Ph.D. Seminar: Algebraic K-theory

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1 Introduction

1.1 What is K-theory

K-theory is a family of abelian groups

$$\{K_n(\mathcal{C})\}_{n\in\mathbb{Z}},$$

where $\mathcal{C}$ can be different kind of mathematical objects: a ring with unit, a topological space, a scheme, a stack, a category...

For a ring R, these abelian groups are built out of finitely generated projective R-modules. For a scheme X, they are built out of algebraic vector bundles over X. So in these settings K-theory can be thought as a family of invariants studying these objects.

The K_0 group is defined by describing generators and relations. In the case of rings, there are also explicit definitions of the K_1 and K_2 groups. The definition of higher K-groups uses homotopical methods. The rough idea is that, given the object $\mathcal{C}$, one constructs a topological space $K(\mathcal{C})$, called the K-space, and the the K-groups are defined as its homotopy groups

$$K_n(\mathcal{C}) \coloneqq \pi_n(K(\mathcal{C})).$$

Especially because homotopical methods are involved, K-theory is usually difficult to compute. One of the most famous is Quillen's computation of the K-groups of finite fields. Computational difficulties aside, algebraic K-theory is important for its numerous connections with Number Theory, Geometric Topology and Algebraic Geometry, some of which we will explore in this seminar.

1.2 Some motivation and history

K-theory was first introduced by Grothendieck in the late '50s as part of his studies in Intersection Theory, where it played a central role in the formulation of the Grothendieck-Riemann-Roch Theorem.

The first definition was that of the K-group of a scheme, $K(X)$ (later denoted by $K_0(X)$), with the K standing for "Klasse". Given a scheme X, the K-group of X is the free abelian group generated by the isomorphism classes $[E]$, for E an algebraic vector bundle over X, modulo the relations

$$[E] = [E'] + [E'']$$

for $0 \to E' \to E \to E'' \to 0$ a short exact sequence of algebraic vector bundles over X.

In the affine case $X = Spec(R)$, we get the definition of the K-group of a ring, $K(R)$ (later denoted by $K_0(R)$). In this case, an algebraic vector bundle over X is equivalent to a finitely generated projective R-module and every short exact sequence splits. Thus, equivalently $K(R)$ is the free abelian group generated by the ismorphism classes $[P]$, for P a finitely generated projective R-module, modulo the relations

$$[P \oplus P'] = [P] + [P'].$$

This is the explicit description of a universal construction, called group completion, applied to the abelian monoid of isomorphism classes of finitely generated projective R-modules with the direct sum.

After Grothendieck's definition of the K-group for schemes, Atiyah and Bott applied the same ideas in Algebraic Topology, replacing the scheme X with a topological space and applying group completion to the abelian monoid of (real or complex) vector bundles over X with the direct sum, obtaining the definition of topological K-theory. In the real case it is denoted by $KO(X)$, while in the complex case by $KU(X)$. An important fact in the topological context is that real vector bundles are classified in homotopy-theoretic terms: for X a connected paracompact topological space, there exists a bijection

$$\mathrm{VB}_{\mathbb{R}}(X)^{iso} \longleftrightarrow [X, BO(n)],$$

where the left hand side is the set of isomorphism classes of real vector bundles over X and the right hand side is the set of homotopy classes of maps from X to the infinite real Grassmannian of n-planes. A consequence of this is that we have a group isomorphism

$$KO(X) \cong [X_+, BO]_*,$$

where $X = X \coprod *$, $BO \coloneqq \varinjlim_n BO(n)$ and $[\_\,,\,\_]_*$ denotes the homotopy classes of pointed maps. This suggests the following definitions. For $n \geq 0$, we define the real K-groups of X

$$KO^{-n}(X) \coloneqq KO(\Sigma^n(X_+)) = [\Sigma^n(X_+), BO]_*,$$

where Σ is the reduced suspension endofunctor on the homotopy category of pointed spaces, which is left adjoint to the loop space functor Ω. Given a closed subspace $A \subset X$, we also define the real relative K-groups of the pair (X, A), for $n \geq 0$,

$$KO^{-n}(X, A) \coloneqq KO^{-n}(C_i),$$

where $A_+ \xrightarrow{i_+} X_+ \to C_i$ is a homotopy cofiber. Recall that a homotopy cofiber gives rise to a homotopy cofiber sequence

$$A_+ \xrightarrow{i_+} X_+ \to C_i \to \Sigma A_+ \xrightarrow{\Sigma i_+} \Sigma X_+ \to \Sigma C_i \to \Sigma^2 A_+ \to \dots$$

and applying $[\_ , BO]_*$ we obtain a long exact sequence. Thus, we have a long exact sequence of real K-groups of X

$$\cdots \to KO^{-2}(A) \to KO^{-1}(X, A) \to KO^{-1}(X) \to KO^{-1}(A) \to KO^0(X, A) \to KO^0(X) \to KO^0(A).$$

Now, a crucial result is Bott Periodicity Theorem , which states that there exists a weak homotopy equivalence

$$BO \simeq \Omega^8 BO.$$

Using the adjunction $\Sigma \dashv \Omega$, we obtain that the real K-groups are periodic of period 8. This allows to extend the definition of real K-groups of X and of (X, A) also for positive indexes. It can be proved that the corresponding family of functors $\{KO^n\}_{n \in \mathbb{Z}}$ satisfies all the axioms of a generalized cohomology theory: it is homotopy invariant, the excision property holds and there are long exact sequences associated to a pair (which is the long exact sequence above).

In the case of complex vector bundles similar results hold, replacing $BO(n)$ with the infinite complex Grassmannian of n-planes $BU(n)$. There is also complex version of Bott Periodicity Theorem, which in this case gives period 2.

The existence of higher topological K-groups, whose main property is to fit into long exact sequences, gave rise to the question of whether similar phenomenona occurs for the Grothendieck group of schemes (now referred to as algebraic K-theory, to distinguish it from the topological version).

Initially, some explicit definitions of K_1 and K_2 for a ring R were introduced by Bass and Milnor rispectively, together with a relative version for the datum of an ideal $I \subset R$, which have the property to fit into long exact sequences.

In the late '60 and early '70s several definitions of higher algebraic K-groups were proposed, but the one widely accepted was the one of Quillen. Quillen's homotopical approach of introducing a K-space as explained in the previous paragraph, is motivated by the fact that homotopy theory of topological spaces gives a machinery to obtain long exact sequences: a fiber sequence induces a long exact sequence on the homotopy groups.

In fact, there are several definitions of higher algebraic K-theory by Quillen. The first one, which just defines the higher K-groups of a ring, constructs the K-space using a purely topological costruction, called the $+$-construction. This is a very ad-hoc construction, made with the purpose of following the general homotopical approach, and with the aim of getting back the K_0, K_1 and K_2 of rings previously defined. But already from the first definition of K_0, it was suspected that a definition of higher algebraic K-theory could be possible also for more general kind of objects beyond rings and schemes, such as symmetric monoidal categories and exact categories. This motivated Quillen to look for more general constructions and the main outcome is the K-space associated to an exact category, obtained via the Q-construction. It can be shown that it agrees with the $+$-construction, via the $S^{-1}S$ construction for symmetric monoidal categories. Another defintion of K-space is given by the Waldhausen S-constuction for Waldhausen categories, which are a kind of categories with a notion of cofibrations and weak equivalences. All these constructions are proven to give the same K-groups, when they are all simultaneously defined.

