VARIOUS BACKGROUND ON THE LANGLANDS CORRESPONDENCE

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Overview. The seminar this term is an attempt to collect some of the basic results and objects that sometimes appear implicitly in talks related to the Langlands program, but are often either taken as assumed background or are mentioned only very briefly. As this covers a range of things, from number theory, representations of groups (complex, real, p-adic, finite or profinite) and geometry that are not always covered in lecture courses, few people may have seen all of this, but many of you might have seen some aspect.

All of these ideas start with abelian groups, in which case the arithmetic statements are usually referred to as class field theory and the geometric statements are statements on the relation of curves and their Jacobians. Already in the arithmetic situation, the only known way to get analytic properties of Zeta- and L-functions is to use the geometric objects appearing in class field. We will spend a large part of the seminar on these results (Talk 1-5), where Talk 5 already explains how geometry can help to prove results of this type. Talks 6 and 7 give an idea on how the Fargues-Fontaine-curve allows to use these geometric ideas to give a new proof of the main theorem of local class field theory.

Starting from talk 8 we will pass to non-abelian situations, mainly GL_2 and GL_n and indicate why in the statements for general groups the notion of dual groups appears.

Talk 1: Local Class Field Theory

Summary: The aim of this talk is to give an overview of the main results of local class field theory. In particular, we will discuss the explicit approach via Lubin-Tate formal group laws.
Main Reference: [Mi20b, Chapter I].
More detailed description: As a motivation, state the local and global Kronecker-Weber Theorem without proof, and deduce

$$\widehat{\mathbb{Q}_p^\times} \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p),$$

see Chapter V, (1.9) Corollary and (1.10) in [Neu99]. Recall the definitions of local and global fields, see for example Definition 7.48, Remark 7.49 and first lines of Chapter 8 in [Mi20a]. Recall the definition of Frobenius elements and state the main theorems of local class field theory (Local Reciprocity Law and Local Existence Theorem), see Chapter I, Theorem 1.1 and Theorem 1.4 in [Mi20b]. From now on, we will only consider finite extensions of $\mathbb{Q}_p$. In the rest of the talk, sketch the explicit approach to local class field theory using Lubin-Tate formal group laws. More precisely: Recall the definition of formal group laws, see Definition 2.3 and Definition 2.6 in [Mi20b, Ch. I]. Briefly discuss Remark 2.4 and Example 2.5 (b) of [Mi20b, Ch. I]. State Definition 2.9 and discuss Example 2.10. Define Lubin–Tate groups as in Summary 2.20 in [Mi20b, Ch. I] (without proof). Explain the construction of K_π following §3 until 3.7 of [Mi20b, Ch. I]. State Theorems 3.9 and 4.8 in [Mi20b, Ch. I] (without proof).

Talk 2: Global Class Field Theory

Summary: The aim of this talk is to give an overview of the main results of global class field theory. As an applications, we will discuss the relation between 1-dimensional Galois representations and Hecke characters.
Main reference: [Mi20b, Chapter V].
More detailed description: Define the ring of adeles $\mathbb{A}_K$, the group of ideles $\mathbb{I}_K$ and

state their basic properties, see e.g. §1 in [Lan94, Ch. VII]. Define the idele class group and explain the relation between ideles and ray class groups, see Proposition 4.6 in [Mi20b, Ch. V]. Discuss norm maps between idele groups, see the discussion before Proposition 4.12 in [Mi20b, Ch. V]. State the main theorems of global class field theory (without proof), see Proposition 5.2, Theorem 5.3, Theorem 5.5 in [Mi20b, Ch. V]. Define Hecke L-functions and Artin L-functions of number fields, see §2 in [Lan94, Ch. XII]. State Artin's conjecture, see Conjecture 5.1 in [deS03]. Use class field theory to show that Artin L-functions of 1-dimensional Galois representations correspond to Hecke L-functions of finite Hecke characters, see §2, Theorem 2 in [Lan94, Ch. XII]. If time permits, you can also give other applications, e.g. you could deduce the Theorem of Kronecker–Weber from global class field theory, another option would be to deduce the quadratic reciprocity law from Artin Reciprocity; see, for example, [Cox, Theorem 8.8, Theorem 8.12].

