Measures and Duality

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These are notes for a talk I am giving in the PhD-seminar on *Local Langlands for* GL_2 taking place in the summer term 2023 in Essen. The main reference for this talk is [BH06].

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1 Notation

We fix a non-archimedean local field F with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p} \subseteq \mathfrak{o}$, uniformizer $\varpi \in \mathfrak{p}$, units $U_F = \mathfrak{o}^\times$ and residue field $\mathbf{k} = \mathfrak{o}/\mathfrak{p}$ of characteristic p and cardinality q. We consider the algebraic subgroups

$$B \coloneqq \left\{\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}\right\}, T \coloneqq \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right\}, N \coloneqq \left\{\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\right\}, N' \coloneqq \left\{\begin{pmatrix} 1 & 0 \\ c' & 1 \end{pmatrix}\right\}, Z \coloneqq \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right\} \subseteq \mathrm{GL}_2$$

called the standard Borel, the standard torus, the unipotent radical of the standard Borel the unipotent radical of the opposite of the standard Borel (relative to the standard torus) and the center, respectively. We also consider the matrix $w \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(F)$ and the standard Iwahori subgroup

$$I \coloneqq \mathrm{GL}_2(\mathfrak{o}) \times_{\mathrm{GL}_2(\mathbf{k})} B(\mathbf{k}) = \left\{\begin{pmatrix} a & c \\ c' & b \end{pmatrix} \;\middle|\; a, b \in U_F, c \in \mathfrak{o}, c' \in \mathfrak{p}\right\} \subseteq \mathrm{GL}_2(F).$$

Finally, G, G_1, G_2 always denote locally profinite groups and X, Y denote locally profinite sets.

2 Some group theory

The reference for this section is [BH06, Sections 7.2, 7.3].

Proposition 2.1 (Iwasawa decomposition)**.** *We have*

$$\mathrm{GL}_2(F) = B(F)\,\mathrm{GL}_2(\mathfrak{o}).$$

Corollary 2.2. *The quotient* $B(F)\backslash \mathrm{GL}_2(F)$ *is compact.*

Proposition 2.3 (Cartan decomposition)**.** *We have*

$$\mathrm{GL}_2(F) = \bigsqcup_{a,b \in \mathbf{Z}, a \le b} \mathrm{GL}_2(\mathfrak{o}) \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} \mathrm{GL}_2(\mathfrak{o}).$$

Corollary 2.4. *For any compact open subgroup* $K \subseteq \mathrm{GL}_2(F)$ *the quotient* G/K *is countable.*

Proposition 2.5 (Iwahori decomposition)**.** *The multiplication map*

$$(I \cap N'(F)) \times (I \cap T(F)) \times (I \cap N(F)) \to I$$

is a bijective homeomorphism, and the same is true for any reordering of the three factors.

Remark 2.6. Note that we have $I \cap N'(F) = \ker(N(\mathfrak{o}) \to N(\mathbf{k}))$, $I \cap T(F) = T(\mathfrak{o})$ and $I \cap N(F) = N(\mathfrak{o})$.

Remark 2.7. All of the above decompositions can be appropriately generalized from GL_2 to the setting of reductive groups.

3 Haar measures

The reference for this section is [BH06, Sections 3.1 - 3.4 and 7.4 - 7.6].

Definition 3.1. • We denote the space of compactly supported and locally constant $\mathbf{C}$-valued functions on X by $C_c^\infty(X)$.

- A *measure on* X is a $\mathbf{C}$-linear map
$$I\colon C_c^\infty(X) \to \mathbf{C}$$
such that $I(f) \in \mathbf{R}_{\geq 0}$ for all $f \in C_c^\infty(X)$ that are valued in $\mathbf{R}_{\geq 0}$.
- Given an open subspace $X' \subseteq X$ we have $C_c^\infty(X') \subseteq C_c^\infty(X)$ via extension by 0, and using this we can restrict measures on X to measures on X'.

We will typically denote a measure by a symbol like μ and the associated map by
$$f \mapsto I_\mu(f) = \int_X f(x)\,\mathrm{d}\mu(x).$$

Remark 3.2. Given a measure μ on X and a compact open subset $A \subseteq X$ we write $\mu(A) \coloneqq I_\mu(1_A)$. Then μ is uniquely determined by $\mu(A)$ for all A. In fact, giving a measure μ on X is equivalent to giving a function
$$\mu\colon \big\{\text{compact open subsets of } X\big\} \to \mathbf{R}_{\geq 0}$$
such that $\mu(A \sqcup B) = \mu(A) + \mu(B)$ for all disjoint A, B.

Remark 3.3. The subset $\mathrm{Meas}(X) \subseteq C_c^\infty(X)^*$ of measures on X is stable under addition and multiplication by scalars in $\mathbf{R}_{\geq 0}$. In other words, it is a cone.