2 Overview of the seminar

The idea for this seminar is to study the main constructions in algebraic K-theory together with some results connecting it to Algebraic Geometry and Number Theory.

We will start with Grothendieck's definition of the K_0 group of rings and schemes. Here we will also see that these are in fact examples of a notion of Grothendieck group of symmetric monoidal categories and exact categories.

Right after this we will explore the relation between the K_0 group and the Chow ring of a smooth algebraic variety over a field k: a consequence of the Grothendieck-Riemann-Roch Theorem allows to prove that they are isomorphic, once we tensor with $\mathbb{Q}$. This will take a couple of talks, since we will start from the definition of the Chow group. Anyway, the idea for this part is not to give many proofs, but rather to focus on the main steps and constructions that allow to reach the final result.

Then, we will proceed with the explicit definitions of Bass' $K_1(R)$ and Milnor's $K_2(R)$, for a ring R. Definitions and proofs of the results for this part only require some Linear Algebra and Group Theory.

In the next four talks we study the various definitions of higher algebraic K-theory: the $+$-construction, the Q-construction and the S-construction. We will also at least sketch the proof that we can recover the higher K-groups of a ring defined with the $+$-construction from the ones defined with the Q-construction. The first in this row of talks is a preliminary one, about the classifying space of a category, which is a constructions that allows to transform a category into a topological space. This is a fundamental notion for all the variuos definitions of the K-space.

Next, we will study some properties of higher algebraic K-theory. These properties regard either the Q-construction or the S-construction.

Another talk is dedicated to present the ideas behind Quillen's computation of the K-groups of finite fields, one of the few computations of K-groups that are known until now.

Then, there are couple of proposals for some topics connecting algebraic K-theory to Number Theory: a result stating the equivalence between the Kummer-Vandiver Conjeture and the vanishing of some K-groups of $\mathbb{Z}$, or the Merkurjev-Suslin Theorem, which relates the K_2 of a field to some étale cohomology groups.

To conclude, we present a more modern perspective on the topic using the language of infinity categories, which allows to view algebraic K-theory as a universal invariant.

3 List of the talks

3.1 Talk 0 (15/10/25): Introduction

(Linda)

I will present the topic of the seminar following the introduction above and briefly explain the content of each talk.

3.2 Talk 1 (22/10/25): The Grothendieck group of rings and schemes

There are two main methods to define the Grothendieck group of some kind of given objects: it is the free abelian group generated by the objects, modulo the relations generated either by a monoidal operation on the objects, or by a notion of short exact sequences of objects.

The first method is the one underlying the definition of the *group completion* $M^{-1}M$ of an abelian monoid M. Give the definition as a universal construction, that is, as the left adjoint functor to the forgetful functor from the category of abelian groups to the one of abelian monoids.

Describe also its explicit construction with generators and relations [Wei13, §II.1]. State (and quickly prove) the properties of group completion listed in [Wei13, §2 Prop. 1.1]. Remark that the group completion of a semiring is a ring. Do some examples, such as the group completion of: $(\mathbb{N}, +)$, $(\mathbb{N}, \cdot)$, isomorphism classes of finite G-sets for a finite group G, isomorphism classes of finite dimensional complex representations of a finite group G [Wei13, §2 Ex. 1.4, 1.5, 1.6].

Define the K_0 *group of a ring with unit* R, as the group completion of the abelian monoid $\mathrm{P}(R)^{iso}$ of isomorphism classes of finitely generated projective R-modules, with the direct sum of R-modules. Explain why we restrict to finitely generated R-modules via the Eilenberg swindle trick. Describe the ring structure on $K_0(R)$ in case R is commutative. Explain that K_0 is a functor from (commutative) rings to abelian groups (commutative rings) [Wei13, §II.2]. Do some examples, such as: $K_0(R) = \mathbb{Z}$ for R a field or a PID or a local ring, $K_0(R) = \mathbb{Z} \oplus Cl(R)$ for R a Dedekind domain [Sri13, §II Ex. 1.1, 1.2]. Present some useful reductions to compute the K_0 of a ring: $K_0(R_1 \times R_2) \cong K_0(R_1) \times K_0(R_2)$, $K_0(\varinjlim_i R_i) \cong \varinjlim_i K_0(R_i)$, $K_0(R) \cong K_0(R/I)$ for $I \subset R$ a nilpotent ideal, Morita invariance $K_0(R) \cong K_0(\mathrm{Mat}_n(R))$ [Wei13, §II pg 75, 2.1.6, Lemma 2.2, Ex. 2.7.2].

Now pass to the K_0 group of schemes. As a motivation, start by rephrasing the definition K_0 of a ring, using short exact sequences [Wei13, §II Def. 7.1, Ex 7.1.1]. Explain why this definition is equivalent to the one above (all the short exact sequences split). Recall that, given R a commutative ring, for $X = Spec(R)$, the category of algebraic vector bundles $\mathbf{VB}(X)$ (finite locally free $\mathcal{O}_X$-modules) is equivalent to the category of finitely generated projective R-modules $\mathbf{P}(R)$ [Wei13, §I Ex. 5.1.2]. Define the K_0 *group of a scheme* X, as the free abelian group generated by the set of algebraic vector bundles $\mathrm{VB}(X)$ modulo the relations given by the short exact sequences of $\mathcal{O}_X$-modules [Wei13, Def. 7.1, Ex. 7.1.3]. Notice that this definition is not equivalent to the group completion of the abelian monoid of isomorphism classes of algebraic vector bundles over X. Describe the ring structure on $K_0(X)$ [Wei13, §II 7.4.2]. Explain that K_0 is a functor from schemes to commutative rings.

We also briefly discuss these construction from a more abstract perspective, introducing a categorical point of view. Notice that $\mathrm{P}(R)^{iso}$ is the set of isomorphism classes of objects of $\mathbf{P}(R)$ and that the monoidal structure on $\mathrm{P}(R)^{iso}$ is induced by the symmetric monoidal structure on $\mathbf{P}(R)$ given by the direct sum of R-modules. So, more generally, one can define the K_0 *group of a symmetric monoidal category* [Wei13, §II Def. 5.1.2]. On the other hand, $\mathrm{VB}(X)$ is the set of objects $\mathbf{VB}(X)$, which is an exact category (intuitively, a category that carries a notion of short exact sequences). So, more generally, one can define the K_0 *group of an exact category* [Wei13, §II Def. 7.1]. Notice that an exact category is also symmetric monoidal category with monoidal product given by the direct sum. In case it is also split exact, then the two constructions of K_0 coincide [Wei13, §II Ex. 7.1.1]. We start to introduce this because this point of view is fundamental for the construction of higher algebraic K-theory, but we can be not too precise for the moment. In particular you can avoid to give the precise definitions of symmetric monoidal categories and exact categories (for these we can simply think at abelian categories, but notice that $\mathbf{P}(R)$ is not abelian but just an exact category!).

