{Mod}^{\mathrm{pc}}_{\Lambda(H)} \arrow[d, "\Lambda(G)\hat{\otimes}_{\Lambda(H)}(\cdot)"] \\
\mathrm{Rep}^{\infty}_E(G) \arrow[r, "\check{(\cdot)}"'] & \mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}
\end{tikzcd}$$

of functors is commutative.

Proof. As for (i), note that $\Lambda(G)\hat{\otimes}_{\Lambda(H)}M$ as defined above is a pseudocompact E-vector space together with a continuous $\Lambda(H)$-bihomomorphism $\Lambda(G) \times M \to \Lambda(G)\hat{\otimes}_E M \to \Lambda(G)\hat{\otimes}_{\Lambda(H)}M$. We show that it satisfies the universal property considered in [11], p. 446. Let $f : \Lambda(G) \times M \to C$ be a continuous $\Lambda(H)$-bihomomorphism into a pseudocompact E-vector space C. By the universal property of $\hat{\otimes}_E$ it gives rise to a unique continuous E-linear map $g : \Lambda(G)\hat{\otimes}_E M \to C$ satisfying $g(\delta\lambda \otimes m) = g(\delta \otimes \lambda m)$ for all $\delta \in \Lambda(G)$, $\lambda \in \Lambda(H)$ and $m \in M$. Since the kernel of g is closed we obtain $U \subseteq \ker(g)$. Therefore, g induces a unique continuous $\Lambda(H)$-bihomomorphism $h : \Lambda(G)\hat{\otimes}_{\Lambda(H)}M \to C$ whose composition with $\Lambda(G) \times M \to \Lambda(G)\hat{\otimes}_{\Lambda(H)}M$ is equal to f.

As for (ii), let $V \in \mathrm{Rep}^{\infty}_E(H)$ and consider the exact sequence

$$0 \longrightarrow U \longrightarrow \Lambda(G)\hat{\otimes}_E\check{V} \longrightarrow \Lambda(G)\hat{\otimes}_{\Lambda(H)}\check{V} \longrightarrow 0$$

in $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$. Applying the functor $\check{(\cdot)}$ we obtain the exact sequence

$$0 \longrightarrow (\Lambda(G)\hat{\otimes}_{\Lambda(H)}\check{V})\check{} \longrightarrow (\Lambda(G)\hat{\otimes}_E\check{V})\check{} \longrightarrow \check{U} \longrightarrow 0$$

in $\mathrm{Rep}^\infty_E(G)$ where $(\Lambda(G)\hat{\otimes}_E\check{V})\check{} \cong C^\infty_c(G,E)\otimes_E V \cong C^\infty_c(G,V)$ by (9). Therefore, it suffices to see that $f \in \mathrm{ind}^G_H(V) \subseteq C^\infty_c(G,V) \cong (\Lambda(G)\hat{\otimes}_E\check{V})\check{}$ if and only if $U \subseteq \ker(f)$. Using that $E[G] \subseteq \Lambda(G)$ is dense, this is readily checked. $\square$

Under the special circumstances encountered in the situation of parabolic induction, the previous results extend to possibly non-compact groups.

Proposition 4.6. *Let H be a closed subgroup of G and assume that there is a compact open subgroup G_0 of G with $G = G_0H$. The functors $\mathrm{ind}^G_H : \mathrm{Rep}^\infty_E(H) \to \mathrm{Rep}^\infty_E(G)$ and $\Lambda(G)\otimes_{\Lambda(H)}(\cdot) : \mathrm{Mod}_{\Lambda(H)} \to \mathrm{Mod}_{\Lambda(G)}$ respect admissibility and coadmissibility, respectively. There is an isomorphism of functors*

$$\check{(\cdot)} \circ \mathrm{ind}^G_H \cong \Lambda(G)\otimes_{\Lambda(H)}\check{(\cdot)} : \mathrm{Rep}^\infty_E(H)^a \to \mathcal{C}_G.$$

Proof. That the functor ind^G_H preserves admissibility is true whenever the quotient G/H is compact (cf. [38], I.5.6). Set $H_0 = G_0 \cap H$. As $G_0\backslash G \cong H_0\backslash H$ as right H-spaces, the decompositions

$$\Lambda(G) \cong \bigoplus_{h\in H_0\backslash H} \Lambda(G_0)h \quad \text{and} \quad \Lambda(H) \cong \bigoplus_{h\in H_0\backslash H} \Lambda(H_0)h$$

show that the natural map $\Lambda(G_0)\otimes_{\Lambda(H_0)}\Lambda(H) \to \Lambda(G)$ is an isomorphism of $(\Lambda(G_0),\Lambda(H))$-bimodules. Thus, $\Lambda(G)\otimes_{\Lambda(H)} M \cong \Lambda(G_0)\otimes_{\Lambda(H_0)} M$ as $\Lambda(G_0)$-modules, proving that the functor $\Lambda(G)\otimes_{\Lambda(H)}(\cdot)$ preserves coadmissibility (cf. Proposition 1.9 (ii)).

If V is an object of $\mathrm{Rep}^\infty_E(H)$ then evaluation at $1 \in G$ is an H-equivariant homomorphism $\mathrm{ind}^G_H(V) \to V$. It induces a homomorphism $\check{V} \to \mathrm{ind}^G_H(V)\check{}$ of $\Lambda(H)$-modules and hence a homomorphism $\Lambda(G)\otimes_{\Lambda(H)}\check{V} \to \mathrm{ind}^G_H(V)\check{}$ of $\Lambda(G)$-modules. We need to see that it is bijective if V is admissible.

Note that restriction to G_0 is a G_0-equivariant isomorphism $\mathrm{ind}^G_H(V) \cong \mathrm{ind}^{G_0}_{H_0}(V)$. As seen above, $\Lambda(G)\otimes_{\Lambda(H)}\check{V} \cong \Lambda(G_0)\otimes_{\Lambda(H_0)}\check{V}$ over $\Lambda(G_0)$. Thus, we may assume $G = G_0$ to be compact. In this case, $\check{V}$ is finitely generated over $\Lambda(H)$ (cf. Proposition 1.9) and $\Lambda(G)\otimes_{\Lambda(H)}\check{V} \cong \Lambda(G)\hat{\otimes}_{\Lambda(H)}\check{V}$ by [11], Lemma 2.1 (ii). With these identifications the bijectivity of the above map was shown in the course of the proof of Lemma 4.5 (ii). $\square$

For compact groups, part (i) of the following theorem is contained in [28], Lemma 5.5.

Theorem 4.7. *Let H be a closed subgroup of G and assume that there is a compact open subgroup G_0 of G such that $G = G_0H$.*

(i) For any integer $i \geq 0$ there is an isomorphism of functors

$$\Lambda(G) \otimes_{\Lambda(H)} E^i_H(\cdot) \cong E^i_G(\Lambda(G) \otimes_{\Lambda(H)} (\cdot)) : \mathcal{C}_H \to \mathcal{C}_G.$$

(ii) For any integer $i \geq 0$ there is an isomorphism of functors

$$\chi_G \otimes_E (S^i \circ \mathrm{ind}^G_H) \cong \mathrm{ind}^G_H \circ (\chi_H \otimes_E S^{i-\dim(G/H)})$$

from $\mathrm{Rep}^\infty_E(H)^a$ to $\mathrm{Rep}^\infty_E(G)^a$.

Proof. As for (i), set $H_0 = H \cap G_0$ and consider the $(\Lambda(G_0), \Lambda(H))$-bimodule isomorphism $\Lambda(G_0) \otimes_{\Lambda(H_0)} \Lambda(H) \to \Lambda(G)$. By the proof of [28], Lemma 5.5, the $\Lambda(H_0)$-module $\Lambda(G_0)$ is flat. As a consequence, the $\Lambda(H)$-module $\Lambda(G)$ is flat and the functor $\Lambda(G) \otimes_{\Lambda(H)} (\cdot) : \mathcal{C}_H \to \mathcal{C}_G$ of Proposition 4.6 is exact. One can then directly follow the arguments of [31], Lemma 6.3 (ii) and Proposition 6.4, to see that the natural $\Lambda(G)$-linear maps

$$\begin{array}{rcll} \mathrm{Ext}^i_{\Lambda(G)}(\Lambda(G) \otimes_{\Lambda(H)} M, \Delta(G)) & \xrightarrow{\varphi} & \mathrm{Ext}^i_{\Lambda(H)}(M, \Delta(G)) & \text{and} \\ \Lambda(G) \otimes_{\Lambda(H)} \mathrm{Ext}^i_{\Lambda(H)}(M, \Delta(H)) & \xrightarrow{\psi} & \mathrm{Ext}^i_{\Lambda(H)}(M, \Delta(G)) & \end{array}$$

are both bijective. Assertion (ii) follows from (i), Proposition 4.6 and Corollary 3.15. $\square$

The following statements are immediate consequences of Theorem 4.7. The purity assertion follows from [23], III.4.2 Theorem 6.

Corollary 4.8. *Let H be a closed subgroup of G and assume that there is a compact open subgroup G_0 of G with $G = G_0H$. If $V \in \mathrm{Rep}^\infty_E(H)^a$ then*

$$j_G(\mathrm{ind}^G_H(V)) = j_H(V) \quad \text{and} \quad d_G(\mathrm{ind}^G_H(V)) = d_H(V) + \dim(G/H).$$

Moreover, V is pure (resp. Cohen-Macaulay) over H if and only if $\mathrm{ind}^G_H(V)$ is pure (resp. Cohen-Macaulay) over G. $\square$

If H is a closed subgroup of G we finally denote by *res* : $\mathrm{Rep}^\infty_E(G) \to \mathrm{Rep}^\infty_E(H)$ the *restriction* functor. There is a natural transformation

$$res \circ S^i_G \longrightarrow S^i_H \circ res \tag{10}$$

of δ-functors that we call restriction, too, and which is constructed as follows. Fix a compact open subgroup G_0 of G and let $Q^\bullet \to E \to 0$ be a resolution of E be finitely generated free $\Lambda(G_0)$-modules. Given $V \in \mathrm{Rep}^\infty_E(G)$ the G-representation $S^i_G(V)$ is the i-th cohomology group of the complex $\varinjlim_N \mathrm{Hom}_{\Lambda(N)}(Q^\bullet, \check{V})$ where N runs through the open subgroups of G_0. Note that $\mathrm{Hom}_{\Lambda(N)}(Q^\bullet, \check{V}) = \mathrm{Hom}^{\mathrm{cont}}_{\Lambda(N)}(Q^\bullet, \check{V})$ for any N because the $\Lambda(N)$-modules $Q^\bullet$ are finitely generated and free. By the arguments given in the proof of Theorem 4.1 the cohomology groups of the complex $\mathrm{Hom}^{\mathrm{cont}}_{\Lambda(H\cap N)}(Q^\bullet, \check{V})$ are canonically isomorphic to $\mathrm{Ext}^\bullet_{\Lambda(H\cap N)}(E, \check{V})$. Thus, the natural transformation (10) is simply induced by the natural transformations $\mathrm{Hom}^{\mathrm{cont}}_{\Lambda(N)}(Q^\bullet, \cdot) \to \mathrm{Hom}^{\mathrm{cont}}_{\Lambda(H\cap N)}(Q^\bullet, \cdot)$.