Remark 3.4. We have a natural isomorphism
$$C_c^\infty(X) \otimes C_c^\infty(Y) \cong C_c^\infty(X \times Y), \qquad f_1 \otimes f_2 \mapsto \big((x,y) \mapsto f_1(x)f_2(y)\big).$$
Thus, given measures μ and ν on X and Y we can define a measure $\mu \otimes \nu$ on $X \times Y$ by declaring
$$I_{\mu\otimes\nu}(f_1 \otimes f_2) \coloneqq I_\mu(f_1) I_\nu(f_2).$$
For a general function $f \in C_c^\infty(X \times Y)$ we then have
$$\int_{X\times Y} f(x,y)\,\mathrm{d}(\mu\otimes\nu)(x,y) = \int_X \int_Y f(x,y)\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x) = \int_Y \int_X f(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y).$$

Definition 3.5. • We define two actions λ_G and ρ_G of G on $C_c^\infty(G)$, that are respectively given by
$$(\lambda_G(g)f)(x) = f(g^{-1}x) \quad \text{and} \quad (\rho_G(g)f)(x) = f(xg)$$
for $g, x \in G$ and $f \in C_c^\infty(G)$. Note that both representations are smooth.

- A measure μ on G is called a *left (resp. right) Haar measure* if $\mu \neq 0$ and $I_\mu(\lambda_G(g)f) = I_\mu(f)$ (resp. $I_\mu(\rho_G(g)f) = I_\mu(f)$) for all $g \in G$ and $f \in C_c^\infty(G)$.

Proposition 3.6. *There exists a left Haar measure for G. It is unique up to a factor $c \in \mathbf{R}_{>0}$.*

Moreover, if μ is a left Haar measure on G then the measure on G given by
$$f \mapsto \int_G f(x^{-1})\,\mathrm{d}\mu(x)$$
is a right Haar measure for G.

Lemma 3.7. *Let $H \subseteq G$ be an open subgroup and let μ be a left Haar measure on G. Then the restriction of μ to H is a left Haar measure on H.*

Definition 3.8. Let μ be a left Haar measure for G. For $g \in G$, the measure on G given by
$$f \mapsto \int_G f(xg)\,\mathrm{d}\mu(x)$$
is again a left Haar measure, so that there exists a unique constant $\delta_G(g) \in \mathbf{R}_{>0}$ such that
$$\delta_G(g) \int_G f(xg)\,\mathrm{d}\mu(x) = \int_G f(x)\,\mathrm{d}\mu(x)$$
for all $f \in C_c^\infty(G)$. Then we define the *modular character of G (or module of G)* as the map $\delta_G\colon G \to \mathbf{R}_{>0}$. It is in fact a continuous character (and even trivial on any compact open subgroup $K \subseteq G$).

We call G *unimodular* if $\delta_G = 1$. This is equivalent to saying that a left Haar measure on G is also a right Haar measure (and the other way around). If G is unimodular, we also just say Haar measure without specifying left or right.

Lemma 3.9. *Let μ_1 be a left Haar measure on G_1 and let μ_2 be a left Haar measure on G_2. Then $\mu_1 \otimes \mu_2$ is a left Haar measure on $G_1 \times G_2$.*

Lemma 3.10. *Suppose we are given an action $\phi\colon G_1 \to \mathrm{Aut}(G_2)$ so that we can form the semidirect product $G_1 \ltimes G_2$. Suppose furthermore that we are given left Haar measures μ_1 and μ_2 on G_1 and G_2 respectively. Then $\mu_1 \otimes \mu_2$ is a left Haar measure on $G_1 \ltimes G_2$.*

If we moreover define $\delta_\phi\colon G_1 \to \mathbf{R}_{>0}$ by the formula

$$\delta_\phi(g) \int_{G_2} f(\phi(g)(x)) \,\mathrm{d}\mu_2(x) = \int_{G_2} f(x) \,\mathrm{d}\mu_2(x)$$

for $g \in G_1$ and $f \in C_c^\infty(G_2)$, then we have

$$\delta_{G_1 \ltimes G_2}(g_1, g_2) = \delta_{G_1}(g_1) \cdot \delta_{G_2}(g_2) \cdot \delta_\phi(g_1^{-1})$$

for $(g_1, g_2) \in G_1 \ltimes G_2$.

Example 3.11. We now give a few important examples.