Define the G_0-group of a noetherian ring and of a noetherian scheme [Wei13, §II Def. 6.2, Def. 6.2.5]. Describe the *Cartan homomorphism* from K_0 to G_0 in both cases. Mention the theorem stating that the Cartan homomorphism is a ring isomorphism in case of separated regular noetherian schemes [Wei13, §II Thm. 8.2]. Also discuss briefly the proof: the key fact is a theorem of Serre stating that, for a regular scheme X, any coherent $\mathcal{O}_X$-module has a finite resolution of vector bundles over X. The advantage of introducing the G_0-group is that sometimes it has better properties then the K_0 group, for example it is possible to define a pushforward for proper maps. Then, for regular schemes, the last theorem allows to deduce the same properties for the K_0-group. We will need this to discuss the relation between K-theory and the Chow group in the next talks.

3.3 Talk 2 (29/10/25): The Chow group and the Chern classes

The aim for the next two talks is to study the relation between $K_0(X)$ and $CH(X)$, the Chow group of a scheme X, which is an invariant built out of the closed irreducible subschemes. The final goal is to look at the following result: for X a smooth algebraic variety over a field k, there exists a ring homomorphism, called the Chern character map,

$$ch : K_0(X) \to CH(X) \otimes \mathbb{Q},$$

which is an isomorphism of $\mathbb{Q}$-algebras, after tensoring with $\mathbb{Q}$. No prerequisites will be assumed, so we start with the definition of the Chow group. In this first talk, we start introducing the Chow group, some of its properties and the Chern classes of algebraic vector bundles, which we will need to define the Chern Character.

Given k a field, by an algebraic variety over k we mean a finite type, separated k-scheme. Define $CH_i(X)$, the *Chow group of dimension* i of a variety X, as the free group generated by closed irreducible subvarietes $Z \subset X$ of dimension i, whose elements are called *algebraic cycles*, modulo *rational equivalence* [Ful98, §1.3]. Rational equivalence may be defined in two equivalent ways: as the one generated by divisors of a rational function on a $i+1$-dimensional subvariety of X [Ful98, §1.4], or as the relation induced by "$\mathbb{P}^1$-homotopy" [Ful98, §1.6, Prop. 1.6] (see also [EH16, §1.2.2]). Taking the direct sum over i, we get the *Chow group of* X

$$CH(X) := \bigoplus_{i \geq 0} CH_i(X),$$

which, by construction, is graded by dimension. In case X is smooth and equidimensional of dimension d, any irreducible subvariety of dimension i has codimension $d-i$. This allows to grade the Chow group also by codimension

$$CH^*(X) := \bigoplus_{i=1}^{d} CH^i(X),$$

where $CH^i(X) := CH_{d-i}(X)$. With these hypothesis on X, this is just a reindexing of the grading. We will keep the convention of the grading by codimension because it is more convenient. For $f : X \to Y$ a proper morphism, we have group homomorphisms

$$f_* : CH_i(X) \to CH_i(Y),$$

called *pushforward maps* [Ful98, §1.4]. For $f : X \to Y$ a flat morphism of smooth schemes, we have a graded group homomorphism

$$f^* : CH^*(Y) \to CH^*(X),$$

called the *pullback map* [Ful98, §1.7]. For X smooth, Fulton proved in [Ful98, §6] that $CH^*(X)$ has a structure of graded ring satisfying a number of properties, among which the projection formula

$$f_*(f^*(a)b) = af_*(b).$$

The product is called is called *intersection product* because in good cases it is indeed given by the intersection of subvarieties, taking account of a multiplicity. More precisely, given $Y \subset X$ an irreducible subvariety, with generic point $x \in Y$, the *multiplicity* of Y in X is the positive integer

$$m(Y; X) := lenght_{\mathcal{O}_{X,x}}(\mathcal{O}_{Y,x}),$$

and $m(Y;X)[Y]$ is the *algebraic cycle associate to* Y [Ful98, §1.5]. For two irreducible subvarieties $Z, W \subset X$ that *intersect properly*, that is, all the irreducible components T of $Z \cap W$ are such that $codim_X T = codim_X V + codim_X W$, the intersection product of the corresponding classes is given by

$$[Z][W] \coloneqq \sum_{T \subset Z \cap W} m(T; Z, W)[T],$$

where the sum runs over all the irreducible components T of $Z \cap W$, and $m(T; Z, W)$ is a positive integer called *intersection multiplicity at* T. These in general are not just the multiplicities $m(T;X)$ above, but one has to take in account of some correction terms, which are expressed in *Serre's Tor formula*:

$$m(T; Z, W) = \sum_{i=0}^{d} (-1)^i lenght_{\mathcal{O}_{X,x}}(Tor_i^{\mathcal{O}_{X,x}}(\mathcal{O}_{Z,x}, \mathcal{O}_{W,x})),$$

where $x \in T$ is the generic point [EH16, Thm. 2.7]. Notice that truncating this formula to $i = 0$ we get $m(T;X)$. The peculiarity of rational equivalence consists in *Chow's moving lemma*, which states that, given any two closed irreducible subvarieties $Z, W \subset X$, there always exists another closed irreducible subvariety $Z' \subset X$ such that Z is rational equivalent to Z' and Z' and W intersect properly, so the above formula completely describes the intersection product. We will see that Serre's Tor formula appears very natural when compared with the product in $K_0(X)$.

The next tool we need to discuss are the *Chern classes*. From now on X is always a smooth algebraic variety over k. Define the *first Chern class* of a line bundle L over X, $c_1(L) \in CH^1(X)$ [EH16, §1.4]. It doesn't depend on the isomorphism class of L, so it determines a function on the Picard group of X

$$c_1 : Pic(X) \to CH^1(X),$$

which is nothing but the isomorphism between the Cartier divisors and the Weil divisors, since the rational equivalence on codimension 1 cycles is exactly the equivalence induced by principal Weil divisors [GW20, Thm. 11.40] (see also [Wei13, §I Construction 5.8]). This isomorphism commutes with pullbacks. The crucial result that allows to define Chern classes is the Projective Bundle Formula. Given E an algebraic vector bundle over X of rank n, consider $q : \mathbb{P}(E) \to X$ the projectivization of E [GW20, §8.8]. Consider the first Chern class of $\mathcal{O}_{\mathbb{P}(E)}(-1)$, the tautological line bundle on $\mathbb{P}(E)$,

$$x \coloneqq c_1(\mathcal{O}_{\mathbb{P}(E)}(-1)) \in CH^1(\mathbb{P}(E)).$$

The Projective Bundle Formula states that $CH^*(\mathbb{P}(E))$ is a free module over $CH^*(X)$ of rank n, via $q^* : CH^*(X) \to CH^*(\mathbb{P}(E))$, generated by the elements $1, x, x^2, \ldots, x^{n-1}$. Thus, we have the relation

$$x^n + \sum_{i=1}^{n} q^*(\alpha_i) x^{n-i} = 0,$$

where $\alpha_i \in CH^i(X)$ are uniquely determined [EH16, Thm. 5.9]. The *Chern classes of* E, $c_i(E) \in CH^i(X)$, for $i \geq 1$, are defined as [EH16, Def. 5.10]

$$c_i(E) \coloneqq \begin{cases} \alpha_i & \text{for } i = 1, \ldots, n \\ 0 & \text{for } i > n. \end{cases}$$

