§ The Weil conjectures

(PhD seminar proposal)

The Weil conjectures are four famous conjectures, proposed by Weil around 1949, regarding the number of $\mathbb{F}_{q^m}$-points of a variety X defined over $\mathbb{F}_q$, where $q = p^n$ for p a prime number and $n \in \mathbb{Z}_{\geq 1}$.
They give informations about the "zeta function" of X, which is a formal power series with coefficients in $\mathbb{Q}$ defined as follows:

Def Let X be a variety over $\mathbb{F}_q$:

(i) Set $N_m := \# X(\mathbb{F}_{q^m})$, where $X(\mathbb{F}_{q^m}) = \mathrm{Hom}_{\mathbb{F}_q}(\mathrm{Spec}\, \mathbb{F}_{q^m}, X)$

(ii) The zeta function of X is
$$Z_X(t) := \exp\left(\sum_{m \geq 1} N_m \frac{t^m}{m} \right) \in \mathbb{Q}[[t]]$$

Now we can state the Weil conjectures

Conjectures (Weil)

1) Rationality: $Z_X(t) \in \mathbb{Q}(t)$. Precisely, $\exists P_0, \dots, P_{2d} \in \mathbb{Q}[t]$ such that
$$Z_X(t) = \frac{P_1(t) \cdot \ldots \cdot P_{2d-1}(t)}{P_0(t) \cdot \ldots \cdot P_{2d}(t)}$$

2) Functional equation: Z_X satisfies the functional equation
$$Z_X\left(\frac{1}{q^d t}\right) = \pm q^{d\chi/2} t^{\chi} Z_X(t),$$
where χ is the Euler characteristic of X

3) Riemann hypothesis: Assume $P_i(0) = 1$ and factor $P_i(t) = \prod_{j=1}^{\deg P_i} (1 - \alpha_{ij} t)$. Then:

- $P_0(t) = 1 - t$
- $P_{2d}(t) = 1 - qt$
- $|\alpha_{ij}| = q^{i/2} \quad \forall i,j$

4) Betti numbers: Suppose $\exists\, \mathcal{Y} \in \mathrm{Sch}_{\mathcal{O}_K}$, where $K/\mathbb{Q}$ finite, $\mathfrak{p} \in \mathrm{Spec}\, \mathcal{O}_K$ over p, s.t.

- $\mathcal{Y}$ flat, projective, gen. smooth $/\mathcal{O}_K$
- $\mathcal{Y} \otimes k(\mathfrak{p}) \cong X$

Then $\deg P_i$ is the i-th Betti number of $(\mathcal{Y} \times_{\mathcal{O}_K} \mathbb{C})(\mathbb{C})$

$\vdash$

Why should you care?

These conjectures motivated Grothendieck and his school to develop the language of schemes. Moreover, they have profound implications, for example the Ramanujan-Petersson conjecture (which gives informations about the growth of the Fourier coefficents of some modular forms)

What theory would you learn?

Mainly étale cohomology. An overview of the seminar could be:

- étale morphisms
- general facts about cohomology of sheaves
- more specific facts about étale cohomology (needed for the proof)
- proof of the conjectures

Which prerequisites should you have?

Not many, I think, probably the first two courses in Algebraic Geometry you had are more than enough.
Some familiarity with homological algebra would be helpful, but there would be one/two talks covering the necessary material.

Possible references:

- Milne - Lectures in Étale Cohomology
- Freitag, Kiehl - Étale Cohomology and the Weil Conjectures

nice introduction by Dieudonné!

p-adic Hodge Theory

Motivation. The objective of this seminar would be to introduce the theory of p-adic Galois representations and the basics of p-adic Hodge theory. Some motivation for studying these concepts can be found in the study of the absolute Galois group of $\mathbb{Q}$: $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Although this group evades direct study, one comes across some special distinguished subgroups $(G_{\mathbb{Q}})_p < G_{\mathbb{Q}}$ for p a prime, which then can be identified with $G_{\mathbb{Q}_p}$. The study of these latter groups, or more generally, of the absolute Galois groups G_K, where K is a finite extension of $\mathbb{Q}_p$, can allow us to deduce more information about $G_{\mathbb{Q}}$ by using the information we obtain at each prime.