- We denote by μ_F the Haar measure on F that is normalized by $\mu_F(\mathfrak{o}) = 1$. We then have $\mu_F(a + \mathfrak{p}^n) = q^{-n}$ for any $a \in F$ and $n \in \mathbf{Z}$.
- A Haar measure $\mu_{F^\times}$ on $F^\times$ is given by the formula
$$\int_{F^\times} f(x) \,\mathrm{d}\mu_{F^\times}(x) \coloneqq \int_F f(x) \|x\|^{-1} \,\mathrm{d}\mu_F(x).$$
- A (left and right) Haar measure $\mu_{\mathrm{GL}_2(F)}$ on $\mathrm{GL}_2(F)$ is given by the formula
$$\int_{\mathrm{GL}_2(F)} f(x) \,\mathrm{d}\mu_{\mathrm{GL}_2(F)}(x) \coloneqq \int_{M_2(F)} f(x) \|\det(x)\|^{-2} \,\mathrm{d}\mu_F^{\otimes 4}(x).$$
In particular $\mathrm{GL}_2(F)$ is unimodular.
- As we have $N(F) \cong N'(F) \cong F$, $Z(F) \cong F^\times$ and $T(F) \cong F^\times \times F^\times$, we also know what their Haar measures are.
- The group $B(F)$ can be written as a semidirect product $B(F) = T(F) \ltimes N(F)$ so that we obtain a left Haar measure $\mu_{B(F)} = \mu_{T(F)} \otimes \mu_{N(F)}$ for it.
  However, $B(F)$ is not unimodular. Its modular character is given by
$$\delta_{B(F)}(g) = \|a^{-1}b\|, \qquad g = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}.$$
- The restriction of the Haar measure on $\mathrm{GL}_2(F)$ to $\mathrm{GL}_2(\mathfrak{o})$ (and consequently also the one to I) coincides with $\mathrm{d}\mu_F^{\otimes 4}$ as $\|\det(x)\| = 1$ for $x \in \mathrm{GL}_2(\mathfrak{o})$. The restricted Haar measure μ_I on I can also be expressed as
$$\int_I f(x) \,\mathrm{d}\mu_I(x) = \int_{N'} \int_T \int_N f(n'tn) \,\mathrm{d}\mu_N(n) \,\mathrm{d}\mu_T(t) \,\mathrm{d}\mu_{N'}(n')$$
using the Iwasawa decomposition.

4 The contragredient representation

Definition 4.1. We define a functor $(\text{-})^\vee\colon \mathsf{Rep}(G)^{\mathrm{op}} \to \mathsf{Rep}(G)$ as follows: For a smooth G-representation π we have the dual space $V_\pi^* = \mathrm{Hom}_{\mathbf{C}}(V_\pi, \mathbf{C})$ that is equipped with a G-action

$$\pi^*\colon G \to \mathrm{Aut}_{\mathbf{C}}(V_\pi^*), \qquad g \mapsto \Big(\varphi \mapsto \big(\pi^*(g)\varphi\colon v \mapsto \varphi(\pi(g^{-1})v)\big)\Big).$$

However, the representation π^* is not necessarily smooth. We thus define $\pi^\vee \coloneqq (\pi^*)^\infty$. We call $\pi^\vee$ the *contragredient* or the *smooth dual* of π.

We denote the natural evaluation pairing $V_{\pi^\vee} \times V_\pi \to \mathbf{C}$ by $\langle \text{-}, \text{-} \rangle$. It induces a natural morphism of abstract G-representations

$$\delta_\pi\colon \pi \to \pi^{\vee *}, \qquad v \mapsto \big(\varphi \mapsto \langle \varphi, v \rangle = \varphi(v)\big)$$

that automatically has image inside $\pi^{\vee\vee} \subseteq \pi^{\vee *}$.

Proposition 4.2. *We have the following properties (where π always denotes a smooth G-representation):*

- *The natural morphism $V_{\pi^\vee}^K \to (V_\pi^K)^*$ is an isomorphism.*
- *The evaluation morphism $\delta_\pi \colon \pi \to \pi^{\vee\vee}$ is injective.*
- *The morphism δ_π is surjective (i.e. an isomorphism) if and only if π is admissible.*
- *The functor $(\text{-})^\vee \colon \mathsf{Rep}(G)^{\mathrm{op}} \to \mathsf{Rep}(G)$ is exact and faithful.*
- *If $\pi^\vee$ is irreducible, then so is π. The converse is true whenever π is admissible.*

5 The duality theorem

In this section we fix a closed subgroup $H \subseteq G$.

Idea 5.1. We would like to understand how taking duals interacts with inducing representations from H to G. Given a smooth representation σ of H one could expect to have a natural isomorphism

$$\mathrm{Ind}_H^G(\sigma^\vee) \cong \left(\text{c-Ind}_H^G \sigma\right)^\vee$$

that is induced by a (G-invariant) pairing of the form

$$V_{\mathrm{Ind}_H^G(\sigma^\vee)} \times V_{\text{c-Ind}_H^G \sigma} \to \mathbf{C}, \qquad (f, f') \mapsto \int_{H\backslash G} \langle f(x), f'(x) \rangle \, \mathrm{d}\mu(x).$$

Note that the function $\langle f(x), f'(x) \rangle$ is indeed well-defined and compactly supported on $H\backslash G$.