Notice that, for a line bundle L, the defintion of $c_1(L)$ coincides with the previous one. The *total Chern class of* E is defined as

$$c(E) \coloneqq 1 + \sum_{i=1}^{n} c_i(E) \in CH^*(X).$$

It has the following properties: it commutes with pullbacks and it holds the Whitney formula

$$c(E) = c(E')c(E''),$$

for any $0 \to E' \to E \to E'' \to 0$ short exact sequence of algebraic vector bundles over X [EH16, Thm. 5.3]. Notice that $c(E)$ is invertible in $CH^*(X)$ with respect to the product structure, because is 1 in degree 0. The Whitney formula implies that we have a group homomorphism $c : K_0(X) \to CH^*(X)$ (considering the additive structure on $K_0(X)$ and the multiplicative structure on $CH^*(X)$), which is not a ring homomorphism. By projecting on components, we also have the functions (which are not group homomorphisms)

$$c_i : K_0(X) \to CH^i(X).$$

3.4 Talk 3 (05/11/25): The Chern character isomorphism

This talk is the follow up of the previous one. We will define the Chern character homomorphism and use (a consequence of) the Grothendieck-Riemann-Roch Theorem to deduce that it is an isomorphism of $\mathbb{Q}$-algebras after tensoring with $\mathbb{Q}$.

In this talk X is always a smooth algebraic variety over k. Recall from Talk 1 that we have the Cartan isomorphism $G_0(X) \cong K_0(X)$. In the last talk we constructed the Chern classes, which are functions $K_0(X) \to CH^i(X)$. Composing with the Cartan isomorphism, we can see the Chern classes as functions

$$c_i : G_0(X) \to CH^i(X).$$

Recall that they are not group homomorphism, but now we will see that they induce a group homomorphism on the i^{th}-graded part of a filtration on $G_0(X)$. For any $Z \subset X$ irreducible closed subvariety, the coherent $\mathcal{O}_X$-module $\mathcal{O}_Z \coloneqq \mathcal{O}_X/\mathcal{I}_Z$, where $\mathcal{I}_Z$ is the ideal defining Z, gives an element $[\mathcal{O}_Z] \in G_0(X)$. Define the *the coniveau or topological filtration* on $G_0(X)$

$$\cdots \subset F^{i+1}_{top}G_0(X) \subset F^i_{top}G_0(X) \subset \cdots \subset G_0(X),$$

such that $F^i_{top}G_0(X)$ is generated by the classes $[\mathcal{F}]$, for $\mathcal{F}$ the coherent $\mathcal{O}_X$-module such that $codim_X(supp(\mathcal{F})) \geq i$. Equivalently, $F^i_{top}G_0(X)$ is the subgroup of $G_0(X)$ generated by the classes $[\mathcal{O}_Z]$, for $Z \subset X$ irreducible closed subvariety with $codim_X Z \geq i$ [Ful98, Ex. 15.1.5]. Thus the i^{th}-graded part $Gr^iG_0(X) \coloneqq F^i_{top}G_0(X)/F^{i+1}_{top}G_0(X)$ is generated by the classes $[\mathcal{O}_Z]$, for $Z \subset X$ irreducible closed subvariety with $codim_X Z = i$. It holds that if $\sum_k n_k Z_k$ is an algebraic cycle of codimension i, that is $codim_X Z_k = i$ for each k, such that $\sum_k n_k[Z_k] = 0$ in $CH^i(X)$, then

$$\sum_k n_k[\mathcal{O}_{Z_k}] \in F^{i+1}_{top}G_0(X).$$

This implies that we have a group homomorphism

$$cl : CH^i(X) \to gr^iG_0(X),$$

such that $[Z] \mapsto [\mathcal{O}_Z]$.

The Grothendieck-Riemann-Roch Theorem implies that, if $codim_X Z = i$, then

$$\begin{aligned} c_j([\mathcal{O}_Z]) &= 0 \qquad \text{if } j < i \\ c_i([\mathcal{O}_Z]) &= (-1)^i(i-1)![Z]. \end{aligned} \tag{1}$$

It follows that the i^{th} Chern class induces a function

$$c_i : gr^i G_0(X) \to CH^i(X),$$

which is also proven to be a group homomorphism. By definition of cl and by 1, we see that

$$cl \circ c_i = (-1)^i (i-1)! id$$
$$c_i \circ cl = (-1)^i (i-1)! id.$$

Hence, tensoring with $\mathbb{Q}$, c_i induces an isomorphism of $\mathbb{Q}$-vector spaces

$$c_i \otimes \mathbb{Q} : gr^i G_0(X) \otimes \mathbb{Q} \xrightarrow{\simeq} CH^i(X) \otimes \mathbb{Q},$$

with inverse $cl \otimes \mathbb{Q}$.

Now, we construct the Chern character. Recall that the total Chern class $c : K_0(X) \to CH^*(X)$ is just a group homomorphism, but not a ring homomorphism. The Chern character recombines the Chern classes in order to get a ring homomorphism. The crucial result to define the Chern character is the Splitting Principle [Wei13, Thm. 5.9]. It states that, given E an algebraic vector bundle over X of rank n, there exists $f : Fl(E) \to X$ a morphism of smooth varieties over k, such that the algebraic vector bundle f^*E over $Fl(E)$ is completely splitted, that is, it has a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = f^*E$$

where E_i are algebraic vector bundles over $Fl(E)$ of rank i. The flag variety $Fl(E)$ is obtained taking the projectivization of E, $q : \mathbb{P}(E) \to X$, which is such that its tautological bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ is a subbundle of q^*E [GW20, Eq. 13.8.1], and iterating the construction considering $\mathbb{P}(E)$ with the quotient bundle $q^*E/\mathcal{O}_{\mathbb{P}(E)}(-1)$, which has rank $n-1$. The successive quotients $L_i \coloneqq E_i/E_{i-1}$, for $i = 1, \ldots, n$, are line bundles over $Fl(E)$. In $K_0(Fl(E))$ we have that

