Quillen Q-Construction and the "$+ = Q$" Theorem

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These are notes on the Quillen Q-Construction and the "$+ = Q$" Theorem based on [Wei13, Chapter IV, §6 and §7]. I would like to thank Linda Carnevale for compiling a wonderful programe for the seminar.

Last time, we have seen how to define higher K-groups using the + and the S^{-1}-constructions, where the former works for rings R and the latter works for symmetric monoidal categories.
Today we are going to look at the Quillen Q-construction, which is used to construct higher K-groups for exact categories.
We'll start by defining the Quillen Q-construction and afterwards, we'll compare the construction to constructions of higher K-theory that we've seen so far; i.e. the +- and S^{-1}-constructions.

Recall:

Definition 0.1 ([Wei13, Ch. II, Def 7.0])**.** An *excact category* is a pair $(\mathcal{C}, \mathcal{E})$, where

- $\mathcal{C}$ additive category
- $\mathcal{E}$ family of sequences of the form

$$0 \to B \xrightarrow{i} C \xrightarrow{j} D \to 0 \tag{$*$}$$

such that $\mathcal{C}$ admits a full embedding into an Abelian category $\mathcal{A}$ such that

- $\mathcal{E}$ is the class of $\mathcal{A}$-short exact sequences in $\mathcal{C}$.
- $\mathcal{C}$ is closed under extensions in $\mathcal{A}$, i.e. if $B, D \in \mathcal{C}$ and $(*)$ exact in $\mathcal{A}$, then $C \in \mathcal{C}$ (up to isomorphism).

Morphisms i in $(*)$ are called *admissible monomorphisms*. Morphisms j in $(*)$ are called *admissible epimorphisms*. ⌟

1 The Quillen Q-Construction

The Q construction is essentially an auxillary category used as an intermediat step to define higher K-groups.

Definition 1.1 ([Wei13, Ch. IV, Def. 6.1])**.** $\mathcal{A}$ exact category. Define category $Q\mathcal{A}$ with same objects as $\mathcal{A}$ and morphisms are diagrams

$$A \xleftarrow[j]{} B_2 \xhookrightarrow{i} B$$

where j admissible epimorphism and i admissible monomorphism in $\mathcal{A}$. Two diagrams are equivalent if we have a diagram

$$\begin{array}{ccccc} A & \overset{}{\underset{j}{\twoheadleftarrow}} & B_2 & \overset{i}{\hookrightarrow} & B \\ \| & & \downarrow{\scriptstyle\cong} & & \| \\ A & \underset{j'}{\twoheadleftarrow} & B_2' & \overset{i'}{\hookrightarrow} & B. \end{array}$$

Composition of $A \twoheadleftarrow B_2 \hookrightarrow B$ and $B \twoheadleftarrow C_2 \hookrightarrow C$ is $A \twoheadleftarrow C_1 \hookrightarrow C$ as in

$$\begin{array}{ccccccc} & & C_1 & \hookrightarrow & C_2 & \hookrightarrow & C \\ & & \downarrow\!\!\!\downarrow & \lrcorner & \downarrow\!\!\!\downarrow & & \\ A & \twoheadleftarrow & B_2 & \hookrightarrow & B & & \end{array}$$

$\lrcorner$

Remark 1.2. If $A \hookrightarrow B$ is an admissible monic in $\mathcal{A}$, we get a morphism $A \overset{\text{id}}{\twoheadleftarrow} A \hookrightarrow B$ in $Q\mathcal{A}$.

If $C \twoheadrightarrow B$ is an admissible epi in $\mathcal{A}$, we get a morphism $B \twoheadleftarrow C \overset{\text{id}}{\hookrightarrow} C$ in $Q\mathcal{A}$. $\lrcorner$

Proposition 1.3 ([Wei13, Ch. IV, Prop. 6.2])**.** *The geometric realization $BQ\mathcal{A}$ is a connected CW complex with $\pi_1(BQ\mathcal{A}) \cong K_0(\mathcal{A})$. The element of $\pi_1(BQ\mathcal{A})$ corresponding to $[A] \in K_0(\mathcal{A})$ is represented by the loop $0 \hookrightarrow A \twoheadrightarrow 0$.*

Proof. $BQ\mathcal{A}$ is a CW-complex by definition.

The zero-cells of this CW-complex (or the zero smiplices of the nerve of $Q\mathcal{A}$) are the objects of $\mathcal{A}$. Since we have a path induced by $0 \hookrightarrow A$ for $A \in \mathcal{A}$, $BQ\mathcal{A}$ is connected.

