Lectures on moduli of Higgs bundles

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Introduction

These are lecture notes of a minicourse on moduli of Higgs bundles given at the summer school on "Recent Perspectives on Hodge Theory" organized by Luigi Lombardi, Luca Tasin and Paolo Stellari.

The original plan for the lectures was changed, when less than two weeks before the summer school two independent proofs of the $P = W$ conjecture by Hausel, de Cataldo and Migliorini from [1] appeared in preprint form: a first one by Maulik and Shen [2] and shortly after this a second one by Hausel, Mellit, Minets and Schiffmann [3].

It therefore seemed appropriate to give some background on the $P = W$-conjecture and to explain some of the geometric results that are key components of the proofs of the conjecture, both of which include ideas originating from the geometric Langlands correspondence and in particular make use of the geometry of Hecke correspondences for Higgs bundles.

There have been many further developments since then. The list of proofs of the conjecture has been extended further by Maulik, Shen and Yin[4], a variant of the conjecture has been proposed in [5] and with the Bourbaki talk of Hoskins[6] an exposition of the two proofs mentioned above has appeared. We hope that the lecture notes might still serve as introduction to this circle of ideas.

[1] Mark Andrea de Cataldo, Tamás Hausel, and Luca Migliorini. Topology of Hitchin systems and Hodge theory of character varieties: the case A_1. *Ann. of Math. (2)*, 175(3):1329–1407, 2012. URL https://doi.org/10.4007/annals.2012.175.3.7

[2] Davesh Maulik and Junliang Shen. The $P = W$ conjecture for GL_n, 2022. URL https://arxiv.org/abs/2209.02568

[3] Tamás Hausel, Anton Mellit, Alexandre Minets, and Olivier Schiffmann. $P = W$ via H_2, 2022. URL https://arxiv.org/abs/2209.05429

[4] Davesh Maulik, Junliang Shen, and Qizheng Yin. Perverse filtrations and fourier transforms, 2023. URL https://arxiv.org/abs/2308.13160

[5] Yakov Kononov, Weite Pi, and Junliang Shen. Perverse filtrations, Chern filtrations, and refined BPS invariants for local $\mathbb{P}^2$. *Adv. Math.*, 433:29, 2023. DOI: 10.1016/j.aim.2023.109294

[6] Victoria Hoskins. Two proofs of the $P = W$ conjecture. *Séminaire Bourbaki*, 76 no 1213, 2023. URL https://www.bourbaki.fr/TEXTES/Exp1213-Hoskins.pdf

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Lecture 1: An introduction to the P=W conjecture

Moduli spaces of Higgs bundles have a double relation to Hodge theory, the guiding theme of this summer school. First, Simpson emphasized that the rich geometric structure of these moduli spaces can be viewed as a geometric analog of a Hodge structure on a non-abelian cohomology group, expressed in terms of a $\mathbb{C}^*$ action on a non-linear space.

Second, and this will be more important for these lectures, these moduli spaces provide a series of varieties that admit two natural algebraic structures, both of which induce Hodge structures on their cohomology. The $P = W$ conjecture formulates an unexpected relation between the Hodge structure coming from one of the algebraic structures and a geometric filtration coming from the other one.

In this first lecture, I would like to explain where the $P = W$ conjecture came from. To do this we will start with one of the simplest examples of a variety that admits two different algebraic structures and explain how a series of such varieties was constructed as moduli spaces of Higgs bundles. The surprising approach of Hausel and Rodriquez-Villegas to computing the cohomology of these spaces will then lead us to a first version of the conjecture.

A SIMPLE EXAMPLE of a complex variety admitting two different natural algebraic structures – one affine and one contracting to a projective variety – is given by the cotangent bundle to an elliptic curve.

If E is an elliptic curve over the complex numbers, its cotangent bundle T^*E is trivial, i.e., $T^*E \cong \mathbb{C} \times E$ because E is a group variety. Moreover, any complex elliptic curve E can be described as the quotient of the complex plane by a lattice

$$E \cong \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}),$$

in particular it is topologically a torus $S^1 \times S^1$. Therefore there exist diffeomorphisms

$$T^*E \cong \mathbb{C} \times \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \cong \mathbb{R}^2 \times (S^1)^2 \cong \mathbb{C}^* \times \mathbb{C}^*$$

and the product $\mathbb{C}^* \times \mathbb{C}^*$ is an affine algebraic variety.

The cohomology of the cotangent bundle $H^*(T^*E) \cong H^*(E)$ is isomorphic to the cohomology of E and thus carries a pure Hodge structure:

$$H^*(T^*E) \cong H^*(E) = \begin{cases} H^{1,1}(E) & \text{in degree 2} \\ H^{1,0}(E) \oplus H^{0,1}(E) & \text{in degree 1} \\ H^{0,0}(E) & \text{in degree 0} \end{cases}$$

where all listed groups are 1-dimensional $\mathbb{C}$-vector spaces.

In contrast to this, the cohomology $H^*((\mathbb{C}^*)^2)$ of the affine variety $(\mathbb{C}^*)^2$ has a mixed Hodge structure:

$$H^*((\mathbb{C}^*)^2) \cong H^*(\mathbb{C}^*) \otimes H^*(\mathbb{C}^*) = \begin{cases} H^{2,2}((\mathbb{C}^*)^2) & \text{in degree 2} \\ H^{1,1}((\mathbb{C}^*)) \oplus H^{1,1}((\mathbb{C}^*)) & \text{in degree 1} \\ H^{0,0}((\mathbb{C}^*)^2) & \text{in degree 0} \end{cases}$$

It has weights 2 in degree 1 and 4 in degree 2.

Remark. The above argument for T^*E also applies to the cotangent bundle $T^*\operatorname{Pic}^0_C$ to the Jacobian Pic^0_C of any smooth projective curve of genus g, or any abelian variety, where we again find a differomorphism $T^*\operatorname{Pic}_C \cong (\mathbb{C}^*)^{2g}$ as Pic_C is a complex torus of real dimension $2g$.

As the weight filtration on cohomology, which appeared implicitly in the above description of $H^{p,q}((\mathbb{C}^*)^2)$ is responsible for the W in the $P = W$ conjecture, let us briefly recall Deligne's construction of the mixed Hodge structure on non-compact or non-smooth varieties.

Aside on the mixed Hodge structure of smooth varieties.

Deligne used compactifications and resolutions of singularities to induce mixed Hodge structures on the cohomology of varieties from the usual Hodge structure on smooth projective varieties.

For a smooth complex variety X, choose a smooth compactification $j\colon X \hookrightarrow \overline{X}$ and denote by $\iota\colon Z = \overline{X} \setminus X \hookrightarrow \overline{X}$ the complement. Then the compactly supported cohomology of X and $\overline{X}$ are related by the long exact sequence

$$\to H_c^{*-1}(Z,\mathbb{Q}) \to H_c^*(X,\mathbb{Q}) \to H_c^*(\overline{X}) \to H_c^*(Z,\mathbb{Q}) \to$$

which comes from the sequence of sheaves

$$0 \to j_!\underline{\mathbb{Q}} \to \underline{\mathbb{Q}} \to \iota_{Z,*}\underline{\mathbb{Q}} \to 0.$$

If Z is smooth of codimension c, the dual exact triangle

$$\to \iota_Z^!\underline{\mathbb{Q}} \to \underline{\mathbb{Q}} \to \mathbb{R}j_*\underline{\mathbb{Q}} \to$$

satisfies $\iota^!\mathbb{Q} = \mathbb{Q}[2c]$ and gives rise to the Gysin sequence

$$\to H^{*-2c}(Z,\mathbb{Q}) \to H^*(\overline{X},\mathbb{Q}) \to H^*(X,\mathbb{Q}) \to H^{*-2c+1}(Z,i^!\mathbb{Q}) \to .$$

As $\iota^!\mathbb{Q}$ is a shift of the dualizing complex on Z, $H^*(Z,i^!\mathbb{Q})$ is the dual of a shift of the cohomology of Z and comes with a pure Hodge structure, as does the cohomology of $\overline{X}$. The cohomology $H^*(X)$ inherits a mixed Hodge structure from this sequence.

If Z is not smooth, we can stratify it into smooth strata and use the same argument inductively.

In the example of $X = \mathbb{C}^*$ and $\overline{X} = \mathbb{P}^1$ the image of the restriction map $H^*(\overline{X}) \to H^*(X)$ is very small – it is only $H^0(X)$ – and the first cohomoloy of $H^*(X)$ comes from the boundary $Z = \{0,\infty\}$, the cohomology of which also surjects onto $H^2(\mathbb{P}^1)$, so the Hodge structure on $H^1(\mathbb{C}^*)$ coincides with the one of $H^2(\mathbb{P}^1)$.

It turns out that this structure does not depend on the choice of the compactification. In particular, the pure part of the cohomology $H^*(X)$ is the image of $H^*(\overline{X}) \to H^*(X)$ for any compactification $\overline{X}$ of X.

THE UPSHOT of this discussion is that the cohomology $H^*(X)$ inherits an increasing weight filtration

$$0 \subseteq W_0H^*(X) \subseteq W_1H^*(X) \subseteq \cdots \subseteq H^*(X)$$

such that the associated graded pieces carry (pure) Hodge structures.

Let us give a more precise statement of Deligne's result, combining Théorème 8.2.4, Proposition 8.2.5 of [7] and Corollaire 3.2.17 from [8].

[7] Pierre Deligne. Théorie de Hodge. III. *Publ. Math., Inst. Hautes Étud. Sci.*, 44: 5–77, 1974. DOI: 10.1007/BF02685881

[8] Pierre Deligne. Théorie de Hodge. II. (Hodge theory. II). *Publ. Math., Inst. Hautes Étud. Sci.*, 40:5–57, 1971. DOI: 10.1007/BF02684692

Theorem. *(Deligne)*

1. *The cohomology of any complex algebraic variety X admits an increasing, rational weight filtration*

$$0 \subseteq W_0 \subseteq \cdots \subseteq W_{2n} = H^n(X, \mathbb{Q})$$

and a decreasing Hodge filtration

$$0 \subseteq F^n \subseteq F^{n-1} \subseteq \cdots \subseteq F^0 = H^n(X, \mathbb{C})$$

such that for all j the filtration $F^\bullet$ induces a pure Hodge structure of weight j on the subquotient $W_j/W_{j-1} \otimes_{\mathbb{Q}} \mathbb{C}$.

These filtrations are functorial with respect to morphisms of varieties and coincide with the filtrations defined by the pure Hodge structure on smooth projective varieties.

2. *If X is smooth only Hodge structures of weight $\geq j$ occur in $H^j(X)$. Moreover, given a smooth proper compactification $X \subset \overline{X}$ the image of $H^n(\overline{X}) \to H^n(X)$ is equal to the weight n part $W_n H^n$.*

3. *If X is proper only Hodge structures of weight $\leq j$ occur in $H^j(X)$. Moreover if $p\colon \widetilde{X} \to X$ is a proper morphism and $\tilde{X}$ is smooth and proper, then* $\ker(H^j(X) \to H^j(\widetilde{X})) = W_{j-1}H^j(X)$, *i.e. the weight j part of $H^j(X)$ is the image of $H^j(X) \to H^j(\widetilde{X})$.*

The W-filtration of the $P = W$ conjecture is the weight filtration on cohomology obtained through the above theorem.

Let us now try to understand the spaces appearing in the conjecture, which generalize our toy example.

A modular explanation of the toy example

In our example of the cotangent bundle of an elliptic curve, the diffeomorphisms $T^*E \cong (\mathbb{C}^*)^2$, or $T^*\operatorname{Pic}_C \cong (\mathbb{C}^*)^{2g}$ are not as strange as it seems, because these admit a modular interpretation. Namely, the space $(\mathbb{C}^*)^{2g} = \operatorname{Hom}(\pi_1(C), \mathbb{C}^*)$ is the space of one-dimensional representations of the fundamental group $\pi_1(C)$ of C and the Riemann-Hilbert correspondence identifies these with line bundles together with a flat connection

$$(\mathbb{C}^*)^{2g} = \operatorname{Hom}(\pi_1(C), \mathbb{C}^*) \cong \{(\mathcal{L}, \nabla\colon \mathcal{L} \to \mathcal{L} \otimes \Omega_C) \mid \nabla \text{ flat connection}\}.$$

The space of line bundles with a flat connection comes equipped with a forgetful map to the Jacobian Pic^0_C

$$\begin{array}{c} \{(\mathcal{L}, \nabla\colon \mathcal{L} \to \mathcal{L} \otimes \Omega_C) \mid \nabla \text{ flat connection}\}\ . \\ \Big\downarrow{\scriptstyle (\mathcal{L},\nabla)\mapsto\mathcal{L}} \\ \operatorname{Pic}^0_C \end{array}$$

This morphism is a torsor for $H^0(C, \Omega_C)$, because any two flat connections differ by a holomorphic differential, i.e. an element of $H^0(C, \Omega_C) \cong H^1(C, \mathcal{O}_C)^*$, which is the fiber of cotangent space to Pic_C.

To relate this fibration directly with the cotangent bundle, there is a trick to consider the Leibniz rule for connections:

This trick can also be understood more conceptually as saying that a connection defines a module structure under the sheaf of differential operators $\mathcal{D}$ on C, which is a filtered algebra whose associated graded

$$\mathrm{gr}\, \mathcal{D} \cong \mathrm{Sym}^\bullet T_C$$

is the symmetric algebra of vector fields – the push forward of the structure sheaf of T^*C to C. For any filtered algebra the Rees construction defines a $\mathbb{C}^*$-equivariant family of algebras over $\mathbb{A}^1$ that degenerates the algebra into its associated graded.

$$\nabla(fs) = f\nabla(s) + s \otimes df$$

where $f \in \mathcal{O}_C(U), s \in \mathcal{L}(U)$ and decorate the term which is not $\mathcal{O}_C$-linear with a parameter $\lambda \in \mathbb{C}$ and define a λ-connection on $\mathcal{L}$ to be a map $\nabla \colon \mathcal{L} \to \mathcal{L} \otimes \Omega_C$ satisfying:

$$\nabla(fs) = f\nabla(s) + \lambda s \otimes df.$$

the space of λ-connections is then fibered over the affine line $\mathbb{A}^1$:

$$\begin{array}{l} M^0_{1,Hodge} := \left\langle (\mathcal{L}, \lambda, \nabla) \,\middle|\, \mathcal{L} \in \mathrm{Pic}^0_C, \lambda \in \mathbb{C}, \nabla \text{ a } \lambda-\text{connection} \right\rangle \\ \quad\Big\downarrow \\ \quad\mathbb{A}^1 \end{array}$$

where the fiber over 1 is the moduli space of flat connections $M^0_{1,DR}$ and the fiber over 0 is the cotangent bundle $T^*\mathrm{Pic}_C =: M^0_{Dol,1}$. Moreover this fibration is equivariant with respect to an action of the multiplicative group $\mathbb{C}^*$ and induces isomorphisms on cohomology

The subscripts *Dol* for Dolbeaut, *DR* for de Rham, *Hodge* and *Betti* are chosen to highlight the parallel structures found classically through comparison isomorphisms on the cohomology of projective varieties and the homeomorphisms of the moduli spaces of bundles, which we can view as different geometric incarnations of the cohomology set $H^1(C, \mathrm{GL}_n)$.

$$H^*(T^*\mathrm{Pic}_C) \cong H^*(M^0_{1,Hodge}) \cong H^*(M^0_{1,DR}) \cong H^*(\mathrm{Rep}(\pi_1(C), \mathbb{C}^*)).$$

THE MIRACLE is, that this picture generalizes to $\mathrm{Rep}(\pi_1(C), \mathrm{GL}_n)$ and vector bundles of higher rank, to principal bundles for a reductive group and the corresponding representations of the fundamental group and even to representations of fundamental groups of higher dimensional projective varieties. We will state the version for GL_n next.

