IRREDUCIBLE CUSPIDAL REPRESENTATIONS I

NOTATION : F non-arch. local field, $\underset{\varpi}{\mathcal{O}} \subseteq F$ ring of integers , $k = \mathcal{O}/\varpi$

$$G = GL_2(F) \supseteq B = N \rtimes T$$
$$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \quad N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

GOAL : Classify all irreducible $\underbrace{\text{smooth representations}}_{\mathrm{Rep}(G)}$ of $G/\mathbb{C}$.

LAST WEEK : $(-)_N : \mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(T)$
$(\pi, V) \longmapsto (\pi_N, V_N)$

maximal quotient of V on which N acts trivially.

* (π, V) is CUSPIDAL if $V_N = 0$.

* π non-cuspidal $\Leftrightarrow$ $\pi \cong$ subrep. of $\mathrm{Ind}_B^G \chi$, for some character $\chi : T \to \mathbb{C}^\times$.

$\Downarrow$ admissible

$\hookrightarrow$ irred. non-cuspidal reps. are :
- $\mathrm{Ind}_B^G \chi$
- $\phi \circ \det$
- $\phi \cdot St_G$

CONVENTION : "cuspidal" = cuspidal smooth rep. $\in \mathrm{Rep}(G)$

TODAY :
* representations $\rightsquigarrow$ modules over the "Hecke algebra"
* new definition of cuspidality
* first "example" of an irreducible cuspidal rep.

§1. THE HECKE ALGEBRA.

H finite group $\rightsquigarrow$ representations of H $\leftrightarrow$ modules over $\mathbb{C}[H]$

replace by Hecke algebra

REMARK: Can define Hecke algebra etc. also for any (unimodular) locally profinite group G, but we mainly care about $GL_2(F)$.

Fix a Haar measure μ on G.

$I_\mu : C_c^\infty(G) \longrightarrow \mathbb{C}$

$f \longmapsto \int_X f(x)\,d\mu(x)$.

DEF.: $f_1, f_2 \in C_c^\infty(G) = \{ f : G \longrightarrow \mathbb{C}$ locally constant, compact support $\}$

$$\rightsquigarrow (f_1 * f_2)(g) := \int_G f_1(x) f_2(x^{-1}g)\,d\mu(x)$$

"CONVOLUTION PRODUCT"

$$\rightsquigarrow \mathcal{H}(G) := (C_c^\infty(G), *)$$

"HECKE ALGEBRA" of G

$\hookrightarrow$ associative $\mathbb{C}$-algebra

REMARKS:
- $\mathcal{H}(G)$ is NOT commutative
- $\mathcal{H}(G)$ has NO unit element
- $*$ depends on μ, but: if μ, ν Haar measures $\Rightarrow \exists c > 0$, s.t. $\nu = c\mu \rightsquigarrow f \longmapsto c^{-1} f$ $\mathbb{C}$-algebra isomorphism between the corresponding Hecke alg.

"EXAMPLE": H discrete group $\rightsquigarrow \int_H f(h)\,d\mu(h) := \sum_{h\in H} f(h)$

$$\rightsquigarrow \mathcal{H}(H) \xrightarrow{\sim} \mathbb{C}[H]$$
$$f \longmapsto \sum_{h\in H} f(h)h$$

($\Rightarrow \mathcal{H}(H)$ has a unit element)

DEF: $K \leq G$ compact open $\rightsquigarrow e_K(g) := \begin{cases} \mu(K)^{-1}, & g\in K \\ 0, & g\notin K \end{cases}$

$\rightsquigarrow e_K \in \mathcal{H}(G)$

PROPOSITION: (1) e_K is an idempotent, i.e. $e_K * e_K = e_K$

(2) $f\in\mathcal{H}(G)$, then $e_K * f = f \Leftrightarrow f(kg) = f(g)\ \forall k\in K, \forall g\in G$

(3) $e_K * \mathcal{H}(G) * e_K$ is a subalgebra of $\mathcal{H}(G)$ with unit element e_K.

proof:

(1) $$e_K * e_K(g) = \int_G \underbrace{e_K(x)}_{\substack{= \\ 0 \\ \text{iff } x\notin K}} e_K(x^{-1}g)\,d\mu(x)$$