For the rest of this section we let $\mathbb{G}$ be a connected reductive group over $\mathbb{Q}_p$ and $G = \mathbb{G}(\mathbb{Q}_p)$ its group of $\mathbb{Q}_p$-rational points. We set

$$c(\mathbb{G}) = \min_{\mathbb{P}\subsetneq\mathbb{G}}\{\dim(\mathbb{N})\}$$

where $\mathbb{P}$ runs through the set of proper parabolic $\mathbb{Q}_p$-subgroups of $\mathbb{G}$ and $\mathbb{N}$ denotes the unipotent radical of $\mathbb{P}$. If $\mathbb{G}$ is $\mathbb{Q}_p$-split then by [5], Proposition 14.18, the constant $c(\mathbb{G})$ depends only on the root system Φ of $\mathbb{G}$. Indeed, if $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ denotes a basis of Φ and if $1 \leq i \leq r$ then we let Ψ_i be the set of positive roots whose expressions in terms of Δ have a non-zero contribution from α_i. We then have

$$c(\mathbb{G}) = \min_{1\leq i\leq r}\{|\Psi_i|\}.$$

If $\mathbb{G}$ is $\mathbb{Q}_p$-split and Φ is irreducible then the values of $c(\mathbb{G})$ can be read off from the tables in [9] and [34], Appendix. We record them in the following list.

Φ	A_ℓ	B_ℓ	C_ℓ	D_ℓ	E_6	E_7	E_8	F_4	G_2
$c(\mathbb{G})$	ℓ	$2\ell-1$	$2\ell-1$	$2\ell-2$	16	27	57	15	5

Given a parabolic $\mathbb{Q}_p$-subgroup $\mathbb{P}$ of $\mathbb{G}$ we denote by $\mathbb{P} = \mathbb{M}\mathbb{N}$ its Levi decomposition and by P, M and N the groups of $\mathbb{Q}_p$-rational points of $\mathbb{P}$, $\mathbb{M}$ and $\mathbb{N}$, respectively. Following standard terminology, an E-linear irreducible admissible smooth G-representation V is called *supercuspidal* if it is not isomorphic to any subquotient of $\mathrm{ind}_P^G(\sigma)$ where $\mathbb{P}$ is a proper parabolic $\mathbb{Q}_p$-subgroup of $\mathbb{G}$ and where σ is an E-linear admissible smooth M-representation viewed as a representation of P via inflation along $P \to P/N \cong M$ (cf. [1], I.3). Note that as one of the main results of [1], a representation is supercuspidal if and only if it is *supersingular* in the sense of [1], I.5 (cf. [loc.cit.], I.5 Theorem 5).

Theorem 4.9. *Let $G = \mathbb{G}(\mathbb{Q}_p)$ be as above.*

(i) If $\mathbb{P} = \mathbb{M}\mathbb{N}$ is a parabolic $\mathbb{Q}_p$-subgroup of $\mathbb{G}$ and if σ is a supercuspidal representation of M then any subquotient V of $\mathrm{ind}_P^G(\sigma)$ satisfies $d_G(V) \leq \dim(G/P) + d_M(\sigma)$.

(ii) If $V \in \mathrm{Rep}_E^\infty(G)^a$ is irreducible and not supercuspidal then either V is finite dimensional over E or $c(\mathbb{G}) \leq d(V)$.

Proof. If V is a subquotient of $\mathrm{ind}_P^G(\sigma)$ then $d_G(V) \leq d_G(\mathrm{ind}_P^G(\sigma))$ by (3). It follows from the general theory of reductive groups over local fields that $G = G_0P$ for some compact open subgroup G_0 of G (cf. [36], 3.3.2). Further, since N acts trivially on σ we obtain

$$d_G(\mathrm{ind}_P^G(\sigma)) = d_P(\sigma) + \dim(G/P) = d_M(\sigma) + \dim(G/P) \tag{11}$$

from Corollary 4.2 and Corollary 4.8. This proves assertion (i).

Under the assumptions of (ii) we make use of a deep classification result of Abe, Henniart, Herzig and Vignéras (cf. [1], Theorem I.3.3 and Corollary II.7.2) building on previous work of Herzig and Abe. If the root system is irreducible there are two possible cases. Either V is isomorphic to a representation of the form $\mathrm{ind}_P^G(\sigma)$ for some proper parabolic $\mathbb{Q}_p$-subgroup $\mathbb{P}$ of $\mathbb{G}$. In this case (11) shows that V has dimension greater than or equal to $\dim(G/P) = \dim(N) \geq c(\mathbb{G})$. Since V is not supercuspidal, the only other possibility is that V is a twist of a generalized Steinberg representation corresponding to a parabolic $\mathbb{Q}_p$-subgroup $\mathbb{P}$ of $\mathbb{G}$. This is infinite dimensional if and only if $\mathbb{P} \neq \mathbb{G}$. In this case the assertion follows from Proposition 5.5 below. □

Remark 4.10. (i) The results of this section together with the above mentioned classification results show that in order to determine the dimensions of the objects of $\mathrm{Rep}_E^\infty(G)^a$ which are of finite length it suffices to consider supercuspidal representations. In the next section we are going to treat the case $G = \mathrm{GL}_2(\mathbb{Q}_p)$.

(ii) If the root system of $\mathbb{G}$ is reducible or if V is supercuspidal we do not know if there are dimension bounds as in Theorem 4.9. If $\mathbb{G} = \mathbb{G}_1 \times \mathbb{G}_2$ is the direct product of two non-trivial connected semisimple $\mathbb{Q}_p$-split groups $\mathbb{G}_1$ and $\mathbb{G}_2$, for example, and if $V = V_1 \otimes_E V_2$ where V_1 is a supercuspidal G_1-representation and V_2 is the Steinberg representation of G_2, then V is irreducible but not supercuspidal. By Corollary 4.4 we have $d_G(V) = d_{G_1}(V_1) + d_{G_2}(V_2)$ so that we are confronted with the same problem as in (i).

5 Examples

We continue to assume that p is a prime number, that E is a field of characteristic p and that G is a p-adic Lie group of dimension $d = \dim(G)$. As a first result we will give an explicit description of the duality character χ_G of G. Let $\mathfrak{g}$ denote the Lie algebra of G over $\mathbb{Q}_p$ endowed with the adjoint action of G. It gives rise to the $\mathbb{Q}_p$-valued character $\bigwedge^d \mathfrak{g}$ of G. As in section 2 we denote by $\delta_G : G \to \mathbb{Q}_p^\times$ the locally constant modulus character of G and define the $\mathbb{Q}_p$-valued character $\mathfrak{d}_G$ of G by $\mathfrak{d}_G = \bigwedge^d \mathfrak{g} \otimes_{\mathbb{Q}_p} \delta_G$.

The following theorem shows how to describe the duality character χ_G in terms of the character $\mathfrak{d}_G$. Its proof relies heavily on results of Schneider and Teitelbaum and shows a partial compatibility with the duality theory for locally analytic representations developed in [31].

Theorem 5.1. *The character $\mathfrak{d}_G$ takes values in $\mathbb{Z}_p^\times$. The duality character χ_G of G coincides with the composition of $\mathfrak{d}_G^{-1} : G \to \mathbb{Z}_p^\times$ with the canonical homomorphisms $\mathbb{Z}_p^\times \to \mathbb{F}_p^\times \to E^\times$.*

Proof. If $|\cdot|$ denotes the normalized absolute value of $\mathbb{Q}_p$ then $\delta_G = |\bigwedge^d \mathfrak{g}|$ by [8], III.3.16, Corollaire à la Proposition 55. This proves the first assertion.

As in section 1 one can define the Iwasawa algebra $\Lambda_{\mathbb{Z}_p}(G)$ of G with coefficients in $\mathbb{Z}_p$ and the bimodule $\Delta_{\mathbb{Z}_p}(G) \cong \prod_{g \in G/G_0} g\Lambda_{\mathbb{Z}_p}(G_0)$ over $\Lambda_{\mathbb{Z}_p}(G)$. In this situation, the analog of Proposition 3.2 holds true with formally the same proof (relying once again on [24], Lemma 2.3 and [31], Lemma 2.2). The formation of $\Lambda_{\mathbb{Z}_p}(G)$, $\Delta_{\mathbb{Z}_p}(G)$ and $\mathrm{Ext}^d_{\Lambda_{\mathbb{Z}_p}(G)}(\mathbb{Z}_p, \Delta_{\mathbb{Z}_p}(G))$ commutes with reduction modulo p. Indeed, the cases of $\Lambda_{\mathbb{Z}_p}(G)$ and $\Delta_{\mathbb{Z}_p}(G)$ are both formally reduced to the standard fact that $\Lambda_{\mathbb{Z}_p}(G_0)/p\Lambda_{\mathbb{Z}_p}(G_0) \cong \Lambda_{\mathbb{F}_p}(G_0)$ for any profinite group G_0. For the extension groups one is reduced to the case that G is a uniform pro-p group by using Proposition 3.2 and its analog over $\mathbb{Z}_p$. In this case, the $\Lambda_{\mathbb{Z}_p}(G)$-module $\mathbb{Z}_p$ admits a $\mathbb{Z}_p$-linearly split resolution by finitely generated free $\Lambda_{\mathbb{Z}_p}(G)$-modules (cf. [25], Chapitre V, (2.2.2.3)). Since it is $\mathbb{Z}_p$-split, this resolution remains exact after reduction modulo p. Moreover, the extension groups $\mathrm{Ext}^\bullet_{\Lambda_{\mathbb{Z}_p}(G)}(\mathbb{Z}_p, \Lambda_{\mathbb{Z}_p}(G))$ are free $\mathbb{Z}_p$-modules because G is a Poincaré duality group (cf. [35], Theorem 5.1.5). By standard arguments we obtain that the formation of extension groups commutes with reduction modulo p, as claimed. As a consequence, it suffices to see that the left action of G on $\mathrm{Ext}^d_{\Lambda_{\mathbb{Z}_p}(G)}(\mathbb{Z}_p, \Delta_{\mathbb{Z}_p}(G))$ is given by $\mathfrak{d}_G^{-1}$.

