§ Talk 2. Smooth representations of locally profinite groups.

§ 1. Locally profinite groups.

• Definition 1. A locally profinite group is a topological group G s.t. every open neighbourhood $\mathcal{U}$ of $1 \in G$ contains a compact open subgroup K of G.

• Examples 2.

(1) Any discrete group (or in general, any profinite group) is loc. prof.

(2) If G is loc. prof. and H is a closed subgroup of G, then H is loc. prof.

(3) If G is loc. prof. and H is a closed normal subgroup of G, then G/H is loc. prof.

• Let F be a non-Archimedean field, $\mathcal{O}_F$ its ring of integers (which is a DVR), $\mathfrak{p}$ the maximal ideal of $\mathcal{O}_F$ and $k = \mathcal{O}_F/\mathfrak{p}$ the residue class field. We assume that k is finite and set $q := |k|$.

Let $\alpha \in \mathcal{O}_F$ be s.t. $\alpha\mathcal{O}_F = \mathfrak{p}$. Then, every $x \in F^\times$ can be written as $x = u\alpha^n$ for some $u \in \mathcal{O}_F^\times =: \mathcal{U}_F$ and some $n \in \mathbb{Z}$ (we use the notation $n =: v_F(x)$). This allows us to define a norm

$$\|x\| = q^{-n} = q^{-v_F(x)}, \quad \|0\| = 0,$$

and thus a metric space topology in F, relative to which it is complete.

Moreover, with this topology, F is a top. field. The fractional ideals

$$\mathfrak{p}^n = \varpi^n \mathcal{O}_F = \{ x \in F : \|x\| \leq q^{-n} \}, \quad n \in \mathbb{Z},$$

are open subgroups of F and give a fundamental system of open neighbourhoods of 0 in F (i.e. for every open $\mathcal{U} \ni 0$, there $\exists\, n \in \mathbb{Z}$ s.t. $\mathfrak{p}^n \subseteq \mathcal{U}$).

One checks that each $\mathfrak{p}^n$ is compact, which means:

• Proposition 3. The group $(F, +)$ is locally prof., and F is the union of its compact open subgroups.

• Similarly, we consider now the multiplicative group $(F^\times, \cdot)$. This is again loc. profinite, with the congruence unit groups

$$\mathcal{U}_F^n := 1 + \mathfrak{p}^n, \quad n \geq 1$$

being compact open and a fund. system of open neighb. of 1 in $F^\times$.

• Let $n \geq 1$ be an integer. The F-v.s. $F^n = F \times \cdots \times F$ with the product topology can be seen to be a loc. prof. group.

In particular, the matrix ring $M_n(F)$ is a locally profinite group under addition, in which mult. of matrices is continuous.

The group $G = GL_n(F)$ is an open subset of $M_n(F)$, and since inversion of matrices is continuous, G is a top. group. The subgroups

$$K = GL_n(\mathcal{O}_F), \quad K_j = 1 + \mathfrak{p}^j M_n(\mathcal{O}_F), \quad j \geq 1$$

are compact open and give a fund. system of open neighb. of 1 in G.

Therefore, G is loc. prof.

In general, if V is an F-v.s. of finite dimension n, choosing a basis gives us an isom. $V \simeq F^n$, which we use to give V a topology (which is independent of the basis). Our previous remarks apply to the algebra $\mathrm{End}_F(V)$ and the group $\mathrm{Aut}_F(V)$.

• Proposition 4. Let G be a loc. prof. group, and $\psi: G \to \mathbb{C}^\times$ a group hom. TFAE:

(1) ψ is cont.,

(2) $\ker \psi$ is open in G.

If ψ satisfies these conditions and G is the union of its compact open subgroups, then $\mathrm{im}\, \psi \subseteq S^1$.

Proof/

(2) $\Rightarrow$ (1)/ Clear, using that if $g \in \psi^{-1}(\{z\})$ for some $z \in \mathbb{C}^\times$, then $\psi^{-1}(\{z\}) = g \cdot \ker \psi$, and since the map $G \to G$, $x \mapsto gx$ is a homeom., $\psi^{-1}(\{z\})$ is open.

