Hind Souly

Talk 11: Representations of the Weil group 2

§ Abstract Machinery:

Let G be a profinite group.

Let $K_0 G$ be the Grothendieck group of G, i.e. the free abelian gp on symbols $[\rho]$, where ρ goes through the set of iso. classes of irreducible smooth reps of G. (a.k.a. gp of virtual reps)

Rmk: We view the set of iso. classes of finite dimensional smooth reps of G as contained in $K_0 G$. If ρ is such rep., then we write

$$\rho = \underbrace{\rho_1 \oplus \dots \oplus \rho_r}_{\text{irred.}}$$

$$\rightsquigarrow \quad [\rho] = \sum_{i=1}^{r} [\rho_i] \in K_0 G.$$

Abuse of notation: drop the brackets and write $[\rho] = \rho$.

$\exists$ dim. map $K_0 G \longrightarrow \mathbb{Z}$ defined in a natural way.

Define $$\tilde{K}_0 G := \bigcup_{\substack{H \leq G \\ \text{open}}} K_0 H$$

and denote its elements as pairs (H, ρ), where $H \leq G$ open and $\rho \in K_0 H$

- $\Gamma(G) := \mathrm{Hom}(G, \mathbb{C}^\times) \subseteq K_0 G$

so $$\tilde{\Gamma}(G) := \bigcup_{\substack{H \leq G \\ \text{open}}} \Gamma(H) \subset \tilde{K}_0 G$$

Def: Let A be an abelian gp, G a profinite gp.

a) An induction constant on G (with values in A) is a function

$$\mathcal{F} : \tilde{K}_0 G \longrightarrow A$$

s.t.: $\forall H \leq G$ open, $\mathcal{F}|_{K_0 H}$ is a gp homomorphism

and if $H \subseteq J$ are open subgps of G and $(H, \rho) \in K_0 H$ has dimension 0, then $\mathcal{F}(J, \mathrm{Ind}_H^J \varrho) = \mathcal{F}(H, \varrho)$

b) A division on G (with values in $\mathcal{A}$) is a function
$$\mathcal{D} : \tilde{\Gamma}(G) \longrightarrow \mathcal{A}.$$

* An induction constant $\mathcal{F}$ gives rise to a division $\partial\mathcal{F}$ via restriction, and $\partial\mathcal{F}$ is called the boundary of $\mathcal{F}$.
* A division $\mathcal{D}$ on G is said to be pre-inductive on G if it is of the form $\mathcal{D} = \partial\mathcal{F}$, for some induction constant $\mathcal{F}$ on G.

Lemma 1: Let G be profinite, $\mathcal{D}$ a division on G. Assume $\exists$ a family $\mathcal{H}$ of open normal subgps H of G, s.t.:
- $G \longrightarrow \varprojlim_{H\in\mathcal{H}} G/H$ is an isomorphism
- the restriction $\mathcal{D}_{G/H}$ of $\mathcal{D}$ to $\tilde{\Gamma}(G/H)$ is pre-inductive on G/H for all $H \in \mathcal{H}$.

Then $\mathcal{D}$ is pre-inductive on G. If $\mathcal{F}$ is the induction constant on G with boundary $\mathcal{D}$, then $\mathcal{D}_{G/H}$ is the boundary of $\mathcal{F}|_{\tilde{R}_0 G/H}$.

§ Main statement: Existence of the Local constant:

Let E/F be a finite separable extension.
For $\psi \in \hat{F}$, we set $\psi_E = \psi \circ \mathrm{Tr}_F^E \in \hat{E}$
And we let $\mathcal{G}^{ss}(F) = \bigcup_{n\geq 1} \mathcal{G}_n^{ss}(F)$

where $\mathcal{G}_n^{ss}(F)$: iso classes of semisimple reps of dim n.

Thm 1: Let $E|F$ run through finite extensions inside $\bar{F}$, and let $\psi \in \hat{F}$, $\psi \neq 1$. Then $\exists$ a unique family of functions

$$\mathcal{G}^{ss}(E) \longrightarrow \mathbb{C}[q^s, q^{-s}]^\times$$
$$\rho \longmapsto \varepsilon(\rho, s, \psi_E)$$

satisfying the following properties:

(i) If $\chi \in \hat{\mathcal{I}}^\times$, then

$$\varepsilon(\chi \circ a_E, s, \psi_E) = \varepsilon(\chi, s, \psi_E)$$

(ii) If $\rho_1, \rho_2 \in \mathcal{G}^{ss}(E)$, then

$$\varepsilon(\rho_1 \oplus \rho_2, s, \psi_E) = \varepsilon(\rho_1, s, \psi_E) \cdot \varepsilon(\rho_2, s, \psi_E)$$