$$[f^*E] = [L_1] + \cdots + [L_n].$$

By the Whitney Formula we get that

$$c_i(f^*[E]) = e_i(c_1([L_1]), \ldots, c_1([L_n])),$$

where e_i is the i^{th} elementary symmetric polynomial in n variables. Notice that, since $Fl(E)$ is obtained as an iteration of projectivizations of algebraic vector bundles, by the Projective Bundle Formula, we have that $f^* : CH^*(X) \to CH^*(Fl(E))$ is injective. Since $c_i(f^*[E]) = f^*c_i([E])$, then we can see the elementary symmetric polynomials in $c_1([L_1]), \ldots, c_1([L_n])$ as elements in $CH^*(X)$. Recall that any symmetric polynomial in some coefficient ring R is uniquely a polynomial with coefficients in R of the elementary symmetric polynomials. Consider the symmetric polynomial in $c_1([L_1]), \ldots, c_1([L_n])$ with coefficients in $\mathbb{Q}$

$$\sum_{k=1}^{n} exp(c_1[L_k]) = \sum_{i \geq 0} \frac{1}{i!}(c_1([L_1])^i + \cdots + c_n([L_n])^i) = \sum_{i \geq 0} \frac{1}{i!} N_i(c_1([L_1]), \ldots, c_n([L_n])),$$

where N_i is the i^{th} power sum symmetric polynomial. Notice that this is a polynomial because the degree is bounded by the dimension of $Fl(E)$). It follows from the above discussion that it can be seen as an element of $CH^*(X) \otimes \mathbb{Q}$ and it defines $ch([E])$, the *Chern character of E*. This defines a ring homomorphism

$$ch : K_0(X) \to CH^*(X) \otimes \mathbb{Q},$$

which we can also see as a ring homomorphism from $G_0(X)$ by precomposing with the Cartan homomorphism. Denote by ch_i the projection on $CH^i(X)\otimes\mathbb{Q}$. Notice that the symmetric polynomial N_i is expressed as a polynomial of the elementary symmetric polynomials as

$$N_i = r_i e_i + Q(e_1, \dots, e_n),$$

for some $r_i \in \mathbb{Q}^\times$ and Q a polynomial in $n-1$ variables with coefficients in $\mathbb{Q}$. This, together with 1, shows that ch_i on $F^i_{top}G_0(X)$ is a non zero rational multiple of c_i. The fact that $c_i \otimes \mathbb{Q} : Gr^i G_0(X) \otimes \mathbb{Q} \to CH^i(X) \otimes \mathbb{Q}$ is an isomorphism implies that

$$ch \otimes \mathbb{Q} : K_0(X) \otimes \mathbb{Q} \to CH^*(X) \otimes \mathbb{Q}$$

is also an isomorphism [Ful98, Ex. 15.2.16].

3.5 Talk 4 (12/11/25): K_1 and K_2 of a ring.

In this chapter we study Bass' definition of $K_1(R)$ and Milnor's definition of $K_2(R)$ for R a ring with unit. These are very explicit abelian groups, whose definition and related results only use some Linear Algebra and Group Theory. They were considered the correct notion of higher K-groups of R because they fit in some exact sequences together with the $K_0(R)$. The Quillen's definition of higher K-groups of R (for all $n \geq 0$), that we will see in the next talks, is much more abstract, but is proven to coincide with these low degree K-groups. An interesting feature of these explicit construction of $K_1(R)$ and $K_2(R)$ is that, despite from what may appear from their very first definition, they are also related to the category $\mathbf{P}(R)$ of finitely generated projective R-modules. In fact, they only depend on it. In particular, notice that this allows to easily deduce the Morita invariance property also for $K_1(R)$ and $K_2(R)$.

Define the K_1 *group of a ring with unit* R, as the abelianization of $GL(R) \coloneqq \varinjlim_n GL_n(R)$ [Wei13, §III Def. 1.1]. Explain that K_1 is a functor from rings to abelian groups. Present Withead's Lemma, which provides an explicit description of the commutator subgroup of $GL(R)$: it is the subgroup $E(R) \subset GL(R)$ generated by the elementary matrices [Wei13, §III Lemma 1.3.3] (see also [Sri13, Prop. 1.5]). Do some examples, such as: $K_1(F) = F^\times$, for F a field, and $K_1(\mathbb{Z}) = \mathbb{Z}^\times = \{1, -1\}$ [Wei13, §III Ex. 1.3.5]. Discuss the homological interpretation of $K_1(R)$:

$$K_1(R) \cong \mathrm{H}_1(GL(R), \mathbb{Z}),$$

where the right hand side is the first group homology of $GL(R)$ acting trivially on $\mathbb{Z}$ [Wei13, §III 1.6.2]. From this follows the relation between $K_1(R)$ and finitely generated projective R-modules:

$$K_1(R) \cong \varinjlim_{P \in t\mathbf{P}} \mathrm{H}_1(\mathrm{Aut}(P), \mathbb{Z}),$$

where $t\mathbf{P}$ is some category defined from $\mathbf{P}(R)$ [Wei13, §Cor. 1.6.3].

Now pass to the K_2. Define the K_2 group of a ring with unit R, the kernel of the group homomorphism $St(R) \to E(R)$, where $St(R)$ is the *Steinberg group* [Wei13, §III Def. 5.2]. Present Steinberg's theorem, which proves that $K_2(R) \subset St(R)$ is the center, hence it is an abelian group [Wei13, §III Thm. 5.2.1]. Explain that K_2 is a functor from rings to abelian groups. Discuss the homological interpretation of $K_2(R)$:

$$K_2(R) \cong \mathrm{H}_2(E(R), \mathbb{Z}),$$

where the right hand side is the second homology group of $E(R)$ acting trivially on $\mathbb{Z}$. This follows from the theory of central extensions, since $St(R)$ is the universal central extension of $E(R)$ [Wei13,

§III Thm 5.5] (see also [Sri13, Cor. 1.12]). From this follows the relation between $K_2(R)$ and the finitely generated projective R-modules:

$$K_2(R) \cong \varinjlim_{P \in t\mathbf{P}} H_2([\mathrm{Aut}(P), \mathrm{Aut}(P)], \mathbb{Z}),$$

where the right hand side is the second group homology of the commutator subgroup of $\mathrm{Aut}(P)$ acting trivially on $\mathbb{Z}$ [Wei13, §III Prop. 5.6]. Do some example, such as: $K_2(\mathbb{Z})$ is cyclic of order 2 [Wei13, §III Ex. 5.2.2], $K_2(F) = F^\times \otimes F^\times / < x \otimes (1-x) | x \in F^\times \setminus \{1\} >$ (this is Matsumoto's Theorem [Wei13, §III Thm. 6.1], which inspired the definition of Milnor's K-theory).