We make free use of the notation introduced in [31]. In particular, $D(G, \mathbb{Q}_p)$ denotes the $\mathbb{Q}_p$-algebra of locally analytic $\mathbb{Q}_p$-valued distributions on G and $\mathcal{D}_{\mathbb{Q}_p}(G) = C_c^{\mathrm{an}}(G, \mathbb{Q}_p)'_b$ as a $(D(G, \mathbb{Q}_p), D(G, \mathbb{Q}_p))$-bimodule. By the proof of [31], Proposition 6.5, the right action of G on $\mathcal{D}_{\mathbb{Q}_p}(G)$ makes $\mathrm{Ext}^d_{D(G,\mathbb{Q}_p)}(\mathbb{Q}_p, \mathcal{D}_{\mathbb{Q}_p}(G))$ a left $D(G, \mathbb{Q}_p)$-module of $\mathbb{Q}_p$-dimension one on which the action of G is given by the character $\mathfrak{d}_G^{-1}$. Further, by [31], Proposition 2.3 we have the analog of our Proposition 3.2, i.e. for any compact open subgroup N of G there is a natural $D(N, \mathbb{Q}_p)$-linear isomorphism $\mathrm{Ext}^d_{D(G,\mathbb{Q}_p)}(\mathbb{Q}_p, \mathcal{D}_{\mathbb{Q}_p}(G)) \to \mathrm{Ext}^d_{D(N,\mathbb{Q}_p)}(\mathbb{Q}_p, D(N, \mathbb{Q}_p))$. In particular, the

$\mathbb{Q}_p$-vector space on the right is one dimensional with its left N-action given by $\mathfrak{d}_G^{-1}|_N = \mathfrak{d}_N^{-1}$. Arguing as in Proposition 3.13, there is an isomorphism

$$\varinjlim_N \operatorname{Ext}^d_{D(N,\mathbb{Q}_p)}(\mathbb{Q}_p, D(G,\mathbb{Q}_p)) \cong \mathfrak{d}_G^{-1} \otimes_{\mathbb{Q}_p} C_c^\infty(G,\mathbb{Q}_p)$$

of left and right G-representations.

Let N be any compact open subgroup of G. We note that there is a homomorphism $\Lambda_{\mathbb{Z}_p}(N) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to D(N,\mathbb{Q}_p)$ of $\mathbb{Q}_p$-algebras which is faithfully flat by [30], Theorem 5.2. Since the ring $\Lambda_{\mathbb{Z}_p}(N)$ is noetherian (cf. [25], Chapitre V, Proposition 2.2.4), the trivial $\Lambda_{\mathbb{Z}_p}(N)$-module $\mathbb{Z}_p$ admits a resolution $P^\bullet \to \mathbb{Z}_p \to 0$ by finitely generated free $\Lambda_{\mathbb{Z}_p}(N)$-modules P^i. As is shown in the proof of [31], Proposition 6.5, the induced complex

$$D(N,\mathbb{Q}_p) \otimes_{\Lambda_{\mathbb{Z}_p}(N)} P^\bullet \longrightarrow D(N,\mathbb{Q}_p) \otimes_{\Lambda_{\mathbb{Z}_p}(N)} \mathbb{Z}_p \longrightarrow 0$$

is an exact resolution of the trivial $D(N,\mathbb{Q}_p)$-module $D(N,\mathbb{Q}_p) \otimes_{\Lambda_{\mathbb{Z}_p}(N)} \mathbb{Z}_p \cong \mathbb{Q}_p$ by finitely generated free $D(N,\mathbb{Q}_p)$-modules. On the other hand, there is a G-biequivariant homomorphism

$$\varinjlim_N \operatorname{Ext}^d_{\Lambda_{\mathbb{Z}_p}(N)}(\mathbb{Z}_p, \Lambda_{\mathbb{Z}_p}(G)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow \varinjlim_N \operatorname{Ext}^d_{D(N,\mathbb{Q}_p)}(\mathbb{Q}_p, D(G,\mathbb{Q}_p)).$$

It is the direct limit of the homomorphisms

$$\operatorname{Ext}^d_{\Lambda_{\mathbb{Z}_p}(N)}(\mathbb{Z}_p, \Lambda_{\mathbb{Z}_p}(G)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow \operatorname{Ext}^d_{D(N,\mathbb{Q}_p)}(\mathbb{Q}_p, D(G,\mathbb{Q}_p)), \tag{12}$$

coming from the $(\Lambda_{\mathbb{Z}_p}(G), \Lambda_{\mathbb{Z}_p}(G))$-bimodule homomorphisms $\Lambda_{\mathbb{Z}_p}(G) \to D(G,\mathbb{Q}_p)$ and the natural isomorphisms

$$\operatorname{Hom}_{\Lambda_{\mathbb{Z}_p}(N)}(P^\bullet, D(G,\mathbb{Q}_p)) \cong \operatorname{Hom}_{D(N,\mathbb{Q}_p)}(D(N,\mathbb{Q}_p) \otimes_{\Lambda_{\mathbb{Z}_p}(N)} P^\bullet, D(G,\mathbb{Q}_p)).$$

By Proposition 3.13 it suffices to show that the maps in (12) are bijective. By decomposing $\Lambda_{\mathbb{Z}_p}(G)$ and $D(G,\mathbb{Q}_p)$ into direct sums of free modules over $\Lambda_{\mathbb{Z}_p}(N)$ and $D(N,\mathbb{Q}_p)$, respectively, we are further reduced to showing that the natural map

$$\operatorname{Ext}^d_{\Lambda_{\mathbb{Z}_p}(N)}(\mathbb{Z}_p, \Lambda_{\mathbb{Z}_p}(N)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow \operatorname{Ext}^d_{D(N,\mathbb{Q}_p)}(\mathbb{Q}_p, D(N,\mathbb{Q}_p)) \tag{13}$$

is bijective. However, this is just the canonical map

$$\begin{aligned} & \operatorname{Ext}^d_{\Lambda_{\mathbb{Z}_p}(N)}(\mathbb{Z}_p, \Lambda_{\mathbb{Z}_p}(N)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\ \longrightarrow \quad & D(N,\mathbb{Q}_p) \otimes_{\Lambda_{\mathbb{Z}_p}(N)} \operatorname{Ext}^d_{\Lambda_{\mathbb{Z}_p}(N)}(\mathbb{Z}_p, \Lambda_{\mathbb{Z}_p}(N)) \end{aligned} \tag{14}$$

into the base extension followed by the canonical isomorphism

$$D(N,\mathbb{Q}_p) \otimes_{\Lambda_{\mathbb{Z}_p}(N)} \operatorname{Ext}^d_{\Lambda_{\mathbb{Z}_p}(N)}(\mathbb{Z}_p, \Lambda_{\mathbb{Z}_p}(N)) \cong \operatorname{Ext}^d_{D(N,\mathbb{Q}_p)}(\mathbb{Q}_p, D(N,\mathbb{Q}_p))$$

(cf. [6], Chapitre X, §6, No. 7, Proposition 10 (b); here we use once more that $D(N, \mathbb{Q}_p) \otimes_{\Lambda_{\mathbb{Z}_p}(N)} \mathbb{Z}_p \cong \mathbb{Q}_p$). As recalled above, the ring homomorphism $\Lambda_{\mathbb{Z}_p}(N) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to D(N, \mathbb{Q}_p)$ is faithfully flat, whence the map (14) is injective and so is the map (13). Now we may assume that N is an open uniform pro-p subgroup of G. In this case N is a Poincaré duality group by [35], Theorem 5.1.5, and both sides of (13) are one dimensional vector spaces over $\mathbb{Q}_p$. This completes the proof. $\square$

Corollary 5.2. *If G is open in the group of $\mathbb{Q}_p$-rational points of a connected reductive group over $\mathbb{Q}_p$ then the duality character χ_G of G is trivial.*

Proof. Let $\mathfrak{g}$ denote the Lie algebra of G. It is the direct sum of its center and its derived Lie algebra. Therefore, the adjoint action of $\mathfrak{g}$ on $\bigwedge^d \mathfrak{g}$ is trivial. Since the action of G on $\bigwedge^d \mathfrak{g}$ is algebraic, it is trivial by Zariski density. By Theorem 5.1 the character $\chi_G = (\bigwedge^d \mathfrak{g} \otimes_{\mathbb{Q}_p} |\bigwedge^d \mathfrak{g}|)^{-1}$ is then trivial, too. $\square$

In what follows we assume that $G = \mathbb{G}(\mathbb{Q}_p)$ is the group of $\mathbb{Q}_p$-rational points of a connected reductive algebraic group $\mathbb{G}$ over $\mathbb{Q}_p$. As before we denote by $\mathbb{P}$ a parabolic $\mathbb{Q}_p$-subgroup of $\mathbb{G}$ with Levi decomposition $\mathbb{P} = \mathbb{M}\mathbb{N}$ and denote by P, M and N the respective groups of $\mathbb{Q}_p$-rational points.

Corollary 5.3. *If $\mathbb{G}$ is $\mathbb{Q}_p$-split and if $\mathbb{P}$ is a $\mathbb{Q}_p$-Borel subgroup of $\mathbb{G}$ with associated set of positive roots Φ^+ then*

$$\chi_P(mn) = \prod_{\alpha \in \Phi^+} \alpha(m)^{-1} |\alpha(m)|^{-1} \mod p$$

for all $m \in M$ and $n \in N$.

Proof. Since N is the union of its open pro-p subgroups the restriction of χ_P to N is trivial. Therefore, the assertion is a direct consequence of Theorem 5.1 and the weight space decomposition of the Lie algebra of G. $\square$

As our next example, we compute the higher smooth duals of the principal series representations of G. Let χ be a smooth E-valued character of $M \cong P/N$, viewed as a smooth E-valued character of P via inflation. Taking into account Theorem 5.1 the following result is formally the same as [31], Proposition 6.5. It is a direct consequence of Corollary 3.16, Theorem 4.7, Corollary 4.8 and Corollary 5.2.

Proposition 5.4. *The smooth principal series representation $\mathrm{ind}_P^G(\chi)$ of G over E is Cohen-Macaulay of dimension $\dim(G/P)$. The smooth E-linear G-representation $S_G^{\dim(G/P)}(\mathrm{ind}_P^G(\chi))$ is isomorphic to $\mathrm{ind}_P^G(\chi_P \chi^{-1})$.* $\square$

By E we also denote the trivial E-valued character of P. The *special representation* $\mathrm{Sp}_P(G,E)$ of G with respect to P is defined by the exact sequence

$$\bigoplus_{P\subsetneqq Q} \mathrm{ind}_Q^G(E) \longrightarrow \mathrm{ind}_P^G(E) \longrightarrow \mathrm{Sp}_P(G,E) \longrightarrow 0,$$

in which the left hand sum runs over the set of groups of $\mathbb{Q}_p$-rational points of the parabolic $\mathbb{Q}_p$-subgroups of $\mathbb{G}$ properly containing $\mathbb{P}$.