(1) $\Rightarrow$ (2)/ Let $\mathcal{U} \subseteq \mathbb{C}^\times$ be an open neighb. of 1. Then $\psi^{-1}(\mathcal{U})$ is open in G and contains 1, hence $\exists K \leq G$ compact open subgroup s.t. $\psi(K) \subseteq \mathcal{U}$. But if we choose $\mathcal{U}$ sufficiently small, it contains no non-trivial subgroup of $\mathbb{C}^\times$, so $\psi(K) = \{1\}$.
In general, if $g \in \ker(\psi)$, $g \cdot K \subseteq \ker \psi$ is an open neighb., showing that $\ker(\psi)$ is open.

For the last statement, one sees that S^1 is the maximal compact subgroup of $\mathbb{C}^\times$. If $K \leq G$ is a compact subgroup, by continuity $\psi(K) \subseteq \mathbb{C}^\times$ also is compact, and thus contained in S^1. ⊠

• <u>Definition 5</u>. A <u>character</u> of a loc. prof. gp. G is a continuous group hom. $\chi: G \longrightarrow \mathbb{C}^\times$. We write 1_G, or 1, for the trivial (constant) character. Further, a character χ is said to be <u>unitary</u> if $\operatorname{im} \chi \subseteq S^1$.

By Prop. 4, if G is the union of its open compact subgroups, every character of G is unitary.

• Let $\widehat{F} := \{\chi: F \to \mathbb{C}^\times \mid \chi \text{ a character of } F\}$, which is a group under multiplication. Since F is the union of its compact open subgroups $\mathfrak{p}^n$, every $\chi \in \widehat{F}$ is unitary.

On the other hand, if $\chi \in \widehat{F} \setminus \{1\}$, then there is a least integer $d \in \mathbb{Z}$ s.t. $\mathfrak{p}^d \subseteq \ker \chi$. Such a d is called the <u>level of χ</u>.

• <u>Proposition 6.</u> (Additive duality). Let $\psi \in \widehat{F} \setminus \{1\}$ of level d.

(1) Let $a \in F$. Then, the map $a\psi: F \to \mathbb{C}^\times$ given by $x \mapsto \psi(ax)$ is a character of F, and if $a \neq 0$, it has level $d - v_F(a)$.

(2) The map $a \mapsto a\psi$ is a group isom. $F \cong \widehat{F}$.

<u>Proof</u>)

(1) Trivial.

(2) The map is clearly an inj. group hom. For surj., if $\theta \in \widehat{F}$ has level c, one finds a sequence $\{u_n\}$ of elements of $\mathcal{U}_F$ s.t. the character $\theta_n := u_n \alpha^{d-c} \psi$ agrees with θ on $\mathfrak{p}^{d-c}$ and $u_{n+1} \equiv u_n \pmod{\mathfrak{p}^n}$. This Cauchy sequence converges to some $u \in \mathcal{U}_F$, and $\theta = u\alpha^{d-c}\psi$. $\boxtimes$

- For the multiplicative case $G = F^\times$, we cannot use the same reasoning as before. Still, if χ is a non-trivial char. on $F^\times$, we define its <u>level</u> as the least integer $n \geq 0$ s.t. $\mathcal{U}_F^{n+1} \subset \ker \chi$.
(By Prop. 4, χ vanishes on $\mathcal{U}_F^m$ for some $m \geq 0$, since $\ker \chi$ is open).

Of course, there are non-unitary characters of χ, such as $x \longmapsto \|x\|$.
In contrast to the additive case, F has a maximal compact subgroup, $\mathcal{U}_F$.
In order to define the characters on $F^\times$, we will use the following iso.: if $1 \leq m < n \leq 2m$, then:

$$\begin{aligned} \mathfrak{p}^m/\mathfrak{p}^n &\xrightarrow{\sim} \mathcal{U}_F^m/\mathcal{U}_F^n \\ x + \mathfrak{p}^n &\longmapsto (1+x)\cdot \mathcal{U}_F^n, \end{aligned}$$

and thus $(\mathfrak{p}^m/\mathfrak{p}^n)^\wedge \cong (\mathcal{U}_F^m/\mathcal{U}_F^n)^\wedge$.