The non-abelian Hodge isomorphism

Let us introduce the objects appearing in the generalization of the geometry explained in the previous section to bundles of any rank. These constructions are due to Hitchin, Donaldson, Corlette and Simpson.

To fix the notation of the moduli problems appearing in the statement we only give the category of objects parameterized, as in all cases there is a natural notion of families of objects parameterized by a scheme T. To distinguish the category of objects from the set of isomorphism classes we use pointed brackets $\langle\rangle$ instead of $\{\}$ which we reserve for sets.

For fixed rank $n \in \mathbb{N}$ and degree $d \in \mathbb{Z}$ we will denote by

$$\mathrm{Bun}^d_n := \left\langle \mathcal{E} \,\middle|\, \begin{array}{c} \mathcal{E} \text{ vector bundle on } C \\ \text{of rank } n \text{ and degree } d \end{array} \right\rangle$$

the stack of vector bundles of fixed rank n and degree d on C, i.e., formally for a scheme T we define $\mathrm{Bun}_n^d(T)$ to be the category of rank n vector bundles on $C \times T$ so that for all $t \in T$ the restriction to $C \times t$ is of degree d.

The stack of Higgs bundles

$$\mathrm{Higgs}_n^d := \left\langle (\mathcal{E}, \phi) \,\middle|\, \mathcal{E} \in \mathrm{Bun}_n^d, \phi \colon \mathcal{E} \to \mathcal{E} \otimes \Omega_C \right\rangle$$

parameterizes vector bundles together with an Ω_C-valued endomorphism. We denote by

$$\mathcal{M}_{m,Dol}^d := \left\langle (\mathcal{E}, \phi) \,\middle|\, (\mathcal{E}, \phi) \in \mathrm{Higgs}_n^d \text{ semistable} \right\rangle$$

the substack stack of semistable Higgs bundles, i.e., those Higgs bundles $(\mathcal{E}, \phi)$ such that for all proper subbundles $0 \subsetneq \mathcal{E}' \subsetneq \mathcal{E}$ that are preserved by ϕ (meaning $\phi(\mathcal{E}') \subseteq \mathcal{E}' \otimes \Omega_C$) we have

$$\frac{\deg(\mathcal{E}')}{\mathrm{rank}(\mathcal{E}')} \leq \frac{\deg(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}.$$

This substack admits a good moduli space, called moduli space of Higgs bundles on C, that we denote by $M_{n,Dol}^d$.

Similarly we denote by

$$M_{n,Betti}^d := \left\{ (A_i, B_i)_{i=1,\dots g} \in GL_n^{2g} \,\middle|\, \prod_{i=1}^{g} [A_i, B_i] = e^{\frac{2\pi i}{n} d} \,\mathrm{id}_n \right\} /\!/ \,\mathrm{GL}_n \,.$$

the moduli space of (semisimple) representation of $\pi_1(C)$. Here we included the so called twisted version for any integer $d \in \mathbb{Z}$.

Theorem 1 (Non-abelian Hodge correspondence). *For a smooth projective curve C and integers n, d there exists a homeomorphism of the moduli spaces of semistable Higgs bundles and the moduli space of n-dimensional representations of $\pi_1(C)$:*

$$M_{n,Dol}^d \simeq M_{n,Betti}^d.$$

Remark. 1. The moduli stack of Higgs bundles can be viewed as cotangent stack to the moduli of vector bundles, generalizing the picture we saw for $T^*\,\mathrm{Pic}$ in the toy example. We will see in the next lecture that the tangent space to the moduli of vector bundles at a vector bundle $\mathcal{E}$ is computed by $H^1(C, \mathrm{End}(C))$ which by Serre duality is dual to $H^0(C, \mathrm{End}(\mathcal{E}) \otimes \Omega_C)$. So the moduli problem for Higgs bundles is the one expected from the cotangent bundle to the moduli stack of vector bundles.

There is a little bit of care needed to make this precise, as deformation theory involves both H^0 and H^1, and dualizing the cotangent complex doesn't behave nicely with respect to cohomological degrees.

2. The structure of a cotangent bundle to the moduli space of vector bundles was one of the motivations when Hitchin introduced this moduli space, as it gives rise to a series of completely integrable systems, which are rare. We will come back to this structure of $M_{n,Dol}^d$ in the third lecture.

3. One application of the above result is that it gives an approach to computing the cohomology of the space of representations of the fundamental group $M^d_{n,Betti}$, which despite of its more explicit description seems to be much harder to analyze, as the equations given by the commutator of matrices are complicated.

4. If rank n and degree d are coprime, the moduli spaces above are smooth. We will also see in the third lecture why in this situation, in contrast to $\mathcal{M}_{Betti}$, the cohomology groups of $\mathcal{M}^d_{n,Dol}$ and $M^d_{n,Dol}$ carry pure Hodge structures, even though these spaces are not proper.

5. There is an analog of the space of λ-connections, $M^d_{n,Hodge}$ over $\mathbb{A}^1$ that generalizes the picture we saw for 1-dimensional representations above, which also explains the definition of $M^d_{n,Betti}$. This is surprising, as vector bundles of non-zero degree do not admit flat connections. There are several approaches to this.

   First, Hitchin [9] generalized an argument of Atiyah-Bott [10] to understand bundles of non-zero degree as follows. Viewing a vector bundle as GL_n-torsor, one can pass to the corresponding PGL_n-torsor. This also works for Higgs bundles and of the arguments work for PGL_n-Higgs bundles. Connections on the associated PGL_n-bundle correspond to projectively flat connections on the GL_n-bundle. It turns out that the moduli stacks of GL_n-bundles define torsors over the corresponding PGL_n versions, i.e., the PGL_n-versions are obtained by passing to a quotient under the action given by the tensor product with bundles of rank 1, which defines an action of the GL_1-version of the moduli spaces on the GL_n version. This allows to deduce the GL_n results form a combination of the PGL_n results and the case of line bundles. In particular Hitchin and Atiyah-Bott explain how the obstruction to lifting representations of $\pi_1(C)$ into PGL_n to GL_n leads to a representation of the universal central extension of $\pi_1(C)$ which in the end gives rise to the above definition of $M^d_{n,Betti}$.

   An algebraic alternative to this method, that directly gives a space $\mathcal{M}^d_{n,Hodge}$, was introduced by Simpson[11]. In this article he generalized his higher-dimensional version of the non-abelian Hodge correspondence to a version that works for bundles on proper Deligne-Mumfod stacks. This is convenient to obtain a version of λ-connections for bundles of non-zero degree. In our situation, after choosing a point $p \in C$, we can replace our curve C by a variant $f_n \colon C(\frac{1}{n}p) \to C$, called a root stack, whose fundamental group is generated by the free group $\pi_1(C \setminus \{p\})$ on $2g$ generators with the additional relation that the monodromy around p has order n – which is the group we would like to see if we want to interpret $M^d_{n,Betti}$ directly as representations of a fundamental group.

   Geometrically the curve $C(\frac{1}{n}p) \to C$ is obtained from C by replacing a disc D around p by the stack $[\tilde{D}/\mu_n]$ obtained from

[9] Nigel Hitchin. The self-duality equations on a Riemann surface. *Proc. Lond. Math. Soc. (3)*, 55:59–126, 1987a. DOI: 10.1112/plms/s3-55.1.59

[10] Michael F. Atiyah and Raoul Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. R. Soc. Lond., Ser. A*, 308:523–615, 1983. ISSN 0080-4614. DOI: 10.1098/rsta.1983.0017

[11] Carlos Simpson. Local systems on proper algebraic V-manifolds. *Pure Appl. Math. Q.*, 7(4):1675–1759, 2011. DOI: 10.4310/PAMQ.2011.v7.n4.a27

the quotient of a disc by the rotation given by the n-th roots of unity, where we view $\tilde{D} \to D$ as the ramified cover of order n. This replaces p by a stacky point that has an automorphism of order n.

This curve comes with a line bundle $\mathcal{O}(\frac{1}{n}p)$ whose n-th power is the pull back of $\mathcal{O}_C(p)$. It turns out that the assignment $\mathcal{E} \mapsto f_n^*\mathcal{E} \otimes \mathcal{O}(-\frac{1}{n}p)^{-d}$ identifies Higgs bundles of degree d on C with those Higgs bundles of degree 0 on $C(\frac{1}{n}p)$ for which the automorphism group of the stacky point acts on the fiber of the bundle at the stacky point by multiplication by $e^{\frac{2\pi i d}{n}}$. In this way the non-abelian Hodge correspondence for bundles of degree 0 on $C(\frac{1}{n}p)$ gives the correspondence for bundles of degree d on C and in particular the moduli space of λ-connections on $C(\frac{1}{n}p)$ defines a natural version of $M^d_{n,Hodge}$.

The $P = W$ conjecture arose from an attempt to compute the cohomology of the moduli space of Higgs bundles. As mentioned above, algebraic geometric tools seemed to provide more tools to compute the cohomology of these moduli spaces, which are closely related to moduli of vector bundles on curves.

However, it took some time to find tools that are good enough to do computations beyond the examples of small rank. Tamas Hausel and Fernando Rodriguez-Villegas thus tried something very surprising.

Aside on the Weil conjectures

The famous Weil conjectures, proven by Deligne, imply that to compute the cohomology of a smooth projective variety X defined say over the rationals $\mathbb{Q}$ it suffices to count the number of points of the variety for reductions of the variety over sufficiently many finite fields.

More precisely, if q is a prime number then the numbers $\#X(\mathbb{F}_{q^n})$ determine the dimensions $H^i(X(\mathbb{C}), \mathbb{Q})$, where roughly any power q^i showing up in the counting process corresponds to a (i,i) class in $H^{2i}(X)$.

The basic example is probably $\mathbb{P}^n(\mathbb{F}_q) = q^n + q^{n-1} + \cdots + q + 1$ which nicely corresponds to the 1-dimensional cohomology groups in even degree and vanishing cohomology in odd degree. Similarly for an elliptic curve E over a finite field one knows that

$$E(\mathbb{F}_q) = q - (\alpha_1 + \alpha_2) + 1$$

where $|\alpha_i| = \sqrt{q}$.

The basic idea underlying this structure is the Lefschetz trace formula from topology expressing the number of fixed points of a sufficiently nice endomorphism $f\colon X \to X$ in terms of the trace of f^* on $H^*(X, \mathbb{Q})$. Over finite fields the $\mathbb{F}_q$-rational points are the fixed points of the Frobenius endomorphism[12] $x \to x^q$ and this turns out

[12] There are several different Frobenius endomorphisms that can be deduced from $x \to x^q$, either acting as element of the Galois group of $\overline{\mathbb{F}}_q$ or geometrically. These lead to different signs in the exponents below.

to automatically satisfy the transversality condition required for the Lefschetz trace formula.

The miracle allowing to argue backwards is the content of Deligne's theorem, proving that the eigenvalues of Frobenius on the i-th cohomology of a smooth proper variety have absolute value $q^{\frac{i}{2}}$.

IF YOU DO THE SAME FOR NON PROPER smooth varieties, this doesn't work so nicely, e.g. for the affine line $\mathbb{A}^1$ you get q points, and so the point counting in our example $(\mathbb{C}^*)^{2g}$ gives

$$(q-1)^{2g} = q^{2g} - 2g \cdot q^{2g-1} \pm \cdots + 1,$$

which looks like the cohomology we would like to get, but the powers of q appear in the wrong cohomological degrees.

Worse, in general, as the cohomology classes can appear in "wrong" cohomological degrees and there are signs involved in the trace formula, cancellations will usually occur for varieties that are not proper.

Remark. This structure motivated the search for mixed Hodge structures on the cohomology of complex varieties, that would mimic the appearance of Frobenius eigenvalues of different absolute value in one cohomological degree, as these define a natural weight filtration on the etale cohomology groups.

Remark. In étale cohomology, naturality of the Frobenius endomorphism implies that the cup product is compatible with Frobenius and thus the weight filtration is multiplicative. The same is known for the weight filtration of complex varieties.

The origin of the P=W conjecture

Although character varieties are not compact, Tamas Hausel and Fernando Rodriguez-Villegas[13] realized that they nevertheless could count the number of points of $M^d_{n,Betti}$ over finite fields $\mathbb{F}_q$ in terms of representation theory of $\mathrm{GL}_n(\mathbb{F}_q)$. They observed that the answers turned out to be polynomials in q and for varying n they found a way to assemble these into an explicit generating series.

[13] Tamás Hausel and Fernando Rodriguez-Villegas. Mixed Hodge polynomials of character varieties. With an appendix by Nicholas M. Katz. *Invent. Math.*, 174(3):555–624, 2008. DOI: 10.1007/s00222-008-0142-x

Comparing their formula for $n = 2$ with the known result for the cohomology of moduli spaces of rank 2 Higgs bundles they made an amazing conjecture combining:

1. a guess to which cohomological degree the powers of q in the formula should contribute

2. a guess which cancellations occur to obtain a conjectural formula for $H^*(M_{Betti})$ and even the weight-filtration $W_\bullet$, i.e., the mixed-Hodge structure on the cohomology of this smooth affine variety.

The first check on this conjectural formula was that the generating series consists of rational functions whose poles magically cancel at least numerically, when you expand the series to get formulas for the cohomology of the moduli spaces.

The second check was a comparison with the published formula for rank 3 bundles obtained by Gothen[14], which did not match, but let to the discovery of a simple mistake in the published formula, which then matched the conjecture (the corrected formula can for example be found in the Appendix of [15]).

By now this conjecture has been proven as a result of a combination of a result of Oliver Schiffmann[16] and one of Anton Mellit[17]. More recently another approach to the counting of Higgs bundles appeared in work of Hongjie Yu[18].

[14] Peter B. Gothen. The Betti numbers of the moduli space of stable rank 3 Higgs bundles on a Riemann surface. *Int. J. Math.*, 5(6):861–875, 1994. ISSN 0129-167X. DOI: 10.1142/S0129167X94000449

[15] Oscar García-Prada, Jochen Heinloth, and Alexander Schmitt. On the motives of moduli of chains and Higgs bundles. *J. Eur. Math. Soc. (JEMS)*, 16(12):2617–2668, 2014. DOI: 10.4171/JEMS/494

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Hausel and Rodriguez-Villegas not only gave a conjectural formula, but they also observed surprising symmetries in the structure of their conjectural description of the weight filtration of their cohomology groups which looked like a tilted version of a hard Lefschetz-symmetry, which they called "curious hard Lefschetz".