Proposition 5.5. *The special representation* $\mathrm{Sp}_P(G,E)$ *is pure of dimension* $\dim(G/P)$*. The surjection* $\mathrm{ind}_P^G(E) \to \mathrm{Sp}_P(G,E)$ *induces an isomorphism in the quotient category* $\mathrm{Rep}_E^\infty(G)^a_{\dim(P)}/\mathrm{Rep}_E^\infty(G)^a_{\dim(P)+1}$.

Proof. Since $\mathrm{ind}_P^G(E)$ is Cohen-Macaulay of dimension $\dim(G/P)$ (cf. Proposition 5.4), the $\Lambda(G)$-module $\mathrm{ind}_P^G(E)\check{}$ is pure of grade $\dim(P)$ (cf. Remark 3.7). It follows from [23], III.4.2 Proposition 9, that so is its submodule $\mathrm{Sp}_P(G,E)\check{}$. Let W denote the kernel of the surjection $\mathrm{ind}_P^G(E) \to \mathrm{Sp}_P(G,E)$ so that $\check{W}$ is a $\Lambda(G)$-submodule of $\oplus_{P\subsetneqq Q}\mathrm{ind}_Q^G(E)\check{}$. By (3) and Proposition 5.4 we have $j(\check{W}) \geq \min_{P\subsetneqq Q}\dim(Q) > \dim(P)$. Thus, $\check{W} \in \mathcal{C}_G^{\dim(P)+1}$ and $W \in \mathrm{Rep}_E^\infty(G)^a_{\dim(P)+1}$. □

The special representations $\mathrm{Sp}_P(G,E)$ were shown to be irreducible by Große-Klönne (cf. [20], Corollary 4.3), Herzig (cf. [22], Theorem 7.2) and Ly (cf. [26], Théorème 3.1). If $\mathbb{P}$ is a Borel subgroup of $\mathbb{G}$ then the special representation $\mathrm{Sp}_P(G,E)$ is usually called the *Steinberg representation* of G over E and will be denoted by St_G.

Proposition 5.6. *If* $G = \mathrm{GL}_3(\mathbb{Q}_p)$ *then the Steinberg representation of* G *over* E *is not Cohen-Macaulay.*

Proof. We let P_1 and P_2 be the groups of $\mathbb{Q}_p$-rational points of the two distinct proper parabolic subgroups of $\mathbb{G}$ which properly contain a fixed Borel subgroup $\mathbb{P}$. Since P_1 and P_2 generate the group G there is an exact sequence

$$0 \longrightarrow E \longrightarrow \mathrm{ind}_{P_1}^G(E) \oplus \mathrm{ind}_{P_2}^G(E) \longrightarrow \mathrm{ind}_P^G(E) \longrightarrow \mathrm{St}_G \longrightarrow 0$$

in $\mathrm{Rep}_E^\infty(G)^a$. As above, we denote by W the kernel of the surjection $\mathrm{ind}_P^G(E) \to \mathrm{St}_G$ and consider the short exact sequence

$$0 \longrightarrow E \longrightarrow \mathrm{ind}_{P_1}^G(E) \oplus \mathrm{ind}_{P_2}^G(E) \longrightarrow W \longrightarrow 0.$$

Using Corollary 3.16 and Proposition 5.4, the long exact sequence obtained by applying the δ-functor $(S_G^i)_{i\geq 0}$ yields

$$\begin{aligned} S_G^1(W) &\cong E, \\ S_G^2(W) &\cong \mathrm{ind}_{P_1}^G(\chi_{P_1}) \oplus \mathrm{ind}_{P_2}^G(\chi_{P_2}) \quad \text{and} \\ S_G^i(W) &= 0 \text{ for } i \notin \{1,2\}. \end{aligned}$$

Moreover, by Proposition 5.4 we have

$$S^i(\mathrm{ind}_P^G(E)) \cong \begin{cases} 0, & \text{if } i \neq \dim(G/P) = 3, \\ \mathrm{ind}_P^G(\chi_P), & \text{if } i = 3. \end{cases}$$

Considering the short exact sequence $0 \to W \to \mathrm{ind}_P^G(E) \to \mathrm{St}_G \to 0$ and analyzing the associated long exact sequence we obtain $S^i(\mathrm{St}_G) = 0$ for $i \notin \{2,3\}$, $S^2(\mathrm{St}_G) \cong E$, as well as a short exact sequence

$$0 \longrightarrow \mathrm{ind}_{P_1}^G(\chi_{P_1}) \oplus \mathrm{ind}_{P_2}^G(\chi_{P_2}) \longrightarrow S^3(\mathrm{St}_G) \longrightarrow \mathrm{ind}_P^G(\chi_P) \longrightarrow 0$$

In particular, St_G is not Cohen-Macaulay. □

Proposition 5.7. *If $G = \mathrm{GL}_2(\mathbb{Q}_p)$ then the Steinberg representation of G is Cohen-Macaulay of dimension one. Up to isomorphism, $S_G^1(\mathrm{St}_G)$ is the unique non-split extension*

$$0 \longrightarrow E \longrightarrow S_G^1(\mathrm{St}_G) \longrightarrow \mathrm{ind}_P^G(\chi_P) \longrightarrow 0$$

of E and $\mathrm{ind}_P^G(\chi_P)$.

Proof. Consider the exact sequence $0 \to E \to \mathrm{ind}_P^G(E) \to \mathrm{St}_G \to 0$. Using Corollary 3.16, Proposition 5.4 and analyzing the associated long exact sequence obtained by applying the δ-functor $(S_G^i)_{i\geq 0}$ we obtain $S_G^i(\mathrm{St}_G) = 0$ unless $i = 1$. In particular, the Steinberg representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E is Cohen-Macaulay. For $i = 1$ we obtain a short exact sequence

$$0 \longrightarrow E \longrightarrow S_G^1(\mathrm{St}_G) \longrightarrow \mathrm{ind}_P^G(\chi_P) \longrightarrow 0.$$

If this sequence was split we would obtain $S^1S^1(\mathrm{St}_G) \cong \mathrm{ind}_P^G(E)$ by Corollary 3.16 and Proposition 5.4. However, $S^1S^1(\mathrm{St}_G) \cong \mathrm{St}_G$ by Corollary 3.15 and [37], Proposition 3.9, because St_G is Cohen-Macaulay of dimension one. Now the Steinberg representation is irreducible whereas $\mathrm{ind}_P^G(E)$ is not. Thus, we arrive at a contradiction. Finally, note that up to isomorphism there is only one non-split extension as above (cf. Proposition 5.3 and [18], Proposition 4.3.13 (2), bearing in mind the different conventions concerning ind_P^G which are used in [17] and [18]). □

Remark 5.8. Over fields of characteristic zero the Steinberg representation is known to be self-dual, i.e. isomorphic to its 0-th smooth dual (cf. [12], §9.10 for the case of GL_2). Proposition 5.6 and Proposition 5.7 show that its behavior in natural characteristic is different. More importantly, we find that even on Cohen-Macaulay representations the smooth duality functors do not preserve irreducibility. Further, the higher smooth duals of an irreducible representation are not necessarily concentrated in a single degree. Both phenomena are in contrast to the Zelevinsky conjecture in characteristic zero (cf. [33], Theorem III.3.1 and Corollary III.3.2) which is also formulated in terms of certain Ext-duals.

Let now again G be an arbitrary p-adic Lie group and let G_0 be a (not necessarily compact) open subgroup of G. Given an E-linear representation V of G_0 we denote by $I_{G_0}^G(V)$ the E-vector space of functions $f : G \to V$ satisfying $f(gh) = h^{-1}f(g)$ for all $g \in G$ and $h \in G_0$. Left translation by elements of G gives $I_{G_0}^G(V)$ the structure of an E-linear G-representation.

If $V \in \mathrm{Rep}_E^\infty(G_0)$ we let $\mathrm{Ind}_{G_0}^G(V) := S^0(I_{G_0}^G(V)) \in \mathrm{Rep}_E^\infty(G)$ denote the subspace of smooth vectors in $I_{G_0}^G(V)$. Recall that we defined $\mathrm{ind}_{G_0}^G(V)$ to be the G-subrepresentation of $I_{G_0}^G(V)$ consisting of all functions whose support is compact modulo G_0. Since the G_0-representation V is smooth we automatically have $\mathrm{ind}_{G_0}^G(V) \subseteq \mathrm{Ind}_{G_0}^G(V)$. If G/G_0 is compact then $\mathrm{ind}_{G_0}^G(V) = \mathrm{Ind}_{G_0}^G(V) = I_{G_0}^G(V)$.

Lemma 5.9. *Let G_0 be an open subgroup of G. If $V \in \mathrm{Rep}_E^\infty(G_0)$ then the G-representation $\mathrm{ind}_{G_0}^G(V)\check{}$ is isomorphic to $I_{G_0}^G(\check{V})$. In particular, the latter is naturally an object of $\mathrm{Mod}_{\Lambda(G)}^{\mathrm{pc}}$ and there is an isomorphism $S_G^0 \circ \mathrm{ind}_{G_0}^G \cong \mathrm{Ind}_{G_0}^G \circ S_{G_0}^0$ of functors $\mathrm{Rep}_E^\infty(G_0) \to \mathrm{Rep}_E^\infty(G)$.*

Proof. We define the E-bilinear map $\langle \cdot, \cdot \rangle : \mathrm{ind}_{G_0}^G(V) \times I_{G_0}^G(\check{V}) \to E$ by $\langle f, F \rangle := \sum_{g \in G/G_0} F(g)(f(g))$. It gives rise to a G-equivariant E-linear map $I_{G_0}^G(\check{V}) \to \mathrm{ind}_{G_0}^G(V)\check{}$. Evaluation at a fixed system of coset representatives of G/G_0 yields E-linear isomorphisms $\mathrm{ind}_{G_0}^G(V) \cong \oplus_{g \in G/G_0} V$ and $I_{G_0}^G(\check{V}) \cong \prod_{g \in G/G_0} \check{V}$ under which the above G-homomorphism corresponds to the isomorphism $\prod_{g \in G/G_0} \check{V} \cong (\oplus_{g \in G/G_0} V)\check{}$. This proves the first assertion. The second assertion is a consequence of the obvious relation $\Sigma_G^0(I_{G_0}^G(\check{V})) = \mathrm{Ind}_{G_0}^G(\Sigma_{G_0}^0(\check{V}))$. $\square$

Remark 5.10. If G_0 is open of finite index in G then the functors $\mathrm{ind}_{G_0}^G$ and $\mathrm{Ind}_{G_0}^G$ on $\mathrm{Rep}_E^\infty(G_0)$ coincide. Further, for any smooth E-valued character χ of G there is an isomorphism of functors $\chi \otimes_E \mathrm{ind}_{G_0}^G(\cdot) \cong \mathrm{ind}_{G_0}^G(\chi|_{G_0} \otimes_E (\cdot))$ (cf. [38], §I.5.2). In this special situation the statement of the preceding lemma is in accordance with that of Theorem 4.7 (ii) for $i = 0$.

