§ Jaquet-Langlands correspondence & (Spectral Side of the) Trace Formula

Over the last weeks we discussed and proved the local Langlands conjecture for GL_2. In the last two talks we want to give an outlook towards more general Langland's-type phenomena: The **Jaquet-Langlands corresp.** This is a special instance of Langland's functoriality. It will require us to consider both the global world and more general reductive groups.
Here is the statement:

1. Jaquet-Langlands Corresp.

(D_v is a skew-field with centre $\mathbb{Q}_v$, $[D_v : \mathbb{Q}_v] = 2^2 = 4$)

Theorem (Local Jaquet-Langlands)
Let D_v be the unique non-split quaternion algebra over $\mathbb{Q}_v$, v some place of $\mathbb{Q}$. Then

$$\exists! \left\{ \begin{matrix} \text{irred. smooth rep's of } D_v^\times(\mathbb{Q}_v) \\ \text{of } \dim \neq 1 \end{matrix} \right\} \xleftrightarrow{1:1} \{\text{irred. discrete series rep's of } GL_2(\mathbb{Q}_v)\}$$

(cuspidal or $\phi \cdot St_G$)

Theorem: (Global Jaquet-Langlands)
Let D be a quaternion algebra $/\mathbb{Q}$, let S be the finite(!) set of places v s.t. $D \otimes \mathbb{Q}_v \not\cong M_2(\mathbb{Q}_v)$. Then there is a unique bijection

$$\left\{ \begin{matrix} \text{irreducible automorphic} \\ \text{(automatically cuspidal) rep's of} \\ D^\times(\mathbb{A}) \\ \text{of } \dim \neq 1 \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \begin{matrix} \text{irred. automorphic cuspidal rep's } \pi \text{ of} \\ GL_2(\mathbb{A}) \\ \text{s.t. } \pi_v \text{ belongs to the discrete series } \forall s \in S \end{matrix} \right\}$$

(~ smooth)

s.t. $\pi'_v = \pi_v \quad \forall v \notin S$

Remarks:

- $\mathbb{A} := \prod'_p \mathbb{Q}_p \times \mathbb{R} := \{(x_v) \in \prod_v \mathbb{Q}_v \mid x_p \in \mathbb{Z}_p$ for all but fin. many $p\}$ is called the ring of **adeles** of $\mathbb{Q}$. With the subspace topology inherited from $\mathbb{A} \subseteq \prod_v \mathbb{Q}_v$, $\mathbb{A}$ becomes a locally compact Hausdorff topological space

  ↳ $\prod_v \mathbb{Q}_v$ is not locally compact, hence no good for our purposes

$\rightsquigarrow$ π (irred.) rep' of $G(\mathbb{A}) \longleftrightarrow \pi = \bigotimes \pi_v$, $\pi_v \in \mathrm{Rep}(G_v)$

The terminology regarding representations of $G(\mathbb{A})$ will be discussed later

(2) It is true that $\{$Characters of $D^\times(\mathbb{A})\} = \{$Characters of $GL_2(\mathbb{A})\}$.
The reason we exclude them in the Global JL-correspondence is that characters of $GL_2(\mathbb{A})$ are never cuspidal

(3) We will stick with $\mathbb{Q}$ for concreteness, however all statements in this talk hold true over any number field

(4) There are similar statements for the groups SL_2, PGL_2 resp. $SD^\times$ (= units of norm one), $PD^\times$. In fact, for the rest of the talk we will stick to SL_2. This doesn't simplify anything significantly but gets rid of some technicalities in some definitions.
In any case, note that in dealing with SL_2 instead of GL_2 we don't loose much: GL_2 is basically just SL_2 + a central torus ← always acts via a character

$\Rightarrow$ rep's of GL_2 are essentially just twists of rep's of SL_2 by characters

2. Spectral theory

Fix
- G locally compact, unimodular top. group
- $\Gamma \subseteq G$ a discrete subgroup

Ex: $SL_2(\mathbb{Z}) \subseteq SL_2(\mathbb{R})$, $\{1\} \subseteq SL_2(\mathbb{Q}_p)$, $SL_2(\mathbb{Q}) \subseteq SL_2(\mathbb{A})$

We consider $L^2(\Gamma\backslash G) := \{ f: G \to \mathbb{C} : f(\gamma g) = f(g)\ \forall \gamma \in \Gamma, \int_{\Gamma\backslash G} |f|^2 d\mu < \infty \}$

($L^2(\Gamma\backslash G)$: Hilbert space; μ: Haar meas. on G)

and the **regular rep'** R of G on $L^2(\Gamma\backslash G)$, $(R(g)\cdot f)(h) = f(hg)$

(R unitary, i.e. preserves the inner product !)