$0 \Leftrightarrow x^{-1}g\notin K$

$\| \Leftrightarrow g\notin K$

$$= \int_K \mu(K)^{-1} e_K(x^{-1}g)\,d\mu(x) = \begin{cases} 0, & \text{if } g\in K \\ \int_K \mu(K)^{-2}\,d\mu(x) = \mu_K(K)^{-1}, & \text{if } g\in K. \end{cases}$$

$$= e_K(g).$$

(2) $f \in \mathcal{H}(G)$, $k \in K$, $g \in G$

$$e_K * f\,(kg) = \int_G e_K(x) f(x^{-1}kg)\, d\mu(x)$$

$$\overset{x \leftrightarrow kx}{=} \int_G \underbrace{e_K(kx)}_{= e_K(x)} f(x^{-1}g)\, \underbrace{d\mu(kx)}_{= \mu(x)} = e_K * f(g)$$

so $e_K * f = f \implies f(kg) = f(g)$

$$\implies e_K * f\,(g) = \int_G \underbrace{e_K(x)}_{= 0 \text{ if } x \notin K} f(x^{-1}g)\, d\mu(x) = \int_K \mu(K)^{-1} \overbrace{f(x^{-1}g)}^{= f(g)}\, d\mu(x) = f(g)$$

(3) ✓

<u>NOTE</u>: $\underbrace{e_K * \mathcal{H}(G) * e_K}_{=: \mathcal{H}(G,K)} = \{ \underbrace{f \in \mathcal{H}(G)}_{\ni e_K} \mid f(k_1 g k_2) = f(g) \ \forall g \in G,\ k_1, k_2 \in K \}$

Moreover, $\mathcal{H}(G) = \bigcup_{\substack{K \leq G \\ \text{compact} \\ \text{open}}} \mathcal{H}(G,K)$ since $f \in \mathcal{H}(G)$ are locally constant so f constant on KgK $\forall g$ for K small enough.

DEF.: A (left) $\mathcal{H}(G)$-module M is called SMOOTH, if

$$\mathcal{H}(G) * M = M.$$

$$\Updownarrow$$

$\forall m \in M \ \exists K \leq G$ compact open, s.t. $e_K * m = m$

$$= \bigcup_K \mathcal{H}(G,K) * M$$

$\rightsquigarrow$ $\mathcal{H}(G)\text{-Mod}$:= category of smooth left $\mathcal{H}(G)$-modules.

$$\mathrm{Hom}_{\mathcal{H}(G)}(M_1, M_2) := \{ M_1 \longrightarrow M_2 \ \mathcal{H}(G)\text{-homs.} \}.$$

$(\pi, V) \in \mathrm{Rep}(G)$ $\rightsquigarrow$ $\mathcal{H}(G)$-module structure?

$f \in \mathcal{H}(G)$, $v \in V$, set

$$\pi(f)v := \int_G f(g)\pi(g)v \, d\mu(g) \in V$$

$$= \sum_{g \in G/K} \int_{gK} \underset{= f(g)}{f(gk)} \underset{= \pi(g)}{\pi(gk)} v \, d\mu(gk)$$

choose $K \leq G$, s.t. $v \in V^K$, $f \in \mathcal{H}(G,K)$

$$= \mu(K) \sum_{g \in G/K} f(g)\pi(g)v$$

EXAMPLE: $v \in V^K$, $\pi(e_K)v = \mu(K) \sum_{g \in G/K} \overbrace{e_K(g)}^{=0 \text{ for } g \notin K} \pi(g)v$

$$= \mu(K) e_K(1) \pi(1) v = v.$$

PROPOSITION : $(\pi, V) \in Rep(G)$

- the operation $(f, v) \longmapsto \pi(f)v$ gives V the structure of a smooth $\mathcal{H}(G)$-module.
- $(\pi', V') \in Rep(G)$, $\phi : V \longrightarrow V'$ G-hom., then ϕ is an $\mathcal{H}(G)$-homomorphism, i.e. $\phi \circ \pi(f) = \pi'(f) \circ \phi$.

proof: $\pi(f_1 * f_2) = \pi(f_1)\pi(f_2)$:

$$\pi(f_1 * f_2)v = \int_G (f_1 * f_2)(g)\,\pi(g)v\,d\mu(g)$$

$$= \int_G \int_G f_1(x) f_2(x^{-1}g)\,d\mu(x)\,\pi(g)v\,d\mu(g)$$

$$\overset{h = x^{-1}g}{=} \int_G \int_G f_1(x) f_2(h)\,\pi(xh)\,v\,d\mu(x)\,d\mu(h)$$

$$= \int_G f_1(x)\pi(x) \int_G f_2(h)\pi(h)v\,d\mu(h)\,d\mu(x) = \pi(f_1)\pi(f_2)v.$$

smooth : $\forall v \in V$, $\exists K \leq G$ compact-op., s.t. $v \in V^K$

$$\Rightarrow \quad \pi(e_K)v = v \quad \rightarrow \quad \mathcal{H}(G) * V = V.$$