(iii) If $\rho \in \mathcal{G}^{ss}_n(E)$, $F \subset K \subset E$ is a tower of finite extensions,

then:
$$\frac{\varepsilon(\mathrm{Ind}_{W_E}^{W_K} \rho, s, \psi_K)}{\varepsilon(\rho, s, \psi_E)} = \frac{\varepsilon(\mathrm{Ind}_{W_E}^{W_K} \mathbb{1}_E, s, \psi_K)^n}{\varepsilon(\mathbb{1}_E, s, \psi_E)^n}$$

- If $\rho \in \mathcal{G}^{ss}(F)$, we call $\varepsilon(\rho, s, \psi)$ the Langlands-Deligne local constant of ρ (relative to $\psi \in \hat{F}$ and s)

We enumerate some of its interesting properties:

Prop: Let $\psi \in \hat{F}$, $\psi \neq 1$ and $\rho \in \mathcal{G}^{ss}(F)$. Then:

a) $\exists\, n(\rho, \psi) \in \mathbb{Z}$ s.t.

$$\varepsilon(\rho, s, \psi) = q^{n(\rho,\psi)(\frac{1}{2}-s)}\, \varepsilon(\rho, \tfrac{1}{2}, \psi)$$

b) Let $a \in F^\times$. Then:

$$\varepsilon(\rho, s, a\psi) = \det \rho(a)\, \|a\|^{\dim(\rho)(s-\frac{1}{2})}\, \varepsilon(\rho, s, \psi)$$

$$n(\rho, a\psi) = n(\rho, \psi) + v_F(a) \dim \rho$$

c) We have, moreover, a functional equation:

$$\varepsilon(\rho, s, \psi)\ \varepsilon(\check{\rho}, 1-s, \psi) = \det \rho(-1)$$

d) $\exists\, n_\rho \in \mathbb{Z}$ s.t if $\chi \in \hat{F}^\times$ of level $k \geq n_\rho$,

then $\varepsilon(\chi \otimes \rho, s, \psi) = \det \rho(c(\chi))^{-1} \varepsilon(\chi, s, \psi)^{\dim \rho}$

for any $c(\chi) \in F^\times$ s.t. $\chi(1+x) = \psi(c(\chi)x)$,

$x \in \mathfrak{p}^{[\frac{k}{2}]+1}$.

Let $L|F$ be finite + Galois, $G = \mathrm{Gal}(L|F)$

By LCFT, we know that $G^{ab} = (G_F / G_L)^{ab} = F^\times / N_F^L(L^\times)$

and hence we may view $\tilde{\Gamma}(G)$ as the set of

pairs (E, χ), where E runs through the intermediate fields

$F \subseteq E \subseteq L$ and χ through the characters of $E^\times$ which are trivial

on $N_E^L(L^\times)$.

We assume the next result to prove the assertions above:

Thm 2: Let $L|F$ be a finite Galois extension with galois gp $G = \mathrm{Gal}(L|F)$. $\Rightarrow \exists\, \psi \in \hat{F}$, $\psi \neq 1$, s.t the following division on G

$$\mathcal{D}_\psi^{L|F} : (E, \chi) \longmapsto \varepsilon(\chi, s, \psi_E)$$
$$\tilde{\Gamma}^\wedge(G)$$

is pre-inductive on G.

- We want to get rid of the restriction on ψ.

By Lemma 1 and Thm 2, the division $\mathcal{D}_\psi^{L|F} : (E, \chi) \mapsto \varepsilon(\chi, s, \psi_E)$ is preinductive on $G_F = \varprojlim_{L|F \text{ finite Galois}} \mathrm{Gal}(L|F) = \varprojlim_{L|F \text{ finite Galois}} G_F / G_L$

Moreover, $\mathcal{D}^{L|F}_{\varphi}$ is the boundary of the induction constant

$$(G_E, \rho) \longmapsto \varepsilon(\rho, s, \psi_E)$$

Now let $a \in F^\times$ and define the function

$$(E, \rho) \longmapsto \det \rho(a) \, \|a\|_E^{(s-\frac{1}{2})\dim \rho}, \quad (E,\rho) \in \widetilde{K}_0 \mathcal{R}_F.$$

Clearly, this is an induction constant on $\mathcal{G}_F$.