In order to study these groups, we make use of *Galois representations*

$$\rho : G_K \to \mathrm{Aut}_E(V),$$

where E is a field and V is an E-vector space. Although we may consider several possibilities for E, we will focus on the case $E = \mathbb{Q}_p$, which leads to the most information. These are the *p-adic Galois representations.*

Overview. Some of the concepts we would see in the seminar are:

- Hodge-Tate representations and decompositions.
- Étale φ-modules.
- Witt vectors.
- Admissible representations.
- Period rings.

Prerequisites. It would be useful to have some background on representation theory and local class field theory.

Goal. This could depend a bit on the structure of the talks and how far we actually get through the material, so there is no specific goal in mind besides understanding the theory.

Applications. One of the more interesting applications resides in the relationship between de Rham and étale cohomology of smooth projective varieties X over $\mathbb{Q}_p$. Thanks to the previously mentioned *period rings*, there is a way to stablish an isomorphism between these two cohomologies:

$$H^i_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \cong H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}},$$

where B_{dR} is the mentioned period ring. Besides this, there are more applications regarding the study of the representations of certain Galois groups, or of the Tate modules of abelian varieties.

References

1 Brinon, O. & Conrad, B., *CMI Summer School notes on p-adic Hodge theory*, 2009.

2 Fontaine, J. & Ouyang, Y., *Theory of p-adic Galois Representations.*

p-divisible groups

Motivation. In classical group theory, a group G is said to be *p-divisible* if the multiplication by p map $[p] : G \to G$ is surjective. One of the most interesting examples is the Prüfer group, defined as the quotient $\mathbb{Q}_p/\mathbb{Z}_p$. In fact, we can express this group as the following inductive limit:

$$\mathbb{Q}_p/\mathbb{Z}_p \cong \varinjlim_k \frac{1}{p^k}\mathbb{Z}/\mathbb{Z}.$$

Interestingly, this group has another property: it is *p-primary torsion*, i.e. every element of $\mathbb{Q}_p/\mathbb{Z}_p$ is a torsion element with order a power of p.

Similarly, if E/K is an elliptic curve, then its p-primary torsion can be written as

$$E(\overline{K})[p^\infty] = \varinjlim_k E(\overline{K})[p^k] \cong \varinjlim_k \left(\mathbb{Z}/p^k\mathbb{Z} \times \mathbb{Z}/p^k\mathbb{Z}\right),$$

where the last isomorphism holds if p is invertible in K. Note that each of the $E(\overline{K})[p^k]$ is a finite group of rank p^{hk}, where $h = 2\dim E = 2$.

These examples motivate the following construction.

Definition 1. A *p-divisible group* over an affine scheme S is an inductive system

$$G = \varinjlim_{v\in\mathbb{N}} G_v,$$

where the G_v are finite flat commutative groups over S such that there exists a natural number h (the *height* of G) satisfying:

(1) G_v has order $p^h v$,

(2) for each v there exists an exact sequence $0 \to G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{[p^v]} G_{v+1}$, with $[p^v]$ being the multiplication by p^v map.

Overview. Here are some of the topics we would cover in this seminar:

- Finite flat group schemes.
- Grothendieck topologies and fpqc sheaves.
- Formal groups.
- Hodge-Tate decompositions.

Prerequisites. In order to go faster through the first sections, it would be useful if we can assume several facts regarding group schemes.

Goal. A nice objective would be to try to understand *Grothendieck-Messings deformation theory*. In summary, in this theory one assigns certain objects to p-divisible groups, which are called *Dieudonné crystals*, and then shows that there is an antiequivalence of categories between the category of p-divisible groups over a field K and the category of Dieudonné crystals. The nice consequence of this

is that working with Dieudonné crystals is usually nicer than with p-divisible groups.
Applications. Some useful places where p-divisible groups appear are local class field theory, moduli spaces of abelian varieties... For more precise examples, the reader may check Jakob Stix's notes.