At that time Mark de Cataldo and Luca Migliorini had been working on a new proof of the so called decomposition theorem of Beilinson–Bernstein–Deligne–Gabber. This is one instance where a similarly shifted hard Lefschetz symmetry shows up naturally, called the perverse filtration $P_\bullet H^*(\mathcal{M}_{Dol})$ – we will give some background on this in Lecture 3. Together with Tamas Hausel, they proved that for $n = 2$ this filtration indeed agrees with the weight filtration on the cohomology of the Betti moduli space and conjectured that this should be a general phenomenon:

Conjecture (P=W conjecture)**.** [19] *Under the non-abelian Hodge isomorphism the perverse filtration on $H^*(\mathcal{M}_{Dol})$ agrees with the weight filtration on $H^*(\mathcal{M}_{Betti})$, more precisely*

$$P_i H^*(\mathcal{M}_{Dol}) = W_{2i} H^*(\mathcal{M}_{Betti}).$$

[19] Mark Andrea A. De Cataldo, Tamás Hausel, and Luca Migliorini. Topology of Hitchin systems and Hodge theory of character varieties: the case A_1. *Ann. Math. (2)*, 175(3):1329–1407, 2012. DOI: 10.4007/annals.2012.175.3.7

Lecture 2: Generators of the cohomology ring

The recent proofs of the P=W conjecture all rely on the fact that multiplicative generators of the cohomology rings of $\mathcal{M}^d_{n,Dol}$ and $M^d_{n,Dol}$ (in the case that (n,d) are coprime) are known, i.e., the cohomology is generated by Künneth components of Chern classes of the universal bundle on the moduli stack. Moreover, the modular description also allows to compare them to classes on $\mathcal{M}^d_{n,Betti}$. These results are the topic of this second lecture.

We will start by recalling why the space of stable Higgs bundles on a curve is smooth and use these computations of deformation complexes to sketch Markman's argument to construct multiplicative generators of the cohomology ring of this moduli space.

We will end the lecture with Shende's result computing the Hodge weights of the generators in $H^*(\mathcal{M}_{Betti})$.

Smoothness of the moduli spaces for (n,d) coprime

Let us begin by recalling the argument for smoothness of our moduli spaces. The basic idea is to compute the tangent space to the moduli problem. For varieties X, the points of the tangent bundle are

$$TX(k) = X(k[\epsilon]/(\epsilon^2)).$$

This description can be used as definition of the tangent space of a moduli problem, or an algebraic stack. We will do this below for moduli of vector bundles and moduli of Higgs bundles.

As it does not require an additional effort and will be useful in our outlook into the proofs of the P=W conjecture, let us also include the version for meromorphic Higgs bundles, i.e. given an (effective) divisor D on C, we denote by $\mathrm{Higgs}_n^{d,D}$ the moduli stack

$$\mathrm{Higgs}_n^{d,D} := \left\langle (\mathcal{E},\phi) \,\middle|\, \begin{matrix} \mathcal{E} \in \mathrm{Bun}_n^d \\ \phi\colon \mathcal{E} \to \mathcal{E} \otimes \Omega_C(D) \end{matrix} \right\rangle$$

and by

$$\mathcal{M}_n^{d,D} := \left\langle (\mathcal{E},\phi) \,\middle|\, \begin{matrix} (\mathcal{E},\phi) \in \mathrm{Higgs}_n^{d,D} \\ (\mathcal{E},\phi) \text{ semistable} \end{matrix} \right\rangle$$

the substack of semistable Higgs bundles – semistability being defined as before – which again admits a good moduli space $M_n^{d,D}$.

Lemma. *Let C be a smooth projective curve, Bun_n the stack of vector bundles on C and Higgs_n the stack of Higgs bundles of rank n on C.*

1. *The tangent space (stack) of Bun_n at $\mathcal{E} \in \mathrm{Bun}_n(k)$ is*

$$T_{\mathcal{E}} \mathrm{Bun}_n = [H^1(C, \mathrm{End}(\mathcal{E}))/H^0(C, \mathrm{End}(\mathcal{E}))].$$

2. *The tangent space (stack) of $\mathrm{Higgs}_n^{d,D}$ at $(\mathcal{E},\phi) \in \mathrm{Higgs}_n^{d,D}(k)$ is*

$$T_{(\mathcal{E},\phi)} \mathrm{Higgs}_n^{d,D} = \left[\mathbb{H}^1\big(C, \mathrm{End}_\bullet(E,\phi)\big)/\mathbb{H}^0(C, \mathrm{End}_\bullet(E,\phi))\right].$$

   Where $\mathrm{End}_\bullet(E,\phi)$ is the 2-term complex of vector bundles on C given by:

$$\mathrm{End}(\mathcal{E}) \xrightarrow{[\_,\phi]} \mathrm{End}(\mathcal{E}) \otimes \Omega_C(D).$$

The gray quotient symbols express that for a stack, the tangent space should again be considered as a stack, because deformations of $\mathcal{E}$ over $k[\epsilon]/(\epsilon^2)$ can have more automorphisms than $\mathcal{E}$ and this difference is exactly given by the H^0 term.

This is relevant, because we can check smoothness by checking that tangent spaces are of constant dimension and for Bun_n this changes $\dim H^1$ to $\dim H^1 - \dim H^0$ which is the negative of an Euler charactersitic and thus automatically constant in flat families.

Proof. There are many different proofs of this result. On this occasion we opt for a computation using cocycles as it is the most elementary.

1. Given a vector bundle $\mathcal{E}$ on C, pick an affine open cover $C = \cup U_i$ such that $\mathcal{E}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$ admits a trivialization. Choosing a trivialization gives cocycles $g_{ij} \in \mathrm{GL}_n(\mathcal{O}_C(U_{ij}))$, where $U_{ij} = U_i \cap U_j$.

   Then for any deformation $\widetilde{\mathcal{E}}$ on $C \times \operatorname{Spec} k[\epsilon]/(\epsilon^2)$ of $\mathcal{E}$ – i.e., $\widetilde{\mathcal{E}}$ is a vector bundle on $C \times \operatorname{Spec} k[\epsilon]/(\epsilon^2)$ equipped with an identification of its restriction to $C \times \operatorname{Spec} k$ with $\mathcal{E}$ – we can lift the trivialization of $\mathcal{E}|U_i$ to one of $\widetilde{\mathcal{E}}|_{U_i}$ and the different possible choices of lifts differ by elements in $\epsilon \cdot \mathrm{End}(\mathcal{O}_{U_i}^{\oplus n}) = \epsilon \cdot \mathrm{Mat}_{n,n}(\mathcal{O}_C(U_i))$.

   This way the cocycle $\widetilde{g}_{ij} \in \mathrm{GL}_n(\mathcal{O}_C(U_{ij})[\epsilon]/(\epsilon^2)) = \mathrm{GL}_n(\mathcal{O}_C(U_{ij})) + \epsilon \,\mathrm{Mat}_{n,n}(\mathcal{O}_{U_{ij}})$ is of the form $\widetilde{g}_{ij} = g_{ij} + \epsilon A_{ij}$ and conversely any such lift defines a deformation if and only if $\widetilde{g}_{ij}$ satisfies the cocycle condition on threefold intersections. Computing these cocycles, we find[20] that the obstruction to extend a given cocycle g_{ij} to $\widetilde{g}_{ij}$ is an element of $H^2(C, \mathrm{End}(\mathcal{E})) = 0$, the different choices are given by $[A_{ij}] \in H^1(C, \mathrm{End}(\mathcal{E}))$ and automorphisms of $\widetilde{\mathcal{E}}$ that restrict to the identity on $\mathcal{E}$ are parametrized by $H^0(C, \mathrm{End}(\mathcal{E}))$.

[20] It is helpful to do this computation, if you have not seen it before.

2. The above computation generalizes for Higgs bundles $(\mathcal{E}, \phi)$, where a local trivialization now expresses $\phi|_{U_i}$ as element of $(\mathrm{End}(\mathcal{O}_C^{\oplus n}) \otimes \Omega_C(D))(U_i) = \mathrm{Mat}_{n,n}(\Omega_C(D)(U_i))$.

   Given an extension $\widetilde{g}_{ij} = g_{ij} + \epsilon A_{ij}$ and a lift $\tilde{\phi}|_{U_i} = \phi|_{U_i} + \epsilon M_i$ with $M_i \in \mathrm{Mat}_{n,n}(\Omega_C(D)(U_i))$ we get the additional condition that $\tilde{\phi}|_{U_i}$ has to define a section of $H^0(C, \mathrm{End}(\widetilde{\mathcal{E}}) \otimes \Omega_C(D))$, which is an equation for the commutator of ϕ_i and g_{ij}, so now we get an obstruction in $H^1(C, \mathrm{End}(\mathcal{E}) \otimes \Omega_C(D))$ and the possible lifts $(\widetilde{\mathcal{E}}, \widetilde{\phi})$ are controlled by the first hypercohomology of the complex $\mathrm{End}(\mathcal{E}) \to \mathrm{End}(\mathcal{E}) \otimes \Omega_C(D)$:

$$\mathbb{H}^1(C, \mathrm{End}(\mathcal{E}) \to \mathrm{End}(\mathcal{E}) \otimes \Omega_C(D)).$$

   Again the automorphisms of $(\widetilde{\mathcal{E}}, \widetilde{\phi})$ that restrict to the identity on $\mathcal{E}$ are given by

$$H^0(C, \mathrm{End}(\mathcal{E}) \to \mathrm{End}(\mathcal{E}) \otimes \Omega_C(D)) = \mathrm{Ker}(H^0(C, \mathrm{End}(\mathcal{E})) \to H^0(C, \mathrm{End}(\mathcal{E}) \otimes \Omega_C(D)).$$

□

Remark. 1. The same computation can be used, to check the infinitesimal lifting criterion for smoothness for the stacks, where one will find that a possible obstruction would define a non-zero class in H^2 of the complexes appearing in the computation.

2. In the computation for Higgs bundles the hypercohomology of the complex $\mathrm{End}(\mathcal{E}) \to \mathrm{End}(\mathcal{E}) \otimes \Omega_C(D)$ will in general have a non-trivial second cohomology group $\mathbb{H}^2$, so that neither the dimension $\dim \mathbb{H}^1$ nor the difference $\dim \mathbb{H}^1 - \dim \mathbb{H}^0$, the dimension of the tangent stack to Higgs_n need to be constant. However, the complex

$$\mathrm{End}(\mathcal{E}) \to \mathrm{End}(\mathcal{E}) \otimes \Omega_C$$

   is self-dual with respect to Serre duality so

$$\mathbb{H}^2(C, \mathrm{End}(\mathcal{E}) \to \mathrm{End}(\mathcal{E}) \otimes \Omega_C) \cong \mathbb{H}^0(C, \mathrm{End}(\mathcal{E}) \to \mathrm{End}(\mathcal{E}) \otimes \Omega_C)^\vee$$

   and the latter group is the dimension of the automorphism group of the Higgs bundle $(\mathcal{E}, \phi)$. So the stack of Higgs bundles is smooth exactly at the bundles whose automorphism group is of minimal dimension, i.e., that only admit the central scalar automorphisms.

   As stable bundles only admit central scalar automorphisms, this implies that in the case that rank n and degree d are coprime both the moduli stack and the moduli space of semistable Higgs bundles are smooth.

3. The situation improves for meromorphic Higgs bundles. If D is an effective divisor, then Serre duality identifies the second cohomology $\mathbb{H}^2$ of the complex $\mathrm{End}(\mathcal{E}) \to \mathrm{End}(\mathcal{E}) \otimes \Omega_C(D)$ with the dual of

$$\mathbb{H}^0(C, \mathrm{End}(\mathcal{E})(-D) \to \mathrm{End}(\mathcal{E}) \otimes \Omega_C)$$

and elements in this group define endomorphisms of $(\mathcal{E}, \phi)$ that vanish at D. However as a non-zero endomorphism f factors as

$$\mathcal{E} \twoheadrightarrow \mathrm{im}(f) \hookrightarrow \mathcal{E}(-D) \subset \mathcal{E}$$

the sheaf $\mathrm{im}(f)$ defines both a quotient and a subsheaf of $\mathcal{E}$, so by semistability it has to have the same slope as $\mathcal{E}$ but then $\mathrm{im}(f)(D) \subset \mathcal{E}$ is a destabilizing subsheaf.

Corollary 2. *The stack of vector bundles* Bun_n^d *on* C*, and the moduli spaces of stable vector bundles* $N_n^{d,st}$*, stable Higgs bundles* $M_n^{d,st}$ *and of semistable meromorphic Higgs bundles* $M_n^{d,D,sst}$ *are smooth and we have*

$$\begin{aligned}
\dim \mathrm{Bun}_n^d &= n^2(g-1) && \\
\dim N_n^{d,st} &= n^2(g-1)+1 && \text{if } g>1\\
\dim M_n^{d,st} &= 2(n^2(g-1))+2 && \text{if } g>1\\
\dim M_n^{d,D,st} &= 2(n^2(g-1))+n^2\deg(D)+1 && \text{if } g>1.
\end{aligned}$$

Proof. In the above lemma and the remark following, we saw that smoothness of the stacks in question follows from the vanishing of the second cohomology groups appearing in the deformation complexes, which happens for moduli of bundles and for semistable meromorphic Higgs bundles if $D > 0$. For stable Higgs bundles we indicated above why smoothness still follows for stable bundles, although the second cohomology group does not vanish. For Bun_n^d the given dimension is the negative of the Euler characteristic of $\mathrm{End}(E)$, which is computed by the Riemann-Roch formula.

The same argument applies for the space of stable Higgs bundles, where the Euler characteristic of $\mathrm{End}_\bullet(\mathcal{E}, \phi)$ is

$$2(n^2(g-1))$$

and we know that H^0 is one dimensional as stable bundles have only central automorphisms, so also H^2 is one dimensional, giving the missing $+2$ in the formula. For semistable meromorphic Higgs bundles the corresponding H^2 vanishes, so here the stack of semistable objects would again be smooth, for the moduli space of stable objects the dimension increases by 1 by removing the 1-dimensional central automorphism group of objects.

The only missing part is then to check that the stacks of (semi)-stable Higgs bundles are non-empty, which follows from the known corresponding statement for (semi)-stable vector bundles.

The latter result can for example be shown by computing the dimension of the closed substack of unstable bundles. □

Generators of the cohomology ring

Smoothness of the moduli spaces allows to identify generators of the cohomology ring $H^*(\mathcal{M}_{Dol})$, following a method introduced by Beauville in the case of vector bundles on curves, that Markman managed to generalize for Higgs bundles.

The basic idea of this approach is beautiful and easy to remember. If X is a smooth proper variety of complex dimension d, its cohomology satisfies Poincaré duality and therefore the cohomology class of the diagonal $\Delta \subseteq X \times X$

$$[\Delta] \in H^{2d}(X \times X) \cong \oplus_{i=0}^{2d} H^i(X) \otimes H^{2d-i}(X)$$

decomposes under the Künneth decomposition as

$$[\Delta] = \sum_j \alpha_j \otimes \check{\alpha}_j$$

where the α_j form a basis of the cohomology that is orthogonal with respect to the Poincaré pairing. In particular, to show that a collection of cohomology classes generates the cohomology ring, it suffices to show that the class of the diagonal can be expressed in terms of these classes.

Remark. If X is a smooth, but not compact variety – e.g., $M^d_{n,Dol}$ – an analogous result holds for the pure part $H^*_{pure}(X) \subseteq H^*(X)$, in the sense that if $X \hookrightarrow \overline{X}$ is a smooth compactification and $\Delta \subset \overline{X} \times X$ is the diagonal, then the Künneth components of $[\Delta] \in H^*(\overline{X}) \otimes H^*(X)$ generate the pure part of $H^*(X)$, because by Deligne's construction of the mixed Hodge structure on $H^*(X)$ the pure part is the image of $H^*(\overline{X}) \to H^*(X)$ and the product with the diagonal cycle in $\overline{X} \times X$ still computes the restriction map $H^*(\overline{X}) \to H^*(X)$.

Note that it this description it is important to keep one of the two factors $\overline{X}$ compact.