Fix $\psi_F \in \widehat{F}$ of level 1. For any $a \in F$, we define

$$\begin{aligned} \psi_{F,a} : F &\longrightarrow \mathbb{C}^\times \\ x &\longmapsto \psi_F(a(x-1)) \end{aligned}$$

we then get:

- <u>Proposition 7.</u> Let $\psi \in \widehat{F}$ have level 1. Let $m, n \in \mathbb{Z}$ s.t. $0 \leq m < n \leq 2m+1$.
The map $a \longmapsto \psi_{F,a}|_{\mathcal{U}_F^{m+1}}$ induces an iso.

$$\mathfrak{p}^{-n}/\mathfrak{p}^{-m} \xrightarrow{\sim} \left(\mathcal{U}_F^{m+1}/\mathcal{U}_F^{n+1}\right)^\wedge$$

<u>Sketch of proof</u>) One checks that this map is the composition

$$\begin{aligned} \mathfrak{p}^{-n}/\mathfrak{p}^{-m} &\xrightarrow{\sim} \left(\mathfrak{p}^{m+1}/\mathfrak{p}^{n+1}\right)^\wedge \xrightarrow{\sim} \left(\mathcal{U}_F^{m+1}/\mathcal{U}_F^{n+1}\right)^\wedge \\ a + \mathfrak{p}^{-m} &\longmapsto a\psi|_{\mathfrak{p}^{m+1}} \\ & \qquad\qquad b\psi|_{\mathfrak{p}^{m+1}} \longmapsto b\psi(\cdot + 1)|_{\mathcal{U}_F^{m+1}} \end{aligned}$$

$\boxtimes$

§ 2. Smooth representations of loc. prof. groups.

• Definition 8. Let G be a loc. prof. group, and let (π, V) be a representation of G, i.e. V is a $\mathbb{C}$-v.s. and $\pi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ a group homomorphism.

(a) We say that (π, V) is smooth if $\forall v \in V$, $\exists K \subseteq G$ compact open subgroup (depending on v) s.t. $\pi(x)v = v$. Equivalently,

$$V = \bigcup_{K} \underbrace{V^K}_{\text{points of } V \text{ fixed by } K}$$

running over all compact open subgroups of G

(b) A smooth rep. (π, V) is called admissible if V^K is finite-dim'l. for all $K \subseteq G$ compact open subgroups.

(c) If (π, V) is a smooth rep., then any G-stable subspace $W \subseteq V$ defines smooth repr. $G \to \mathrm{Aut}_{\mathbb{C}}(W)$ and $G \to \mathrm{Aut}_{\mathbb{C}}(V/W)$. We say that (π, V) is irreducible if $V \neq 0$ and V has no G-stable subspace W s.t. $0 \neq W \neq V$.

(d) Let (π_1, V_1), (π_2, V_2) smooth repr. of G. We define the set of maps as

$$\mathrm{Hom}_G(\pi_1, \pi_2) = \{ f : V_1 \to V_2 \ \mathbb{C}\text{-linear map} \mid f \circ \pi_1(g) = \pi_2(g) \circ f \ \ \forall g \in G \}.$$

Using this, we can define a category $\mathrm{Rep}(G)$ of smooth reps. of G, which is in fact abelian.

(e) Two smooth reps. (π_1, V_1), (π_2, V_2) of G are isomorphic, or equivalent, if there exists a $\mathbb{C}$-isom. $f : V_1 \to V_2$ s.t.

$$f \circ \pi_1(g) = \pi_2(g) \circ f \quad \forall g \in G.$$

• <u>Example 9</u>. A character χ of G is a smooth rep. $\chi: G \to \mathbb{C}^\times = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$.
In fact, there is a bijection

$$\left\{ \begin{matrix} \text{1-dim'l smooth} \\ \text{reps. of } G \end{matrix} \right\} \Big/ \cong \quad \xleftrightarrow{1-1} \quad \widehat{G}.$$