Talk 3: Harmonic Analysis on Global Fields

Summary: The aim of this talk is to give an introduction to harmonic analysis on locally compact abelian groups with an emphasize on groups of number theoretic significance. As application of the adelic Poisson summation formula, we will deduce Riemann–Roch for curves over finite fields.
Main Reference: [Po15].
More detailed description: Recall the existence of Haar measures on locally compact abelian groups and their compatibility in short exact sequences, see Definition 2.12, Theorem 2.13 and §3.2 in [Po15]. Define (unitary) characters and the Pontryagin dual of a locally compact abelian group, see Definition 3.3, Definition 3.4 and Definition 3.7 of [Po15]. Give the following examples: the additive group of local and global fields, the additive group of the adeles $(\mathbb{A}, +)$, see table before Example 3.10 in [Po15] (cf. Theorem 4.4, Proposition 5.1 and Corollary 5.4). Explain the Fourier inversion formula and Plancherel's theorem for locally compact abelian groups, see Definition 3.11 until Corollary 3.14 of [Po15]. Describe the self-dual Haar measure explicitly in the case of local fields, see Proposition 4.8 of [Po15]. Introduce the Tamagawa measure on $\mathbb{A}$ and state that the volume of $\mathbb{A}/K$ is 1, see Proposition 5.5 of [Po15]. Introduce Schwartz–Bruhat functions and sketch the proof of the adelic Poisson summation formula, see Definition 4.6 and Theorem 5.7 of [Po15]. Explain that the Poisson summation formula for function fields implies Riemann–Roch for curves over finite fields, see §5.6 of [Po15].

Talk 4: Tate's Thesis

Summary: The aim of this talk is to explain the key ideas of Tate's thesis. As motivation, we will first discuss the functional equation of the Riemann zeta function. Afterwards, we will prove the functional equation of global zeta functions using the Poisson summation formula on the adeles and apply it to Hecke L-functions.
Main reference: [Po15].
More detailed description: As a starting point state the functional equation of the Riemann zeta function, see Theorem 1.1 in [Po15]. Recall the Fourier inversion formula, the Poisson summation formula and explain that the functional equation of the theta function is a direct consequence of the Poisson summation formula, see Definition 1.3 to Theorem 1.7 in [Po15]. To motivate the Fourier-theoretic approach in Tate's thesis, sketch the proof of the functional equation of the Riemann zeta function, see 1.4. of [Po15]. Introduce the (multiplicative) Haar measure on the ideles, see §5.9 of [Po15]. Define the global zeta integral of a Schwartz–Bruhat function and sketch the proof of the functional equation of global zeta integrals, see Theorem 5.16 of [Po15]. Sketch the proof of the functional equation for Hecke L-functions for number fields, see §5.11 of [Po15]. If you want, you can restrict to the case $K = \mathbb{Q}$, i.e. Dirichlet characters, see Theorem 6.3.4 in [Dei12]. If time permits, explain how

to deduce the analytic class number formula from a volume computation on the ideles, see §7.5. and §7.6. of [RV99].

Talk 5: Geometric unramified class field theory

1. Let $K = \mathcal{O}_{X,\eta}$ be the function field of a smooth, proper, geometrically connected curve X over $\mathrm{Spec}(\mathbb{F}_q)$, $q = p^a$. Recall the statement of unramified global class field theory for K :

$$\mathrm{GL}_1(K)\backslash\widehat{\mathrm{GL}_1(\mathbb{A}_K)}/\mathrm{GL}_1(\mathcal{O}) \simeq (\mathrm{Gal}(\overline{K}/K)^{\mathrm{unr}})^{\mathrm{ab}},$$

(the hat denotes profinite completion) given by $(a_x) \mapsto \prod_x \mathrm{Frob}_x^{a_x}$. Here $\mathcal{O} = \prod_{x\in|X|} \widehat{\mathcal{O}}_{X,x}$ (product runs over all closed points).

The aim of the talk is to geometrize both sides of the isomorphism in 1. and sketch a proof of a categorical and geometric version, which works over any base field (including the geometric situation $k = \mathbb{C}$).