Beauville observed that in the case of the moduli space of stable vector bundles, this can be used as follows. Let M^d_n denote the moduli space of vector bundles of rank n and degree d, which are assumed to be coprime for now. In this case we saw that the moduli space is smooth and we can computed the dimension of the moduli stack by using the Riemann Roch formula for the Euler characteristic

$$\chi(\mathrm{End}(\mathcal{E})) = 0 + n^2(1-g).$$

As stable bundles satisfy $H^0(C, \mathrm{End}(\mathcal{E})) = 1$ the moduli space is of dimension $n^2(g-1) - 1$.

As the rank and degree are coprime, there exists a universal bundle $\mathcal{E}_{univ}$ on $\mathcal{M}^d_n \times C$. Moreover, for any two stable bundles $\mathcal{E}_1, \mathcal{E}_2$ we have that

$$H^0(C, \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)) = \begin{cases} k & \text{if } \mathcal{E}_1 \cong \mathcal{E}_2 \\ 0 & \text{if } \mathcal{E}_1 \neq \mathcal{E}_2 \end{cases}$$

Thus we can describe the class of the diagonal in terms of bundles, by applying this computation to the universal family as follows: Consider the projections

$$\begin{tikzcd} M_n^d \times M_n^d \times C \arrow[r, shift left, "pr_{13}, pr_{23}"] \arrow[r, shift right] \arrow[d, "pr_{12}"] & M_n^d \times C\, . \\ M_n^d \times M_n^d & \end{tikzcd}$$

Then the complex

$$\mathbb{R}pr_{12,*}(\mathrm{Hom}(pr_{13}^*\mathcal{E}_{univ}, pr_{23}^*\mathcal{E}_{univ})) = [\mathcal{V}_0 \xrightarrow{d} \mathcal{V}_1]$$

can be described as complex of vector bundles on $M_n^d \times M_n^d$ that fiberwise computes cohomology $H^*(C, \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2))$ and H^0 of this complex is non-zero exactly along the diagonal Δ, i.e. Δ is the first degeneracy locus of the differential d and moreover

$$\dim \mathcal{V}_1 - \dim \mathcal{V}_0 = -\chi(\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) = n^2(g-1) = \dim M_n^d - 1.$$

Thus the degeneracy locus is of the expected codimension and therefore can be expressed by the Thom-Porteous formula in terms of the Chern classes of the complex. By the Riemann-Roch formula the Chern classes of the complex can be expressed in terms of the Künneth components of the Chern classes

$$c_i(\mathcal{E}_{univ}) \in H^{2i}(M_n^d \times C) \cong \oplus_{j=0}^2 H^{2i-j}(M_n^d) \otimes H^j(C).$$

This proves that the Chern classes of the universal bundle generate the cohomology of the moduli space.

Corollary 3 (Atiyah-Bott)**.** [21] *The cohomology ring $H^*(M_n^d)$ of the moduli space of stable vector bundles of coprime rank and degree n, d is generated by the Künneth components of the Chern classes of the universal bundle $\mathcal{E}_{univ}$.*

[21] Michael F. Atiyah and Raoul Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. R. Soc. Lond., Ser. A*, 308:523–615, 1983. ISSN 0080-4614. DOI: 10.1098/rsta.1983.0017

Remark (Cohomology of Bun_n^d)**.** For the stack of bundles Bun_n^d the cohomology ring $H^*(\mathrm{Bun}_n^d)$ turns out to be freely generated by the Chern classes of the universal bundle, but this requires a different argument.

Remark (Alternative generators)**.** Instead of Künneth components of the Chern classes $c_i(\mathcal{E}_{univ}) \in H^*(M_n^d \times C)$ it is often convenient to use the coefficients $\mathrm{ch}_i(\mathcal{E}_{univ})$ of the Chern character $\mathrm{ch}(\mathcal{E}_{univ})$ and replace the Künneth components by using a basis $\{\gamma_j\} \in H^*(C)$ and take the images

$$\mathrm{ch}_i(\gamma) := pr_{M_n^d,*}(pr_C^*\gamma \cup \mathrm{ch}_i(\mathcal{E}_{univ})) = \int_\gamma \mathrm{ch}_i(\mathcal{E}_{univ}).$$

This is for example helpful, because in this normalization the Riemann-Roch formula allows to compute the behavior of the classes under correspondences.

As the difference of the two sets of generators can be expressed as homogeneous expressions in terms of Chern classes of lower degree, they fortunately appear in the same step of the weight filtration.

Aside: Parabolic bundles

The results both for the cohomology of the moduli space and the cohomology of the moduli stack of bundles admit variants for moduli spaces and stacks of parabolic vector bundles. These moduli problems were introduced by Mehta and Seshadri[22]. As these moduli spaces also appear in the proofs of the $P = W$ conjecture, let us briefly indicate this in the case of bundles with full flags. A full (quasi)-parabolic structure $V_\bullet$ on a vector bundle $\mathcal{E}$ on C at a point $p \in C$ is a flag of subspaces

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathcal{E}|_p$$

of the fiber $\mathcal{E}|_p$ of $\mathcal{E}$ at p.

For any finite subset $S \subset C(K)$ let us denote by $\mathrm{Bun}_{n,S}^{para}$ the stack of vector bundles equipped with full parabolic structures at the points $p \in S$, so that the forgetful morphism $\mathrm{Bun}_{n,S}^{para} \to \mathrm{Bun}_n$ is a smooth fiber bundle with fibers isomorphic to products of $m = |S|$ copies of the flag variety $Fl_n = \mathrm{GL}_n / B_n$, because for any family of vector bundles $\mathcal{E}_T$ over $C \times T$, the restrictions $\mathcal{E}_{p,T} := \mathcal{E}_T|_{p \times T}$ of the bundle to a point $p \in C$ is a vector bundle on T and can thus be trivialized Zariski-locally on T. The choice of a trivialization identifies the space of parabolic structures on the bundle with the flag manifold.

The analog of the tautological classes in the cohomology $H^*(\mathrm{Bun}_{n,S}^{para})$ are the Künneth components of the Chern classes $c_i(\mathcal{E}_{univ})$ of the universal bundle together with the Chern classes of the universal line bundles $\mathcal{V}_{i,p}/\mathcal{V}_{i-1,p}$ on $\mathrm{Bun}_{n,S}^{para}$ that are defined by the parabolic structures and which determine the Chern classes of $\mathcal{V}_{i,p}$.

To mimic the above argument for the generation of the cohomology ring of the moduli space of spaces of stable parabolic bundles, one first needs to find an appropriate stability condition that takes the relative position of subbundles and the chosen flags into account. Usually one chooses a numerical stability condition for which semistable parabolic bundles are automatically stable. In this situation one again finds that for bundles of the same rank and degree, the only possible morphisms are isomorphisms. To express this cohomologically we replace sheaves of morphisms by their parabolic counterpart as follows.

Given two n-dimensional vector spaces V, W equipped with full flags $V_\bullet, W_\bullet$, let us denote by $\mathrm{Hom}_{filt}(V_\bullet, W_\bullet) \subseteq \mathrm{Hom}(V, W)$ the linear subspace of maps $f \colon V \to W$ that respect the flag, i.e., that satisfy $f(V_i) \subseteq W_i$ for all i. In standard coordinates these are the maps corresponding to upper triangular matrices, so this space has dimension $\frac{n(n+1)}{2}$.

Given two bundles with parabolic structures $(\mathcal{E}_1, V_{1,\bullet}), (\mathcal{E}_2, V_{2,\bullet})$ the sheaf of homomorphisms $\phi \colon \mathcal{E}_1 \to \mathcal{E}_2$ that respect the parabolic structure, i.e. that induce maps $V_{1,i} \to V_{2,i}$ can be described as kernel

$$\mathcal{H}om_{para}(\mathcal{E}_1, \mathcal{E}_2) = \mathrm{Ker}(\mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2) \to \mathrm{Hom}(\mathcal{E}_1|_p, \mathcal{E}_2|_p)/\mathrm{Hom}_{filt}(\mathcal{E}_1|_p, \mathcal{E}_2|_p).$$

[22] V. B. Mehta and C. S. Seshadri. Moduli of vector bundles on curves with parabolic structures. *Math. Ann.*, 248:205–239, 1980. DOI: 10.1007/BF01420526

The Euler characteristic of this sheaf is

$$\chi(\mathcal{H}\mathrm{om}_{para}(\mathcal{E}_1,\mathcal{E}_2)) = \chi(\mathcal{H}\mathrm{om}(\mathcal{E}_1,\mathcal{E}_2)) - |S|\cdot\frac{n(n-1)}{2} = n^2(1-g) - |S|\cdot\frac{n(n-1)}{2}$$

which again is minus the dimension of the moduli stack of parabolic bundles and the difference to the moduli space is 1, so the above argument applies, noting that here the characteristic classes of $\mathbb{R}\pi_*(\mathrm{Hom}_{para}(pr_{1,3}^*\mathcal{E}_{univ}, pr_{23}^*\mathcal{E}_{univ}))$ can be expressed in terms of the above mentioned generators by the Riemann-Roch formula.

The idea of Markman's argument

Eyal Markman found a beautiful way to adapt Beauville's argument for moduli spaces of vector bundles on curves to moduli spaces of Higgs bundles and more generally to moduli spaces of sheaves on symplectic or Poisson surfaces[23,24].

To see how this works, let us consider the naive analog of the above argument, i.e. let us consider the universal Higgs bundles $(\mathcal{E}_{univ}, \phi_{univ})$ on $\mathrm{Higgs}_n \times C$, the projection map

$$\begin{array}{c} \mathrm{Higgs}_n^d \times \mathrm{Higgs}_n^d \times C\ . \\ \Big\downarrow \pi \\ \mathrm{Higgs}_n^d \times \mathrm{Higgs}_n^d \end{array}$$

and compute the complex

$$\mathbb{R}\pi_*(\mathrm{Hom}_{Higgs}(pr_{1,3}^*\mathcal{E}_{univ}, pr_{23}^*\mathcal{E}_{univ})).$$

As non-trivial homomorphisms of stable Higgs bundles of the same rank and degree will only appear on the diagonal, the diagonal should again be related to a degeneracy locus.

THIS APPROACH faces several obstructions:

1. The space of stable Higgs bundles is not proper.
2. The complex computing homomorphisms of Higgs bundles is a two-term complex

   $$[\mathrm{Hom}(pr_{13}^*\mathcal{E}_{univ}, pr_{23}^*\mathcal{E}_{univ}) \xrightarrow{[\,,\phi]} \mathrm{Hom}(pr_{13}^*\mathcal{E}_{univ}, pr_{23}^*\mathcal{E}_{univ}) \otimes \Omega_C]$$

   so applying $\mathbb{R}\pi_*$ will end up giving a 3-term complex

   $$\mathcal{V}_0 \xrightarrow{d_0} \mathcal{V}_1 \xrightarrow{d_1} \mathcal{V}_2$$

   on $\mathrm{Higgs}_n^d \times \mathrm{Higgs}_n^d$.

Markman managed to resolve these obstructions. For the second problem, he uses that – as we saw above – the 3-term complex is self-dual and that both d_0, d_1 drop in rank exactly over the diagonal (see Lemma 4 [25], this problem disappears for meromorphic bundles). This allows to save the argument for the Thom-Porteous formula, even though there are 3 bundles involved.

The problem that the moduli space is not proper is addressed using two different ingredients:

[23] Eyal Markman. Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces. *J. Reine Angew. Math.*, 544:61–82, 2002. DOI: 10.1515/crll.2002.028

[24] Eyal Markman. Integral generators for the cohomology ring of moduli spaces of sheaves over Poisson surfaces. *Adv. Math.*, 208(2):622–646, 2007. DOI: 10.1016/j.aim.2006.03.006

[25] Eyal Markman. Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces. *J. Reine Angew. Math.*, 544:61–82, 2002. DOI: 10.1515/crll.2002.028

1. The cohomology of $H^*(\mathcal{M}_{Dol})$ carries a pure Hodge structure and thus for any compactification $\mathcal{M}_{Dol} \subset \overline{\mathcal{M}}_{Dol}$ the restriction map on cohomology is surjective. We will see an argument for this purity in the next lecture.

2. Using a modular compactification $\mathcal{M}_{Dol} \subset \overline{\mathcal{M}}_{Dol}$ (interpreting Higgs bundles as sheaves on the projective completion of the cotangent bundle of C) the argument expressing the diagonal class in terms of Chern classes works over $\overline{\mathcal{M}}_{Dol} \times \mathcal{M}_{Dol}$.

Remark. 1. The argument works better for a version of stable Higgs bundles where Ω_C is replaced by $\Omega_C(D)$ for some effective divisor D on C. In this situation, the corresponding complex

$$[\mathrm{Hom}(pr_{13}^*\mathcal{E}_{univ}, pr_{23}^*\mathcal{E}_{univ}) \xrightarrow{[\_,\phi]} \mathrm{Hom}_{Higgs}(pr_{13}^*\mathcal{E}_{univ}, pr_{23}^*\mathcal{E}_{univ}) \otimes \Omega_C(D)]$$

is no longer self-dual, but in this case Serre duality shows that for stable Higgs bundles H^2 vanishes, so the above complex is again a 2-term complex and we can then apply the original approach of Beauville.

2. We will see the modular compactification of $\mathcal{M}_{Dol}$ used by Markman in the next lecture.

Theorem 4 (Markman). *The cohomology rings of the moduli spaces of semistable Higgs bundles $M^d_{n,Dol}$ and meromorphic Higgs bundles $M^{d,D}_{n,Dol}$ of coprime rank n and degree d are generated by the Künneth components of the Chern classes of the universal bundle.*

Remark. As Chern classes always define Hodge-Tate classes and the multiplicative generators are obtained from a Künneth decomposition where one factor is given by the cohomology of the base curve C, Markman's result on the generation of the cohomology ring by universal classes gives a more refined result on the Hodge structure of $H^*(\mathcal{M}_{Dol})$: The cohomology is built out of the Hodge structure of C and Hodge-Tate classes.

Remark. The argument can also be adapted to parabolic Higgs bundles, using a suitable version of the modular compactification that we will see in the next lecture. This has been worked out recently by Lee and Lee[26].

[26] Jia Choon Lee and Sukjoo Lee. Generators for the cohomology of the moduli space of irregular parabolic higgs bundles, 2024. URL `https://arxiv.org/abs/2402.05380`

Shende's argument for the weights of the generators

The weight filtration on the cohomology of a variety is multiplicative with respect to the cup product. Thus a natural first step in understanding the $P = W$ is to figure out in which step of the weight filtration the generators of the cohomology appear. This is now a result of Vivek Shende [27], which shows that in contrast to the weights in the cohomology of the moduli space of Higgs bundles, the weights of the Künneth components do not drop.

[27] Vivek Shende. The weights of the tautological classes of character varieties. *Int. Math. Res. Not.*, 2017(22):6832–6840, 2017. DOI: 10.1093/imrn/rnv363

Theorem 5 (Shende)**.** *Under the non-abelian Hodge isomorphism*

$$H^*(M^d_{n,Dol}, \mathbb{Q}) \cong H^*(M^d_{n,Betti}, \mathbb{Q})$$

all Künneth components of the i-th Chern class of the universal bundle are mapped to cohomology classes of weight $2i$*.*

Sketch of the argument. The first insight is that the space of representations of the fundamental group of $\pi_1(C)$ is the space of local systems on $C(\mathbb{C})$. If we consider $C(\mathbb{C})$ as a simplicial space $\Sigma_\bullet$ for any simplicial decomposition of $C(\mathbb{C})$, we can identify local systems on $C(\mathbb{C})$ with local systems on the simplicial scheme $\Sigma_\bullet$. This is a simplicial scheme in which all Σ_i are finite disjoint unions of $\mathrm{Spec}(\mathbb{C})$ and consequently all cohomology groups of this space have weight 0. The key point of this perspective is, that the universal local system now defines an algebraic vector bundle on the simplicial scheme $M_{Betti} \times \Sigma_\bullet$, whereas there is no algebraic model of this vector bundle on $M_{Betti} \times C$.