• <u>Example 10</u>. Assume that G is compact, and thus profinite. Let (π, V) be an irred. smooth rep. of G. We claim that V is fin. dim'l.
Indeed, if $v \in V$, $v \neq 0$, then $\exists K \subseteq G$ compact open subgroup s.t. $v \in V^K$.
But then, the subspace spanned by the finite set (since $[G:K] < +\infty$)

$$\{ \pi(g) v \ : \ g \in G/K \}$$

is G-stable and non-trivial, so irreducibility means that it spans V.
Further, if $K' := \bigcap_{g \in G/K} g K g^{-1}$, then it is an open normal subgroup of G of finite index acting trivially on V. This means that V is an irred. rep. of the finite discrete group G/K'.

• <u>Proposition 11</u>. Let G be loc. prof., and (π, V) a smooth rep. of G. TFAE:

(1) V is the sum of its irred. G-subspaces.

(2) V is the direct sum of a family of irred. G-subspaces.

(3) Any G-subspace of V has a G-complement in V.

If these conditions are satisfied, we say that (π, G) is <u>G-semisimple</u>.

• <u>Lemma 12</u>. Let G loc. prof., and $K \subseteq G$ compact open subgrp. Let (π, V) be a smooth rep. of G. Then V is the sum of its irreducible K-subspaces (i.e. V is <u>K-semisimple</u>).

• <u>Proposition 13</u>. A sequence
$$U \longrightarrow V \longrightarrow W$$
of maps between smooth reps. of G is exact if and only if
$$U^K \to V^K \to W^K$$
is exact for all $K \leq G$ compact open subgroups.

• <u>Proposition 14</u>. Let $(\pi, V) \in \mathrm{AbsRep}(G)$, an abstract rep. of G (i.e. not necessarily smooth). Define
$$V^\infty := \bigcup_K V^K \quad \to \text{going over all compact open subgrps. of } G$$
V^∞ is a G-stable subspace of V, and the map
$$\pi^\infty : G \to \mathrm{Aut}_{\mathbb{C}}(V^\infty), \quad g \longmapsto \pi(g)|_{V^\infty}$$
is a grp. hom. s.t. (π^∞, V^∞) is a <u>smooth</u> rep. of G.
This defines a functor
$$(-)^\infty : \mathrm{AbsRep}(G) \longrightarrow \mathrm{Rep}(G),$$
which is right-adjoint to the forgetful functor $\mathrm{Rep}(G) \to \mathrm{AbsRep}(G)$.
In other words, if $(\pi, V) \in \mathrm{Rep}(G)$ and $(\sigma, W) \in \mathrm{AbsRep}(G)$, then
$$\mathrm{Hom}_G(V, W) = \mathrm{Hom}_G(V, W^\infty).$$
Note, that, in particular, $(-)^\infty$ is left-exact.

§3. <u>Frobenius reciprocity and Schur's lemma</u>.

• Let G be loc. prof., and H a closed subgrp. (and hence also loc. prof.).
Let (σ, W) be a smooth rep. of H. We define a space X of functions $f : G \to W$ satisfying the following properties:

$\overbrace{}^{\text{the automorphism } \sigma(h): W \to W \text{ acting on } f(g) \in W}$

(1) $f(hg) = \sigma(h) f(g) \quad \forall h \in H, g \in G$.

(2) there exists a compact open subgroup K of G (depending on f) s.t. $f(gx) = f(g) \quad \forall g \in G, x \in K$.

Now, we define a group hom. $\Sigma : G \to \mathrm{Aut}_{\mathbb{C}}(X)$ by

$$\underbrace{\Sigma(g) f}_{\in X, \text{ so a map } G \to W} : x \longmapsto f(xg) \quad \forall g, x \in G, f \in X.$$

Note that $\Sigma(g) f$ satisfies (2) for gKg^{-1}, where K is the corresponding compact open subgroup for f.