To conclude we see how these groups fit in a long exact sequence together with $K_0(R)$. Define the relative versions of these groups $K_1(R, I)$ and $K_2(R, I)$, for the datum of an ideal $I \subset R$ [Wei13, §III Def. 2.2, Def. 5.7]. Discuss the existence of an exact sequence [Wei13, §III Prop. 2.3, Thm. 5.7.1]

$$\begin{array}{ccccc} K_2(R,I) & \longrightarrow & K_2(R) & \longrightarrow & K_2(R/I) \\ & & & & \downarrow \\ K_1(R,I) & \longrightarrow & K_1(R) & \longrightarrow & K_1(R/I) \\ & & & & \downarrow \\ K_0(R,I) & \longrightarrow & K_0(R) & \longrightarrow & K_0(R/I), \end{array}$$

where $K_0(R, I) = K_0(I)$ is the K_0 group of an ideal, defined in [Wei13, §II Exercise 2.3].

3.6 Talk 5 (19/11/25): The classifying space of a small category

This is a preliminary talk for the next three, where we will see several definitions of higher K-theory: Quillen's +-construction, Quillen's Q-construction and Waldhausen's S-construction. In these definitions is crucial the categorical point of view already glimpsed in Talk 1, where we saw the definitions of the K_0 group of a symmetric monoidal category and of an exact category. Analogously, higher K-theory groups are invariants associated to some kind of categories. All these definitions of higher K-theory follow a general recipe: given some kind of category $\mathcal{C}$, ones construct a topological space (in fact, a CW-complex) $K(\mathcal{C})$, called the K-space, and the K-theory groups are defined as its homotopy groups:

$$K_n(\mathcal{C}) \coloneqq \pi_n(K(\mathcal{C})) \quad \text{for } n \geq 0.$$

The tool that allows to produce a topological space starting from a category is the classifying space construction.

The classifying space construction consists in two steps, passing through simplicial sets. Recall the definitions of the category of simplicial objects Δ, the category of simplicial sets $SSet \coloneqq Psh(\Delta, Set)$ and a simplicial object in a category $\mathcal{C}$, $\Delta^{op} \to \mathcal{C}$. We have the examples of the simplicial object in small categories and in topological spaces

$$Cat \leftarrow \Delta^{op} \to Top.$$

A general theorem in category theory guarantees that these functors factor through the Yoneda embedding $Y : \Delta^{op} \to SSet$ and that the second functor in the decomposition admits a right adjoint. So we obtain all the functors in the diagram below:

$$\begin{array}{ccccc} & & \Delta^{op} & & \\ & \swarrow & \downarrow{\scriptstyle Y} & \searrow & \\ Cat & \underset{N}{\overset{h}{\rightleftarrows}} & SSet & \underset{Sing}{\overset{|\_|}{\rightleftarrows}} & Top, \end{array}$$

called, h the homotopy category, N the nerve, $|\_|$ the geometric realization and $Sing$ the singular simplex functor. For explicit descriptions of the nerve and geometric realization functors see [Wei13, §IV Def. 3.1.4]. The *classifying space* is the composition of the nerve and geometric realization construction:

$$B\mathcal{C} \coloneqq |N\mathcal{C}|.$$

For an explicit description of $B\mathcal{C}$ see [Wei13, §IV Recipe 3.1.1].

Do the example of BG the classifying space of a group G. This is a model for the Eilenberg-Mac lane space $K(G,1)$, that is, its homotopy groups are such that $\pi_1(BG) = G$ and $\pi_n(BG) = 0$ for any $n \neq 1$ [Wei13, §IV Ex. 3.1.2]. These properties characterize BG up to homotopy equivalences. Do the example of BM the classifying space of a monoid M and the property that $\pi_1(BM) \cong M^{-1}M$ [Wei13, §IV Ex. 3.4.2].

It's interesting to see how some categorical properties get translated into topological properties. For example, a category with an initial (or final) object is such that its classifying space is contractible [Wei13, §IV Ex. 3.2.2]. A functor induces a cellular map on the classifying spaces and a natural transformation of functors induces an homotopy between the associated maps. As a consequence, a pair of adjoint functors induces a pair of homotopy inverse maps [Wei13, §IV 3.2].

To conclude, present Quillen's Theorems A and B [Wei13, §IV Thm. 3.7, Thm. 3.8]. Theorem A gives sufficient conditions for a functor to induce an homotopy equivalence on the classifying spaces. Theorem B describes, under certain hypothesis, the homotopy fiber of the map induced by a functor on the classifying spaces as the classifying space of a category. For a definition of homotopy fiber look at [Wei13, §IV, 1.2]. These results will be useful later, for example to prove the "$+ = Q$" Theorem.

3.7 Talk 6 (26/11/25): The Quillen + and $S^{-1}S$-constructions

In this talk we will construct a K-theory space $K(R)$ associated to a ring with unit R, whose homotopy groups define the K-groups of R

$$K_n(R) \coloneqq \pi_n(K(R)),$$

such that $K_0(R)$, $K_1(R)$ and $K_2(R)$ coincide with the ones already defined.

Quillen's first approach to construct $K(R)$ was to start from $BGL(R)$, the classifying space of the group $GL(R) = \varinjlim_n GL_n(R)$, and modify it ad hoc, via the so called $+$-construction, so that its low degrees homotopy groups coincide with the low degrees K-groups of R. In the next talk we will see that the K-groups obtained applying the Q-construction to the exact category $\mathbf{P}(R)$ of finitely generated projective R-modules coincides with the one defined via the $+$-construction (the "$+ = Q$" Theorem). This comparison theorem is actually a consequence of a more general one that compares the Q-construction with the $S^{-1}S$-construction for symmetric monoidal categories, which can be thought as a categorical version of group completion. So, in this talk we will also study the $S^{-1}S$ construction and see its connection with the $+$-construction.

Define the *$+$-construction* for the datum of a connected based CW-complex X and a perfect normal subgroup $P \subset \pi_1(X)$ [Wei13, §IV Def. 1.4.1]. It is a purely topological construction that determines, up to homotopy equivalences, another connected based CW-complex X^+ with an acyclic map $X \to X^+$, such that $\pi_1(X^+) \cong \pi_1(X)/P$. A theorem of Quillen guarantees the existence of X^+ and its universal property, up to homotopy equivalence [Wei13, §IV Thm. 1.5]. The proof is purely topological: it's obtained by attaching in an appropriate way some 2-cells and 3-cells to X (you don't need to give a proof, but in case you are interested, look at [Sri13, Thm. 2.1]). Then, applying this construction to $X = BGL(R)$ and $P = E(R) \subset GL(R)$ we obtain the CW-complex $BGL(R)^+$ such that $\pi_1(BGL(R)^+) \cong GL(R)/E(R) = K_1(R)$. The fact that the map $BGL(R) \to BGL(R^+)$ is acyclic also guarantees that $\pi_2 BGL(R)^+ \cong H_2(E(R), \mathbb{Z}) \cong K_2(R)$ [Wei13, §IV Prop. 1.7] (see also [Wei13, Prop 2.5, Cor 2.6]). With this description we also obtain the homological interpretation of $K_3(R)$

$$K_3(R) \coloneqq \pi_3(BGL(R)^+) \cong \mathrm{H}_3(St(R), \mathbb{Z}),$$

where the right hand side is the third homology group of $E(R)$ acting trivially on $\mathbb{Z}$ [Sri13, Cor. 2.6]. Notice that to explain all this you need to first recall the topological notions of homotopy fiber, acyclic space and acyclic maps [Wei13, §IV, 1.2, Def. 1.3, Def. 1.4].