Thus, $(E,\rho) \mapsto \det \rho(a) \, \|a\|_E^{(s-\frac{1}{2})\dim \rho} \cdot \varepsilon(\rho, s, \psi_E)$

is also an induction constant.

The boundary of this latter is

$$(E, \chi) \longmapsto \chi(a) \, \|a\|_E^{(s-\frac{1}{2})} \, \varepsilon(\chi, s, \psi_E) = \varepsilon(\chi, s, a\psi_E)$$

$\rightsquigarrow$ Prop 1 (b)

Hence, this division is pre-inductive and the boundary of the ind. const. $(E, \rho) \mapsto \varepsilon(\rho, s, a\psi_E)$.

So Thm 2 holds for all $\psi \in \hat{F}$, $\psi \neq 1$ and this proves Thm 1 for reps of Galois gps.

Prop 1 a) has already been discussed in Giulio's talk.

Next goal: Extend these results to reps of Weil gps.

Fix $\varpi \in F$ a uniformizer. Let $\phi \in \hat{F}^\times$ be unramified.

Write $\phi(\varpi) = q^{-s(\phi)}$, for some $s(\phi) \in \mathbb{C}$.

For $E|F$ finite, $\varpi_E \in E$ uniformizer, we also have

$$\phi_E(\varpi_E) = q_E^{-s(\phi)}$$

Hence, if $\chi \in \hat{E}^\times$, we have

$$\varepsilon(\chi\phi_E, s, \psi_E) = \varepsilon(\chi, s+s(\phi), \psi_E) \qquad (*)$$

boundary

Claim 1: Let $(G_E, \rho) \in \tilde{K}_0 G_F$, let $\phi \in \hat{F}^\times$ be unramified + of finite order. Then $\varepsilon(\phi_E \otimes \rho, s, \psi_E) = \varepsilon(\rho, s+s(\phi), \psi_E)$

Pf: They are both induction constants with the same boundary. Hence, they are equal. ∎

Now let $\mathbb{1}_E$ be the trivial character of the Weil gp W_E.

and define
$$\lambda_{E|F}(s,\psi) = \frac{\varepsilon(\mathrm{Ind}_{W_E}^{W_F} \mathbb{1}_E, s, \psi)}{\varepsilon(\mathbb{1}_E, s, \psi_E)}$$

Claim 2: $\lambda_{E|F}(s,\psi)$ is constant in s.

Pf: If $\phi \in \hat{F}^\times$ is unram + of finite order

$$\Rightarrow \phi \otimes \mathrm{Ind}_{W_E}^{W_F} \mathbb{1}_E \simeq \mathrm{Ind}_{W_E}^{W_F} \phi_E.$$

$$\Rightarrow \lambda_{E|F}(s,\psi) = \frac{\varepsilon(\mathrm{Ind}_{W_E}^{W_F} \phi_E, s, \psi)}{\varepsilon(\phi_E, s, \psi_E)}$$
$$\overset{(*)}{=} \frac{\varepsilon(\mathrm{Ind}_{W_E}^{W_F} \mathbb{1}_E, s+s(\phi), \psi)}{\varepsilon(\mathbb{1}_E, s+s(\phi), \psi_E)}$$
$$= \lambda_{E|F}(s+s(\phi), \psi)$$

$\rightsquigarrow$ $\lambda_{E|F}(s+\xi, \psi) = \lambda_{E|F}(s,\psi)$ for all roots of unity $\xi \in \mathbb{C}$.

But $\lambda_{E|F}(s,\psi) = \text{const} \cdot \times (q^{\frac{1}{2}-s})^{\text{power}}$

so this proves our claim. ∎

$\rightsquigarrow$ $\lambda_{E|F}(\psi)$ is called the Langlands constant.

$$\rightsquigarrow \frac{\varepsilon(\mathrm{Ind}_{W_E}^{W_K} \rho, s, \psi_K)}{\varepsilon(\rho, s, \psi_E)} = \lambda_{E|K}(\psi_K)^n$$

for $\rho \in \mathcal{G}_n^{ss}(E)$

Func. equation in Prop

$\overset{\sim}{\underset{n=2}{\to}}$ $\lambda_{E|F}(\Psi)^2 = \det(\mathrm{Ind}_{E|F} 1_E)(-1)$

has order 2

$\Rightarrow$ $\lambda_{E|F}(\Psi)$ is a 4-th root of unity.