$\phi \circ \pi(f) = \pi'(f) \circ \phi$:

$$(\phi \circ \pi(f))(v) = \phi\Big(\mu(K) \sum_{g \in G/K} f(g)\pi(g)v\Big)$$

choose K fixing v and f

$$= \mu(K) \sum_{g \in G/K} f(g) \overbrace{\phi(\pi(g)v)}^{=\pi'(g)\phi(v)} = (\pi'(f)\circ\phi)(v).$$

$\square$

<u>EXAMPLES:</u> • $(\pi, V) = (\lambda, \mathcal{H}(G))$

so the $\mathcal{H}(G)$-module $\mathcal{H}(G)$ should correspond to the rep. $(\lambda, \mathcal{H}(G))$

for $\phi, f \in \mathcal{H}(G)$, $\lambda(\phi) f = \phi * f$

$$\Big(\int_G \phi(x)\lambda(x) f \, d\mu(x)\Big)(g) = \int_G \phi(x) f(x^{-1}g)\, d\mu(x)$$

• $(\pi, V) = (\mathcal{H}(G), \rho)$

$$\rho(\phi) f(g) = f * \hat{\phi}(g), \quad \hat{\phi}(g) := \phi(g^{-1})$$

$$\int_G \phi(x)\rho(x) f(g)\, d\mu(x) \qquad\qquad \int_G f(x)\hat{\phi}(x^{-1}g)\, d\mu(x)$$

$$\int_G \phi(x) f(gx)\, d\mu(x) \overset{x \mapsto g^{-1}x}{=\!=} \int_G f(x)\phi(g^{-1}x)\, d\mu(x)$$

PROPOSITION: M smooth $\mathcal{H}(G)$-module

$\exists!$ G-hom. $\pi: G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(M)$, such that

$(\pi, M) \in \mathrm{Rep}(G)$ and $\pi(f)m = f * m \quad \forall f \in \mathcal{H}(G), m \in M$.

Moreover, $M' \in \mathcal{H}(G)\text{-Mod}$, with assoc. G-rep. (π', M'),

then any $\mathcal{H}(G)$-hom. $M \longrightarrow M'$ is a G-hom. $\pi \to \pi'$.

proof: $\varphi : \mathcal{H}(G) \underset{\mathcal{H}(G)}{\otimes} M \twoheadrightarrow M$ surjective (since M is smooth)

Claim: φ is injective

$\sum_{i=1}^{r} f_i \otimes m_i \in \ker(\varphi)$, choose K s.t. $f_i \in \mathcal{H}(G,K) \ \forall i$

and $m_i \in e_K * M \ \forall i$

then $0 = e_K \otimes \varphi(\sum f_i \otimes m_i)$

$= e_K \otimes \left(\sum f_i * m_i\right) = \sum_{i=1}^{r} f_i \otimes m_i$ $\square$ Claim

$\Rightarrow \overbrace{\mathcal{H}(G) \underset{\mathcal{H}(G)}{\otimes} M}^{\text{smooth } G\text{-rep}} \cong M$ $\mathcal{H}(G)$-iso

$\rightsquigarrow (\lambda, \mathcal{H}(G))$

take $m \in M$, $K \leq G$. s.t. $e_K * m = m$

then $\pi(g)m = (\lambda(g)e_K) * m = \mu(K)^{-1} \mathbb{1}_{gK} * m$

$\pi(g)(e_K \otimes m) = (\lambda(g)e_K) \otimes m$

$$\rightsquigarrow \pi(f)m = \int_G f(g)\pi(g)m\, d\mu(g) = \int_G f(g)\mu(K)^{-1}\mathbb{1}_{gK} * m\, d\mu(g)$$

$$\parallel$$

$$f * m = \underbrace{\left(\int_G f(g)\mu(K)^{-1}\mathbb{1}_{gK}\, d\mu(g)\right)}_{=f} * m$$

$$\phi : M \longrightarrow M' \qquad \phi(\pi(g)m) = \phi(\mu(K)^{-1}\mathbb{1}_{gK} * m) = \pi'(g)\phi(m) \qquad \square$$

<u>RMK.</u>: In particular $\quad \mathrm{Rep}(G) \xrightarrow{\sim} \mathcal{H}(G)\text{-Mod}.$

$(\pi, V) \in \mathrm{Rep}(G)$, $\quad \pi(e_K) : V \twoheadrightarrow V^K \quad$ kernel $= V(K)$

$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \parallel$ linear span of $v - \pi(k)v$, $k \in K$, $v \in V$.