Finally, we treat the supersingular representations of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ constructed by Barthel and Livné (cf. [3]) and classified by Breuil (cf. [10]). Let $Z \cong \mathbb{Q}_p^\times$ denote the center of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Set $G_0 = Z \cdot \mathrm{GL}_2(\mathbb{Z}_p)$ and let V denote a finite dimensional E-linear representation of $\mathrm{GL}_2(\mathbb{F}_p)$. We view V as a smooth representation of G_0 via inflation along $\mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(\mathbb{F}_p)$ and by letting $p \in Z$ act trivially. Set

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \alpha = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \text{ and } G_n = G_0 \cap \alpha^n G_0 \alpha^{-n}$$

for any $n \in \mathbb{Z}$. We let $K_0 = \mathrm{GL}_2(\mathbb{Z}_p)$ and for $m \geq 1$ denote by K_m the kernel of the reduction homomorphism $\mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(\mathbb{Z}_p/p^m\mathbb{Z}_p)$. By T

we denote the subgroup of G consisting of diagonal matrices. By N and $\overline{N}$ we denote the subgroups of G consisting of upper and lower triangular unipotent matrices, respectively. Setting

$$T_0 = T \cap G_0,\ N_n = \alpha^n (N \cap G_0)\alpha^{-n} \text{ and } \overline{N}_n = \alpha^{-n}(\overline{N} \cap G_0)\alpha^n$$

we have the Iwahori decomposition $G_n = \overline{N}_0 T_0 N_n$ whenever $n \geq 1$. Further, $w\alpha w^{-1}Z = \alpha^{-1}Z$ and $G_0 = G_{-1} \coprod N_0 w G_{-1}$ so that

$$G_0\alpha G_0 = G_0\alpha^{-1}G_0 = \alpha^{-1}G_0 \coprod N_0 w\alpha^{-1}G_0 = \alpha^{-1}G_0 \coprod N_0\alpha G_0. \tag{15}$$

Multiplying by α^n from the left we obtain $\alpha^n G_0 \alpha G_0 = \alpha^{n-1}G_0 \coprod N_n \alpha^{n+1} G_0$ for all $n \in \mathbb{Z}$.

Theorem 5.11. *Assume $G = \mathrm{GL}_2(\mathbb{Q}_p)$. If G_0 and V are as above then $S^i_G(\mathrm{ind}^G_{G_0}(V)) = 0$ for any integer $i \geq 2$.*

Proof. By Lemma 5.9 we have $\mathrm{ind}^G_{G_0}(V)\check{} \cong I^G_{G_0}(\check{V})$ and need to see that $\Sigma^i(I^G_{G_0}(\check{V}) = 0$ for $i \geq 2$.

For any integer n we denote by $\check{V}_n$ the $\alpha^n G_0 \alpha^{-n}$-representation whose underlying E-vector space is $\check{V}$ and such that $(\alpha^n g \alpha^{-n})\check{v} = g\check{v}$ for all $g \in G_0$ and $\check{v} \in \check{V}$. With this notation the Cartan decomposition $G = \coprod_{n\geq 0} G_0\alpha^n G_0$ induces the G_0-equivariant Mackey decomposition $I^G_{G_0}(\check{V}) \cong \prod_{n\geq 0}\mathrm{ind}^{G_0}_{G_n}(\check{V}_n)$, sending a function $f \in I^G_{G_0}(\check{V})$ to the family $(f_n)_{n\geq 0}$ of functions $f_n \in \mathrm{ind}^{G_0}_{G_n}(\check{V}_n)$ defined by $f_n(g) = f(g\alpha^n)$. For any integer $m \geq 0$ it induces the decomposition

$$\mathrm{Ext}^i_{\Lambda(K_m)}(E, I^G_{G_0}(\check{V})) \cong \prod_{n\geq 0} \mathrm{Ext}^i_{\Lambda(K_m)}(E, \mathrm{ind}^{G_0}_{G_n}(\check{V}_n)). \tag{16}$$

Since K_m is normal in G_0 and $\mathrm{ind}^{G_0}_{G_n} \cong \mathrm{ind}^{G_0}_{G_nK_m} \circ \mathrm{ind}^{G_nK_m}_{G_n}$, the exactness of the induction functors implies

$$\mathrm{Ext}^i_{\Lambda(K_m)}(E, \mathrm{ind}^{G_0}_{G_n}(\check{V}_n)) \cong \mathrm{ind}^{G_0}_{G_nK_m}(\mathrm{Ext}^i_{\Lambda(K_m)}(E, \mathrm{ind}^{G_nK_m}_{G_n}(\check{V}_n))). \tag{17}$$

Further, restriction to K_m induces a $\Lambda(K_m)$-linear bijection $\mathrm{ind}^{G_nK_m}_{G_n}(\check{V}_n) \cong \mathrm{ind}^{K_m}_{G_n\cap K_m}(\check{V}_n)$ so that by Shapiro's lemma

$$\mathrm{Ext}^i_{\Lambda(K_m)}(E, \mathrm{ind}^{G_nK_m}_{G_n}(\check{V}_n)) \cong \mathrm{Ext}^i_{\Lambda(G_n\cap K_m)}(E, \check{V}_n). \tag{18}$$

Let $\ell \geq m$. Under the identifications (16), (17) and (18) the restriction map $\mathrm{Ext}^i_{\Lambda(K_m)}(E, I^G_{G_0}(\check{V})) \to \mathrm{Ext}^i_{\Lambda(K_\ell)}(E, I^G_{G_0}(\check{V}))$ is the direct product of the maps

$$\mathrm{ind}^{G_0}_{G_nK_m}(\mathrm{Ext}^i_{\Lambda(G_n\cap K_m)}(E, \check{V}_n)) \longrightarrow \mathrm{ind}^{G_0}_{G_nK_\ell}(\mathrm{Ext}^i_{\Lambda(G_n\cap K_\ell)}(E, \check{V}_n)),$$

obtained by composing a function on the left with the G_nK_ℓ-equivariant restriction map $\mathrm{Ext}^i_{\Lambda(G_n\cap K_m)}(E,\check{V}_n) \to \mathrm{Ext}^i_{\Lambda(G_n\cap K_\ell)}(E,\check{V}_n)$. We will see that if $i \geq 2$ then there is an integer ℓ such that these restriction maps are zero for all $n \geq 0$.

By Proposition 3.8 have $\varinjlim_{\ell\geq m} \mathrm{Ext}^i_{\Lambda(G_n\cap K_\ell)}(E,\check{V}_n) = 0$ for $i \geq 1$ and each $n \geq 0$ individually. However, the vanishing of this direct limit might not be uniform in n. Note that all occurring extension groups are finite dimensional over E so that we can at least choose $\ell > m$ with the property that the above restriction maps are zero for all $n < m$. We claim that if $i \geq 2$ and $n \geq m$ then the restriction map is zero for $\ell = m+1$. This will complete the proof.

Setting $T_m = K_m \cap T$ and assuming $m \geq 1$ we have the decomposition $K_m = \overline{N}_m T_m N_m$. If $n \geq m$ then we also have the decomposition $G_n \cap K_m = \overline{N}_m T_m N_n$. Conjugation with α^{-n} yields $\alpha^{-n}(G_n \cap K_m)\alpha^n = \overline{N}_{m+n} T_m N_0$ and therefore

$$\mathrm{Ext}^i_{\Lambda(G_n\cap K_m)}(E,\check{V}_n) \cong \mathrm{Ext}^i_{\Lambda(U_{mn})}(E,\check{V}) \quad \text{with} \quad U_{mn} = \overline{N}_{m+n} T_m N_0.$$

If $m \geq 2$ then K_m is a uniform pro-p group (cf. [15], Theorem 5.2). In the particular case of $\mathrm{GL}_2(\mathbb{Q}_p)$ the same is true of the groups U_{mn} if $2 \leq m \leq n$. Indeed, by [15], Theorem 4.5, it suffices to show that U_{mn} is powerful, i.e. that every commutator is contained in the closure of the subgroup generated by the p^ε-th powers of U_{mn}. Here $\varepsilon = 1$ or $\varepsilon = 2$ according to whether p is odd or even. Since the groups $\overline{N}_{m+n}$, T_m and N_0 are commutative, it suffices to check the following finite number of cases. Letting $c \in p^{m+n}\mathbb{Z}_p$, $a, d \in 1 + p^m\mathbb{Z}_p$ and $b \in \mathbb{Z}_p$ we have

$$\begin{aligned}
\left[\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right] &= \begin{pmatrix} 1 & 0 \\ (1-\frac{d}{a})c & 1 \end{pmatrix} \in \overline{N}_{2m+n} = \overline{N}^{p^m}_{m+n}, \\
\left[\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right] &= \begin{pmatrix} 1 & (1-\frac{a}{d})b \\ 0 & 1 \end{pmatrix} \in N_m = N_0^{p^m}, \text{ and} \\
\left[\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right] &= \begin{pmatrix} 1+bc & b^2c \\ -c^2b & 1+bc+(bc)^2 \end{pmatrix} \in \alpha^{-n}K_{m+n}\alpha^n,
\end{aligned}$$

where $\alpha^{-n}K_{m+n}\alpha^n = \alpha^{-n}(\overline{N}_{m+n}T_{m+n}N_{m+n})\alpha^n \subseteq \alpha^{-n}(\overline{N}^{p^\varepsilon}_m T^{p^\varepsilon}_m N^{p^\varepsilon}_n)\alpha^n \subseteq U^{p^\varepsilon}_{mn}$.