To recover also $K_0(R)$, one defines the *K-space of R* as

$$K(R) \coloneqq K_0(R) \times BGL(R)^+,$$

the disjoin union of copies of $BGL(R)^+$ indexed on $K_0(R)$ [Wei13, §IV Def. 1.1.1].

Now pass to the $S^{-1}S$ construction for symmetric monoidal categories. First recall the definition of a symmetric monoidal category S [Wei13, §II Def. 5.1]. Notice that BS is an homotopy associative and commutative h-space, that is, a monoid in the homotopy category of topological spaces, and so $\pi_0(BS)$ is an abelian monoid. [Wei13, §IV, 4]. Define the category $S^{-1}S$ [Wei13, §IV Def. 4.2, Rmk. 4.2.2]. This can be thought as a categorical version of the group completion construction. Describe its symmetric monoidal structure and notice that $\pi_0(BS^{-1}S)$ is an abelian group [Wei13, §IV Rmk. 4.2.2]. In case every morphism in S is an isomorphism (if it is not, we can take $isoS$ the subcategory of S given by the isomorphisms [Wei13, §IV Def. 4.1]), define the *K-space of S*

$$K(S) \coloneqq BS^{-1}S,$$

which is a CW-complex based at the identity, and the *K-groups of S* for any integer $n \geq 0$

$$K_n(S) \coloneqq \pi_n(K(S)).$$

Explain that a monoidal functor induces group homomorphisms on the K-groups [Wei13, §IV Def. 4.3]. Prove that $K_0(S) = \pi_0(BS^{-1}S)$ is the group completion of the abelian monoid $\pi_0(BS)$ and deduce that $K_0(S)$ coincides with the Grothendieck's group of a symmetric monoidal category defined in Talk 1 [Wei13, §IV Lemma 4.3.1].

The relation with the +-construction is described in [Wei13, §IV Cor. 4.11.1]: for $S = iso\mathbf{P}(R)$, the connected component of the identity of $BS^{-1}S$ is homotopically equivalent to $BGL(R)^+$. In other words,

$$BS^{-1}S \simeq K_0(R) \times BGL(R)^+ = K(R).$$

Sketch the proof of this (see also [Sri13, Thm. 7.4]).

3.8 Talk 7 (03/12/25): The Quillen Q-construction and the "$+ = Q$" Theorem

In this talk we study the Quillen Q-construction that associates to an exact category $\mathcal{A}$ another category $Q\mathcal{A}$. This is used to define a K-theory space taking the classifying space, and hence K-groups taking homotopy groups. This is the generalization of the Grothendieck group of an exact category that we roughly discussed in Talk 1. Moreover, analogously to the Grothendieck groups, it holds that for split exact categories the definitions of K-groups via the $S^{-1}S$ and Q-constructions coincide. Applying it to the category $\mathbf{P}(R)$ of finitely generated projective modules, we obtain the "$+ = Q$" Theorem.

First recall the definition of an exact category $\mathcal{A}$ [Wei13, §II Def. 7.0]. Define the category $Q\mathcal{A}$ [Wei13, §IV Def. 6.1]. Prove that its classifying space $BQ\mathcal{A}$ is a connected CW-complex such that $\pi_1(BQ\mathcal{A}) \cong K_0(\mathcal{A})$, where the right hand side is the Grothendieck group of an exact category defined in Talk 1 [Wei13, §IV Prop. 6.2]. This motivates the definition of the *K-space of* $\mathcal{A}$:

$$K(\mathcal{A}) := \Omega BQ\mathcal{A},$$

and the *K-groups of* $\mathcal{A}$, for any integer $n \geq 0$,

$$K_n(\mathcal{A}) := \pi_n(K(\mathcal{A})).$$

Explain that an exact functor induces group homomorphisms on the K-groups [Wei13, §IV Def. 6.3]. Applying this to certain exact categories, we obtain the K and G-groups of rings and schemes [Wei13, §IV Def. 6.3.2, Def. 6.3.3, Def. 6.3.4].

Now the goal is to prove [Wei13, §IV Thm. 7.1] (see also [Sri13, Thm 7.7]), which states that if $\mathcal{A}$ is a split exact category and $S := iso\mathcal{A}$, then there exists an homotopy equivalence

$$\Omega BQ\mathcal{A} \simeq BS^{-1}S.$$

Taking homotopy groups, it follows that, for any integer $n \geq 0$,

$$K_n(\mathcal{A}) \cong K_n(S).$$

Applying this to $\mathcal{A} = \mathbf{P}(R)$, we get the "$+ = Q$" Theorem [Wei13, §IV Cor. 7.2], which states the existence of an homotopy equivalence

$$\Omega BQ\mathbf{P}(R) \simeq K_0(R) \times BGL(R)^+,$$

and hence that the different definitions of K-theory of rings coincide.

3.9 Talk 8 (10/12/25): The Waldhausen S-construction

In this talk we present the third construction of higher algebraic K-theory: the Waldhausen S-construction. This construction is applied to Waldhausen categories, which are catgeories with a notion of cofibrations and weak equivalences. The reference for this is [Wei13, §IV, 8].

This construction is probably useful to discuss the localization sequence of Thomason and Throbaugh [Ful98] and the modern definition of K-theory of stable infinity categories.

3.10 Talk 9 (17/12/25): Properties of higher algebraic K-theory

In this talk we present some properties of higher K-theory of abstract exact or Waldhausen categories. The statement of these properties often sounds quite abstract and technical, but, on the other hand, their applications to the examples of G and K-theory of schemes consist in some typical properties of a cohomology theory. This should convince us that the complicated Quillen and Waldhausen constructions described in Talks 7 and 8 actually give rise to good theories.

The idea for this talk is not to give a detailed exposition of the proofs, but rather to have an overview of the properties, from the general statements to their applications to examples, underlying which results can be first obtained in the easier context of exact categories, what are their limitations and how the Waldhausen context helps in solving them.

We follow [Wei13, §V]. In blue there are some suggestions for the exposition or what to say about proofs.

1. **Additivity**
This property can be formulated with the same statement both for exact or Waldhausen categories. You can restrict to the exact setting and mention that analogous defintions and results hold in the Waldhausen context. First recall that an exact functor F between exact or Waldhausen categories induces a map between the K-space (for the Quillen or Waldhausen construction respectively), and hence group homomorphisms F_* on the K-groups. To state the theorem, we need the definition of a *short exact sequence* of exact functors [Wei13, Def. 1.1]. The *Additivity Theorem* is [Wei13, Thm. 1.5]. It states that, given a short exact sequence of exact functors $0 \to F' \to F \to F'' \to 0$, then $F_* = F'_* + F''_*$.

   If you want to tell something about the proof, you can discuss the idea of considering the diagram described there. To do that, you need to talk about the Universal Example [Wei13, §1.1.1].