As the i-th Chern class of a bundle can be defined as pull back of a class on the classifying stack $B\,\mathrm{GL}_n$, Chern classes are always pure of weight $2i$. Thus for the simplicial universal bundle on $M_{Betti} \times \Sigma_\bullet$, which defines a morphism $M_{Betti} \times \Sigma_\bullet \to B\,\mathrm{GL}_n$, the weights of the Künneth components of the Chern classes also have weight $2i$.

To implement this idea, one faces several problems:

1. We are interested in moduli spaces of vector bundles for which rank and degree are coprime and these do not correspond to local systems on a curve.

2. We need to relate the Chern classes of the simplicial local system with those of the universal Higgs bundle.

Shende resolves the first problem by replacing the structure group GL_n of vector bundles by PGL_n, as in our discussion of Hitchin's approach to the non-abelian Hodge correspondence after Theorem 1. In this group, the representations with monodromy $e^{\frac{2\pi i}{n}d}$ define PGL_n-local systems and all but the first Chern classes can be obtained as pull-back from the PGL_n moduli space. To obtain the full result, one notes that the morphism of the moduli spaces in question is a quotient by the action given by tensoring with 1 dimensional representations. For rational cohomology the corresponding spectral sequence degenerates because the Künneth components of the first Chern class restrict to generators of the cohomology of the fibers and one can thus apply the Theorem of Leray-Hirsch.

As for the non-abelian Hodge theorem one can alternatively replace the curve C by a root stack, which also admits a simplicial realization. This allows to avoid the passage to PGL_n-representations.

For the second problem this approach is also helpful, as one can first compare the simplicial bundle with its geometric realization to get to the bundle on $M_{Betti} \times C$. The analytic Riemann-Hilbert isomorphism then identifies this with the universal bundle on $M_{DR} \times C$. Finally this is the restriction of the universal bundle on

$M_{Hodge} \times C$ and this restricts to the universal bundle on $\mathcal{M}_{Dol} \times C$. As the restriction induces isomorphisms on the cohomology of $M_{Dol}, M_{Hodge}, M_{DR}$, this finally gives the result. As before one can alternatively first prove the statement for PGL_n-bundles and deduce the statement for vector bundles from this, which is what you find in Shende's article. □

Remark. Note that in the argument showing that the cohomology of $M^d_{n,Dol}$ is generated by the universal classes, we used that the cohomology is pure. The above result shows that purity fails for the Betti moduli space. As the Künneth components of the same weight appear in both odd and even cohomological degree, this shows that there are lots of cancellations in the point counting for the character variety.

This also illustrates why the extra structure on $M^d_{n,Dol}$ is helpful to prove results on the cohomology of $M^d_{n,Betti}$, as we now know that this cohomology is also generated by the tautological classes.

Lecture 3: Geometry of moduli of Higgs bundles

The key geometric structure used to study the geometry of the moduli space $\mathcal{M}_{Dol}$ of Higgs bundles on curves is Hitchin's fibration, which is most easily described on the level of the moduli functor (or moduli stack):

$$\begin{aligned} h\colon \mathrm{Higgs}^d_n &\to \mathcal{A} := \oplus_{i=1}^n H^0(C, \Omega_C^{\otimes i}) \\ (\mathcal{E}, \phi\colon \mathcal{E} \to \mathcal{E} \otimes \Omega_C) &\mapsto \mathrm{charpol}(\phi) = ((-1)^{n-i}\,\mathrm{Tr}(\wedge^i \phi)). \end{aligned}$$

Here we consider the vector space $\oplus_{i=1}^n H^0(C, \Omega_C^{\otimes i})$ as affine space, called the Hitchin base. The morphism then sends a Higgs bundle to the coefficients of the characteristic polynomial of the Higgs field ϕ, which are sections of $\Omega_C^{\otimes i}$.

The BNR-correspondence

How to think about the fibers of Hitchin's fibration? For any point $p \in C$ the Higgs field restricts to an endomorphism of the fiber of E and evaluating the characteristic polynomial $h(E, \phi)$ at p, fixes the characteristic polynomial of the fiberwise endomorphism.

For a single vector space V the equivalence between

1. Endomorphisms $\phi\colon V \to V$
2. $k[t]$-module structures $k[t] \times V \to V$ on V
3. Coherent sheaves $\mathcal{T}$ on $\mathbb{A}^1 = \operatorname{Spec} k[t]$ with $H^0(\mathbb{A}^1, \mathcal{T}) = V$

given by viewing an endomorphism ϕ as defining multiplication by t, i.e., $t \cdot v := \phi(v)$. This 3rd description gives a geometric interpretation of endomorphisms ϕ with fixed characteristic polynomial $p(t)$, because the Cayley-Hamilton theorem identifies these with $k[t]/(p(t))$-modules and these correspond to sheaves supported on $\mathrm{Spec}(k[t]/(p(t))) \subseteq \mathbb{A}^1$, which is a fancy way of identifying the generalized eigenspace for an eigenvalue λ with the part of the sheaf supported at $\lambda \in \mathbb{A}^1$.

This correspondence globalizes. A Higgs bundle $(\mathcal{E}, \phi\colon \mathcal{E} \to \mathcal{E} \otimes \Omega_C)$ can equivalently be described as $(\mathcal{E}, \phi\colon \mathcal{T}_C \otimes \mathcal{E} \to \mathcal{E})$ and this is equivalent to equipping $\mathcal{E}$ with a module structure for the $\mathcal{O}_C$-algebra given by the symmetric algebra $\mathrm{Sym}^\bullet \mathcal{T}_C = \oplus_{i=0}^\infty \mathcal{T}_C^{\otimes i}$. On every open $U \subset C$ on which we can find a trivialization of $\mathcal{T}_C$, the symmetric algebra is isomorphic to a polynomial ring and globally the relative spectrum $\mathrm{Spec}_{\mathcal{O}_C} \mathrm{Sym}^\bullet \mathcal{T}_C = T^*C$ is the total space of the cotangent bundle to C.

Thus we find an equivalence between

1. Higgs bundles $(\mathcal{E}, \phi)$ of rank n on C,
2. $\mathrm{Sym}^\bullet \mathcal{T}_C$-module structures on a rank n vector bundle $\mathcal{E}$ and
3. Coherent sheaves $\mathcal{F}$ on $T^*C \xrightarrow{p} C$ such that $p_*\mathcal{F} = \mathcal{E}$ is a torsion free $\mathcal{O}_C$-module of rank n.

Moreover, the Cayley-Hamilton theorem implies that under this correspondence Higgs bundles with characteristic polynomial $a = h(\mathcal{E}, \phi) \in \mathcal{A}$ correspond to sheaves $\mathcal{F}$ on T^*C supported on

$$C_a := \mathrm{Spec}(\mathrm{Sym}^\bullet T_C/(a)) \xrightarrow{p_a} C$$

where $C_a \subset T^*C$ is a one dimensional scheme that is finite of degree n over C. This one-dimensional scheme C_a is called *spectral curve*. Thus Higgs bundles of rank n with fixed characteristic polynomial a correspond to coherent sheaves $\mathcal{F}$ on the spectral curve C_a such that $\mathcal{F}$ is $\mathcal{O}_C$-torsion free of rank n. As the structure sheaf $\mathcal{O}_{C_a}$ of C_a is by construction a locally free $\mathcal{O}_C$ module of rank n, the condition for $\mathcal{F}$ to define a rank n vector bundle on C can equivalently be expressed by asking that $\mathcal{F}$ is torsion free of rank 1 on C_a – here one has to be careful to define the condition of being of rank 1 on the possibly non reduced and reducible curve C_a to mean that at each generic point η_i of C_a the length of $\mathcal{F}$ coincides with the length of the structure sheaf $\mathcal{O}_{C_a,\eta_i}$.

In case C_a is reduced this simply means that $\mathcal{F}$ is generically locallly free of rank 1 on all components of C_a, but if C_a is non-reduced this condition is more subtle. E.g., for $a = 0$ the spectral curve $C_0 \subseteq T^*C$ is the n-th infinitesimal neighborhood of the 0 section of the cotangent bundle and in this case rank n vector bundles supported on the reduced 0-section and line bundles on C_0 are the two extreme possibilities for rank 1 sheaves on C_0.

Let us denote by Coh_{1,C_a} the stack of coherent, torsion free sheaves of rank 1 on C_a. Then the above interpretation of Higgs fields of fixed characteristic polynomial implies the BNR-correspondence:

Theorem 6 (Beauville–Narasimhan–Ramanan,Schaub)**.** *For every $a \in \mathcal{A}$ there is an equivalence of categories between Higgs bundles of rank* *n with characteristic polynomial a and $\mathcal{O}_C$-torsion free sheaves of rank 1 on the spectral curve C_a:*

$$h^{-1}(a) = \langle (\mathcal{E},\phi) \in \mathrm{Higgs}_n \,| h(\mathcal{E},\phi) = a \rangle \cong \mathrm{Coh}_{1,C_a}.$$

Remark. Beauville–Narasimhan–Ramanan observed moreover, that under the above equivalence subsheaves $\mathcal{E}' \subset \mathcal{E}$ of a Higgs bundle $(\mathcal{E},\phi)$ that are preserved under ϕ, i.e. $\phi(\mathcal{E}') \subseteq \mathcal{E}' \otimes \Omega_C$, correspond to $\mathcal{O}_{C_a}$-subsheaves $\mathcal{F}' \subseteq \mathcal{F}$. In particular if C_a is integral, the stability condition for the Higgs bundle $(\mathcal{E},\phi)$ is automatically satisfied, because on integral curves torsion free sheaves of rank 1 do not have saturated proper subsheaves. Thus the fibers of h over these points of $\mathcal{A}$ are compactified Jacobians.

In general, stability of Higgs bundles is defined in terms of Higgs subbundles $(\mathcal{E}',\phi) \subset (\mathcal{E},\phi)$ and these are determined by the subspace $\mathcal{E}'_\eta \subset \mathcal{E}_\eta$ of the generic fiber of $\mathcal{E}$. Under the BNR-correspondence these correspond to $\mathcal{O}_{C_a}$-subsheaves $\mathcal{F}' \subset \mathcal{F}$ defined by choosing $\mathcal{O}_{C_a,\eta_i}$ submodules at all generic points η_i of C_a. Again one needs to be careful, that in case the local ring at η_i is not reduced – e.g. if $\phi = 0$ – there may be many such submodules.

This problem was overlooked in the formulation of Theorem 3.1 of the original article by Schaub [28], but later clarified by Chaudouard-Laumon [29] (see Remarque 4.2 of their article).

[28] Daniel Schaub. Spectral curves and compactification of Jacobians. *Math. Z.*, 227(2):295–312, 1998. DOI: 10.1007/PL00004377

[29] Pierre-Henri Chaudouard and Gérard Laumon. A support theorem for the Hitchin fibration. *Ann. Inst. Fourier*, 66 (2):711–727, 2016. DOI: 10.5802/aif.3023

Example. For a simple example, take $C = \mathbb{P}^1$ and replace the canonical bundle Ω_C by $\mathcal{O}_{\mathbb{P}^1}(2)$ in the definition of Higgs bundles, i.e. consider the moduli problem of pairs $(\mathcal{E},\phi\colon \mathcal{E} \to \mathcal{E} \otimes \mathcal{O}(2))$, in order to have a positive dimensional base.

For bundles of rank 2 bundles the possible characteristic polynomials are parametrized by the affine space $\mathcal{A} = H^0(\mathbb{P}^1,\mathcal{O}(2)) \times H^0(\mathbb{P}^1,\mathcal{O}(4))$ and the reducible spectral curves are defined by the characteristic polynomials that lie in the image of

$$\begin{aligned} H^0(\mathbb{P}^1,\mathcal{O}(2)) \times H^0(\mathbb{P}^1,\mathcal{O}(2)) &\to \mathcal{A} \\ (a,b) &\mapsto (a-t)(b-t) = t^2 - (a+b)t + ab. \end{aligned}$$

Thus the general spectral curve is an elliptic curve, a 2-sheeted, ramified cover of $\mathbb{P}^1$, ramified at 4 points. The reducible spectral curves correspond to the unions of 2 sections of $\mathcal{O}(2)$ and the non-reduced spectral curves are the infinitesimal neighborhoods of sections of $\mathcal{O}(2)$.

The picture visible in this example that:

1. The general spectral curve is smooth.

2. All spectral curves are connected.

3. For integral spectral curves, the fiber of $h^{-1}(a)$ is proper and irreducible that contains an open subset isomorphic to the jacobian of the spectral curve.

4. The fibers over reducible or non-reduced curves are not of finite type.

The last problem, that fibers over non-integral curves are not of finite type, disappears after imposing a semistability condition, as semistable Higgs bundles are known to form a bounded family.

Remark. All of the above properties generalize to Higgs bundles on curves C of genus $g(C) > 1$ and the same is true if we replace Ω_C by $\Omega_C(D)$ for any positive divisor D. This positivity condition on the line bundle $\Omega_C(D)$ is needed only to ensure that a Bertini argument implies connectedness and generic smoothness.

The BNR-correspondence reveals an additional structure of the moduli space of Higgs bundles: Tensoring with line bundles on spectral curves defines an action of the Jacobian of spectral curves on the fibers of Hitchin's morphism:

$$\begin{aligned} \otimes\colon \mathrm{Pic}^0_{C_a} \times \mathrm{Coh}_{1,C_a} &\to \mathrm{Coh}_{1,C_a} \\ (\mathcal{L}, \mathcal{F}) &\mapsto \mathcal{F} \otimes \mathcal{L}. \end{aligned}$$

Here we denoted by $\mathrm{Pic}^0_{C_a}$ the connected component of the Picard scheme of a spectral curve parametrizing line bundles that restrict to bundles of degree 0 on all irreducible components of C_a.

This operation does not change the degree of $\mathcal{F}$ and also preserves subsheaves and therefore it preserves stability and semistability of Higgs bundles. This implies that the action induces an action on the corresponding moduli spaces of (semi)-stable Higgs bundles.

The Picard schemes $\mathrm{Pic}^0_{C_a}$ fit into a family $\mathcal{P}_\mathcal{A} \to \mathcal{A}$, for example because the degree 0 condition defines an open substack $\underline{\mathcal{P}_\mathcal{A}} := \underline{\mathrm{Pic}}^0_{C_\mathcal{A}/\mathcal{A}}$ of the relative moduli stack of locally free sheaves of rank 1 on $C_\mathcal{A}$ over $\mathcal{A}$ and $\mathcal{P}_\mathcal{A}$ is the rigidification of this stack with respect to the constant automorphism group $\mathbb{C}^*$.

Thus we obtain actions:

$$\begin{aligned} \mathrm{act}\colon \underline{\mathcal{P}}_\mathcal{A} \times_\mathcal{A} \mathrm{Higgs}_n &\to \mathrm{Higgs}_n \\ (\mathcal{L}, \mathcal{F}) &\mapsto \mathcal{F} \otimes \mathcal{L}. \end{aligned}$$

and

$$\begin{aligned} \mathrm{act}\colon \mathcal{P}_\mathcal{A} \times_\mathcal{A} M_{Dol} &\to M_{Dol} \\ (\mathcal{L}, \mathcal{F}) &\mapsto \mathcal{F} \otimes \mathcal{L}. \end{aligned}$$

It turns out that the compactness of the fibers of h over integral curves, where we found a description as compactified jacobians, is a general feature of Hitchin's morphism. This was proved in different levels of generality By Hitchin [30], Faltings [31] and Nitsure [32].