This result allows us to determine the E-vector spaces $\mathrm{Ext}^i_{\Lambda(U_{mn})}(E,\check{V})$ for any $i \geq 0$. We let $\mathfrak{m}$ denote the maximal ideal of the ring $\Lambda(U_{mn})$ and endow $\Lambda(U_{mn})$, V and $\check{V}$ with their $\mathfrak{m}$-adic filtrations. Denoting by $\mathrm{gr}(\Lambda(U_{mn}))$, $\mathrm{gr}(V)$ and $\mathrm{gr}(\check{V})$ the associated graded objects there is a functorial isomorphism

$$\mathrm{Tor}_i^{\Lambda(U_{mn})}(E,V) \cong \mathrm{Tor}_i^{\mathrm{gr}(\Lambda(U_{mn}))}(E,\mathrm{gr}(V))$$

(cf. [21], Theorem 3.3'). Dually, we obtain functorial isomorphisms

$$\mathrm{Ext}^i_{\Lambda(U_{mn})}(E,\check{V}) \cong \mathrm{Ext}^i_{\mathrm{gr}(\Lambda(U_{mn}))}(E,\mathrm{gr}(\check{V}))$$

for all $i \geq 0$. Note that $\mathfrak{m}$ is the augmentation ideal of $\Lambda(U_{mn})$. We let $\overline{\nu}$ and ν denote topological generators of $\overline{N}_{m+n}$ and N_0, respectively, and let τ_1 and τ_2 denote topological generators of T_m. Sending u, x, y and z to the classes of $1-\overline{\nu}$, $1-\tau_1$, $1-\tau_2$ and $1-\nu$ in $\mathfrak{m}/\mathfrak{m}^2$, respectively, induces an isomorphism $E[u,x,y,z] \cong \mathrm{gr}(\Lambda(U_{mn}))$ of graded E-algebras (cf. [15], Theorem 7.24). Since $\overline{N}_{n+m}$ and T_m act trivially on $\check{V}$, the elements u, x and y annihilate $\mathrm{gr}(\check{V})$. Therefore, $\mathrm{gr}(\check{V})$ can also be computed through the adic filtration of $\Lambda(N_0)$. By the Künneth formula we obtain E-linear isomorphisms

$$\begin{aligned}
& \mathrm{Ext}^i_{\Lambda(U_{mn})}(E,\check{V}) \\
\cong\; & \oplus_{r+s=i} \mathrm{Ext}^r_{\mathrm{gr}(\Lambda(\overline{N}_{m+n}T_m))}(E,E) \otimes_E \mathrm{Ext}^s_{\mathrm{gr}(\Lambda(N_0))}(E,\mathrm{gr}(\check{V})) \\
\cong\; & \oplus_{r+s=i} \Big(\bigwedge^r \mathrm{Ext}^1_{\mathrm{gr}(\Lambda(\overline{N}_{m+n}T_m))}(E,E)\Big) \otimes_E \mathrm{Ext}^s_{\mathrm{gr}(\Lambda(N_0))}(E,\mathrm{gr}(\check{V})) \\
\cong\; & \oplus_{r+s=i} \Big(\bigwedge^r \mathrm{Ext}^1_{\Lambda(\overline{N}_{m+n}T_m)}(E,E)\Big) \otimes_E \mathrm{Ext}^s_{\Lambda(N_0)}(E,\check{V})
\end{aligned}$$

for all $i \geq 0$ which are compatible with restriction. Note that $(\overline{N}_{m+n}T_m)^p = \overline{N}_{m+n+1}T_{m+1}$ so that the restriction map

$$\begin{tikzcd}
\mathrm{Ext}^1_{\Lambda(\overline{N}_{m+n}T_m)}(E,E) \arrow[r, "\cong"] & \mathrm{Hom}(\overline{N}_{m+n}T_m,E) \arrow[d] \\
\mathrm{Ext}^1_{\Lambda(\overline{N}_{m+n+1}T_{m+1})}(E,E) \arrow[r, "\cong"] & \mathrm{Hom}(\overline{N}_{m+n+1}T_{m+1},E)
\end{tikzcd}$$

is trivial because E has characteristic p. Since $\dim(N_0) = 1$ the claim follows. $\square$

Remark 5.12. The final arguments given in the above proof imply more generally that for $n > m \geq 2$ and any integer $i \geq 1$ the restriction map $\mathrm{Ext}^i_{\Lambda(G_n\cap K_m)}(E,\check{V}_n) \to \mathrm{Ext}^i_{\Lambda(N_n)}(E,\check{V}_n)$ is surjective and that its kernel coincides with the kernel of the restriction map $\mathrm{Ext}^i_{\Lambda(G_n\cap K_m)}(E,\check{V}_n) \to \mathrm{Ext}^i_{\Lambda(G_n\cap K_{m+1})}(E,\check{V}_n)$.

We keep the assumption $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and from now on assume that V is irreducible over G_0. We recall the construction of the G-equivariant *Hecke operator* $T = T_V$ on $\mathrm{ind}^G_{G_0}(V)$ given in [3], section 3. The composition of the natural maps $V^{N_0} \hookrightarrow V \to V_{\overline{N}_0}$ is bijective. It induces an E-linear endomorphism $U = U_V : V \to V_{\overline{N}_0} \cong V^{N_0} \hookrightarrow V$ of V. For $g \in G$ and $v \in V$

we denote by $[g,v]\in \mathrm{ind}_{G_0}^G(V)$ the function with support gG_0 and value v at g. The operator T is then determined by the formula

$$
\begin{aligned}
(19) \qquad T([g,v]) &= \sum_{G_0\alpha G_0=\coprod x\alpha G_0} [gx\alpha, U(x^{-1}v)] \\
&= [gw\alpha, U(wv)] + \sum_{n\in N_0/N_1} [gn\alpha, U(n^{-1}v)],
\end{aligned}
$$

where we made use of the decomposition (15). By [3], Proposition 4, and [10], Théorème 1.1, the G-representation

$$\pi_V = \mathrm{ind}_{G_0}^G(V)/T(\mathrm{ind}_{G_0}^G(V))$$

is irreducible and by [3], Theorem 34, any supersingular smooth E-linear representation of $G=\mathrm{GL}_2(\mathbb{Q}_p)$ with trivial action of $p\in Z\subset G_0$ is of this form. Note that $\mathrm{ind}_{G_0}^G(V)$ and hence π_V admit central characters which are equal to the central character of V. The character of G obtained by composition with the determinant map $\det : G\to \mathbb{Q}_p^\times \cong Z$ will be denoted by $\delta_V : G\to E^\times$.

Theorem 5.13. *If $G=\mathrm{GL}_2(\mathbb{Q}_p)$ and if V is an irreducible representation of G_0 as above then the supersingular G-representation π_V is Cohen-Macaulay of dimension one. There are isomorphisms*

$$S_G^1(\pi_V)\cong \pi_{\check{V}} \cong \pi_V\otimes_E \delta_V^{-1}$$

of E-linear smooth G-representations.

Proof. By Lemma 5.9 we have $S_G^0(\mathrm{ind}_{G_0}^G(V)) \cong \mathrm{Ind}_{G_0}^G(\check{V})$. Identifying $F\in \mathrm{Ind}_{G_0}^G(\check{V})$ with the infinite sum $\sum_{g\in G/G_0}[g,F(g)]$ our first claim is that the operator $S_G^0(T_V)$ is given by $S_G^0(T_V)(F)=\sum_{g\in G/G_0} T_{\check{V}}([g,F(g)])$. Note that $S_G^0(T_V)$ is obtained from the Pontryagin dual $\check{T}_V$ of T_V by restriction to the subspace of smooth vectors in $I_{G_0}^G(\check{V})$. By continuity, $\check{T}_V$ commutes with infinite sum expansions as above, and hence so does $S_G^0(T_V)$. Moreover, by formula (19) the operator $T_{\check{V}}$ has a unique extension to $I_{G_0}^G(\check{V})$ commuting with infinite sums as above. Therefore, it suffices to see that the restriction of $S_G^0(T_V)$ to $\mathrm{ind}_{G_0}^G(\check{V})$ agrees with $T_{\check{V}}$. In fact, this will show that more generally $\check{T}_V = T_{\check{V}}$ as G-equivariant endomorphisms of $I_{G_0}^G(\check{V})$.

Note that U_V is a projection whose image is V^{N_0} and whose kernel coincides with the kernel of the natural map $V\to V_{\overline{N}_0}$. Since the analogous characterization holds for $U_{\check{V}}$ and since $wN_0w=\overline{N}_0$ one deduces that $U_{\check{V}}$ is the transpose of the endomorphism wUw of V. We also note that if $y\in G_0$ has the property that $\alpha y\alpha G_0 = G_0$ then $yG_1=wG_1$. Indeed, the decomposition (15) shows that otherwise $\alpha y\alpha G_0\subseteq \alpha N_0\alpha G_0\subseteq N_0\alpha^2 G_0$ which has trivial intersection with G_0 by the Cartan decomposition.

Now let $g, h \in G$, $v \in V$ and $\check{v} \in \check{V}$. Using the pairing introduced in the proof of Lemma 5.9 we have

$$\begin{aligned} \langle [g,v], S_G^0(T_V)([h,\check{v}])\rangle &= \langle T_V([g,v]), [h,\check{v}]\rangle \\ &= \sum_{G_0\alpha G_0 = \coprod_x x\alpha G_0} \langle [gx\alpha, U_V(x^{-1}v)], [h,\check{v}]\rangle. \end{aligned}$$

The latter sum is zero unless $hG_0 \subseteq gG_0\alpha G_0$. Assuming $hx_0 = gx\alpha$ with $x, x_0 \in G_0$ and using $[h,\check{v}] = [hx_0, x_0^{-1}\check{v}]$ we obtain $\langle [gx\alpha, U_V(x^{-1}v)], [h,\check{v}]\rangle = \check{v}(x_0U_V(x^{-1}v))$. On the other hand, we have

$$\langle [g,v], T_{\check{V}}([h,\check{v}])\rangle = \sum_{G_0\alpha G_0 = \coprod_x x\alpha G_0} \langle [g,v], [hx\alpha, U_{\check{V}}(x^{-1}\check{v})]\rangle,$$

which is zero unless $gG_0 \subseteq hG_0\alpha G_0$. Note that the conditions $gG_0 \subseteq hG_0\alpha G_0$ and $hG_0 \subseteq gG_0\alpha G_0$ are equivalent. Indeed, noting that $G_0\alpha G_0$ contains $\alpha^{-1} \in w\alpha wZ$ and hence is invariant under inversion, the conditions are equivalent to $G_0h^{-1}gG_0 = G_0\alpha G_0 = (G_0\alpha G_0)^{-1} = G_0g^{-1}hG_0$. In this case we may write $gy_0 = hy\alpha$ with $y, y_0 \in G_0$ and obtain $\langle [g,v], T_{\check{V}}([h,\check{v}])\rangle = U_{\check{V}}(y^{-1}\check{v})(y_0^{-1}v)$. Since $\alpha x_0^{-1}y\alpha = x^{-1}y_0 \in G_0$, our above remark implies that we may assume $x_0^{-1}y = x^{-1}y_0 = w$. Since $U_{\check{V}}$ is the transpose of wU_Vw this yields

$$U_{\check{V}}(y^{-1}\check{v})(y_0^{-1}v) = \check{v}(ywU_V(wy_0^{-1}v)) = \check{v}(x_0U_V(x^{-1}v)),$$

thus proving our claim.