The pair (Σ, X) is a smooth rep. of G, denoted the rep. of G smoothly induced by σ, written as

$$(\Sigma, X) = \mathrm{Ind}_H^G \sigma.$$

The map $\sigma \mapsto \mathrm{Ind}_H^G \sigma$ gives a functor $\mathrm{Rep}(H) \to \mathrm{Rep}(G)$.

On the other hand, there is a canonical H-hom.

$$\alpha_\sigma : \mathrm{Ind}_H^G \sigma \longrightarrow W$$
$$f \longmapsto f(1).$$

• Proposition 15 (Frobenius Reciprocity). Let H be a closed subgroup of a loc. prof. group G. For a smooth rep. (σ, W) of H and a smooth rep. (π, V) of G, the canonical map

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) \longrightarrow \mathrm{Hom}_H(\pi, \sigma)$$
$$\phi \longmapsto \alpha_\sigma \circ \phi$$

is an isom. functorial on π, σ.

Proof) Let $f: V \to W$ be an H-morphism. We define a G-morph. $f_*: V \to \mathrm{Ind}\,\sigma$ as

$$f_{\boxtimes}(v) : G \longrightarrow W, \quad g \longmapsto f(\pi(g)v).$$

One then checks that the map $f \mapsto f_{\boxtimes}$ inverts our desired map.

$\boxtimes$

• <u>Proposition 16.</u> The functor $\mathrm{Ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$ is additive and exact.

<u>Proof</u> Let (σ, W) be a smooth rep. of H. We define

$$I(\sigma) := \{ f : G \to W \mid \underbrace{f(hg) = \sigma(h) f(g) \quad \forall h \in H, g \in G}_{\text{first condition of } X} \}$$

This gives a functor $I : \mathrm{Rep}(H) \longrightarrow \mathrm{AbsRep}(G)$. One can clearly check that I is additive and exact, and that

$$\mathrm{Ind}_H^G(\sigma) = I(\sigma)^\infty.$$

Therefore, Ind_H^G is also additive, and by Prop. 14, it is left-exact.

For right-exactness, let (σ, W), (τ, U) smooth reps. of H and $f : W \to U$ an H-surjection. Let $\phi \in I(\tau)^\infty$, and choose a compact open subgroup $K \leq G$ fixing ϕ.
The support of ϕ is a union of cosets HgK, and the value $\phi(g) \in U$ must be fixed by $\tau(H \cap gKg^{-1})$.
By Prop. 13 applied to H and its compact open subgroup $H \cap gKg^{-1}$, the H-morphism $W^{H \cap gKg^{-1}} \longrightarrow U^{H \cap gKg^{-1}}$ is surjective, and thus we can find some $w_g \in W$, fixed by $\sigma(H \cap gKg^{-1})$, and s.t. $f(w_g) = \phi(g)$.
We define a function $\Phi : G \to W$ with the same support as ϕ and $\Phi(hgk) = \sigma(h) w_g$ for each $g \in H \backslash \mathrm{supp}\, \phi / K$. Then Φ is fixed by K, and hence lies in $I(\sigma)^\infty$. Since its image in $I(\tau)^\infty$ is ϕ, this finishes the proof. $\boxtimes$

• <u>Definition 17.</u> Let G be a loc. prof. grp., $H \leq G$ a closed subgrp., and (σ, W) a smooth rep. of H. Consider now the following $\mathbb{C}$-v.s. of functions which are <u>compactly supported modulo H</u>:

$$X_c := \{ f : G \to W,\ f \in X \mid \operatorname{supp} f \text{ is compact in } H\backslash G \}.$$

Equivalently, we may ask that $\operatorname{supp} f \subseteq HC$ for some compact set C in G.

The space X_c is stable under the action of G and provides a smooth rep. of G, which we denote by $c\text{-}\mathrm{Ind}_H^G \sigma$. This provides a functor

$$c\text{-}\mathrm{Ind}_H^G \sigma : \mathrm{Rep}(H) \longrightarrow \mathrm{Rep}(G).$$

It is called <u>compact induction</u>, or <u>smooth induction with compact supports</u>.

• <u>Proposition 18.</u> Let G be loc. prof., and $H \leq G$ a closed subgroup.