Spectral theory: Study of how R decomposes into subrep's ?

Def: An irred. representation (π, V) of G is
- **smooth automorphic** $\iff (\pi, V) \subseteq \mathcal{A}(\Gamma\backslash G) = \{ f: \Gamma\backslash G \to \mathbb{C} \mid f \text{ 'smooth'} \}$
- **discrete series** $\iff (\pi, V) \underset{\text{closed}}{\subseteq} L^2(\Gamma\backslash G) \iff (\pi, V) \subseteq L^2(\Gamma\backslash G) \cap \mathcal{A}(\Gamma\backslash G)$
- **cuspidal** $\iff (\pi, V) \subseteq \mathcal{A}_0(\Gamma\backslash G) = \{ f: \Gamma\backslash G \to \mathbb{C} : f \text{ decays rapidly @ } \infty \}$ ($\mathcal{A}_0 \subseteq L^2(\Gamma\backslash G)$)

Ex: • $G = \mathbb{R}, \Gamma = \mathbb{Z} \implies L^2(S^1) = \widehat{\bigoplus_{n\in\mathbb{Z}}} \mathbb{C}\cdot e^{2\pi i n(-)} \cong L^2(\mathbb{Z})$, $f = \sum_n a_n e^{2\pi i n(-)} \longmapsto (a_n)$

Observe: $L^2(S^1)$ decomposes discretely, all rep's of S^1 appear

• $G = \mathbb{R}, \Gamma = \{1\} \implies L^2(\mathbb{R}) = \int_{x\in\mathbb{R}}^{\oplus} \mathbb{C}\, e^{2\pi i x(-)} \xrightarrow{\sim} L^2(\mathbb{R})$, $f = \int_{\mathbb{R}} a(x) e^{2\pi i x(-)} dx \mapsto a$

Note: $\mathbb{R} \curvearrowright L^2(\mathbb{R})$ is irreducible

Observe: $L^2(\mathbb{R})$ 'decomposes' continously, all rep's of $\mathbb{R}$ appear

Slogan: R knows essentially everything about the rep's of G

2.1. Compact case

Assume that $\Gamma\backslash G$ is compact $\left(\text{Ex: } \begin{matrix} SD^\times(\mathbb{Q}) \subseteq SD^\times(\mathbb{A}) \\ \{1\} \subseteq S^3 \subseteq \mathbb{H}^\times \\ \{1\} \subseteq SD_p^\times(\mathbb{Q}_p) \end{matrix} \right)$

Theorem: (Peter-Weyl)

$$L^2(\Gamma\backslash G) = \widehat{\bigoplus_\pi} m_\pi \cdot \pi := \widehat{\bigoplus_\pi} V_\pi^{\oplus m_\pi}$$

where π runs through (iso-classes) of irreducible (unitary) rep's of G. Moreover, $m_\pi < \infty$

Proof: (Sketch)

Given $f \in C_c^\infty(G)$ we have $R(f) : L^2(\Gamma\backslash G) \to L^2(\Gamma\backslash G)$

i.e. $$(R(f)\phi)(x) := \int_G f(y)(R(y)\phi)(x)\, d\mu(y) = \int_G f(y)\phi(xy)\, dy$$
$$= \int_G f(x^{-1}y)\phi(y)\, dy = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\phi(y)\, dy$$

is an 'integral operator with **kernel**'

$$K_f(x,y) := \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y) \qquad x,y \in \Gamma\backslash G$$

sum is really finite because supp f is cpt.

In this case $R(f) : L^2(\Gamma\backslash G) \longrightarrow L^2(\Gamma\backslash G)$ is a 'compact operator'. (i.e. a limit of lin. operators with fin. dim. image)

(This is true more generally provided that $K_f \in L^2(\Gamma\backslash G \times \Gamma\backslash G)$ ← violated e.g. for $\{1\} \subseteq \mathbb{R}$)

Fact: (**Spectral thm for cpt. operators**)

(i) If $R(f) \neq 0$ then $R(f)$ has an eigenvalue $\lambda \neq 0$

(ii) Each eigenspace E_λ of $R(f)$ for $\lambda \neq 0$ is fin. dim.