[30] Nigel Hitchin. Stable bundles and integrable systems. *Duke Math. J.*, 54: 91–114, 1987b. DOI: 10.1215/S0012-7094-87-05408-1

[31] Gerd Faltings. Stable G-bundles and projective connections. *J. Algebr. Geom.*, 2(3):507–568, 1993

[32] Nitin Nitsure. Moduli space of semistable pairs on a curve. *Proc. Lond. Math. Soc. (3)*, 62(2):275–300, 1991. DOI: 10.1112/plms/s3-62.2.275

Theorem 7 (Hitchin, Faltings, Nitsure)**.** *The morphisms $h\colon M^d_{n,Dol} \to \mathcal{A}$ and $h\colon M^{d,D}_{n,Dol} \to \mathcal{A}_D$ are projective, flat and the group scheme $\mathcal{P}_\mathcal{A}$ defines a relative action over $\mathcal{A}$ with affine stabilizers.*

This result on properness of Hitchin's fibration allows to deduce that for coprime rank and degree, the cohomology of moduli of

Higgs bundles is pure, although the space being flat over an affine space is not proper.

Corollary 8 (Purity of $H^*(M^d_{n,Dol})$). *For coprime integers (n,d) the cohomology $H^*(M^d_{n,Dol})$ and $H^*(M^{d,D}_{n,Dol})$ of the moduli spaces of (meromorphic) Higgs bundles carries a pure Hodge structure.*

Proof. We will prove the result by showing that the cohomology groups appear as both the cohomology of a smooth but not proper variety and of a proper but not smooth variety. Combining Deligne's estimates for the weights for these two varieties we then obtain purity:

As the moduli spaces $M^d_{n,Dol}$ and $M^{d,D}_{n,Dol}$ are smooth (Corollary 2), the weights appearing in H^i are $\geq i$.

Hitchin's morphism is equivariant with respect to the $\mathbb{C}^*$ action on $M^d_{n,Dol}$ induced from scaling the Higgs field $t.(\mathcal{E},\phi) := (\mathcal{E}, t\cdot\phi)$ and the contracting action on $\mathcal{A}$ that acts with weight i on $H^0(\mathcal{A},\Omega^i_C)$.

This implies, that the cohomology of the projective 0-fiber coincides with the cohomology of the $H^*(M^d_{n,Dol}) \cong H^*\big(h^{-1}(0)\big)$, for example because the complex $\mathbb{R}h_*\mathbb{Q}$ is $\mathbb{C}^*$ equivariant, and thus $H^*(\mathcal{A},\mathbb{R}h_*\mathbb{Q})$ coincides with the 0 fiber (see e.g., [33]) of the complex which by proper base change coincides with the cohomology of $h^{-1}(0)$. As $h^{-1}(0)$ is proper, Deligne's result shows that the weights appearing in $H^i(h^{-1}(0))$ are $\leq i$.

This shows that the Hodge structure is pure. □

[33] Tom Braden. Hyperbolic localization of intersection cohomology. *Transform. Groups*, 8(3):209–216, 2003. DOI: 10.1007/s00031-003-0606-4

Remark. There are several natural compactifications of the moduli space of Higgs bundles.

1. The natural action of the multiplicative group $\mathbb{C}^*$ on $\mathcal{M}_{Dol}$ used above induces a contracting action on the base $\mathcal{A}$ of Hitchin's fibration, that is of weight i on $H^0(C,\Omega^i_C)$. Note that the scaling action on the base a defines an action on the family of spectral curves $C_{\mathcal{A}}$ over $\mathcal{A}$, that does not change the isomorphism class of the curve C_a, but scales the natural embedding of the spectral curves into T^*C.

   For any $\mathbb{C}^*$-equivariant, proper family $h\colon M \to \mathcal{A}$ the quotient

   $$[M\times\mathbb{C}\smallsetminus(h^{-1}(0)\times\{0\})/\mathbb{C}^*] \to [\mathcal{A}\times\mathbb{C}\smallsetminus(\{0\}\times\{0\})/\mathbb{C}^*] = \mathbb{P}(A\oplus\mathbb{C})$$

   is a proper extension to the weighted projective completion of $\mathcal{A}$.

   In this compcatification the fibers of h being constant along the $\mathbb{C}^*$ orbits, the added limit points along the $\mathbb{C}^*$ orbits for $\lambda\to\infty$ define constant families over $\mathbb{C}^*\cup\{\infty\}\cong\mathbb{C}$.

2. The interpretation of Higgs bundles as sheaves on T^*C allows to compactify $\mathcal{M}_{Dol}$ by embedding the moduli space into a moduli space of sheaves on the projective completion $\mathbb{P}(T^*C\oplus\mathcal{O}_C)$ of T^*C, i.e. a moduli space of sheaves with pure one dimensional support on a projective surface. This is the compactification used by Markman.

Lecture 4: Perverse filtration, the support theorem and Hecke correspondences

Let us finally explain how the perverse filtration comes about. After a short reminder on the motivation, we will give two perspectives on this, the sheaf theoretic one used in the original approach by Alexander Beilinson, Joseph Bernstein, Pierre Deligne and Ofer Gabber and a global one, that was formulated by Mark de Cataldo and Luca Migliorini which can be expressed purely in terms of cup products with Chern classes. The two proofs of the P=W conjecture make use of these perspectives in different ways.

A brief introduction to the decomposition theorem

The notion of perverse sheaves originated in the work of Mark Goresky and Robert MacPherson who wanted to modify the definition of simplicial chain complexes for singular spaces in an intrinsic way so that the failure of Poincaré duality for singular spaces would disappear. This turned out to work very well for algebraic varieties where the resulting cohomology is called intersection cohomology, $IH^\star(X)$. A sheaf-theoretic formulation allowed to extend this to work in a relative situation that not only retains Poincaré duality but also a hard Lefschetz theorem, i.e., for an ample line bundle L on an irreducible d-dimensional projective variety X, the cup product with the Chern class $c_1(L)$ defines isomorphisms

$$\cup c_1(L)^i\colon IH^{d-i}(X,\mathbb{Q}) \xrightarrow{\cong} IH^{d+i}(X,\mathbb{Q}).$$

In particular one can upgrade the filtration on the cohomology groups $IH^*(X)$ defined by the nilpotent operator $\cup c_1(L)$ to an action of the Lie algebra sl_2 such that the cohomological grading agrees – up to a shift by the dimension d of X – with the grading defined by the diagonal element $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{sl}_2$.

Remark. In complex geometry choices of metrics allow to construct the adjoints f of the operator $e = \cup c_1(L)$ lowering the cohomological degree that split the filtration and allow to prove Riemann-Hilbert relations. There are very few situations in which these adjoint operators have been proven to be of geometric origin, but we will encounter one such example in the second proof of the conjecture.

The perverse filtration is a filtration on the cohomology of a smooth variety X that is induced by a projective morphism $f\colon X \to A$, through a relative version of these ideas, which is summarized in the decomposition theorem.

The starting point for the decomposition theorem is Deligne's result that for a smooth projective morphism $f\colon X \to A$ between smooth varieties, not only are the cohomology groups of the fibers $H^i(f^{-1}(a),\mathbb{Q})$ locally constant, but the spectral sequence

$H^i(A, \mathbb{R}^j f_* \mathbb{Q}) \Rightarrow H^{i+j}(X, \mathbb{Q})$ degenerates, because

$$\mathbb{R}f_*\mathbb{Q} \cong \bigoplus_{i=0}^{2d_f} R^i f_* \mathbb{Q}[-i].$$

One way to obtain this result is to prove that there exists a well behaved notion of weights, for which the local systems $R^i f_* \mathbb{Q}$ are pure and to show that this prevents the existence of extensions. An alternative gets available if one manages to prove a hard Lefschetz theorem for the complex $\mathbb{R}f_*\mathbb{Q}$, which also implies the existence of a splitting by homological algebra.

The decomposition theorem generalizes the splitting of the natural filtration of $\mathbb{R}f_*\mathbb{Q}$ to the case where $f\colon X \to Y$ is not smooth, using a filtration on $\mathbb{R}f_*\mathbb{Q}$ for which the subquotients are not necessarily concentrated in a single cohomological degree. This was originally proved using weight structures (Théoreme 6.2.5 in [34]), but Mark de Cataldo and Luca Migliorini later showed[35] that the hard Lefschetz theorem can also be seen as a key structure in the cohomology of smooth projective varieties that can be used to give an alternative proof. Let us first state the result, then unravel the terminology and see how this works in simple examples.

[34] Alexander Beilinson, Joseph Bernstein, Pierre Deligne, and Ofer Gabber. *Faisceaux pervers. Actes du colloque "Analyse et Topologie sur les Espaces Singuliers". Partie I*, volume 100 of *Astérisque*. Paris: Société Mathématique de France (SMF), 2nd edition edition, 2018

[35] Mark Andrea de Cataldo and Luca Migliorini. The Hodge theory of algebraic maps. *Ann. Sci. Éc. Norm. Supér. (4)*, 38(5):693–750, 2005. DOI: 10.1016/j.ansens.2005.07.001

Theorem 9 (Special case of the decomposition theorem)**.** *Let $f\colon X \to A$ be a projective morphism with fibers of dimension $\leq d$ and suppose that the constant complex $\mathbb{Q}[\dim X]$ is self-dual on X (e.g. X smooth), then there exists an isomorphism*

$$\mathbb{R}f_*\mathbb{Q}[\dim X] \cong \bigoplus_{i=-\dim X}^{\dim X} {}^p\mathcal{H}^i(\mathbb{R}f_*\mathbb{Q}[\dim X])[-i],$$

where ${}^p\mathcal{H}^i(\mathbb{R}f_*\mathbb{Q}[\dim X])[-i]$ are the perverse cohomology sheaves associated to a canonical perverse filtration ${}^p\tau_{\leq i}(\mathbb{R}f_*\mathbb{Q}[\dim X])$.

The perverse cohomology sheaves satisfy Poincaré duality and relative hard Lefschetz isomorphisms, i.e., the cup product with the Chern class $c_1(L)$ of a relatively ample bundle on X defines isomorphisms

$$\cup c_1(L)^i \colon {}^p\mathcal{H}^{d-i}(\mathbb{R}f_*\mathbb{Q}[\dim X]) \xrightarrow{\cong} {}^p\mathcal{H}^{d+i}(\mathbb{R}f_*\mathbb{Q}[\dim X]).$$

Moreover, all of the summands ${}^p\mathcal{H}^i(\mathbb{R}f_*\mathbb{Q}[\dim X])$ are semisimple perverse sheaves.

The perverse filtration on cohomology is the filtration on $H^*(X)$ induced by the above ${}^p\tau_{\leq i}\mathbb{R}f_*\mathbb{Q}[\dim X]$ suitably indexed – unfortunately the convention used is not uniform in the papers on the subject.

Before we specify the index-shifts appearing both in the perverse filtration and in the definition of perverse sheaves, let us try to explain where they come from and how one can remember the numbers appearing in these shifts.

Some background on perverse sheaves

As mentioned before, the original aim to introduce intersection cohomology was to rectify the failure of Poincaré duality on singular spaces – I find it helpful to think of these as embedded in a smooth ambient manifold. In terms of sheaf cohomology one could formulate this problem as the attempt to find complexes such that their cohomology satisfies Poincaré duality.

There is the immediate first problem, that Poincaré duality depends on the dimension of the variety in question, i.e. $H^i(X)$ is dual to $H^{2\dim X-i}(X)$. In terms of Verdier-duality, this is reflected in the result that the dualizing complex for a smooth variety is a shifted constant sheaf $\underline{\mathbb{Q}}[2\dim X]$ placed in cohomological degree $-2\dim X$.

Now if $\iota\colon Z \hookrightarrow X$ is a smooth subvariety the cohomology of $\iota_*\underline{\mathbb{Q}}$ computes the cohomology of Z and thus satisfies duality, but the center of symmetry for the cohomology is now $\dim Z$ instead of $\dim X$.

This discrepancy disappears if we shift both sheaves by the dimension of their support, i.e. the cohomology groups of $\underline{\mathbb{Q}}[\dim X]$, the constant sheaf placed in cohomological degree $-\dim X$ are concentrated in cohomological degrees $[-\dim X, \dim X]$ and the cohomology groups of $\iota_*\underline{\mathbb{Q}}[\dim Z]$, are concentrated in cohomological degrees $[-\dim Z, \dim Z]$. Poincaré duality now identifies H^{-i} with the dual of H^i for both sheaves.

THE UPSHOT IS that self-dual complexes in the derived category of sheaves on X want to live in different cohomological degrees, depending on the dimension of their support.

MAGICALLY, THIS SIMPLE RECIPE indeed defines an abelian subcategory $\mathrm{Perv}(X) \subset D^b_c(X)$ of perverse sheaves

$$\mathrm{Perv}(X) = \left\{ K \in D^b(X) \mid \begin{array}{ll} \dim\operatorname{supp}\mathcal{H}^{-i}(\,K\,) \leq i & \text{and} \\ \dim\operatorname{supp}\mathcal{H}^{-i}(\mathbb{D}K) \leq i & \end{array} \right\}.$$

Note that in particular our two examples of the shifted constant sheaves $\underline{\mathbb{Q}}[\dim X]$ and $\iota_*\underline{\mathbb{Q}}[\dim Z]$ supported on smooth varieties, are both elements of $\mathrm{Perv}(X)$.

LET US UNRAVEL the numbers in the definition a bit more: As dimensions of varieties are non-negative, the first condition says that perverse complexes $K \in \mathrm{Perv}(X)$ can only have non-trivial cohomology sheaves in degree ≤ 0 and for $i = 0$ the condition appearing above says that the cohomology sheaves of K in degree 0 have to be skyscraper sheaves supported at points. Duality ensures that skyscraper sheaves in degree 0 are indeed allowed.

Similarly duality ensures that the complexes cannot have cohomology sheaves in degree $< -\dim X$, i.e., cohomology sheaves of a perverse complex on X are concentrated in dimensions $[-\dim X, 0]$.

THE ABELIAN CATEGORY $\mathrm{Perv}(X)$ shares many categorical properties of the category of constructible sheaves in the sense that it is the heart of a t-structure with

$$^{p}D_c(X)^{\leq 0} := \{K \in D^b(X) \mid \dim \operatorname{supp} \mathcal{H}^{-i}(K) \leq i\}$$

and $^{p}D_c(X)^{\geq 0}$ being defined by the same condition for the dual $\mathbb{D}K$. This structure entails, that for all translates $^{p}D_c(X)^{\leq j}$ of $^{p}D_c(X)^{\leq 0}$ every complex K gets equipped with natural truncations $^{p}\tau^{\leq j}K \in {}^{p}D_c(X)^{\leq j}$ and perverse cohomology sheaves $^{p}\mathcal{H}^i(K) \in \operatorname{Perv}(X)$ corresponding to the associated graded pieces of these truncations.

Even more surprisingly, these properties are deduced form general categorical principles using the available properties of $D^b(X)$ and the basic properties of various derived functors attached to open or closed embeddings. This is one reason why the first part of the book of Beilinson, Bernstein, Deligne and Gabber [36] is of categorical rather than geometric nature.