$\rightsquigarrow V^K$ is an $\mathcal{H}(G,K)$-module

<u>PROPOSITION</u>: (1) $(\pi, V) \in \mathrm{Rep}(G)$ irreducible then V^K is either zero or a simple $\mathcal{H}(G,K)$-module

(2)
$$\left\{ \begin{matrix} (\pi, V) \in \mathrm{Rep}(G) \\ \text{irred. s.t.} \\ V^K \neq 0 \end{matrix} \right\} \Big/ \cong \quad \xrightarrow{1-1} \quad \left\{ \begin{matrix} \text{simple} \\ \mathcal{H}(G,K)\text{-modules} \end{matrix} \right\} \Big/ \cong$$

<u>COROLLARY</u>: $(\pi, V) \in \mathrm{Rep}(G)$, $\quad V \neq 0$

Thm (π, V) is irreducible $\iff$ $\forall K \leq G$ compact-open, V^K is either zero or simple as $\mathcal{H}(G,K)$-mod.

SOME MORE IDEMPOTENTS:

$K \leq G$ compact open, $\rho \in \hat{K}$

$$\rightsquigarrow \quad e_\rho(x) := \begin{cases} \frac{\dim \rho}{\mu(K)} \operatorname{tr}(\rho(x^{-1})) & , x \in K \\ 0 & , x \notin K \end{cases}$$

$\rightsquigarrow e_\rho \in \mathcal{H}(G)$

FACTS:
- e_ρ is an idempotent
- $(\pi, V) \in \operatorname{Rep}(G)$, then $\pi(e_\rho) : V \twoheadrightarrow V^\rho$ is the projection onto the ρ-isotypic component

$$V^\rho = \sum_{\substack{W \subseteq V \\ \text{irred. } K\text{-subsp.} \\ \text{on which } K \text{ acts via } \rho}} W$$

- $e_K = e_{\mathbb{1}_K}$.

§2. MATRIX COEFFICIENTS.

DEF.: $(\pi, V) \in \mathrm{Rep}(G)$, $v \in V$, $\ell \in \check{V} = (V^*)^\infty$

$$\rightsquigarrow \quad \gamma_{\ell \otimes v} : g \longmapsto \langle \ell, \pi(g)v \rangle = \ell(\pi(g)v)$$

$$\gamma_{\ell\otimes v} \in C^\infty(G) \qquad \langle , \rangle : \check{V} \times V \longrightarrow \mathbb{C}$$

$C(\pi) :=$ $\mathbb{C}$-v.sp. spanned by $\gamma_{\ell \otimes v}$, $\ell \otimes v \in \check{V} \otimes V$

γ are called "(MATRIX) COEFFICIENTS" of π.

EXAMPLE: $\pi = \mathbb{1}$, $\pi^\vee \cong \mathbb{1}$

$$\langle , \rangle : \mathbb{1} \times \mathbb{1} \longrightarrow \mathbb{C} \quad \rightsquigarrow \quad \gamma_{\ell \otimes v} : g \mapsto \ell \cdot v$$
$$(v, w) \longmapsto v \cdot w$$

$\rightsquigarrow$ $C(\pi) :=$ constant functions

NOTE: $$G \times G \curvearrowright C(\pi), \quad (g_1, g_2) f(h) = f(g_1^{-1} h g_2)$$

$$\check{V} \otimes V \twoheadrightarrow C(\pi) \quad \text{surjective } G \times G\text{-hom.}$$
$$\ell \otimes v \longmapsto \gamma_{\ell \otimes v}$$

$(\pi, V) \in \mathrm{Rep}(G)$ irreducible $\rightsquigarrow$ $Z \curvearrowright V$ via central character ω_π

$$\rightsquigarrow \quad \gamma(zg) = \omega_\pi(z) \gamma(g)$$

$$\sum_i \gamma_{\ell_i \otimes v_i}(zg) = \sum_i \ell_i(\pi(zg) v_i) = \omega_\pi(z) \sum \gamma_{\ell_i \otimes v_i}(g)$$

$\rightsquigarrow$ $\mathrm{supp}(\gamma)$ is invariant under translation by Z.

<u>DEF:</u> $(\pi, V) \in \mathrm{Rep}(G)$ irred.

π is γ-CUSPIDAL if every $\gamma \in C(\pi)$ is compactly supported modulo Z.