Now: $L^2(\Gamma\backslash G) = \widehat{\bigoplus_\pi} m_\pi \pi$:

Since $G \curvearrowright L^2(\Gamma\backslash G)$ is unitary, it suffices to show that there exists a min. closed G-invariant subspace $V \subseteq L^2$

Indeed, pick f s.t. $R(f) \neq 0$, let $\lambda \neq 0$ be an eigenvalue.
Consider the set $\{P \subseteq L^2 \text{ closed } G\text{-inv.} \mid P \cap E \neq 0\}$

and pick P s.t. $M := E \cap P$ is of min. dim. (possible because $\dim E < \infty$)

Pick $\varphi \in M \setminus \{0\}$ and set $V := \overline{\langle G \cdot \varphi \rangle_{\mathbb{C}}}$ ⇝ closed, G-invariant

Claim: V is minimal

Proof: Suppose $W \subseteq V$ is closed and G-invariant $\Longrightarrow V = Q \oplus^{\perp} Q'$

Note: $\varphi \in M \subseteq P \Rightarrow G\varphi \subseteq P \overset{(*)}{\Rightarrow} V \subseteq P \Rightarrow \varphi \in V \cap E \subseteq P \cap E = M \Rightarrow V \cap E = M$

Now: $M = V \cap E \underset{\substack{V, Q, Q', E \\ R(f)\text{-invariant}}}{=} (Q \cap E) \oplus (Q' \cap E)$

Thus ◦ $Q \cap E = M \overset{\text{as in}}{\underset{(*)}{\Longrightarrow}} V \subseteq Q \Rightarrow V = Q$
or ◦ $Q' \cap E = M \Rightarrow V \subseteq Q' \Rightarrow V = Q'$

□

$m_\pi < +\infty$: Pick f s.t. $R(f)|_{V_\pi} \neq 0$ with eigenvalue $\lambda \neq 0$

$$\Longrightarrow \underbrace{\operatorname{Eig}(\lambda, R(f))}_{\text{fin. dim}} \supseteq \bigoplus^{m_\pi} \operatorname{Eig}(\lambda, R(f)|_{V_\pi})$$

□

Remark: Conversely, assume that G is compact. Then any irreducible rep' (π, V) of G occurs in $L^2(G) = \hat{\bigoplus}_{\pi'} m_{\pi'} \pi'$

Proof: Trick: Fix $v^\vee \in V^\vee$

$$\Longrightarrow V \xhookrightarrow{\text{irred.}} L^2_{\Gamma}(G)$$

$$v \longmapsto (g \longmapsto \langle v^\vee, \pi(g) v \rangle)$$

(matrix coefficients)

cont. + $\Gamma \backslash G$ cpt. $\Rightarrow \in L^2$

(Why? ⇝ Being in L^2 is no issue but in general the matrix coeff. have no reason to be left-Γ-invariant !)

□

Warning: However, appearing in $L^2(\Gamma \backslash G)$ is a strong condition

↳ Being automorphic is more than just being a ⊗-product of smooth rep's of the local factors

2.2. Non-compact case

From now on we will stick to the arithmetic situation, e.g.

- $\{1\} \subseteq SL_2(\mathbb{Q}_p)$
- $SL_2(\mathbb{Z}) \subseteq SL_2(\mathbb{R})$
- $SL_2(\mathbb{Q}) \subseteq SL_2(\mathbb{A})$

How does $L^2(\Gamma\backslash G)$ decompose?

$$\rightsquigarrow \quad L^2 = L^2_{disc} \oplus L^2_{cont} = L^2_{cusp} \oplus L^2_{disc\setminus cusp} \oplus L^2_{cont}$$

Theorem: (Gelfand, Piatetski-Shapiro)

$R(f)|_{L^2_{cusp}}$ is a compact operator for all $f \in C_c^\infty(G)$.

In particular, $L^2_{cusp} = \widehat{\bigoplus_{\substack{\pi \\ cusp.}}} m_\pi \cdot \pi$ and $m_\pi < +\infty$

Theorem: (Langlands)

$$L^2(SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A})) = \bigoplus_{\chi=[P,\rho]} L^2_\chi ,$$

where $\chi = [P,\rho]$ runs through pairs $P \subseteq G$ parabolic
ρ cuspidal rep' of P/R_uP
modulo conjugation

$\rightsquigarrow$ There are two options:

- $P = G$, $\rho = \pi$ cuspidal rep of G, $L^2_\chi \cong m_\pi \cdot \pi$
- $P = \begin{pmatrix} * & * \\ & * \end{pmatrix}$, $\rho = \mu$ (cuspidal) rep' of T, $T = \begin{pmatrix} * & \\ & * \end{pmatrix}$