- Let ρ be an irred. smooth rep of W_E.

Let $\Phi \in W_E$ be a Frobenius. $\exists\, k \geq 1$, s.t.

$\rho(\Phi)^k$ commutes with $\rho(W_E)$

$\xrightarrow[\text{Schur's lemma}]{}$ $\rho(\Phi)^k = c$

Let $\chi \in \widehat{W_E}$ be unramified, s.t. $\chi(\Phi)^k = c^{-1}$

$\Rightarrow$ $\chi \otimes \rho(W_E)$ is finite.

$\xRightarrow[\text{d'après Giulio}]{}$ $\chi \otimes \rho$ factors through a rep ρ_0 of G_E.

So by claim 1, we may identify $\varepsilon(\rho, s, \psi_E) = \varepsilon(\rho_0, s - s(\chi), \psi_E)$

$\rightsquigarrow$ so we reduced it to the Galois case.

Notation:

$\mathcal{G}_n^0(F)$: set of iso-classes of irreducible smooth reps of W_F of dimension n.

$\mathcal{G}_n(F)$: set of equiv. classes of n-dim, semisimple, Deligne representations of W_F.

$\mathcal{G}_2^{nr}(F) \subset \mathcal{G}_2^0(F)$: $\rho \in \mathcal{G}_2^{nr}(F)$ if $\exists\, \chi \neq 1$ a character of W_F s.t. $\rho \otimes \chi \cong \rho$.

If $\rho \in \mathcal{G}_2^0(F) \setminus \mathcal{G}_2^{nr}(F)$, then ρ is said to be totally ramified.

Def: A pair $(E|F, \chi)$, where $E|F$ is quadratic + tamely ramified and $\chi \in \hat{E}^\times$ is admissible if:

- χ doesn't factor through $N_{E|F}$
- if $\chi|_{U_E^1}$ does factor through $N_{E|F} \Rightarrow E|F$ is unramified

$\mathbb{P}_2(F)$: set of iso-classes of admissible pairs

If $(E|F, \xi) \in \mathbb{P}_2(F)$, ξ may me regarded as a character of W_E $\rightsquigarrow$ $\rho_\xi = \mathrm{Ind}_{W_E}^{W_F} \xi$

Thm 3: If $(E|F, \xi)$ is an admissible pair, the representation ρ_ξ is irreducible. Moreover, the map $(E|F, \xi) \mapsto \rho_\xi$ induces a bijection

$$\mathbb{P}_2(F) \xrightarrow{\sim} \mathcal{G}_2^0(F) \quad \text{if } p \neq 2$$
$$\mathbb{P}_2(F) \xrightarrow{\sim} \mathcal{G}_2^{nr}(F) \quad \text{if } p = 2.$$

Lemma: Let $(E|F, \xi)$ be an adm. pair. Let $\chi = \chi_{E|F}$ be the non-trivial character of $F^\times$ which is trivial on $N_{E|F}(E^\times)$

$$\Longrightarrow \rho_\xi \cong \chi \otimes \rho_\xi.$$

In particular, $\rho_\xi \in \mathcal{G}_2^{nr}(F) \Longleftrightarrow E|F$ unramified.

Pf: $\chi \otimes \rho_\xi \cong \mathrm{Ind}_{W_F}^{W_E}(\chi_E \otimes \xi)$, where $\chi_E = \chi \circ N_{E|F} = 1$

On the other hand, let $\rho_\xi \in \mathcal{G}_2^{nr}(F)$ and χ the unram. quadratic character of $F^\times$. If $\langle\sigma\rangle = \mathrm{Gal}(E|F)$

$$\Longrightarrow \xi^\sigma / \xi = \chi_E.$$

$$\Longrightarrow \xi|_{U_E^1} \text{ factors through } N_{E|F}$$

Thus, by definition of admissible pairs, $E|F$ is unramified.

Proof of Thm 3:

- Let $(E|F, \xi) \in \mathbb{P}_2(F)$, $\sigma \in Gal(E|F)$, $\sigma \neq 1$.
  ξ doesn't factor through $N_{E|F}$
  $\Rightarrow$ ξ, ξ^σ are distinct.

  Note that the Artin map $W_E^{ab} \cong E^\times$ is $Gal(E|F)$-equivariant.
  $\Rightarrow$ ξ, ξ^σ of W_E are distinct

  $\underset{\text{Mackey theory}}{\Longrightarrow}$ $\rho_\xi = \mathrm{Ind}_{W_E}^{W_F} \xi$ is irreducible.