<u>EXAMPLE:</u> $\pi = \mathbb{1}$ is <u>not</u> γ-cuspidal (and not cuspidal) since the support of every non-zero coefficient is G. $(= GL_2(F))$

<u>PROPOSITION:</u> (1) π γ-cuspidal $\Rightarrow$ admissible

(2) (π, V) irred. admissible. Suppose $\exists\, \gamma \overset{C(\pi)}{\in} \neq 0$ compactly supported modulo Z $\Rightarrow$ γ-cuspidal.

<u>proof:</u> (1) suppose π not admissible

$\rightsquigarrow$ $\dim V^K = \infty$ for some $K \subseteq G$ compact open

$\dim V^K$ is countable $\Rightarrow$ $\dim \check{V}^K = \dim \mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})$ is uncountable

$v \in V^K \rightsquigarrow \Gamma_v : \check{V}^K \longrightarrow C(\pi)$ injective

$\neq 0$ $\qquad \ell \longmapsto \gamma_{\ell, v}$ since $gv, g \in G$ span V

$\rightsquigarrow$ $\mathrm{im}(\Gamma_v) = \{ f \in C(\pi) \mid f(zgk') = \omega_\pi(z) f(g)$ + supp. on a finite U of $Z \backslash G / K \}$

dim $(\Gamma_v(\check{V}^K))$ countable but Γ_v inj. ↯

(2) (π, V) irred. + adm. (Manu's talk: $\Rightarrow \pi^\vee$ irred. + adm.)

$\check{V} \otimes V$ smooth $G\times G$-rep. $\rightsquigarrow$ smooth $\mathcal{H}(G\times G) \cong \mathcal{H}(G)\otimes\mathcal{H}(G)$-module.

$K \leq G$ compact open. $\rightsquigarrow (\check{V}\otimes V)^{K\times K} = (e_K \otimes e_K) * (\check{V} \otimes V)$

$= \check{V}^K \otimes V^K$

K small enough $\rightsquigarrow V^K, \check{V}^K$ fin. dim'l simple $\mathcal{H}(G,K)$-modules
(i.e. small enough s.t. $\neq 0$)

"Jacobson Density thm." $\Rightarrow$ $\check{V}^K \otimes V^K$ simple $/ \mathcal{H}(G,K)\otimes\mathcal{H}(G,K) \cong \mathcal{H}(G\times G, K\times K)$)

$\Rightarrow$ $\check{V}\otimes V$ is irred. adm. $/G\times G$.

$\Rightarrow$ $\gamma: \check{V}\otimes V \longrightarrow C(\pi)$ inj. $\Rightarrow$ iso.

$\Rightarrow$ $C(\pi)$ irred. as $G\times G$-rep.

for $\gamma_{\neq 0} \in C(\pi)$, $\forall \gamma' \in C(\pi)$ is a finite linear combination of elements $(g,h)\gamma$

If γ compactly supported modulo Z

$\Rightarrow$ so γ' is.

$\square$

THEOREM: (π, V) irred. smooth In particular, π is adm.

π is cuspidal $\Leftrightarrow$ π is γ-cuspidal.

proof: $(\Longrightarrow)$ Cartan dec. $\rightarrow$ $T^+ = \left\{ \begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} = t^n \,\middle|\, n \geq 0 \right\}$ set of reps. of $ZK \backslash G / K$, $K = GL_2(\mathcal{O})$

LEMMA: $v \in V$, $\ell \in \check{V}$, $\exists m \geq 0$, s.t. $\gamma_{\ell \otimes v}(t^n) = 0$ $\forall n \geq m$.

$N_1 \subseteq N$ compact open, s.t. $\ell \in \check{V}^{N_1}$ ($\check{V}$ smooth)

$V_N = 0 \Rightarrow v \in V(N)$ and $\exists N_2 \subseteq N$ c. open

$V(N) = \langle v - \pi(n) v \rangle$

$$\int_{N_2} \pi(x) v \, dx = 0$$

$v = \sum v_i - \pi(n_i) v_i$

choose $N_2 \subseteq N$ compact open, containing all n_i

$$\sum_i \int_{N_2} \pi(x)(v_i - \pi(n_i) v_i) \, d\mu(x)$$

$$\int_{N_2} \pi(x) v_i \, d\mu(x) - \int_{N_2} \pi(x n_i) v_i \, d\mu(x) = 0$$

$\Rightarrow \int_{N_0} \pi(x) v \, d\mu(x) = 0$ $\forall N_0 \subseteq N$ compact open. ($N_0 \supseteq N_2$)