Upshot: $L^2 = L^2_{cusp} \oplus \bigoplus_{\mu \in Rep(T)} L^2_\mu$

Slogan: All complications stemm from parabolic subgroups

Rem: $R(f)|_{L^2_\mu}$ can be described in terms of so-called **Eisenstein-series**

(analogoues of $e^{2\pi i x(\cdot)}$ for $L^2(\mathbb{R})$)

3. The trace formula

3.1. Compact Case

Assume that $\Gamma\backslash G$ is compact $\Longrightarrow L^2(\Gamma\backslash G) = \widehat{\bigoplus_\pi} m_\pi \pi$

Recall: • Key part of the proof: Check that $R(f)$ is compact

• $(R(f)\phi)(x) = \int_{\Gamma\backslash G} K_f(x,y)\,\phi(y)\,dy \;,\; K_f(x,y) = \sum_\gamma f(x^{-1}\gamma y)$

Note: $tr(R(f)) = \sum_\pi m_\pi \, tr(R(f)|_\pi) = \sum_\pi m_\pi \, tr(\pi(f))$; on the other hand

Fact for integral operators

$$tr(R(f)) \overset{\text{Fact}}{=} \int_{\Gamma\backslash G} K_f(x,x)\,dx := \int_{\Gamma\backslash G} \sum_\gamma f(x^{-1}\gamma x)\,dx$$

$$= \int_{\Gamma\backslash G} \sum_{\underset{\text{conjugacy class}}{[\gamma]}} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\,dx \qquad \Gamma_\gamma := \text{centraliser of } \gamma \text{ in } \Gamma$$

$$= \sum_{[\gamma]} \int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)\,dx$$

$$\overset{\text{Fubini}}{=} \sum_{[\gamma]} \int_{G_\gamma\backslash G} \int_{\Gamma_\gamma\backslash G_\gamma} f(x^{-1}\underbrace{u^{-1}\gamma u}_{=\gamma} x)\,du\,dx$$

$$= \sum_{[\gamma]} vol(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\,dx$$

Conclusion: $tr(R(f)) = \sum_{[\gamma]} vol(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\,dx = \sum_\pi m_\pi \underbrace{tr(\pi(f))}_{=\hat{f}(\pi)}$

Ex: $\Gamma = \mathbb{Z} \subseteq \mathbb{R} = G$; $\Gamma_\gamma = \Gamma, G_\gamma = G,\ vol(\Gamma_\gamma\backslash G_\gamma) = vol(S^1) = 1,\ m_\pi = 1, \dots$

$\Longrightarrow \quad tr(R(f)) = \sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$

3.2. Non-cpt. case

Want: $\sum_{[\gamma]} J_\gamma^T(f) = \sum_{\pi} J_\pi^T(f) \quad \forall f \in C_c^\infty(Z(\mathbb{A})\backslash GL_2(\mathbb{A}))$

T some truncation parameter

obtained by integrating some function K_f^T on $\Gamma\backslash G \times \Gamma\backslash G$ over the diagonal

↳ agrees with K_f on a big open subset $U \subseteq G$ depending on T

Geometric side

Def.: $\gamma \in SL_2(\mathbb{Q})$ is called

- **elliptic** if the eigenvalues of γ are not rational ($\Rightarrow \gamma$ is semisimple) [↗ γ not contained in a parabolic]
- **hyperbolic** if the eigenvalues of γ are rational and distinct
- **unipotent** if the eigenvalues of γ are rational and coincide

Theorem: (Arthur's (not so) simple trace formula)

Let $f = \prod f_v \in C_c^\infty(SL_2(\mathbb{A}))$ s.t.

$$\int_{G_\gamma(\mathbb{Q}_{v_i})\backslash G(\mathbb{Q}_{v_i})} f(x^{-1}\gamma x)\, dx \overset{(*)}{=} 0 \quad \forall \gamma \in SL_2(\mathbb{Q}) \text{ hyperbolic for at least two places } v_1, v_2.$$

Then

$$tr\left(R(f)|_{L^2_{cusp}}\right) = vol\left(SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A})\right) f(1) + \sum_{\substack{[\gamma] \\ \text{elliptic}}} vol(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\, dx$$

$$- vol\left(SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A})\right) \sum_{\substack{\mu \in Rep(T) \\ \mu^2 = 1}} \mu(f)$$

Proof Idea:

Spectral side: (*) forces all integrals over Eisenstein series to vanish, only those with residues contribute (as for Dirichlet L-fct's, most Eisenstein series are holomorphic) and these are precisely those with $\mu^2 = 1$

Geometric side: (*) forces all extra contributions to vanish except that for $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.