2. State [Sta, Tag 0BQM] in the case of interest here, i.e. $\pi_1^{\mathrm{et}}(X, \overline{\eta}) \simeq \mathrm{Gal}(\overline{K}/K)^{\mathrm{unr}}$. This geometrizes the Galois side.
3. Explain that the double quotient

$$\mathrm{GL}_1(K)\backslash\mathrm{GL}_1(\mathbb{A}_K)/\mathrm{GL}_1(\mathcal{O}),$$

where $\mathcal{O} = \prod_{x\in|X|} \widehat{\mathcal{O}}_{X,x}$ is nothing else than the set of line bundles on X, because $\mathrm{GL}_1(\mathbb{A}_K)/\mathrm{GL}_1(\mathcal{O})$ is a complicated description of divisors and the map $\mathrm{GL}_1(K) = K* \to \mathrm{GL}_1(\mathbb{A}_K)/\mathrm{GL}_1(\mathcal{O})$ maps a function to its divisor.
4. Reformulate 1. as follows: there exists a bijection between isomorphism classes of continuous characters $\chi\colon \pi_1^{\mathrm{et}}(X)^{\mathrm{ab}} \to \overline{\mathbb{Z}}_\ell^*$ and characters $\rho\colon \mathrm{Pic}(X) \to \overline{\mathbb{Z}}_\ell^*$, where ℓ is a prime different from p, with the property that if χ_ρ corresponds to ρ, we have that

$$\rho(\mathcal{O}([x])) = \chi_\rho(\mathrm{Frob}_x).$$

Now let k be a field and let X be a smooth, proper and geometrically connected curve over k. The aim of the rest of the talk is to sketch that there is an equivalence of categories between character Λ-local systems on $\mathrm{Pic}_k(X)$ and rank one étale Λ-local systems on X (Λ could for simplicity be torsion, prime to the characteristic of k or $\mathbb{Z}$ if $k = \mathbb{C}$). Depending on how much time is left one can cover the following points:

1. Introduce $\mathrm{Div}^d_{X/k} \simeq X^{(d)}$ (c.f. [Klei, Definition 3.6, Exercise 3.8, Remark 3.9]; you do not have to prove this), the Picard scheme $\mathrm{Pic}_k(X)$ ([Klei, Definition 4.1]) of X/k and the Abel-Jacobi morphism ([Klei, Definition 4.6]) (see also [To11, Section 1.3]). Explain why for $d \geq 2g-2$ ($d \geq g$ if $g = 0$) the Abel-Jacobi morphism has as geometric fibers projective spaces by the Riemann-Roch theorem ([To11, Proposition 2.1.4]). Recall that projective spaces are geometrically simply connected (For $k = \mathbb{C}$ this is easy, but it also holds in arithmetic situations, where one can reduce to the projective line and use Hurwitz' formula; or argue using Grothendieck's classification of vector bundles c.f. [FF18, Remarque 8.6.2].).
2. Define the notion of a character local system on $\mathrm{Pic}_k(X)$ using the group structure ([FGSV, Definition 3, Bhargav Bhatt, Geometric CFT]). Explain how a character $\overline{\mathbb{Z}}_\ell$-local system on $\mathrm{Pic}_k(X)$, for $k = \mathbb{F}_q$ gives a character $\rho\colon \mathrm{Pic}(X)(\mathbb{F}_q) \to \overline{\mathbb{Z}}_\ell^*$ (see discussion before [FGSV, Theorem 5, Bhargav Bhatt, Geometric CFT]), but you do not have to prove [FGSV, Theorem 5, Bhargav Bhatt, Geometric CFT].
3. State without proof the homotopy exact sequence [Sta, Tag 0C0J].
4. Lead us through the proof that pull back along the Abel-Jacobi morphism induces the desired equivalence of categories following Bhatt's sketch [FGSV, Theorem 6, Bhargav Bhatt, Geometric CFT] (see also [To11, Theorem 2.2.1]).

Talk 6:The Fargues-Fontaine curve

The aim of this talk is to give an overview of the Fargues-Fontaine curve ([FF18]). A nice overview article is [Mo19]. We will not be able to give details about proofs here unfortunately, but it is more important to explain why this object appears when one thinks about geometrizing LCFT.