(1) The functor $c\text{-}\mathrm{Ind}_H^G : \mathrm{Rep}(H) \longrightarrow \mathrm{Rep}(G)$ is additive and exact.

(2) There is a morphism of functors $c\text{-}\mathrm{Ind}_H^G \longrightarrow \mathrm{Ind}_H^G$, which is an isom. iff $H\backslash G$ is compact.

(3) (Frob. reciprocity) Assume that H is open, $(\sigma, W) \in \mathrm{Rep}(H)$ and $(\tau, V) \in \mathrm{Rep}(G)$. There is a functorial isomorphism

$$\mathrm{Hom}_G(c\text{-}\mathrm{Ind}\, \sigma, \tau) \cong \mathrm{Hom}_H(\sigma, \tau).$$

• From now on, we assume that G/K is countable for all compact open subgroups $K \leq G$.

• <u>Lemma 19.</u> Let (π, V) be an irred. smooth rep. of G. Then, the dimension $\dim_{\mathbb{C}} V$ is countable.

<u>Proof</u>) Let $v \in V$, $v \neq 0$. By smoothness, $v \in V^K$ for some compact open subgroup $K \leq G$. Since the subspace spanned by the (countable) set

$$\{\pi(g)v : g \in G/K\}$$

is G-stable and non-zero, it must be V (by irreducibility). $\boxtimes$

• <u>Schur's Lemma.</u> If (π, V) is an irred. smooth rep. of G, then $\mathrm{End}_G(V) = \mathbb{C}$.

<u>Proof.</u>) Let $\phi \in \mathrm{End}_G(V)$, $\phi \neq 0$. Since $\ker \phi$, $\mathrm{im}\, \phi$ are both G-stable subspaces of V (and $\phi \neq 0$), then ϕ must be bijective and invertible. It follows that $\mathrm{End}_G(V)$ is a complex division algebra.

Fix $v \in V$, $v \neq 0$, s.t. the translates $\pi(g)v$ of v span V. Then, any $\phi \in \mathrm{End}_G(V)$ is clearly uniquely determined by $\phi(v) \in V$. Since V has countable $\mathbb{C}$-dim., then $\mathrm{End}_G(V)$ also has countable $\mathbb{C}$-dim.

On the other hand, let $\phi \in \mathrm{End}_G(V)$, $\phi \notin \mathbb{C}$. This means that ϕ is transcendental (i.e. if $P \in \mathbb{C}[X]$, and $P(\phi) = 0 \Rightarrow P = 0$), and that the field $\mathbb{C}(\phi)$ is contained in $\mathrm{End}_G(V)$. But the following subset of $\mathbb{C}(\phi)$ is $\mathbb{C}$-lin. independent:

$$\{(\phi - a)^{-1} \mid a \in \mathbb{C}\},$$

which means that $\mathbb{C}(\phi)$ (and hence $\mathrm{End}_G(V)$) has uncountable dimension over $\mathbb{C}$. This is a contradiction, so ϕ must be in $\mathbb{C}$, and thus $\mathrm{End}_G(V) = \mathbb{C}$. $\boxtimes$

• <u>Corollary 20.</u> Let (π, V) be an irred. smooth rep. of G. The centre Z of G acts on V via a character $\omega_\pi : Z \to \mathbb{C}^\times$, i.e.

$$\pi(z)v = \omega_\pi(z)\cdot v \qquad \forall v \in V,\ z \in Z.$$

ω_π is the <u>central char. of π</u>.

• <u>Corollary 21.</u> If G is abelian, any irred. smooth rep. of G is one-dimensional.

• Lastly, we discuss how semisimplicity behaves with regards to induction.

• <u>Lemma 22.</u> Let G be loc. prof., and $H \leq G$ an open subgroup of G of finite index.

(1) If (π, V) is a smooth rep. of G, then V is G-semisimple if and only if it is H-semisimple.

(2) Let (σ, W) be a semisimple smooth rep. of H. The induced representation $\mathrm{Ind}_H^G \sigma$ is G-semisimple.