[36] Alexander Beilinson, Joseph Bernstein, Pierre Deligne, and Ofer Gabber. *Faisceaux pervers. Actes du colloque "Analyse et Topologie sur les Espaces Singuliers". Partie I*, volume 100 of *Astérisque*. Paris: Société Mathématique de France (SMF), 2nd edition edition, 2018

The categorical part of the theory also includes a description of the simple objects in $\operatorname{Perv}(X)$. Recall that our basic building blocks motivating the definition of our category were shifted constant sheaves on smooth subvarieties. It is natural to enlarge these to local systems, because duality works just as well for these. Let thus $\iota\colon Z \hookrightarrow X$ be a closed subvariety, L an irreducible local system on a smooth, dense open $j\colon U_Z \subseteq Z$. Then there is a unique simple object $i_* j_{!*} L[\dim Z] \in \operatorname{Perv}(X)$, called the middle extension, which is sometimes also called intersection complex $\operatorname{IC}(L)$ of L. It is uniquely characterized by the conditions that it is supported on Z, its restriction to U_Z is $L[\dim Z]$ and furthermore satisfies strict inequalities in the support condition for perverse sheaves in all degrees $> -\dim Z$.

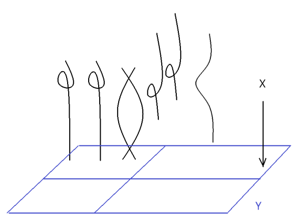
Note that requiring strict inequalities in the support condition for supports smaller than Z is the simplest condition that prohibits the addition of perverse sheaves supported on closed subvarieties of Z. That this simple condition already guarantees that there is a unique perverse extension of $L[\dim Z]$ is another categorical miracle.

In practice, middle extensions are rarely computed directly – a famous exception being Beilinson's article[37], but often one uses that a complex obtained from some geometric construction satisfies these numerical characterizations by the support of cohomology groups.

[37] Alexander Beilinson. How to glue perverse sheaves. K-theory, arithmetic and geometry, Semin., Moscow Univ. 1984-86, Lect. Notes Math. 1289, 42-51 (1987)., 1987

As a simple example, consider a versal deformation of a banana-curve, i.e., the nodal curve obtained by gluing two copies of $\mathbb{P}^1$ along the two points 0 and ∞.

This defines a family $X \to Y$ where $Y = \mathbb{A}^2$ is an affine space. The fibers over the two coordinate axes smoothen out one of the two nodes of the banana curve and are thus isomorphic to a nodal $\mathbb{P}^1$, the general fiber is an elliptic curve.

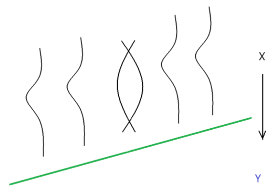


Deformation of the banana curve

The push forward $\mathbb{R}f_*\mathbb{Q}[3]$ is a complex that computes the cohomology of the fibers and thus is concentrated in degrees $[-3,-1]$. Moreover, the relative hard Lefschetz theorem induces an isomorphism

$$^{p}\mathcal{H}^{-1}(\mathbb{R}f_*\mathbb{Q}[3]) \cong {}^{p}\mathcal{H}^{1}(\mathbb{R}f_*\mathbb{Q}[3]).$$

Now the first perverse cohomology sheaf of $\mathbb{R}f_*\mathbb{Q}[3]$ over the 2-dimensional base has to be concentrated in cohomological degrees $[-1,1]$, as the whole complex is concentrated in degrees $[-3,-1]$ it thus has to be a sheaf concentrated in degree -1, so ${}^p\mathcal{H}^{-1}(\mathbb{R}f_*\mathbb{Q}[3]) \cong {}^p\mathcal{H}^1(\mathbb{R}f_*\mathbb{Q}[3]) \cong \mathbb{Q}$ are constant sheaves on $\mathbb{A}^2$. Therefore outside of the origin ${}^p\mathcal{H}^0(\mathbb{R}f_*\mathbb{Q}[3])$ has to coincide with the first cohomology group of the fibers and at the origin, it has to account for the extra summand in $H^2(f^{-1}(0))$. As this cohomology appears over the origin only and ${}^p\mathcal{H}^0(\mathbb{R}f_*\mathbb{Q}[3])$ is concentrated in degrees $[-2,-1]$, it has to be the middle extension of the local system $H^1(f^{-1}(a))$ on the complement of the coordinate axes.



Restriction to diagonal

If we restrict the family $X \to Y$ to the diagonal $\mathbb{A}^1 \subset \mathbb{A}^2$ to get $f_\Delta \colon \mathcal{C} \to \mathbb{A}^1$, the new family still has a smooth total space $\mathcal{C}$, so that the decomposition theorem still applies, but now the extra summand in $H^2(f^{-1}(0))$ appears in cohomological degree 0 of $\mathbb{R}f_{\Delta,*}\mathbb{Q}[2]$, so it becomes a perverse skyscraper sheaf occurring as direct summand of ${}^p\mathcal{H}^0(\mathbb{R}f_{\Delta,*}\mathbb{Q}[2])$.

These examples show that together with basic cohomological information coming from the dimension of the fibers, hard Lefschetz symmetries put strong restrictions on the possible perverse summands appearing in the decomposition theorem for projective morphisms. We will see a bit later, that Ngô found that the additional symmetries of Hitchin's fibrations coming from the action of the relative Picard group put even stronger restrictions on the summands. These will allow to show that for meromorphic Higgs bundles all summands appearing in the decomposition theorem applied to the meromorphic version of Hitchin's fibration arise as middle extensions from the locus of smooth spectral curves.

Back to the perverse filtration on cohomology (sheaf theoretic version)

To define the perverse filtration, let as before X, A be smooth varieties and $f\colon X \to A$ a proper morphism. Then

$$\mathbb{R}f_*\underline{\mathbb{Q}}[\dim X] \cong \bigoplus_{i=-m}^{m} {}^p\mathcal{H}^i(\mathbb{R}f_*\underline{\mathbb{Q}}[\dim X])[-i]$$

for some m, which turns out to be $\dim X \times_A \dim X - \dim X$, which in the case of an equidimensional morphism is equal to the relative dimension.

Thus the cohomology of ${}^p\tau_{\leq j}\mathbb{R}f_*\underline{\mathbb{Q}}[\dim X]$ injects into $H^*(X, \mathbb{Q}[\dim X])$ and the corresponding filtration

$${}^pP_jH^*(X, \mathbb{Q}[\dim(X)]) \subseteq H^*(X, \mathbb{Q}[\dim(X)])$$

is the perverse filtration.

This numbering has the advantage, that it agrees with the weight filtration induced by the relative Lefschetz operator on $Rf_*\underline{\mathbb{Q}}[\dim X]$, which is symmetric around 0 and thus used in the second proof [38].

[38] Tamás Hausel, Anton Mellit, Alexandre Minets, and Olivier Schiffmann. $P = W$ via H_2, 2022. URL https://arxiv.org/abs/2209.05429

Usually the filtration is normalized to induce a filtration of $H^*(X,\mathbb{Q})$ in non-negative degrees $P_0 \subsetneq P_1 \subseteq \cdots \subseteq H^*(X)$, i.e.

$$P_i H^*(X) := \operatorname{im}\left(H^*(A, {}^p\tau_{\leq i+\dim X - m} Rf_*\mathbb{Q}) \to H^*(X)\right).$$

This numbering has the advantage, that the P=W conjecture reads

$$P_i H^*(M^d_{n,Dol}) = W_{2i} H^*(M^d_{n,Dol}).$$

As CONSISTENCY CHECK let us look at case of line bundles again. There we saw, that all multiplicative generators of $H^*(\mathbb{C}^{*2g},\mathbb{Q})$ in degree 1 have weight 2, so that in this case the weight filtration is a renumbering of the cohomological degree. On the other hand Hitchin's morphism in this case is the projection $h\colon T^*\operatorname{Pic}_C \cong \mathcal{A} \times \operatorname{Pic}_C \to \mathcal{A}$, so all direct images $R^i h_*\mathbb{Q}$ are constant sheaves $\bigwedge^i H^1(C,\mathbb{Q})$ on $\mathcal{A}$ which are the perverse summands of $\mathbb{R}h_*\mathbb{Q}$ and thus in this case the perverse filtration coincides with the filtration by cohomological degree.

The perverse filtration on cohomology (cohomological version)

As mentioned above, in the approach of de Cataldo and Migliorini, the decomposition theorem is deduced from a hard Lefschetz theorem. This allows to deduce the following concrete description of the perverse filtration on the cohomology of X with respect to f in the case of a projective base $Y = \overline{A}$.

Theorem 10 (Perverse filtration on cohomology). [39] *Let $f\colon X \to Y$ be a morphism of projective varieties with being X smooth, L an ample class on Y and η a relatively ample class on X. Then the perverse filtration $P_i H^*(X)$ is the filtration induced by the nilpotent operator $\cup f^*L$ on $H^*(X)$, i.e.,*[40]

$$P_i H^r(X) = \sum_{a+b=i+\dim f(X)-r} \operatorname{Ker}((f^*L^{a+1})) \cap \operatorname{im}(f^*(L)^{-b})$$

and the cup product with η satisfies the Hard Lefschetz theorem on the associated graded pieces of the filtration.

For singular X the analog of the result holds when the constant sheaf $\mathbb{Q}[n]$ is replaced by the intersection cohomology complex IC_X.

Remark. The case we are interested in is Hitchin's morphism $h\colon M_{Dol} \to \mathcal{A}$ which is a projective morphism, but the base of the fibration is not compact.

However, we saw already that the $\mathbb{C}^*$ action on $\mathcal{M}_{Dol}$ allows us to define a natural compactification as smooth Deligne-Mumford stacks. For these the intersection cohomology complex of the coarse moduli space is again the shifted constant sheaf $\mathbb{Q}[n]$, so the decomposition theorem again applies as in the case of smooth varieties. And there is a simple relation between the cohomology of $\mathcal{M}_{Dol}$, of its $\mathbb{C}^*$-equivariant cohomology and the cohomology of the compactification. This is used in the article of Hausel, Mellit, Minets and Schiffmann[41]

[39] Mark Andrea de Cataldo and Luca Migliorini. The Hodge theory of algebraic maps. *Ann. Sci. Éc. Norm. Supér. (4)*, 38(5):693–750, 2005. DOI: 10.1016/j.ansens.2005.07.001

[40] We chose the shift in the weights that makes $H^0(X)$ appear in $P_0H^0(X)$, which is convenient for the P=W conjecture.

To unravel the statement, it is instructive to think about the case when f is a smooth morphism, as then $P_i H^*(X) = \oplus_{j\leq i} H^*(Y, R^j f_*\mathbb{C})$ and the sheaves $R^j f_*\mathbb{C}$ are locally constant.

[41] Tamás Hausel, Anton Mellit, Alexandre Minets, and Olivier Schiffmann. $P = W$ via H_2, 2022. URL https://arxiv.org/abs/2209.05429

to reduce the computation of the perverse filtration to the problem over a projective base.

For the final argument, the fact that the base $\mathcal{A}$ of Hitchin's fibration is contractible and thus the geometry of the base does not contribute to the global cohomology groups is essential in both arguments. Versions of these results for fibrations over non contractible spaces appear in the recent article on $P = C$ phenomena[42].

[42] Yakov Kononov, Weite Pi, and Junliang Shen. Perverse filtrations, Chern filtrations, and refined BPS invariants for local $\mathbb{P}^2$. *Adv. Math.*, 433:29, 2023. DOI: 10.1016/j.aim.2023.109294

Methods to analyze supports in the decomposition theorem

The question is of course, how we can get our hands on the abstractly defined perverse filtration.

There is an easy geometric condition for a proper map $f \colon X \to A$ from a smooth n-dimensional variety X ensuring that $\mathbb{R}f_*\mathbb{Q}$ is a perverse sheaf, called semi-smallness. This arises from the observation that proper base change implies that the fiber of $\mathbb{R}f_*\mathbb{Q}$ at a point a is $H^*(f^{-1}(a), \mathbb{Q})$, so that if the fiber has dimension $\leq i$ the cohomology is concentrated in degrees $[0, 2i]$ and moreover $\mathbb{D}Rf_*\mathbb{Q}[n] = \mathbb{R}f_!\mathbb{D}\mathbb{Q}[n] = \mathbb{R}f_*\mathbb{Q}[n]$ is self-dual.

Thus if f is semi-small, i.e. if for all $i \geq 0$ the codimension of

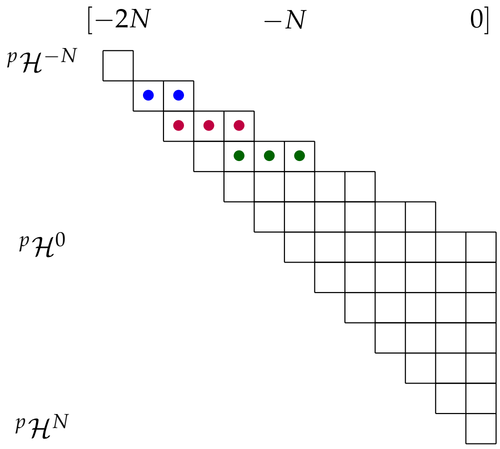
$$A_i := \{a \in A \mid \dim f^{-1}(a) \geq i\} \subseteq A$$

is $\geq 2i$ the complex $\mathbb{R}f_*\mathbb{Q}[n]$ will be a perverse sheaf and if the inequality is strict for all $i > 0$, which in particular implies that f is generically finite, the complex is the middle extension of the sheaf defined by the restriction of $\mathbb{R}f_*\mathbb{Q}$ to the open subset where f is finite. In particular in both of these situations there is only ${}^p\mathcal{H}^0$ appears in the decomposition theorem.

The relative hard Lefschetz theorem give another restriction on the possible summands. For example in the case of Hitchin's fibration the base and fibers of the fibration are of dimension N. A perverse sheaf on A will have cohomology sheaves in cohomological degrees $[-N, 0]$.

The cohomology of the fibers will be concentrated in degree $[0, 2N]$ so the complex $\mathbb{R}h_*\mathbb{Q}[2N]$ will have cohomology sheaves in cohomological degrees $[-2N, 0]$.

So for $i = -N, \dots, N$ the complex ${}^p\mathcal{H}^i(\mathbb{R}h_*\mathbb{Q}[2N])[-i]$ can a priori have cohomology sheaves in degree $[-N + i, i]$ but we already know that these are concentrated in degree $[-2N, 0]$, so for example the top cohomology ${}^p\mathcal{H}^N(\mathbb{R}h_*\mathbb{Q}[2N])[-N]$ has to be an honest sheaf placed in cohomological degree 0 and by the relative hard Lefschetz theorem we get that at the other end ${}^p\mathcal{H}^{-N}(\mathbb{R}h_*\mathbb{Q}[2N])[N]$ is a sheaf placed in cohomological degree $-2N$. For the next complex ${}^p\mathcal{H}^{N-1}(\mathbb{R}h_*\mathbb{Q}[2N])[-N+1]$ we similarly find non trivial cohomology sheaves only in degree $[-1, 0]$ and so forth. I find it helpful to diplay this in a picture where each line represents a perverse cohomology sheaf and we put boxes on the horizontal axis where cohomology sheaves can occur:



The dots in the diagram highlight the degrees in which the Künneth components of $c_1(\mathcal{E}_{univ}), c_2(\mathcal{E}_{univ}), c_3(\mathcal{E}_{univ})$ of the universal bundle $\mathcal{E}_{univ}$ appear according to the P=W conjecture.