1. Roughly motivate the Fargues-Fontaine curve as follows (beware; this is not a historically accurate motivation!): let E be a non-archimedean local field and we want to construct an interesting geometric object X, such that the étale fundamental group of X is the absolute Galois-group of E. From the persective of étale cohomology $\mathrm{Spec}(E)$ looks like a compact Riemann surface (Tate-Nakayama duality, cohomological dimension), so that we would like X to be something like a Riemann surface (which rules out just taking $X = \mathrm{Spec}(E)$). If you want you can motivate the construction of the curve in the equal characteristic case using Drinfeld's lemma. Point out the difference to the story in the previous talk: we are now geometrizing the whole absolute Galois group of E and not just the unramified part! See [S21, Lecture 2].
2. Now let E be a non-archimedean (NA) local field of residue characteristic p. Let F be an algebraically closed, complete NA-field, which is an extension of the residue field of E. Write $\mathcal{O}_F = \{x \in F \colon |x| \leq 1\}$ for the ring of integers and $\varpi \in \mathcal{O}_F$ a pseudo-uniformizer, i.e. $0 < |\varpi| < 1$. Introduce the ramified Witt vectors $W_{\mathcal{O}_E}(\mathcal{O}_F) = \{\sum_{n\geq 0}[x_n]\pi^n \colon x_n \in \mathcal{O}_F\}$ [FF18, Section 1.2]. Introduce $Y_{E,F} = \mathrm{Spa}(W_{\mathcal{O}_E}(\mathcal{O}_F), W_{\mathcal{O}_E}(\mathcal{O}_F)) - V(\pi \cdot [\varpi])$ as a set and draw a picture. You do not have to explain the formalism of adic spaces and how the structure sheaf is defined in detail; maybe just say some words about this category. One can however explain that for $I = [\rho_1, \rho_2] \subset (0,1)$ with $\rho_1, \rho_2 \in |F|$, one can describe $Y_I = \mathrm{Spa}(B_I, B_I^+)$, where B_I is the completion of $W_{\mathcal{O}_E}(\mathcal{O}_F)[\frac{1}{\pi}, \frac{1}{[\varpi]}]$ w.r.t certain norms ([FF18, Exemple 1.6.3]). Explain why the Frobenius φ_F acts on $Y_{E,F}$ in a properly discontinous way ([SW20, Beginning of section 13.5.] or [FS21, Proposition II.1.16]) and define $X_{E,F}^{\mathrm{ad}} = Y_{E,F}/\varphi_F^{\mathbb{Z}}$ ([SW20, Definition 13.5.1] or [FS21, Definition II.1.15]). Define the algebraic curve ([SW20, Definition 13.5.2] or [FS21, Proposition II.2.7]) and state the fundamental properties of this construction ([SW20, Theorem 13.5.3]). State that regardless of the choice of F, we have that
$$H^0(X_{E,F}, \mathcal{O}_{X_{E,F}}) = E$$
(this is very important for geometrizing the automorphic side) and that
$$H^1(X_{E,F}, \mathcal{O}_{E,F}) = 0$$
[FF18, Points 1., 2., 3. in Section 8.2.1.1] (which makes you think that the curve has genus zero c.f. [FF18, Proposition 5.4.2. and Exemple 5.4.3] for the difference - besides the obvious one that the Fargues-Fontaine curve is not of finite type over $\mathrm{Spec}(E)$ - with the projective line).

Depending on how much time is left, you can discuss the following:

3. State the classification of vector bundles on $X_{E,F}$ (see [FF18, Définition 5.6.22] for the definition of the vector bundles $\mathcal{O}_{X_{E,F}}(\lambda)$ and [FF18, Théorème 5.6.26.(3)] for the statement of the classification) and prove that $X_{E,F}$ is geometrically simply connected ([FF18, Théorème 8.6.1]).

Talk 7: Local class field theory via the Fargues-Fontaine curve

The aim here is to give a rough overview of the article [Fa20]. Warning: For this talk, you need to make an effort to make the ideas of background material - formally perfectoid spaces, diamonds, Banach-Colmez spaces appear - digestible without going into too many definitions.