- Injectivity: Let $(E_i|F, \xi_i) \in \mathbb{P}_2(F)$, $i = 1, 2$ and assume $\rho_{\xi_1} \cong \rho_{\xi_2}$.
  - if $E_1|F \cong E_2|F$, we may take $E_1 = E_2 = E$.
    $Res_{E|F}\, \rho_{\xi_1} = \xi_1 \oplus \xi_1^\sigma$, where $Gal(E|F) = \langle \sigma \rangle$
    $\Rightarrow$ $\xi_2 \in \{\xi_1, \xi_1^\sigma\}$
    $\Rightarrow$ $(E|F, \xi_2) \cong (E|F, \xi_1)$.
  - Now suppose $E_1 \neq E_2$ and let $L = E_1 E_2$.
    At least one of the $E_i|F$ is totally ramified.

$\Rightarrow$ $[L:F] = 4$ and the max. unram. sub-ext $E|F$ of $L|F$ has degree 2.

why? Assume $E_2|F$ is totally ramified (it is also tamely ramified by definition of adm. pairs).

$\Rightarrow$ $e(E_2|F) = 2$

$\rightsquigarrow$ $e(E_1|F) \mid 2 = e(E_2|F) \overset{\text{Abhyankar's lemma}}{\Longrightarrow} E_1E_2|E_1$ is unramified

Let $\chi_i = \chi_{E_i|F}$.

Observe that $\rho = \rho_{\xi_i}$ is fixed under tensoring with χ_1 and χ_2 and hence also with $\chi_{E|F}$.

$\Rightarrow \rho \in \mathcal{G}_2^{nr}(F)$

$\xRightarrow{\text{Lemma}}$ $E_i = E$ for $i = 1,2$. $\divideontimes$ $\rightsquigarrow$ injectivity.

- Surjectivity: If $\rho \in \mathcal{G}_2^{nr}(F)$ (recap: Frob. rec + irred.)

  $\Rightarrow \rho \cong \mathrm{Ind}_{W_E}^{W_F} \xi$, where $E|F$ quadratic.

  $\Rightarrow (E|F, \xi)$ is admissible and $\rho \cong \rho_\xi$.

- If ρ is tot. ramified (in this case $p \neq 2$)

  As $\dim \rho = 2$, $\rho | \mathcal{P}_F$ (pro p-gp) decomposes as a sum of characters.

  $\rightarrow \exists\, K|F$ finite + tamely ramified + Galois s.t $\rho | W_F = \theta \oplus \theta'$

- Suppose $\theta \neq \theta' \Rightarrow$ the W_F-stabilizer of θ is θ_L, for some quadratic ext. $L|F$ $\underset{K}{\cap}$

The natural rep of W_L in the θ-isotypic subspace of ρ provides a character ξ of W_L s.t $\rho = \mathrm{Ind}_{W_L}^{W_F} \xi$.

ξ may be viewed as a char. of $L^\times$.

Aim: $(L|F, \xi)$ is admissible.

Let $\sigma \in \mathrm{Gal}(L|F)$, $\sigma \neq 1$. $\xi^\sigma | W_K = \theta'$

$\Rightarrow \xi \neq \xi^\sigma$ and ξ doesn't factor through $N_{L|F}$.

Since ρ is totally ramified, $L|F$ is totally ramified.

Hence, if $\xi|U_L^1$ factors through $N_{L|F}$,

then $\dfrac{\xi^\sigma}{\xi} = \text{triv}$ on $U_L = U_F U_L^1$ and is unramified.

$\Rightarrow \xi^\sigma = \chi_L \xi$ for some $\underset{\neq 1}{\chi} \in F^\times$, unramified.

$\Rightarrow \quad \rho \cong \chi \otimes \rho$ and $\rho \in \mathcal{G}_2^{nr}(F)$ ↯

$\Rightarrow \quad (L|F, \xi)$ is admissible and $\rho \cong \rho_\xi$.

- Need to exclude $\Theta = \Theta'$. Let $L|F$ be the max. unramified sub-ext of $K|F$.

  As $K|L$ is cyclic, Θ admits extension to a char. ξ of W_L, s.t. ξ occurs in $\rho|W_L$.

  i.e. we may have chosen $K|F$ to be unramified.

  However, this would imply $\rho \in \mathcal{G}_2^{nr}(F)$ ↯.

Reference: Bushnell - Henniart, The Local Langlands conjecture for GL_2.