Let us write $I_{G_0}^G(\check{V}) \cong \prod_{n\geq 0} I_n$ where I_n is the subspace of functions supported on $G_0\alpha^nG_0$. The decomposition of $\alpha^nG_0\alpha G_0$ given before Theorem 5.11 shows that we have $T_{\check{V}} = T_{\check{V}}^+ + T_{\check{V}}^-$ where $T_{\check{V}}^+ : \prod_{n\geq 0} I_n \to \prod_{n\geq 0} I_n$ is G_0-equivariant and homogeneous of degree 1 given by

$$T_{\check{V}}^+([g\alpha^n, \check{v}]) = \sum_{x\in N_0/N_1} [g\alpha^nx\alpha, U_{\check{V}}(x^{-1}\check{v})]$$

and $T_{\check{V}}^-$ factors through the projection $\prod_{n\geq 0} I_n \to \prod_{n\geq 1} I_n$ such that the resulting G_0-equivariant map $\prod_{n\geq 1} I_n \to \prod_{n\geq 0} I_n$ is homogeneous of degree -1 given by

$$T_{\check{V}}^-([g\alpha^n, \check{v}]) = [g\alpha^{n-1}, wU_{\check{V}}(w\check{v})].$$

Our second claim is that $T_{\check{V}}^+$ is injective. It suffices to show that the induced map $I_n \to I_{n+1}$ is injective for any $n \geq 0$. In fact, this is part of the proof of the injectivity of $T_{\check{V}}$ on $\mathrm{ind}_{G_0}^G(\check{V})$. Let us recall the argument. Let $F \in I_n$ and write $F = \sum_y [y\alpha^n, F(y\alpha^n)]$ where $G_0\alpha^nG_0 = \coprod_y y\alpha^nG_0$. Since

$G_{n+1} \subseteq G_n$ one sees that $y\alpha^n N_0\alpha G_0 \cap y'\alpha^n N_0\alpha G_0 = \emptyset$ for all $y \neq y'$ as above. Thus, $G_0\alpha^n N_0\alpha G_0 = \coprod_y \coprod_{x\in N_0/N_1} y\alpha^n x\alpha G_0$. Therefore, $T^+_{\check{V}}(F) = 0$ implies $U_{\check{V}}(x^{-1}F(y\alpha^n)) = 0$ for all x and y. Assuming $F(y\alpha^n) \neq 0$ this vector spans a nonzero N_0-invariant subspace of $\check{V}$. Since the action of N_0 factors through the p-group N_0/N_1 this subspace contains a nonzero N_0-invariant vector. However, this vector must be contained in the kernel of $U_{\check{V}}$, contradicting the fact that $U_{\check{V}}$ is the identity on $\check{V}^{N_0}$.

Taking up the notation of the proof of Theorem 5.11 and using [38], I.5.6, there are G_0-equivariant isomorphisms

$$I_n^{K_m} \cong \operatorname{ind}_{G_nK_m}^{G_0}(\check{V}_n^{K_m\cap G_n}) \text{ for all } n, m \geq 0.$$

If $n \geq m$ then $G_n \cap K_m = \overline{N}_m T_m N_n$. Conjugating with α^{-n} we see that $\check{V}_n^{K_m\cap G_n} = \check{V}^{N_0}$ is independent of $n \geq m$. Further, if $n \geq m$ then $G_{n+1}K_m = G_nK_m = \overline{N}_0 T_0 N_m$ and we obtain that the finite dimensions of $I_n^{K_m}$ and $I_{n+1}^{K_m}$ coincide. As a consequence of the above injectivity statement, the map $T^+_{\check{V}} : I_n^{K_m} \to I_{n+1}^{K_m}$ is bijective whenever $n \geq m$.

On the other hand, we claim that for $n > m \geq 1$ the map $T^-_{\check{V}} : I_n^{K_m} \to I_{n-1}^{K_m}$ is zero. Note first that for $n \geq m$ and $F \in I_n^{K_m}$ we have $F(G_0\alpha^n) \subseteq \check{V}^{N_0}$. Indeed, if $y \in G_0$ then $yN_ny^{-1} \subseteq K_m$ because $N_n \subseteq K_m$ and because K_m is normal in G_0. Given $x \in N_0$ this implies $xF(y\alpha^n) = F(y\alpha^n x^{-1}\alpha^{-n}y^{-1}y\alpha^n) = F(y\alpha^n)$. If V is not a twist of the trivial representation then the required vanishing statement follows from the fact that $w\check{V}^{N_0} = \check{V}^{\overline{N}_0}$ is contained in the kernel of $U_{\check{V}}$. Namely, $U_{\check{V}}$ is a projection with image $\check{V}^{N_0}$, $\ker(U_{\check{V}})$ is a direct sum of T_0-weight spaces and the weight spaces $\check{V}^{N_0}$ and $\check{V}^{\overline{N}_0}$ are distinct unless $\check{V}$ is a character. In the latter case, write

$$F = \sum_{g\in G_0/G_{n-1}} \sum_{x\in G_{n-1}/G_n} [gx\alpha^n, F(gx\alpha^n)].$$

Note that the natural map $N_{n-1}/N_n \to G_{n-1}/G_n$ is bijective. Further, the K_m-invariance of F implies $F(gx\alpha^n) = F(gxg^{-1}g\alpha^n) = F(g\alpha^n)$ for all $x \in N_{n-1}$ because $gxg^{-1} \in K_{n-1} \subseteq K_m$ if $n > m$. As a consequence, we obtain

$$\begin{aligned} T^-_{\check{V}}(F) &= \sum_{g\in G_0/G_{n-1}} \sum_{x\in N_{n-1}/N_n} [g\alpha^{n-1}, \alpha^{-(n-1)}x\alpha^{n-1}wU_{\check{V}}(wF(gx\alpha^n))] \\ &= \sum_{g\in G_0/G_{n-1}} \sum_{x\in N_{n-1}/N_n} [g\alpha^{n-1}, F(g\alpha^n)] = 0, \end{aligned}$$

because if $\check{V}$ is a character then $U_{\check{V}}$ is the identity, the action of $N_0 = \alpha^{-(n-1)}N_{n-1}\alpha^n$ on $\check{V}$ is trivial and $p = (N_{n-1} : N_n)$ is zero in E.

Our next claim is that $S_G^0(T_V)$ is injective. Let $F \in S_G^0(\mathrm{ind}_{G_0}^G(V)) \cong \mathrm{Ind}_{G_0}^G(\check{V})$ be contained in the kernel of $S_G^0(T_V)$ and choose $m \geq 0$ such that $F \in I_{G_0}^G(\check{V})^{K_m}$. Writing $F = (F_n)_{n\geq 0}$ with $F_n \in I_n^{K_m}$ we have

$$0 = S_G^0(T_V)(F) = (T_{\check{V}}^-(F_1), T_{\check{V}}^+(F_0) + T_{\check{V}}^-(F_2), \ldots).$$

However, if $n > m > 0$ then $T_{\check{V}}^+(F_{n-2}) + T_{\check{V}}^-(F_n) = T_{\check{V}}^+(F_{n-2})$ by the above vanishing result on $T_{\check{V}}^-$. Therefore, the injectivity of $T_{\check{V}}^+$ yields $F_{n-2} = 0$ and hence $F_n = 0$ for all $n \geq m-1$. As a consequence, the component of $S_G^0(T_V)(F)$ in degree $m-2$ reads $0 = T_{\check{V}}^+(F_{m-3}) + T_{\check{V}}^-(F_{m-1}) = T_{\check{V}}^+(F_{m-3})$. The same arguments and downward induction imply $F = 0$, as claimed. In particular, we obtain $S_G^0(\pi_V) = 0$, as was predicted by Proposition 3.9.

Since the restriction of $S_G^0(T_V)$ to $\mathrm{ind}_{G_0}^G(\check{V}) \subseteq S_G^0(\mathrm{ind}_{G_0}^G(V))$ coincides with $T_{\check{V}}$ we obtain a G-equivariant map $\pi_{\check{V}} \to \mathrm{coker}(S_G^0(T_V))$ that we claim to be a bijection. Let again $F = (F_n)_{n\geq 0}$ with $F_n \in I_{G_0}^G(\check{V})^{K_m}$ for all $n \geq 0$. The surjectivity of the above map will follow once we show that the element $H = (H_n)_{n\geq 0}$ with $H_n = 0$ for $0 \leq n \leq m+1$ and $H_n = F_n$ for $n > m+1$ is contained in the image of $S_G^0(T_V)$. According to the bijectivity result on $T_{\check{V}}^+$ there are elements $H_n' \in I_n^{K_m}$ with $T_{\check{V}}^+(H_n') = H_{n+1}$ for all $n \geq m+1$. Setting $H_n' = 0$ for $n \leq m$ and $H' = (H_n')_{n\geq 0} \in I_{G_0}^G(\check{V})^{K_m}$ we have $S_G^0(T_V)(H') = T_{\check{V}}^+(H') + T_{\check{V}}^-(H') = T_{\check{V}}^+(H') = H$ because of the above vanishing property of $T_{\check{V}}^-$.

Now assume $S_G^0(T_V)(F) \in \mathrm{ind}_{G_0}^G(\check{V})$. The injectivity of the above map will follow once we can show that $F \in \mathrm{ind}_{G_0}^G(\check{V})$. Since $S_G^0(T_V)(F)$ is zero in almost all components this follows from the same arguments that we used in order to prove the injectivity of $S_G^0(T_V)$.