1. Recall the statement of LCFT: let E be a NA local field and W_E the Weil group, then there is an isomorphism
$$\mathrm{Art}_E \colon E^* \simeq W_E^{\mathrm{ab}},$$
which is uniquely characterized via Lubin-Tate theory.

The aim is to geometrize both sides of the above isomorphism.

2. Introduce (affinoid) perfectoid spaces [SW20, Definition 6.1.1, Definition 7.1.2] and introduce Y_S, X_S ([FS21, Definition II.1.15]).
3. Explain the definition of Div^d ([Fa20, Définition 2.6]) and why local systems on Div^1 are relevant for geometrizing continuous characters $\chi\colon W_E^{\mathrm{ab}} \to \Lambda$ ([Fa20, Proposition 3.1]). It would be great if there is time to explain the connection to Lubin-Tate theory [Fa20, Proposition 2.16] (this makes sure that the correspondence we construct geometrically agrees with LCFT).
4. Introduce the Picard-stack [Fa20, Définition 2.4] and describe it as in [Fa20, Discussion after Définition 2.4]. Mention [Fa20, Rémarque 2.5]. Explain how this can be used to geometrize continuous characters over E^*.
5. Introduce the Abel-Jacobi morphism [Fa20, Définition 2.7] and state the main result on simply connetedness of geometric fibers in the equal characteristic case [Fa20, Théorème 5.4 and section 6].

Talk 8: The starting point for other groups: Langlands' classification of representations of real and complex groups

Strangely enough, even though the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ is very small, one of the starting points of a correspondence that gives a version of class field theory for non-abelian groups G instead of $\mathrm{GL}_1 = \mathbb{G}_m$ seems to have been the classification of representations of real reductive groups in terms of representations of the Weil group, due to Langlands [La73].

To get an idea about this, it is probably best to start with the representations of $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ as in [Kn94] and to describe these in terms of representations of Weil-groups of $\mathbb{R}$ and $\mathbb{C}$. This is the first key thing we should take away from the talk.

A general reductive group over $\mathbb{R}$, like $\mathsf{SO}(n)$ may not contain split tori that were important for GL_n. This makes the theory richer, as not every representation is induced from simple subgroups as in the example of GL_n. The main point of the second part of the talk is to introduce the dual group which shows up in this setup and to explain this notion in some of the basic examples. The main statements then take a bit of time to digest. A classical exposition of the ideas of [La73] is [Bo79]. This has later been expanded by Adams-Barbasch-Vogan [ABV]. It would be great if we could see the main protagonists, i.e. the L-groups and spaces of parameters that appear in [ABV, Theorem 1.18].

(Remark: Very recently there has been a geometric approach by Scholze that may finally give an intrinsic explanation of the parametrization. This is certainly beyond a first talk on the subject.)

Talk 9: Adelic descriptions of classical spaces and Hecke operators

You may have seen Hecke operators on modular forms, operators that are usually defined in an elementary way and then been surprised to see adelic quotients and Hecke algebras defined as algebras of functions on double cosets. The first part of this talk should explain why these are related and why complicated looking adelic quotients are sometimes helpful descriptions of simpler quotients. A good starting point is the case of GL_2, where space of lattices and quotients of the upper half plane admit an adelic description [Del, Section 0]. In this language the Hecke-correspondence and Hecke-operators get a group-theoretic description [Del, Section 2.3]. I would find it helpful to turn the presentation of [Del] around and start with the classical picture, i.e. the space of lattices (closely related to elliptic curves) and then explain that the adelic picture helps to understand the relation between Hecke operators for different primes, e.g. why the Hecke-operators for different primes commute or how the algebra generated by Hecke operators decomposes as a product of algebras for different primes.

As a generalization mention the strong approximation theorem and explain why this implies that the adelic quotient can be identified with the quotient of the archimedian places by an arithmetic subgroup.

In the second part we pass to function fields $K = k(C)$ of a smooth curve C and see why there the adelic quotients describe principal bundles on curves and see Hecke operators there.