References

1 Berthelot P., Messing W., *Théorie de Dieudonné Cristalline I*, Journées de Géométrie algébrique, Rennes, 1978 (Astérisque Vol. 63).

2 Berthelot P., Breen L., Messing W., *Théorie de Dieudonné Cristalline I*, Lecture Notes in Mathematics, Springer Verlag.

3 Stix J., *A course on finite flat group schemes and p-divisible groups*, Lecture Notes.

Compactifications of Locally Symmetric Varieties

Locally symmetric varieties are certain double quotients of the form

$$X := \Gamma \backslash G/K,$$

where $G = \mathbf{G}(\mathbb{R})$ denotes the real points of a semisimple algebraic group $\mathbf{G}$, $K \subseteq G$ is a maximal compact subgroup and $\Gamma \subseteq G$ is a discrete subgroup. The simplest example is

$$M := \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{H},$$

where $\mathbb{H} \subseteq \mathbb{C}$ denotes the upper half-plane and $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ through Moebius transforms.

Somewhat surprisingly, despite the fact that all the above groups are real, this quotient is naturally a *complex* algebraic variety; in fact it is just $\mathbb{A}^1(\mathbb{C})$. This is true more generally provided that G/K is a *Hermitean symmetric domain* of classical type and Γ is *arithmetic*. Prominent examples of such X are (the complex points of) *Shimura varieties*.

In any case, locally symmetric spaces are very interesting as they feature prominently in numerous different mathematical areas and can be understood and studied from various angles:

- Moduli Theory: M is the (coarse) moduli space of elliptic curves,
- Automorphic Representation Theory: M carries a line bundle $\mathcal{L}$ and sections of $\mathcal{L}^{\otimes k}$ are precisely modular functions of weight k on $\mathbb{H}$,
- Riemannian Geometry: M is a complete Riemannian manifold with one *end*, a so-called *cusp*.

Note that we may compactify $M \cong \mathbb{A}^1(\mathbb{C}) \subseteq \mathbb{P}^1(\mathbb{C}) =: \overline{M}$. Now, also $\overline{M}$ has a natural interpretation in all of the above pictures:

- Moduli Theory: $\overline{M}$ is the (coarse) moduli space of semistable curves of genus one,
- Automorphic Representation Theory: $\mathcal{L}$ extends to a line bundle $\overline{\mathcal{L}}$ on $\overline{M}$ such that sections of $\overline{\mathcal{L}}^{\otimes k}$ are precisely modular forms of weight k on $\mathbb{H}$,
- Riemannian Geometry: $\overline{M}$ is the geodesic completion of M; the single cusp corresponds precisely to the only proper parabolic subgroup $P \subset \mathrm{SL}_2(\mathbb{R})$.

This picture generalises as follows: X can always be compactified into a normal, proper, algebraic variety $\overline{X}^{BBS}$ by adding finitely many symmetric spaces of lower dimension in the boundary corresponding to the rational parabolic subgroups of $\mathbf{G}$. Moreover, $\overline{X}^{BBS}$ is projective. It is called the *Bailey-Borel-Compactification* of X. Unfortunately, it is usually not smooth and so we will also try to understand different choices of compactifications $\overline{X}$ which are smooth.

The primary goal of this seminar would be to try to understand X and its compactifications as explicitly as possible and ultimately prove the existence and projectivity of the above $\overline{X}$.

To do so, in the first part of the seminar we would spent some time on learning about reductive algebraic groups, their representation theory and also toric varieties. These methods will allow us to understand the complicated geometry of X in terms of discrete data later on.

The tools we would use are mostly from representation theory and/ or discrete/ convex geometry. We will try to avoid Riemannian geometry whenever possible, thought it might be nice to see some parts of this story as well. In any case, we will try to keep the prerequisites minimal (for the most part it should be fine to think about classical varieties instead of schemes for example).