   The Additivity Theorem immediately extends to finite long exact sequences ([Wei13, Cor. 1.2.1]).

   Applications
The main application of the Additivity Theorem is the *Projective Bundle Formula* [Wei13, Thm. 1.5]. It states that for any quasi-projective scheme X and $\mathcal{E}$ an algebraic vector bundle on X of rank r, there is a ring isomorphism $K_*(X)[T]/(T^{r+1}) \cong K_*(\mathbb{P}(\mathcal{E}))$.

   In the proof one writes down a finite long exact sequence of exact functors. This is the main point where Additivity Theorem is used. Also, notice that here $K_*(X)$ is considered with his ring structure, which was not mentioned yet in the previous talks (look at [Wei13, §IV, Ex. 6.6.5])

2. **Resolution**
There are different Resoution Theorems, depending on the context in which they are formulated. In the context of exact categories, there is the *Resolution Theorem* [Wei13, Thm. 3.1]. Roughly, it states that, given $\mathcal{P} \subset \mathcal{H}$ a full exact subcategory of an exact category, such that each object in $\mathcal{H}$ has a $\mathcal{P}$-resolution, then $K_*(\mathcal{P}) \cong K_*(\mathcal{H})$.

   The proof is not so difficult, but abstract. Maybe there is no time to present it.

   Applications
Recall that a theorem of Serre states that, for any X separated noetherian scheme, any coherent sheaf admits a finite resolution of algebraic vector bundles. Then, by Resolution Theorem, it immediately follows that, with these hypothesis on X, $G_*(X) \cong K_*(X)$ [Wei13, Thm. 3.4]. This generalizes the particular case $G_0(X) \cong K_0(X)$, which was already discussed in Talk 1. There are also the following interesting applications of the Resolution Theorem. Let $f : X \to Y$ be a morphism of noetherian schemes of finite flat dimension. There exist a *pullback (or base change) map* on G-theory $f^* : G_*(Y) \to G_*(X)$ [Wei13, §3.6] and, if f is proper, a *pushforward (or transfer) map* on K-theory $f_* : K_n(X) \to K_n(Y)$ [Wei13, Prop. 3.7.1]. Moreover, some *Base Change Formula* and *Projection formula* hold [Wei13, Thm. 3.7.2, Cor 3.7.3].

   Maybe there is no time to talk much about pullback and pushforward maps and their formulas, but are interesting to mention because they are typical features of a cohomology theory.

3. **Devissage**
This is a property for Quillen K-theory of abelian categories. The *Devissage Theorem* is [Wei13, Thm. 4.1]. Roughly, it states that, for $\mathcal{A} \subset \mathcal{B}$ an inclusion of abelian categories, such that each object in $\mathcal{B}$ admits a filtration with subquotients in $\mathcal{A}$, then $K_*(\mathcal{A}) \cong K_*(\mathcal{B})$.

The proof should be not too hard. One should remember the definition of the Q-construction and use a couple of facts about the homotopy theory of the classifying space discussed in Talk 5.

Applications
The Devissage Theorem is usually a tool that allows to perform computations of K-groups. For example, an immediate consequence is that G-theory of rings doesn't see nilpotent elements, i.e. $G_*(R) \cong G_*(R/I)$ for any $I \subset R$ nilpotent ideal. An important application of Devissage Theorem is that, given $Z \subset X$ a closed subscheme of a noetherian scheme, then $G_*(Z) := K_*(M(Z)) \cong K_*(M_Z(X))$, where $M_Z(X)$ is the abelian category of coherent sheaves on X with support in Z [Wei13, Ex. 4.3].

The case of schemes is an exercise in the book, but the proof should be analogue to the one for rings, which is explained in [Wei13, App. 4.4].

4. **Localization**
There are different Localization Theorems, depending on the context in which they are formulated. In the context of abelian categories, there is the *Abelian Localization Theorem* [Wei13, Thm. 5.1]. It states that for $\mathcal{B} \subset \mathcal{A}$ a Serre subcategory of a small abelian category, there is an homotopy fibration sequence $K(\mathcal{B}) \to K(\mathcal{A}) \to \mathcal{A}/\mathcal{B}$, and hence there is a long exact sequence of K-groups.

Maybe just recall the definition of a Serre subcategory. The proof is quite long and technical, you can avoid it.

Applications
From the Abelian Localization Theorem applied to $M_Z(X) \subset M(X)$ and the application to Devissage Theorem, it immediately follows the localization sequence for the G-theory of schemes [Wei13, Ex.6.11]. It states that, given $Z \subset X$ a closed subscheme of a noetherian scheme with open complement U, there is a long exact sequence

$$\cdots \to G_n(Z) \to G_n(X) \to G_n(U) \to G_{n-1}(Z) \to \dots$$

ending with $G_0(Z) \to G_0(X) \to G_0(U) \to 0$.

What about a localization sequence for K-theory of schemes? This should be one of the main results in Thomason-Trobaugh's article [Wei13, Thm. 7.6], which uses the Waldhausen context applied to perfect complexes. I still need to understand what are the issues for K-theory in the exact context and how the Waldhausen context helps in solving them.

3.11 Talk 10 (07/01/26): Computation of the K-theory of finite fields

In this talk we present Quillen's computation of K-theory of finite fields. The higher K-theory groups of a finite field $\mathbb{F}_q$ are completely determined [Wei13, §IV Cor. 1.13]: for $n \geq 1$

$$K_n(\mathbb{F}_q) \cong \begin{cases} \mathbb{Z}/(q^i - 1) & n = 2i - 1 \\ 0 & n \text{ even} \end{cases}$$

The computation is based on the $+$-construction and requires the construction of some operations on K-theory, which are discussed in [Wei13, §IV, 5].

Another reference for this are the notes K-theory of finite fields.

3.12 Talk 11 (14/01/26): The Kummer-Vandiver Conjecture/The Merkurjev-Suslin Theorem

There are two proposals for this talk, both related to some topic in Number Theory.

One proposal is to present the equivalence between the Kummer-Vandiver Conjecture and the vanishing of some K-groups of $\mathbb{Z}$. The Kummer-Vandiver Conjecture states that, given a prime p, the class number of the maximal real subfield of the p^{th}-cyclotomic field is not divisible by p. In [Mas92] it is proven that this is equivalent to the vanishing of all the groups $K_{4n}(\mathbb{Z})$, for $n \geq 1$.

The other proposal is to present the Merkurjev-Suslin Theorem [Sri13, Thm. 8.5], which establishes the group isomorphism, for a field F and an integer $n > 0$,

$$K_2(F) \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathrm{H}^2_{\text{ét}}(F, \mu_n^{\otimes 2}).$$

3.13 Talk 12 (21/01/26): Universality of algebraic K-theory

In this talk we present algebraic K-theory from a more modern point of view using the language of infinity categories. The advantage of this point of view is that it is possible to formulate a universal property for algebraic K-theory. A reference for this are the Lecture Notes on Algebraic K-theory.

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