As the simple perverse sheaves are middle extensions from local systems supported on subvarieties of dimension k placed in degree $-k$, only the full subvariety and divisors can appear in ${}^p\mathcal{H}^{-N+1}$ in the above picture.

Bao Châu Ngô found a very powerful theorem[43] that restricts possible supports much further in the presence of the action of an abelian scheme on the fiber of the map, as happens in the case of Hitchin's fibration. Namely he showed that in this case the action induced from the cap product with classes from the first cohomology of the abelian variety part of the action also induces a free action of the exterior algebra of H^1 giving rise to a string of cohomology sheaves in different summands which restricts cohomological supports further. For Hitchin's fibration, the jacobians P_a of singular spectral curves C_a are abelian groups schemes, so that they are extensions

$$1 \to P_a^{aff} \to P_a \twoheadrightarrow P_a^{ab} \to 1$$

where P_a^{aff} is an affine subgroup and the quotient P_a^{ab} is projective, i.e. an abelian variety. In the case of spectral curves P_a^{ab} turns out to be the jacobian of the normalization of the reduced curve C_a^{red}.

Ngô's theorem requires these affine pieces to appear in sufficiently high codimension only. To formulate this condition for a family of smooth abelian schemes $P_A \to A$ let us denote the fiber over $a \in A$ by P_a and by δ_a the dimension of the affine part $P_a^{aff} \subseteq P_a$. A family of smooth abelian schemes $P_A \to A$ is called δ-regular if for all δ

$$\operatorname{codim}_A\{a \in A \mid \delta_a \geq \delta\} \geq \delta.$$

Theorem 11 (Ngô support theorem). [44] *Let $f\colon M \to A$ be a projective morphism that is equidimensional of relative dimension d equipped with an action of a smooth quasi-projective abelian scheme $P \to A$ that is δ-regular and assume that the shifted constant sheaf $\mathbb{Q}[\dim M]$ on M is self-dual.*

If K is a simple perverse sheaf appearing in the decomposition of ${}^p\mathcal{H}^*(Rf_*\mathbb{Q})$ supported on $Z \subseteq A$, then there exists an open $U \subset A$ such that $U \cap Z \neq \varnothing$ and a non-trivial local system L on $U \cap Z$, such

[43] Báo Châu Ngô. The fundamental lemma for Lie algebras. *Publ. Math., Inst. Hautes Étud. Sci.*, 111:1–169, 2010. DOI: 10.1007/s10240-010-0026-7

[44] Báo Châu Ngô. The fundamental lemma for Lie algebras. *Publ. Math., Inst. Hautes Étud. Sci.*, 111:1–169, 2010. DOI: 10.1007/s10240-010-0026-7

that the middle extension $j_{!*}L[\dim A]$ *appears as a summand of the top cohomology sheaf* $R^{2d}f_*\mathbb{Q}$.

Remark. In the original formulation of the result an additional technical hypothesis on the polarizability of the Tate module of P appeared that has now been proven to be automatic for quasi-projective P.[45]

We have seen a simple example of this phenomenon before, when we looked at the versal deformation of the banana curve, where also the top cohomology detected possible perverse summands. This is useful, as the top cohomology sheaf is determined by the irreducible components of the fibers $f^{-1}(a)$ of f, which can often be controlled.

For the case of Hitchin's fibration for meromorphic Higgs bundles Pierre-Henri Chaudouard and Gérard Laumon showed[46,47] that the support theorem implies that all perverse summands appearing have full support, i.e., the complex $Rh_*\mathbb{Q}$ is completely determined by its restriction to the open part $\mathcal{A}^{sm} \subset \mathcal{A}$ parameterizing smooth spectral curves.

Theorem 12 (Chaudouard-Laumon)**.** *Let D be an effective divisor, n, d coprime integers and $h\colon M^{d,D}_{n,Dol} \to \mathcal{A}$ the corresponding Hitchin fibration. Then all perverse sheaves appearing as summands of ${}^p\mathcal{H}^*(Rh_*\mathbb{Q})$ have full support, i.e., they can be obtained as middle extension from their restriction to the open subscheme $\mathcal{A}^{sm}$ parameterizing smooth spectral curves.*

In the article of Davesh Maulik and Junliang Shen [48] this proof is adapted to the case of parabolic meromorphic Higgs bundles. It implicitly also enters into the proof through the application of a theorem of Yun[49] that proves the compatibility of certain cup products with the perverse filtration by restriction to the open part defined by smooth spectral curves, where the computation is then one for a family of abelian varieties.

Similarly, the proof of Hausel–Mellit–Minets–Schiffmann also uses a reduction to a version of the fibration that again has full support, although this property is not used explicitly.

Outlook into the proofs of the P=W conjecture

The proof of the P=W conjecture share some basic ideas. Both use that in order to prove the conjecture, it suffices to show that the generators appear in the predicted parts of the perverse filtration and that multiplication preserves the perverse weight.

These statements no longer make use of $\mathcal{M}_{Betti}$ and thus make sense for meromorphic Higgs bundles as well. As we recalled above, these moduli spaces have the advantage that the summands appearing in the decomoposition theorem are all determined by the restriction of the fibration to the open subset of $\mathcal{A}$ that parameterizes smooth spectral curves. A property that is directly used in the proof by Maulik-Shen who moreover rely on a result of Yun that again relies on this result in order to reduce one key argument to the case

[45] Giuseppe Ancona and Dragos Fratila. Ngô support theorem and polarizability of quasi-projective commutative group schemes. *Épijournal de Géom. Algébr., EPIGA*, 8:10, 2024. DOI: 10.46298/epiga.2024.12345

[46] Pierre-Henri Chaudouard and Gérard Laumon. A support theorem for the Hitchin fibration. *Ann. Inst. Fourier*, 66 (2):711–727, 2016. DOI: 10.5802/aif.3023

[47] Pierre-Henri Chaudouard and Gérard Laumon. Addendum to: "A support theorem for the Hitchin fibration". *Ann. Inst. Fourier*, 67(3):1005–1008, 2017. DOI: 10.5802/aif.3103

[48] Davesh Maulik and Junliang Shen. The $P = W$ conjecture for GL_n, 2022. URL `https://arxiv.org/abs/2209.02568`

[49] Zhiwei Yun. Langlands duality and global Springer theory. *Compos. Math.*, 148(3):835–867, 2012. DOI: 10.1112/S0010437X11007433

of a fibration by abelian varieties. In the proof of Hausel-Mellit-Minets-Schiffmann this property does not appear explicitly in the argument.

The reduction to meromorphic Higgs bundles is achieved in different ways in the two articles, but it seems to me that for this part of the proofs the reduction steps could be interchanged.

MAULIK AND SHEN rephrase a beautiful argument they had used for SL_n-Higgs bundles[50] to identify the constant sheaf supported on the image of the embedding $M^d_{n,Dol} \hookrightarrow M^{d,p}_{n,Dol}$ of Higgs bundles into meromorphic Higgs bundles as a sheaf of vanishing cycles for a suitable map $M^{d,p}_{n,Dol} \to \mathbb{A}^1$. As vanishing cycles behave nicely with push forwards and are also functorial and thus behave well with respect to morphisms induced from cup products with cohomology classes, this allows them to reduce the $P = W$ conjecture to the statement, that the cup product with the tautological classes for the moduli space of meromorphic Higgs bundles behaves sufficiently well with respect to the perverse filtration.

[50] Davesh Maulik and Junliang Shen. Endoscopic decompositions and the Hausel-Thaddeus conjecture. *Forum Math. Pi*, 9:49, 2021. DOI: 10.1017/fmp.2021.7. Id/No e8

Here another ingredient in their proof is a preprint of Mellit[51] that proves that the curious hard Lefshetz theorem that was conjectured by Hausel and Rodrigues-Villegas indeed holds for the cohomology of the Betti moduli spaces $M^d_{n,Betti}$. As this strong Lefschetz symmetry for both the perverse and the weight filtration already implies that the $P = W$ conjecture already follows if one can prove that one of the filtrations is contained in the other.

[51] Anton Mellit. Cell decompositions of character varieties. Preprint, arXiv:1905.10685 [math.AG], 2019. URL `https://arxiv.org/abs/1905.10685`

As an aside let us mention that in this article many of the arguments move between the moduli spaces of Higgs bundles and the SL_n and PGL_n versions of this space, because some of the results used are only available in the literature for one of these moduli spaces. We already saw a variant of this phenomenon in Shende's argument to compute the weights on the generators of the cohomology of the Betti moduli space. It would be nice to have an exposition of the argument, that is formulated purely in terms of Higgs bundles.

HAUSEL, MELLIT, MINETS AND SCHIFFMANN instead rely on a trick coming from the Springer correspondence that incidentally also appears in the other part of the argument of the proof of Maulik and Shen. This allows them to compare the cohomology of $M^d_{n,Dol}$ with the cohomology of moduli meromorphic, parabolic Higgs bundles.

Hecke correspondences and their Higgs version

A second key ingredient appearing in both proofs of the P=W conjecture, are versions of Hecke correspondences.

For vector bundles these are the correspondences defined by modifications of bundles at a point $p \in C$:

$$\mathrm{Mod}^{d,1}_n := \left\langle (p, \mathcal{E}' \subseteq \mathcal{E}) \;\middle|\; \begin{matrix} p \in C, \mathcal{E} \in \mathrm{Bun}^d_n, \mathcal{E}' \in \mathrm{Bun}^{d-1}_n \\ \mathrm{supp}(\mathcal{E}/\mathcal{E}') = p \end{matrix} \right\rangle$$

which comes equipped with morphisms

$$
\begin{tikzcd}
& \mathrm{Mod}_n^{d,1} \arrow[dl, "{(p,\mathcal{E}'\subset\mathcal{E})\mapsto(p,\mathcal{E}')}"'] \arrow[dr, "{(p,\mathcal{E}'\subset\mathcal{E})\mapsto\mathcal{E}}"] & \\
C \times \mathrm{Bun}_n^{d-1} & & \mathrm{Bun}_n^d
\end{tikzcd}.
$$

These corrrespondences are analogs of the modular description of Hecke operators on moduli of elliptic curves and are the main tool used to formulate the Langlands correspondence. The main advantage of the geometric situation considered here is that the operators occur in a family parameterized by points of the curve C.

Moduli spaces of parabolic bundles are closely related to these spaces, as the datum of a Hecke modification $(p, \mathcal{E}' \subset \mathcal{E})$ is the same as the datum of a 1-dimensional quotient of the fiber of $\mathcal{E}$ at p, whereas a parabolic bundle specifies a flag of quotient spaces.

These geometric correspondences admit versions for Higgs bundles:

$$
\mathrm{Mod}^{d,1}_{\mathrm{Higgs},n} := \left\langle (p, (\mathcal{E}', \phi') \subseteq (\mathcal{E}, \phi)) \;\middle|\; \begin{array}{c} p \in C, \mathcal{E} \in \mathrm{Higgs}_n^d, \mathcal{E}' \in \mathrm{Higgs}_n^{d-1} \\ \mathrm{supp}(\mathcal{E}/\mathcal{E}') = p \end{array} \right\rangle
$$

that commute with the morphism to the Hitchin base $\mathcal{A}$.

When restricting to the fiber over a point $a \in A$ defining a smooth spectral curve C_a, the BNR-correspondence gives a description of the fiber of $\mathrm{Mod}^d_{\mathrm{Higgs},n}$ as

$$
\begin{tikzcd}
& C_a \times \mathrm{Pic}_{C_a} \arrow[dl, "{(\tilde{p},\mathcal{L})\mapsto(\pi(\tilde{p}),\mathcal{L}(-\tilde{p}))}"'] \arrow[dr, "{(\tilde{p},\mathcal{L})\mapsto\mathcal{L}}"] & \\
C \times h^{-1}(a) & & h^{-1}(a)
\end{tikzcd}
$$

which encodes the action of the points of the spectral curve on Pic_{C_a}.

In the article of Hausel-Mellit-Minets-Schiffmann, this perspective is expanded by considering these correspondences as modifications of sheaves on the surface T^*C, but it would be possible to formulate the proof entirely in the language of Higgs bundles.

As correspondences can be used to induce operations on cohomology by pulling back classes from C and Higgs_n^{d-1} and pushing the cup product forward to Higgs_n^d, they give rise to natural cohomological operators lowering the cohomological degree and thus are a natural place to look for operators completing the Lefschetz operators on $Rh_*\mathbb{Q}$ to an sl_2-triple.

THERE ARE IMMEDIATE OBSTRUCTIONS to implementing this strategy:

1. In general, Hecke operators do not respect semi-stability of Higgs bundles.

2. The correspondences are between moduli spaces of bundles of different degrees. In particular, they will usually not preserve the coprimality assumption needed to obtain smooth moduli spaces.

MAULIK AND SHEN only use a part of the correspondence in the variant of the moduli space of parabolic Higgs bundles in which as in the space of Hecke modifications, the point p at which a full flag is added is allowed to vary, i.e. they consider the space

$$M_{n,C}^{d,D,par} := \left\langle (p,\mathcal{E},\phi,V_\bullet) \mid \begin{array}{c} p \in C, (\mathcal{E},\phi) \in \mathrm{Higgs}_n^{d,D,\mathrm{sst}}, (\mathcal{E},V_\bullet) \in \mathrm{Bun}_{n,p}^{para} \\ \text{s.th. } V_\bullet \text{ is } \phi-\text{stable} \end{array} \right\rangle$$

Which comes equipped with a morphism

$$M_{n,C}^{d,D,par} \to C \times M_n^{d,D}.$$

This space was introduced by Yun to analyze a perverse filtration for a closely related problem. The key observation is that on the space of parabolic bundles the universal bundle $\mathcal{E}$ comes equipped with a full flag and considering the above moduli space in which the point in which the flag is added is allowed to vary this family of flags defines a full flag of subbundles of $\mathcal{E}$, i.e. on this space the Chern classes of the universal bundles are expressed in terms of the Chern classes of line bundles. For these it is easier to show that cup products preserve the perverse filtrations and Maulik and Shen manage to use this argument of Yun to deduce the P=W conjecture.

HAUSEL, MELLIT, MINETS AND SCHIFFMANN instead circumvent the obstructions mentioned above in a series of reduction steps. Namely they observe that the problem that Hecke operators do not respect semi-stability disappears on the part of the fibration where the spectral curve is reduced and irreducible. For meromorphic Higgs bundles, they observe that the argument that contracting $\mathbb{C}^*$ actions allow to identify cohomology of fibers can be applied to the composition

$$M_{n,Dol}^{d,D} \to \oplus_{i=1}^n H^0(C,\Omega_C(D)^i) \xrightarrow{\mathrm{Res}} \mathbb{C}^{n|D|-1}$$

obtained by restricting the Higgs field to points in D. For a general fiber of this map they show that Higgs bundles have to be simple and thus reduce to a situation where semi-stability is automatic.

They then show that a Riemann-Roch computation allows to determine the action of Hecke operators on the tautological classes and then construct an sl_2-triple acting on the cohomology of $M_{n,Dol}^d$ that completes the actions of the cup product with the ample class. Interestingly the formula for this sl_2-triple had independently been found by Polishchuk[52] in the case of families of Jacobians of smooth curves.

The final argument is then formulated entirely in terms of global cohomology, bypassing a sheaf-theoretic argument. It would be interesting to see whether the computations which are made for the Hecke correspondences also work in a relative setting to produce an sl_2 action on $Rh_*\mathbb{Q}$ directly and to deduce directly that the grading given by the diagonal element of sl_2 has to induce the perverse filtration.

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