Again we start with $G = \mathrm{GL}_n$ in which case it is elementary to see that the adelic quotient $\mathrm{GL}_n(K)\backslash\mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(\mathcal{O})$ describes the category $\mathrm{Bun}_n(k)$ of vector bundles on C: Taking a bundle, trivializing it over an open subset and locally around all points of the curve, the gluing cocycle gives an element in the double quotient (e.g. [Zhu, Lemma 4.3.1]) and conversely, in the quotient $\mathrm{GL}_n(\mathcal{K}_x)/\mathrm{GL}_n(\mathcal{O}^\wedge_{X,x})$ a matrix computation shows that the quotient taken without completions gives the same set, given (g_x) one can thus define a locally free sheaf $\mathcal{E}_g$ by defining local sections as elements $s \in k(X)^n$ such that $g_x.v \in \mathcal{O}^n_{X,x} \subset k(X)^n$.

(For general groups a similar result holds see [Zhu, Lemma 4.3.1], but the gluing argument is more subtle, called Beauville-Laszlo gluing ([Zhu, Theorem 1.4.2]). With some care, it can be deduced from fpqc-descent, but there are more general versions of the fact that one can sometimes glue objects from completions along closed subschemes.)

Explain that the analog of Hecke corrrespondences for lattices now describes modifications of bundles, i.e. pairs of bundles $\mathcal{E}' \subset \mathcal{E}$ such that $\mathcal{E}/\mathcal{E}' \cong k(x)^m$ for some $1 \leq m \leq n$. (This also links back to the lecture on geometric class field theory.)

If time is left it would be very nice to state the structure of the Hecke algebra, i.e. the Satake isomorphism. Hecke operators come from functions on the double coset $G(\mathcal{O}^\wedge_{X,x})\backslash G(K_x)/G(\mathcal{O}^\wedge_{C,x})$. These double cosets are enumerated by $X_*(T)/W$ (for GL_n these are diagonal matrices with entries powers of a uniformizer, up to permutation by the elementary divisor theorem). Irreducible representations of a reductive group (over $\mathbb{C}$) are similarly enumerated by characters (up to conjugation by the Weyl group) $X^*(T)$ (or dominant characters). In the language of talk 8, the characters of the dual group are given by the cocharacters of G and the Satake isomorphism states that this combinatorial coincidence actually comes from an isomorphism of the Hecke algebra for G and the representation ring of the dual group. Moreover this has a categorical version ([MV07, (13.1)], [Zhu, Theorem 5.3.2 and Theorem 5.2.9]). It would be nice to indicate this statement, but without going into the definitions of perverse sheaves.

Talk 10: Drinfeld's construction of automorphic sheaves for GL_2

To end the seminar we would like to indicate how the first automorphic sheaves were constructed by Drinfel'd. The main plan is to follow Laumon's Bourbaki talk [Lau02] on the GL_n argument that was found much later, but restrict to the case $n = 2$, which reduces to an argument that is very close to Drindel'd's original argument and simplifies all of the constructions. A key technical result [Dr83, Appendix] used in this case is closely related to the proof of geometric class field theory from talk 5.

Start by stating the main result, explaining that for any irreducible rank 2 local system E on a curve there exists a Hecke-eigenshaef sheaf A_E on Bun_2 with eigenvalue E, in particular the trace function of A_E is an automorphic form f_E with Hecke eigenvalues described by the representation defined by E.

For the construction, some background may be helpful: A formula for the function f_E was known on $\mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(\mathcal{O})$, but it was not known whether it is constant on the fibers of the map to $\mathrm{Bun}_n(k)$, i.e. if it was invariant under $\mathrm{GL}_n(k(X))$, this argument used a result of Shalika [Sha74], who explained that f_E could be constructed by a Fourier transform of a function that is a product of local terms. Drinfel'd found a geometric way to rephrase this construction by Fourier transform and tried to find a way to get descent to Bun_2 by an analog of the argument from geometric class field theory.

Laumon's Bourbaki talk explains how one can set up this construction in a way that again works both for curves over $\mathbb{F}_q$ and $\mathbb{C}$. If one restricts this to GL_2 all inductive arguments disappear.

Please talk to the organizers to discuss which points should be highlighted.

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