$\begin{pmatrix} \varpi^a & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi^{-a} & \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \varpi^a * \\ & 1 \end{pmatrix}$

but $\exists m \geq 0$, s.t. $t^a N_2 t^{-a} \subseteq N_1$ $\forall a \geq m$

$$\rightsquigarrow \langle \ell, \pi(t^a) v \rangle = c_1 \int_{N_1} \langle \underbrace{\check{\pi}(x^{-1}) \ell}_{= \ell \text{ since } \ell \in \check{V}^{N_1}}, \pi(t^a) v \rangle \, dx =$$

$\langle \ell, \pi(t^a) v \rangle = \gamma_{\ell \otimes v}(t^a)$

$$= c_1 \cdot \int_{N_1} \underbrace{\langle \check{\pi}(t^{-a})\ell, \pi(t^{-a} x t^{a})}_{= \ell(t^{a} t^{-a} x t^{a} v)} v \rangle \, dx$$

$$= c_2 \cdot \int \langle \check{\pi}(t^{-a})\ell, \pi(x) v \rangle \, dx = 0$$

$t^{-a} N_1 t^{a} \supseteq N_2 \Rightarrow \int = 0$

$\square$

Fix $\underset{\neq 0}{f} = \gamma_{\ell \otimes v} \in C(\pi)$, $K' \trianglelefteq K$ open normal fixing ℓ and v.

Show: supp. compact mod Z

$k_1, \ldots, k_r$ representatives of K/K'.

if $g \in G$, $\exists n \geq 0$, s.t. $ZKgK = ZKt^n K = \bigcup_{i,j} ZK' k_i^{-1} t^n k_j K'$

$\Rightarrow \underbrace{\operatorname{Supp} f}_{\text{compact mod } Z} \subseteq \bigcup_{\text{finite}} ZK' \underbrace{(\operatorname{Supp} f_{ij} \cap T^+)}_{} K'$, f_{ij} = coeff. fct. $x \mapsto f(k_i x k_j^{-1})$

$\overset{\text{Lemma}}{\Longrightarrow}$ γ-cuspidal

$(\Leftarrow)$ (π, V) irred. γ-cuspidal $\Rightarrow$ admissible

$\Rightarrow$ $\check{\pi}$ irred. and admissible

$K_n := 1 + \varpi^n M_2(\mathcal{O})$, $n \geq 1$

take $v \in V$, $n \geq 1$, s.t. $v \in V^{K_n}$, $t = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}$

$\ell \in \check{V}^{K_n} \longrightarrow g \overset{\gamma_{\ell \otimes v}}{\longmapsto} \langle \ell, \pi(g) v \rangle$ compactly supp. mod Z

$\Rightarrow \langle \ell, \pi(t^{a}) v \rangle = \gamma_{\ell \otimes v}(t^{a}) = 0 \quad \forall a \gg 0$

$\dim \check{V}^{K_n} < \infty \Rightarrow \exists c$, such that

$$\gamma_{\ell \otimes v}(t^a) = 0 \quad \forall \ell \in \check{V}^{K_n}, \ a \geq c.$$

$$\Longrightarrow \quad \pi(e_{K_n})\pi(t^a)v = 0$$

for $j \in \mathbb{Z}$, write $N_j = \begin{pmatrix} 1 & p^j \\ 0 & 1 \end{pmatrix}$, $N_j' = \begin{pmatrix} 1 & 0 \\ p^j & 1 \end{pmatrix}$

$$T_n = K_n \cap T \quad \rightsquigarrow \quad K_n = N_n T_n N_n'$$

$$K_n^{(a)} = t^{-a} K_n t^a = N_{n-a} T_n N_{n+a}'$$

$$0 = \pi(e_{K_n})\pi(t^a)v = \pi(t^a)\pi(e_{K_n^{(a)}})v$$

$$= \pi(t^a) \sum_{N_{n-a}/N_a} \pi(x)\pi(e_{K_n^{(a)} \cap K_n})v$$

$$= c \cdot \pi(t^a) \int_{N_{n-a}} \pi(x) v \, dx = 0$$

v is fixed by $K_n^{(a)} \cap K_n$

$$\Longrightarrow \quad v \in V(N)$$

$$\Longrightarrow \quad V \text{ is cuspidal.}$$

$\square$

§3. INTERTWINING.

$K \leq G$ compact open, $\hat{K} := \left\{ \begin{matrix} \text{irred smooth} \\ \text{reps of } K \end{matrix} \right\} / \cong$

DEFINITION: • $i = 1, 2$, $K_i \leq G$ compact open, $\rho_i \in \hat{K}_i$, $g \in G$.

g INTERTWINES ρ_1 with ρ_2 if:

$$\mathrm{Hom}_{K_1^g \cap K_2}(\rho_1^g, \rho_2) \neq 0$$

, where $\rho_1^g : x \mapsto \rho_1(gxg^{-1})$, $K_1^g = g^{-1} K_1 g$.