Using some combinatorics of CW-compelxes / simplicial set, one can show: Since the family of all morphism $0 \hookrightarrow A$ in $Q\mathcal{A}$ induces a maximal lattice, we can present $\pi_1(BQ\mathcal{A})$ as follows:

- Generators: Morphisms in $Q\mathcal{A}$.
- Relations: $[0 \hookrightarrow A] = 1$ for $A \in Q\mathcal{A}$ and $[f] \cdot [g] = [f \circ g]$ for composable morphisms f, g in $Q\mathcal{A}$.

Remains to show that we can reduce the generating set down to $0 \hookrightarrow A \twoheadrightarrow 0$ and that relations correspond with exact sequences.

Generators: Note Composition of $[0 \hookrightarrow B_2]$ and $[B_2 \hookrightarrow B]$ is $[0 \hookrightarrow B_2 \hookrightarrow B]$. Hence

$$1 = [0 \hookrightarrow B_2 \hookrightarrow B] = [B_2 \hookrightarrow B] \cdot \underbrace{[0 \hookrightarrow B_2]}_{=1} = [B_2 \hookrightarrow B].$$

Thus

$$[A \twoheadleftarrow B_2 \hookrightarrow B] = [B_2 \hookrightarrow B] \cdot [A \twoheadleftarrow B_2] = [A \twoheadleftarrow B_2].$$

Now

$$[A \twoheadleftarrow B] \cdot [0 \twoheadleftarrow A] = [0 \twoheadleftarrow A \twoheadleftarrow B] = [0 \twoheadleftarrow B],$$

hence

$$[A \twoheadleftarrow B] = [0 \twoheadleftarrow B] \cdot [0 \twoheadleftarrow A]^{-1}$$

and we get the generators $[0 \twoheadleftarrow A]$ as required.

Relations: Let $A \hookrightarrow B \twoheadleftarrow C$ be a short exact sequence in $\mathcal{A}$. Exactness yields:

$$(C \twoheadleftarrow B) \circ (0 \hookrightarrow C) = 0 \twoheadleftarrow A \hookrightarrow B$$

Hence (using above relation for generators)

$$[C \twoheadleftarrow B] = [C \twoheadleftarrow B][0 \hookrightarrow C] = [0 \twoheadleftarrow A \hookrightarrow B] = [0 \twoheadleftarrow A]$$

Thus

$$[0 \twoheadleftarrow B] \underset{composition}{=} [C \twoheadleftarrow B][0 \twoheadleftarrow C] = [0 \twoheadleftarrow A][0 \twoheadleftarrow C],$$

which is the additivity relation in $K_0(\mathcal{A})$. This also yields that $\pi_1(BQ\mathcal{A})$ is Abelian: We have the exact sequences $A \hookrightarrow A \oplus B \twoheadrightarrow B$ and $B \hookrightarrow A \oplus B \twoheadrightarrow A$ yielding

$$[0 \twoheadleftarrow A][0 \twoheadleftarrow B] = [0 \twoheadleftarrow A \oplus B] = [0 \twoheadleftarrow B][0 \twoheadleftarrow A].$$

All relations in $\pi_1(BQ\mathcal{A})$ are generated by these relations: Given two composable morphisms $A \twoheadleftarrow B_2 \hookrightarrow B$ and $B \twoheadleftarrow C_2 \hookrightarrow C$ in $Q\mathcal{A}$, we have

$$\begin{array}{ccccc}
 & & \ker\varphi' & = & \ker\varphi & & \\
 & & \downarrow & & \downarrow & & \\
 & & B_2 \times_B C_2 & \hookrightarrow & C_2 & \hookrightarrow & C \\
 & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \\
A & \twoheadleftarrow & B_2 & \hookrightarrow & B. & &
\end{array}$$

Hence we get from the exact columns the relations

$$[0 \twoheadleftarrow \ker\varphi] = [0 \twoheadleftarrow B_2 \times_B C_2][0 \twoheadleftarrow B_2]^{-1}$$

and

$$[0 \twoheadleftarrow \ker\varphi] = [0 \twoheadleftarrow C_2][0 \twoheadleftarrow B]^{-1}$$

Hence

$$\begin{aligned}
& & [0 \twoheadleftarrow C_2][0 \twoheadleftarrow B]^{-1} &= [0 \twoheadleftarrow B_2 \times_B C_2][0 \twoheadleftarrow B_2]^{-1} \\
&\Leftrightarrow & [0 \twoheadleftarrow C_2][0 \twoheadleftarrow B]^{-1}[0 \twoheadleftarrow B_2] &= [0 \twoheadleftarrow B_2 \times_B C_2] \\
&\Leftrightarrow & [0 \twoheadleftarrow C_2][0 \twoheadleftarrow B]^{-1}[0 \twoheadleftarrow B_2][0 \twoheadleftarrow A]^{-1} &= [0 \twoheadleftarrow B_2 \times_B C_2][0 \twoheadleftarrow A]^{-1} \\
&\Leftrightarrow & [B \twoheadleftarrow C_2][A \twoheadleftarrow B_2] &= [A \twoheadleftarrow B_2 \times_B C_2] \\
&\Leftrightarrow & [B \twoheadleftarrow C_2 \hookrightarrow C][A \twoheadleftarrow B_2 \hookrightarrow B] &= [A \twoheadleftarrow B_2 \times_B C_2 \hookrightarrow C],
\end{aligned}$$

which is the relation generated by the composition. Thus $K_0(\mathcal{A}) = \pi_1(BQ\mathcal{A})$. ■