Finally, we show that the map $S_G^1(T_V)$ is bijective. Note that as G_0-equivariant maps we have

$$S_G^1(T_V) = \Sigma_G^1(\check{T}_V) = \Sigma_G^1(T_{\check{V}}) = \Sigma_{G_0}^1(T_{\check{V}}^+) + \Sigma_{G_0}^1(T_{\check{V}}^-).$$

As a first step we will prove that $\Sigma_{G_0}^1(T_{\check{V}}^+) = 0$. Let

$$[F] \in S_G^1(\mathrm{ind}_{G_0}^G(V)) = \varinjlim_m \mathrm{Ext}^1_{\Lambda(K_m)}(E, I_{G_0}^G(\check{V})) \cong \varinjlim_m \prod_{n\geq 0} \mathrm{Ext}^1_{\Lambda(K_m)}(E, I_n)$$

be represented by $F = (F_n)_{n\geq 0} \in \prod_{n\geq 0} \mathrm{Ext}^1_{\Lambda(K_{m-1})}(E, I_n)$ with $m > 2$. Let F' be the element obtained by replacing F_n by zero for all $n \leq m$. Note that $\prod_{n\leq m} I_n$ is a finite dimensional G_0-representation and consequently has a trivial $\Sigma_{G_0}^1$ by Proposition 3.8 and Corollary 3.16. This implies that

$[F] = [F']$ and that it suffices to prove the triviality of the map

$$\begin{tikzcd}
\mathrm{Ext}^1_{\Lambda(K_{m-1})}(E, I_n) \arrow[r] & \mathrm{Ext}^1_{\Lambda(K_m)}(E, I_n) \arrow[d] & \\
& \mathrm{Ext}^1_{\Lambda(K_m)}(E, I_{n+1}) \arrow[r] & \mathrm{Ext}^1_{\Lambda(K_{m+1})}(E, I_{n+1})
\end{tikzcd}$$

for all $n > m > 2$. Here the vertical arrows are restriction and the horizontal arrow is induced by $T^+_{\check{V}}$. Since $G_n K_i = G_{n+1} K_i$ for $m-1 \leq i \leq m+1$ the proof of Theorem 5.11 and Remark 5.12 show that equivalently we need to prove the triviality of the map

$$\mathrm{Ext}^1_{\Lambda(N_n)}(E, \check{V}_n) \longrightarrow \mathrm{Ext}^1_{\Lambda(N_{n+1})}(E, \check{V}_{n+1}),$$

induced by $T^+_{\check{V}}$. It is given by the compatible pair of homomorphisms $N_{n+1} \hookrightarrow N_n$ and $U_{\check{V}} : \check{V}_n \to \check{V}_{n+1}$. If V is one dimensional then N_n (resp. N_{n+1}) acts trivially on $\check{V}_n$ (resp. on $\check{V}_{n+1}$) and $U_{\check{V}}$ is the identity map. In this case, the above restriction map can be identified with the restriction map $\mathrm{Hom}(N_n, E) \longrightarrow \mathrm{Hom}(N_{n+1}, E)$ which is zero because $N_{n+1} = N_n^p$ and because E has characteristic p. In general, Poincaré duality shows that there is an isomorphism $\mathrm{Ext}^1_{\Lambda(N_n)}(E, \check{V}_n) \cong (\check{V}_n)_{N_n}$, sending the class of a crossed homomorphism $N_n \to \check{V}_n$ to the class of its value at a topological generator of N_n. Composing with $U_{\check{V}}$ the corresponding crossed homomorphism $N_{n+1} \to \check{V}_{n+1}$ takes values in $\check{V}_{n+1}^{N_{n+1}}$. As recalled above, if V is not one dimensional then the image of $\check{V}_{n+1}^{N_{n+1}}$ in $(\check{V}_{n+1})_{N_{n+1}}$ is zero, as claimed.

On the other hand, the map $\Sigma^1_{G_0}(T^-_{\check{V}})$ is bijective. Proceeding as above it suffices to prove the bijectivity of the map

$$\mathrm{Ext}^1_{\Lambda(N_n)}(E, \check{V}_n) \longrightarrow \mathrm{Ext}^1_{\Lambda(N_{n-1})}(E, \check{V}_{n-1}), \tag{20}$$

induced by $T^-_{\check{V}}$ if $n > m > 2$. In order to make this map explicit, note first that $wU_{\check{V}}w : \check{V}_n \to \check{V}_{n-1}$ is N_n-equivariant. In fact, if $x \in N_n$ and $\check{v} \in \check{V}_n$ then $(wxw - 1)w\check{v} \in \ker(U_{\check{V}})$ so that $wU_{\check{V}}(wx\check{v}) = wU_{\check{V}}(w\check{v}) = xwU_{\check{V}}(w\check{v})$ because N_n acts trivially on $\check{V}_{n-1}$. Therefore, $T^-_{\check{V}} : \mathrm{ind}_{N_n}^{N_{n-1}}(\check{V}_n) \to \check{V}_{n-1}$ is the composition of $\mathrm{ind}_{N_n}^{N_{n-1}}(wU_{\check{V}}w)$ and the norm map $\mathrm{ind}_{N_n}^{N_{n-1}}(\check{V}_{n-1}) \to \check{V}_{n-1}$. Using the behavior of the norm map under the isomorphism

$$\mathrm{Ext}^1_{\Lambda(N_{n-1})}(E, \mathrm{ind}_{N_n}^{N_{n-1}}(\check{V}_n)) \cong \mathrm{Ext}^1_{\Lambda(N_n)}(E, \check{V}_n)$$

of Shapiro's lemma (cf. [27], Proposition 1.6.4) we obtain that the map (20) is the composition of $\mathrm{Ext}^1_{\Lambda(N_n)}(E, wU_{\check{V}}w)$ and the corestriction map $\mathrm{cor} : \mathrm{Ext}^1_{\Lambda(N_n)}(E, \check{V}_{n-1}) \to \mathrm{Ext}^1_{\Lambda(N_{n-1})}(E, \check{V}_{n-1})$.

We let x_0 be a fixed topological generator of N_{n-1} so that the elements x_0^i for $0 \leq i \leq p-1$ form a set of representatives of N_{n-1}/N_n and x_0^p is a topological generator of N_n. Further, we let $\check{v}$ be a non-zero element of $\check{V}_n^{\overline{N}_n} = \check{V}_{n-1}^{\overline{N}_{n-1}}$. Recall from above that we have isomorphisms $\mathrm{Ext}^1_{\Lambda(N_n)}(E, \check{V}_n) \cong (\check{V}_n)_{N_n} \cong \check{V}_n^{\overline{N}_n}$ under which $\check{v}$ is represented by a cocyle f with $f(x_0^p) = \check{v}$. The cocyle $\mathrm{cor}(wU_{\check{V}}w \circ f)$ satisfies

$$\begin{aligned}\mathrm{cor}(wU_{\check{V}}w \circ f)(x_0) &= \sum_{i=0}^{p-2} x_0^{-i} wU_{\check{V}}(wf(x_0^i x_0 x_0^{-(i+1)})) + x_0^{-(p-1)} wU_{\check{V}}(wf(x_0^p)) \\ &= \ x_0^{-(p-1)} wU_{\check{V}}(w\check{v}) = x_0\check{v}.\end{aligned}$$

Here the first equality comes from the definition of the corestriction map on the level of cochains (cf. [27], Chapter I, §5, p. 46). The second equality follows from $f(1) = 0$ because f is a cocyle. The third equality follows from the fact that $U_{\check{V}}$ is the identity on $\check{V}^{N_0} = w\check{V}^{\overline{N}_0}$ and that $x_0^{-p} \in N_n$ acts trivially on $\check{V}_{n-1}$. Now the image of $x_0\check{v}$ in $(\check{V}_{n-1})_{N_{n-1}}$ is the same as that of $\check{v}$, hence is non-zero. This proves the injectivity and hence the bijectivity of the map (20). It also completes the prove of the bijectivity of the map $S_G^1(T_{\check{V}})$.

Let us now analyze the long exact sequence obtained by applying the δ-functor $(S_G^i)_{i\geq 0}$ to the short exact sequence

$$0 \longrightarrow \mathrm{ind}_{G_0}^G(V) \xrightarrow{T_V} \mathrm{ind}_{G_0}^G(V) \longrightarrow \pi_V \longrightarrow 0,$$

noting that T_V is injective by [3], Theorem 19. We have already seen that $S_G^0(T_V)$ is injective and hence that $S_G^0(\pi_V) = 0$. Further, $S_G^i(\pi_V) = 0$ for $i \geq 2$ by Theorem 5.11 and because $S_G^1(T_V)$ is bijective. Therefore, the long exact sequence simply reads

$$0 \longrightarrow S_G^0(\mathrm{ind}_{G_0}^G(V)) \xrightarrow{S_G^0(T_V)} S_G^0(\mathrm{ind}_{G_0}^G(V)) \longrightarrow S_G^1(\pi_V) \longrightarrow 0,$$

and $S_G^1(\pi_V) \cong \mathrm{coker}(S_G^0(T_V)) \cong \mathrm{coker}(T_{\check{V}}) = \pi_{\check{V}}$, as seen above.

For the final formula $\pi_{\check{V}} \cong \pi_V \otimes_E \delta_V^{-1}$ note that $\check{V}$ and $V \otimes_E \delta_V^{-1}$ are irreducible G_0-representations. Let $\omega : T_0 \to E^\times$ denote the highest weight of V, i.e. the character affording the T_0-action on V^{N_0}. There are T_0-equivariant isomorphisms

$$\check{V}^{N_0} \cong (V_{N_0})\check{} \cong (V^{\overline{N}_0})\check{} \cong (\omega^{-1})^{-1} = \omega.$$

Note that we extended the central character of V to a character of G through the determinant map. As a consequence, the central characters of $\check{V}$ and

$V \otimes_E \delta_V^{-1}$ are both equal to δ_V^{-1}. By the classification of the irreducible G_0-representations in [3], Proposition 4, the G_0-representations $\check{V}$ and $V \otimes_E \delta_V^{-1}$ are isomorphic. As a direct consequence, the operators $U_{\check{V}}$ and $U_V \otimes_E 1$ correspond to each other. This implies that under the G-equivariant isomorphisms

$$\mathrm{ind}_{G_0}^G(\check{V}) \cong \mathrm{ind}_{G_0}^G(V \otimes_E \delta_V^{-1}) \cong \mathrm{ind}_{G_0}^G(V) \otimes_E \delta_V^{-1}$$

the operator $T_{\check{V}}$ corresponds to $T_V \otimes_E 1$. This yields the final formula $\pi_{\check{V}} \cong \pi_V \otimes_E \delta_V^{-1}$. $\square$

Remark 5.14. As recalled above, the smooth E-linear $\mathrm{GL}_2(\mathbb{Q}_p)$-representation π_V is irreducible, admissible and supersingular. By [3], Corollary 36, π_V is supercuspidal. On the other hand, Corollary 1.8, Remark 3.7 and Theorem 5.13 show that π_V is not an injective object in $\mathrm{Rep}_E^\infty(\mathrm{GL}_2(\mathbb{Q}_p))$. Once again, this phenomenon is in contrast to the theory of smooth representations in characteristic zero (cf. [13], Theorem 5.4.1) and in more precise form was already observed by Paskunas (cf. [29], Theorem 1.1).

Remark 5.15. In his work on the p-adic local Langlands correspondence Colmez constructed a contragredient operation on smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representations with p-torsion coefficients (cf. [14], IV.4.5). Under the p-adic local Langlands correspondence it gives rise to the usual duality operation on (φ, Γ)-modules with p-torsion coefficients (cf. [14], Théorème IV.4.15). Our results in Proposition 5.4, Proposition 5.7 and Theorem 5.13, together with the corresponding formulae in [14], Proposition IV.4.18, show that on the infinite dimensional irreducible smooth E-linear representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ Colmez' contragredient coincides with the first smooth duality functor $S^1_{\mathrm{GL}_2(\mathbb{Q}_p)}$.

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