Master seminar

p-adic Galois representations

University of Duisburg-Essen, winter term 2020/21

Organizer:	Prof. Dr. Jan Kohlhaase
Day and time:	Tue 14–16
Place:	online (instructions will be given on moodle)

Contents: One of the main objectives of algebraic number theory is to understand the absolute Galois group $G_K = \mathrm{Gal}(K^{\mathrm{sep}}|K)$ of a local or a global field K. We will focus on nonarchimedean local fields of residue characteristic p and study their absolute Galois groups through so-called p-adic representations, i.e. continuous linear actions on finite dimensional p-adic vector spaces. In this setting, large parts of the theory go back to Jean-Marc Fontaine and his school.

The topics covered include examples from algebraic geometry, étale φ-modules, the tilting equivalence for perfectoid fields, étale (φ,Γ)-modules, the formalism of period rings, Hodge-Tate representations, de Rham representations, crystalline representations, semistable representations, filtered isocrystals and the main theorems of p-adic Hodge theory.

1. Generalities and examples [03.11. + 10.11.]: [3], Chapter I; [4], §§3.1-3.6 might also be helpful; categories of (continuous) representations; ℓ-adic representations (we will later assume that $\ell = p$ is the residue characteristic of a nonarchimedean local field); tensor products, symmetric/exterior powers and duals; cyclotomic character and Tate twists; Tate modules of elliptic curves and abelian varieties; ℓ-adic cohomology of proper smooth varieties (as a motivation try to summarize the Weil conjectures; there are better sources online); (wild) inertia group of a local field: definition and structure; state the ℓ-adic monodromy theorem and its converse

2. Étale φ-modules [17.11. + 24.11.]: [3], §§2.2-2.3; [4], §§2.1-2.2 and §§3.10-3.26; [1], §III.1; here E is a (nonarchimedean local) field of characteristic p; the aim is to establish the equivalences of categories in [3], Theorems 2.21, 2.32 and 2.33; they give descriptions of G_E-representations over $\mathbb{F}_p$, $\mathbb{Z}_p$ and $\mathbb{Q}_p$ in terms of so-called étale φ-modules over varying coefficient rings (you will need the notion of a Cohen ring); the mod-p setting in Theorem 2.21 should be proven in detail; explain the dévissage argument and passage to the limit needed to pass from $\mathbb{F}_p$ to $\mathbb{Z}_p$

3. Étale (φ,Γ)-modules [01.12. + 08.12.]: [4], Chapter 4; [3], Chapter 4; [2], §13; [1], §III.2; here K is a nonarchimedean local field of characteristic zero; the aim is to establish the equivalence of categories in [3], Theorem 4.22 (see also [4], Theorem 4.20); the main idea is that $H = \mathrm{Gal}(K^{\mathrm{sep}}|K[\zeta_{p^\infty}])$ can be identified canonically with the absolute Galois group of a nonarchimedean local field E of characteristic p (this was first observed by

Fontaine and Wintenberger using their field of norms functor; explain how it can be deduced from Scholze's tilting construction); the open subgroup $\Gamma \cong \mathbb{Z}_p$ of $\mathrm{Gal}(K[\zeta_{p^\infty}]|K)$); the Γ-action on E lifts to a φ-invariant action on a suitable Cohen ring (cf. [4], 4.4-4.5 for the case $K = \mathbb{Q}_p$); étale (φ, Γ)-modules; the equivalence of categories is then a formal consequence of the equal characteristic result for E proven in talk 2

4. The formalism of period rings [15.12. + 12.01.]: [3], §2.1 and §§3.5.2-3.5.3; semilinear B-representations; classification in terms of continuous cohomology; regular (F, G)-rings; the admissible B-representations form a Tannakian category; the K^{sep}-admissible representations are precisely the discrete ones (explain in detail how this is a reformulation of Hilbert's Theorem 90); the study of C-admissible representations (with $C = \widehat{K^{\mathrm{sep}}}$) is called Sen theory; state the main result of Sen theory in [3], Proposition 3.55; explain why the cyclotomic character is not C-admissible using [3], Corollary 3.56; this and the geometric examples in talk 1 motivate the construction of more elaborate period rings

5. Hodge-Tate and de Rham representations [19.01. + 26.01.]: [2], §§2.3-2.4 and §4; [3], §§5.1-5.2; if you know some German then the thesis [6] of Wahlers might be helpful; Hodge-Tate representations and Hodge-Tate weights; reminder on Witt vectors; construction of the ring B_{dR}^{+} (the notation $\mathcal{R}$ is nowadays superseded by the notation $\mathfrak{o}_C^{\flat}$); the isomorphism $\mathrm{gr} B_{\mathrm{dR}} \cong B_{\mathrm{HT}}$; de Rham representations; state the comparison isomorphism [3], Theorem 5.33, of Faltings/Tsuji; any de Rham representation is Hodge-Tate but not conversely

6. Crystalline and semistable representations [02.02. + 09.02.]: [2], §9 (more accurate); [3], §6; construction of A_{cris}; the inclusion $A_{\mathrm{cris}} \subset B_{\mathrm{dR}}^{+}$; the rings B_{cris}^{+} and B_{cris}; G_K-action, Frobenius and filtration on B_{cris}; B_{cris} is $(\mathbb{Q}_p, G_K)$-regular; crystalline representations; the Frobenius on B_{cris}; the ring B_{st} with the actions of φ, G_K and N; B_{st} is $(\mathbb{Q}_p, G_K)$-regular; semistable representations; crystalline implies semistable implies de Rham; admissible filtered (φ, N)-modules; [2], Example 9.1.12, might help to illustrate some of the theory; the main theorems of p-adic Hodge theory (cf. [3], Theorems A and B)

References

[1] L. Berger: An introduction to the theory of p-adic representations, *Geometric aspects of Dwork theory. Vol. I*, (255-292), Walter de Gruyter, 2004

[2] O. Brinon, B. Conrad: Notes on p-adic Hodge theory, *notes from the CMI Summer School*, preprint, 2009

[3] J.-M. Fontaine, Y. Ouyang: Theory of p-adic Galois representations, preprint

[4] J. Kohlhaase: p-adic Galois representations, *lecture notes*, 2015

[5] P. Schneider: Galois representations and (φ, Γ)-modules, *Cambridge Studies in Advanced Mathematics* **164**, Cambridge University Press, 2017

[6] D. Wahlers: Der Ring der p-adischen Perioden, *Bachelor thesis*, University of Duisburg-Essen, 2017