• $K \leq G$ compact open, $(\pi, V) \in \mathrm{Rep}(G)$, $\rho \in \hat{K}$

π CONTAINS ρ / ρ OCCURS IN π if

$$\mathrm{Hom}_K(\rho, \pi) \neq 0.$$

PROPOSITION: (π, V) irreducible smooth rep. of G, containing ρ_1, ρ_2.

Then $\exists g \in G$ which intertwines ρ_1 with ρ_2.

proof: $V = \bigoplus_{\rho \in \hat{K}_i} V^\rho$ $\quad$ π contains $\rho_1, \rho_2 \Rightarrow V^{\rho_i} \neq 0$, $i = 1, 2$

$\downarrow e_2$

V^{ρ_2}

$V^{\rho_1} \neq 0$, π irred. $\Rightarrow$ $\pi(g^{-1}) V^{\rho_1} = V^{\rho_1^g}$ span V

$\rightsquigarrow$ $\exists g \in G$, s.t. $e_2 \circ \pi(g^{-1})$ induces a non-zero map $V^{\rho_1} \longrightarrow V^{\rho_2}$

$$\begin{tikzcd} V \arrow[r, "\pi(g^{-1})"] & V \arrow[r, two heads, "e_2"] & V^{\rho_2} \\ V^{\rho_1} \arrow[u, phantom, "\cup"] \arrow[dr, bend right] & \sum_{\tilde g\in G} V^{\rho_1^{\tilde g}} \arrow[u, phantom, "="] & \\ & V^{\rho_1^g} \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

$\Rightarrow \mathrm{Hom}_{K_1^g \cap K_2}(\rho_1^g, \rho_2) \neq 0$

IDEA: look at isotypical components and find a $g \in G$ such that the map

$$\begin{tikzcd} V \arrow[r, "g^{-1}"] & V \arrow[r, two heads] & V^{\rho_2} \\ V^{\rho_1} \arrow[u, phantom, "\cup"] \arrow[urr, "\neq 0"'] & & \end{tikzcd}$$

induces a non-zero map $\square$

CLAIM: g intertwines ρ_1 with ρ_2 $\iff$ g^{-1} intertwines ρ_2 with ρ_1

ρ_1^g, ρ_2 are semisimple as reps. of $K_1^g \cap K_2$

$\Rightarrow \dim \mathrm{Hom}_{K_1^g \cap K_2}(\rho_1^g, \rho_2) = \dim \mathrm{Hom}_{K_1^g \cap K_2}(\rho_2, \rho_1^g) \cong \mathrm{Hom}_{K_1 \cap K_2^{g^{-1}}}(\rho_2^{g^{-1}}, \rho_1)$

DEF.: • ρ_i INTERTWINE in G if $\exists g \in G$, intertwining ρ_1 with ρ_2

⇝ reflexive + symmetric but not transitive.

• (K, ρ) INTERTWINES ρ if it intertwines ρ with itself.

PROPOSITION: $K \leq G$ compact open, $g \in G$, $\rho \in \hat{K}$. TFAE

(1) $\exists f \in e_\rho * \mathcal{H}(G) * e_\rho$, such that $f|_{KgK} \neq 0$

(2) g intertwines ρ.

$e_\rho(x) = \frac{\dim \rho}{\mu(K)} \operatorname{tr} \rho(x^{-1})$

proof: $C^\infty(KgK) \circlearrowleft_{K\times K}$ via $(k_1,k_2)f : x \longmapsto f(k_1^{-1} x k_2)$
smooth

$H := \{(k, g^{-1}kg) \in K\times K \mid k \in K\cap gKg^{-1}\}$

$\rightsquigarrow \quad C^\infty(KgK) \longrightarrow \mathbb{C} \circlearrowleft \mathbb{1}_H \quad H\text{-hom.}$
$f \longmapsto f(g)$

$(k, g^{-1}kg)f(g) = f(k^{-1}g\, g^{-1}kg) = f(g).$

Frob. reciprocity $\rightsquigarrow \quad C^\infty(KgK) \xrightarrow{\circledast} \mathrm{Ind}_H^{K\times K} \mathbb{1}_H$
$= \{\phi : K\times K \to \mathbb{C} \text{ sm. s.t. } \phi(hk_1, g^{-1}hg\, k_2) = \phi(k_1,k_2)\}$