Definition 1.4. Let $\mathcal{A}$ be a small exact category. Define $K\mathcal{A} := \Omega BQ\mathcal{A}$ and

$$K_n(\mathcal{A}) = \pi_n K\mathcal{A} = \pi_{n+1}(BQ\mathcal{A})$$

for $n \geq 0$. ⌟

An exact functor $\mathcal{A} \to \mathcal{B}$ induces a functor $Q\mathcal{A} \to Q\mathcal{B}$. This induces a map $BQ\mathcal{A} \to BQ\mathcal{B}$ and so a map $K_n(\mathcal{A}) \to K_n(\mathcal{B})$.

Isomorphic functors induce the same map on K-groups because they induce isomorphic functors $Q\mathcal{A} \to Q\mathcal{B}$.

Definition 1.5. Let R be a ring with unit.

- Let $\mathbf{P}(R)$ = exact category of finitely generated projective R-modules. Define $K(R) := K\mathbf{P}(R)$ and $K_n(R) := K_n\mathbf{P}(R)$, the *$K$-groups* of R.
- If R is Noetherian. Let $\mathbf{M}(R)$ = category of finitely generated R-modules. Set $G(R) := K\mathbf{M}(R)$ and $G_n(R) := K_n\mathbf{M}(R)$, the *$G$-groups* of R.

⌟

For $n = 0$, these definitions agree with the earlier definitions by Proposition 1.3

2 The "$+ = Q$" Theorem

Let $S = \operatorname{iso}\mathcal{A}$. and consider $\mathcal{A}$ symmetric monoidal using $\oplus$. Then we also defined $K^{\oplus}\mathcal{A} = B(S^{-1}S)$.

We conclude the talk by proving the following Theorem, comparing the constructions of K-theory.

Theorem 2.1 ([Wei13, Ch. IV, Thm. 7.1])**.** *If $\mathcal{A}$ is a split exact category and $S = \operatorname{iso}\mathcal{A}$, then $\Omega BQ\mathcal{A} \simeq B(S^{-1}S)$. Hence $K_n(A) \cong K_n(S)$ for all $n \geq 0$.*

From this we get the $+ = Q$-Theorem, since last time we saw that the S-construction is a $+$-construction for projective R-modules.

Corollary 2.2 ("$+ = Q$",[Wei13, Corollary 7.2])**.** *For every ring R,*

$$\Omega BQP(R) \cong K_0(R) \times BGL(R)^+.$$

Hence $K_n(R) \cong K_nP(R)$ for all $n \geq 0$.

Idea behind proof of 2.1: Find a fibre sequence

$$B(S^{-1}S) \to ? \to BQ\mathcal{A}$$

with ? contractible.

cooking up topological spaces is hard, but we have Quillen Theorem B from Anna's talk.

Idea: Use Quillen Theorem B, i.e. find a category ? such that

$$S^{-1}S \to ? \to Q\mathcal{A}$$

is a nice enough fibre sequence.

Definition 2.3. Define $\mathcal{E}\mathcal{A}$ to be the category with objects the short exact sequence in $\mathcal{A}$. And morphisms $E' = (A' \hookrightarrow B' \twoheadrightarrow C') \to (A \hookrightarrow B \twoheadrightarrow C)$ diagrams

$$\begin{array}{ccccccc}
E'\colon & & A' & \hookrightarrow & B' & \twoheadrightarrow & C' \\
\Big\downarrow & & {\scriptstyle\alpha}\uparrow & & \| & & \uparrow \\
 & & A & \hookrightarrow & B' & \twoheadrightarrow & C'' \\
 & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow \\
E\colon & & A & \hookrightarrow & B & \twoheadrightarrow & C.
\end{array}$$

Two such diagrams are equivalent if there is an isomorphism between them that is the identity at all verticies excepte for C''. ⌟

The right column consists of morphisms in $Q\mathcal{A}$. Hence get a functor

$$t\colon \mathcal{E} \to Q\mathcal{A}, t(A \hookrightarrow B \twoheadrightarrow C) = C.$$

Write $\mathcal{E}_C = t^{-1}(C)$.