The problem is now that we don't know what measure μ we should use to integrate. Maybe we would expect that there exists an essentially unique such measure that is invariant under right translation by G. But typically this is too much to ask for…

Definition 5.2. We set

$$\delta_{H\backslash G} \coloneqq \delta_G|_H \cdot \delta_H^{-1} \colon H \to \mathbf{R}_{>0}.$$

Then we define $C_c^\infty(H\backslash G)$ to be the space of locally constant functions $f \colon G \to \mathbf{C}$ that are compactly supported modulo H and such that

$$f(hg) = \delta_{H\backslash G}(h) f(g)$$

for all $h \in H$ and $g \in G$. Similarly to before we have an action $\rho_{H\backslash G}$ of G on $C_c^\infty(H\backslash G)$ by right translation. $\rho_{H\backslash G}$ is a smooth representation.

Remark 5.3. The notation in the definition is confusing. The elements of $C_c^\infty(H\backslash G)$ are not functions on $H\backslash G$ but rather functions on G that transform in a certain kind under left translation by H.

Proposition 5.4. *Let μ_H be a left Haar measure on H and let μ_G be a right Haar measure on G.*

- *The map*

$$\rho_G \to \rho_{H\backslash G}, \qquad f \mapsto \left(\widetilde{f} \colon x \mapsto \int_H \delta_G(h)^{-1} f(hx) \, \mathrm{d}\mu_H(h) \right)$$

  is a well-defined and surjective morphism of smooth G-representations.
- *The morphism $I_{\mu_G} \colon C_c^\infty(G) \to \mathbf{C}$ factors through $C_c^\infty(H\backslash G)$ to give*

$$I_{\mu_{H\backslash G}} \colon C_c^\infty(H\backslash G) \to \mathbf{C}, \qquad f \mapsto \int_{H\backslash G} f(x) \, \mathrm{d}\mu_{H\backslash G}(x).$$

- *$I_{\mu_{H\backslash G}}$ is the unique (up to a scalar $c \in \mathbf{R}_{>0}$) non-trivial morphism $C_c^\infty(H\backslash G) \to \mathbf{C}$ that satisfies*

$$I_{\mu_{H\backslash G}}(\rho_{H\backslash G}(g) f) = I_{\mu_{H\backslash G}}(f), \qquad f \in C_c^\infty(H\backslash G), g \in G$$

  and

$$I_{\mu_{H\backslash G}}(f) \geq 0, \qquad f \in C_c^\infty(H\backslash G), f \geq 0.$$

Remark 5.5. This is again a warning about confusing notation. $\mu_{H\backslash G}$ is not a measure on $H\backslash G$, although we will think of it as such. And $\int_{H\backslash G} f(x) \, \mathrm{d}\mu_{H\backslash G}(x)$ is not an integral on $H\backslash G$.

Theorem 5.6 (Duality Theorem)**.** *Fix $\mu_{H\backslash G}$ as in Proposition 5.4 and let σ be a smooth representation of H. We have a well-defined G-invariant pairing*

$$V_{\mathrm{Ind}_H^G(\delta_{H\backslash G}\otimes\sigma^\vee)} \times V_{\text{c-Ind}_H^G\,\sigma} \to \mathbf{C}, \qquad (f, f') \mapsto \int_{H\backslash G} \langle f(x), f'(x)\rangle \,\mathrm{d}\mu_{H\backslash G}(x)$$

that is natural in σ. This pairing induces an isomorphism of smooth G-representations

$$\mathrm{Ind}_H^G\big(\delta_{H\backslash G}\otimes\sigma^\vee\big) \cong \big(\text{c-Ind}_H^G\,\sigma\big)^\vee.$$

Example 5.7. We will be intersted in applying Theorem 5.6 in the situation where $G = \mathrm{GL}_2(F)$ and $H = B(F)$ and the representation σ really is a representation of $T(F)$ (that we view as a representation of $B(F)$ that is trivial on $N(F)$). In that case the quotient $H\backslash G$ is compact by Corollary 2.2 so that $\text{c-Ind}_H^G\,\sigma \cong \mathrm{Ind}_H^G\,\sigma$ and $\delta_{H\backslash G} = \delta_H^{-1}$. Thus Theorem 5.6 then gives

$$\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\big(\delta_{B(F)}^{-1}\otimes\sigma^\vee\big) \cong \Big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\,\sigma\Big)^\vee.$$

References

[BH06] Colin J. Bushnell and Guy Henniart. *The Local Langlands Conjecture for* GL(2). Springer, 2006.