At the end of the term there should be some time left to get a peak at how the understanding of X and its compactifications are useful to study automorphic representation theory and/or what $\overline{X}\backslash X$ parametrises from the moduli theoretic perspective.

References

[1] Ash A., Mumford D., Rapoport M., Tai Y., *Smooth Compactifications of Locally Symmetric Varieties* (with the collaboration of P. Scholze), 2^{nd} Edition, Cambridge Mathematical Library, 2010.

Intersection Theory

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The idea is to learn some classical intersection theory by proving one of the following theorems:

Theorem (Grothendieck-Riemann-Roch)**.** *Let $f\colon X \to Y$ be a proper morphism of non-singular varieties. Then for all $\alpha \in K(X)$, we have*

$$\mathrm{ch}(f_*\alpha) \cdot \mathrm{td}(T_Y) = f_*(\mathrm{ch}(\alpha) \cdot \mathrm{td}(T_X)).$$

Theorem (Hirzebruch-Riemann-Roch)**.** *Let E be a vector bundle on a non-singular complex variety X. Then*

$$\chi(X, E) = \int_X \mathrm{ch}(E) \cdot \mathrm{td}(T_X).$$

During the seminar, we will not only learn what the notation in these theorems means but we will also explore some more topics in algebraic geometry:

- blow ups
- intersection products
- Chow groups and Chow rings
- Chern classes

The main reference for the seminar will be either Fulton's book *Intersection Theory* or the book by Eisenbud and Harris *3264 and all that.*

A quadratic Bézout's Theorem

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The classical Bézout's theorem is given as follows:

Theorem (Bézout)**.** *Fix an algebraically closed field k. Let $f_1, \dots, f_n$ be hypersurfaces in $\mathbb{P}^n$, and let d_i be the degree of f_i for each i. Assume that $f_1, \dots, f_n$ have no common components, so that $f_1 \cap \cdots \cap f_n$ is a finite set. Then, summing over the intersection points of $f_1, \dots f_n$, we have*

$$\sum_{points} i_p(f_1, \dots, f_n) = d_1 \cdots d_n,$$

where $i_p(f_1, \dots, f_n)$ is the intersection multiplicity of $f_1, \dots, f_n$ at p.

It is a bit of a pity that this theorem only works for algebraically closed fields. In order to remove that assumption, one needs to move the computation to the Grothendieck-Witt ring of quadratic forms. There one can obtain the following result:

Theorem. *Fix a perfect field k. Let $\sum_{i=1}^n d_i \equiv n+1 \mod 2$, and let $f_1, \dots, f_n$ be hypersurfaces in $\mathbb{P}^n$ of degree $d_1, \dots, d_n$ that intersect transversely. Given an intersection point p of $f_1, \dots, f_n$, let $J(p)$ be the signed voulme of the parallelipiped determined by the gradient vectors of $f_1, \dots, f_n$ at p. Then summing over the interseciton points of $f_1, \dots, f_n$, we have*

$$\sum_{points} \mathrm{Tr}_{k(p)/k}\langle J(p)\rangle = \frac{d_1 \cdots d_n}{2} H \in \mathrm{GW}(k),$$

where H is the hyperbolic form $\langle 1\rangle + \langle -1\rangle$ and $\mathrm{Tr}_{k(p)/k} \colon \mathrm{GW}(k(p)) \to \mathrm{GW}(k)$ is given by post-composing with the field trace.

For k algebraically closed, this gives back the classical Bézout's Theorem. For $k = \mathbb{R}$ or k a finite field, this gives back versions of Bézout's Theorem that were already known. Therefore this theorem not only provides a Bézout's Theorem for all perfect fields but also unifies the statements of the earlier-known Bézout's theorems.

The idea for a seminar on this topic would be to study the proof of this theorem and to, along the way, also learn more about intersection theory over arbitrary (perfect) fields. Furthermore, we will learn about how intersection theory can be done in the quadratic setting.

The main source for the seminar will be Stephen McKean's paper *An arithmetic enrichment of Bézout's Theorem*. The paper is not that long, so that we will have ample time to study the necessary prerequisites to understand the proof.