What do we need to show for Quillen Theorem B: We need to identify the fibres and show that all of the functors induced by morphisms in $Q\mathcal{A}$ induce equivalences on the fibres. Then we'd be done.

What are the fibres?

The endomorphisms of $C \in Q\mathcal{A}$ are (essentially) automorphisms of C in $\mathcal{A}$. Thus a morphism in $\mathcal{E}_C$ is (essentially)

$$\begin{array}{ccccc} A' & \hookrightarrow & B' & \twoheadrightarrow & C \\ \alpha \uparrow \cong & & \cong \downarrow \beta & & \| \\ A & \hookrightarrow & B & \twoheadrightarrow & C \end{array}$$

an isomorphism.

Example 2.4. The assignment $A \in S \mapsto (A \xrightarrow{id} A \to 0) \in \mathcal{E}_0$ induces a homotopy equivalence. ⌟

Lemma 2.5 ([Wei13, Ch. IV, Lemma 7.5 and Remark 7.5.2])**.** *For any $C \in \mathcal{A}$, $\mathcal{E}_C$ is symmetric monoidal and there is a faithful monoidal funcotr $\eta_C \colon S \to \mathcal{E}_C; A \mapsto (A \hookrightarrow A \oplus C \twoheadrightarrow C)$.*

Moreover, this functor is essentially surjective if $\mathcal{A}$ is split exact.

There is a more general construction of S^{-1} detailed in [Wei13, Ch. IV, Definition 4.7.1], such that we can form $S^{-1}\mathcal{E}\mathcal{A}$ when $\mathcal{A}$ is split exact. η_C induces a functor $S^{-1}S \to S^{-1}\mathcal{E}_C$.

Assume that $\mathcal{A}$ is split exact from now on.

Proposition 2.6 ([Wei13, Ch. IV, Prop. 7.6])**.** *Each $S^{-1}S \to S^{-1}\mathcal{E}_C$ is a homotopy equivalence.*

Idea of proof. The proof goes by considering the cofibre of $S^{-1}S \to S^{-1}\mathcal{E}_C$ and a version of Quillen Theorem A from Anna's talk. ■

Lemma 2.7 ([Wei13, Ch. IV, Lemma 7.7])**.** *For each morphism $\varphi \colon C' \to C$ in $Q\mathcal{A}$, there is a canonical functor $\varphi^* \colon \mathcal{E}_C \to \mathcal{E}_{C'}$ and a natural transformation $\eta_E \colon \varphi^*(E) \to E$ from φ^* to the inclusion of $\mathcal{E}_C$ in $\mathcal{E}\mathcal{A}$.*

Construction of φ^*. Represent φ by $C' \twoheadleftarrow C'' \hookrightarrow C$ and take $A \hookrightarrow B \twoheadrightarrow C$ in $\mathcal{E}\mathcal{A}$. Choose a pullback $B' = C'' \times_C B$. Then we get an admissible compositte $B' \twoheadrightarrow C'' \twoheadrightarrow C'$ with kernel A', yielding

$$\varphi^*(A \hookrightarrow B \twoheadrightarrow C) := A' \hookrightarrow B' \twoheadrightarrow C'.$$

■

Theorem 2.8 ([Wei13, Ch. IV, Thm. 7.8 and proof of Thm. 7.1])**.** *The sequence $S^{-1}S \to S^{-1}\mathcal{E}\mathcal{A} \xrightarrow{t} Q\mathcal{A}$ is a homotopy fibration and $\mathsf{E}\mathcal{A}$ is contractible.*

Proof. We want to use Quillen Theorem B to prove the first part. For this, we need to show that the induced base changes φ^* of $0 \hookrightarrow C$ and $0 \twoheadleftarrow C$ induce homotopy equivalences of fibres.

For $0 \hookrightarrow C$: The composition $S^{-1}S \to S^{-1}\mathcal{E}_C \xrightarrow{\varphi^*} S^{-1}\mathcal{E}_0 = S^{-1}S$ is the identity and thus φ^* is a homotopy equivalence.

For $0 \twoheadleftarrow C$: The composition $S^{-1}S \to S^{-1}\mathcal{E}_C \xrightarrow{\varphi^*} S^{-1}\mathcal{E}_0 = S^{-1}S$ sends A to $A \oplus C$ in $S^{-1}S$, which is a homotopy equivalence.

(Contractibility ommitted.) ■

We now get Theorem 2.1 as a consequence of Theorem 2.8.

References

[Wei13] Charles A. Weibel. *The K-book. An introduction to algebraic K-theory.* English. Vol. 145. Grad. Stud. Math. Providence, RI: American Mathematical Society (AMS), 2013. ISBN: 978-0-8218-9132-2.