$\circledast$ is an isomorphism

inverse: $\phi \longmapsto f_\phi \in C^\infty(KgK)$,

$$f_\phi(k_1 g k_2) := \phi(k_1^{-1}, k_2)$$

(1) $\iff \exists f \in e_\rho * \mathcal{H}(G) * e_\rho$, s.t. $f|_{KgK} \neq 0$

$\iff \underbrace{e_\rho * C^\infty(KgK) * e_\rho}_{\cong\, e_\rho * \mathrm{Ind}_H^{K\times K}(\mathbb{1}_H) * e_\rho \,\hat{=}\, \mathrm{Ind}_H^{K\times K}(\mathbb{1}_H)^{\rho\otimes\check{\rho}}} \neq 0$ $\neq 0$

$\iff \mathrm{Hom}_{K\times K}(\rho\otimes\check{\rho}, \mathrm{Ind}_H^{K\times K}(\mathbb{1}_H)) \cong \mathrm{Hom}_H(\rho\otimes\check{\rho}, \mathbb{1}_H) \neq 0$

$\iff$ the rep. $k \longmapsto \rho(k)\otimes\check{\rho}(g^{-1}kg)$ of $K\cap gKg^{-1}$ has a fixed vector
"$\rho\otimes\check{\rho}^{g^{-1}}$"

$\rho \otimes \check{\rho} \longrightarrow \mathbb{1}_H$

$\Longleftrightarrow \mathrm{Hom}_{K \cap gKg^{-1}}(\mathbb{1}, \rho \otimes \check{\rho}^{g^{-1}}) \cong \mathrm{Hom}(\rho, \rho^{g^{-1}}) \neq 0$

$\Longleftrightarrow$ g intertwines ρ. $\square$

---

§4. THE SPHERICAL HECKE ALGEBRA.

$Z \subseteq K \subseteq_{\text{op}} G$ compact mod center. $(\rho, W) \in \hat{K}$

$$\mathcal{H}(G, \rho) := \left\{ f : G \longrightarrow \mathrm{End}_{\mathbb{C}}(W) \,\middle|\, \begin{array}{l} \text{compactly supp mod } Z \\ f(k_1 g k_2) = \rho(k_1) f(g) \rho(k_2) \end{array} \right\}$$

$$f \rightsquigarrow \mathrm{Supp}\, f = \bigcup_{\text{finite}} KgK$$

"SPHERICAL HECKE ALGEBRA" / "INTERTWINING ALGEBRA" of ρ in G

assoc. $\mathbb{C}$-alg. with 1

μ Haar measure on G/Z, $\phi_1, \phi_2 \in \mathcal{H}(G, \rho)$

$$\phi_1 * \phi_2(g) := \int_{G/Z} \phi_1(x)\, \phi_2(x^{-1}g)\, d\mu(x)$$

NOTE: $\exists$ canonical algebra isomorphism

$$e_\rho * \mathcal{H}(G) * e_\rho \cong \mathcal{H}(G, \rho) \otimes \mathrm{End}_{\mathbb{C}}(W).$$

LEMMA: $g \in G$,

$\exists \phi \in \mathcal{H}(G,\rho) \quad \operatorname{Supp} \phi = KgK \iff g$ intertwines ρ.

proof: $f \in \operatorname{End}_{\mathbb{C}}(W)$, $g \in G$

$$\left(kgk' \longmapsto \rho(k) f \rho(k') \right) \in \mathcal{H}(G,\rho)$$

$\iff$ for $k \in K^g \cap K$, we have $f \circ \rho(k) = \rho^g(k) \circ f$

$\iff f \in \operatorname{Hom}_{K^g \cap K}(\rho, \rho^g)$

$\square$

$\rightsquigarrow \Rightarrow \operatorname{Hom}_{K^g \cap K}(\rho, \rho^g) \overset{\text{canonically}}{\cong} \{ f \in \mathcal{H}(G,\rho) \mid \operatorname{supp} f \subseteq KgK \}$.

($\psi : G \longrightarrow \operatorname{End}_{\mathbb{C}}(W)$ $\operatorname{supp} \psi = KgK$, and

$$\psi(kgk') := \rho(k) f \rho(k')$$

$\psi \in \mathcal{H}(G,\rho) \iff \psi(kgk')$

$k \in g^{-1}Kg \cap K \qquad k = g^{-1}k'g \qquad$ then

$$f \circ \rho(k) = \psi(1gk) = \psi(k'g) = \rho(k')f$$
$$= \rho^g(k) f$$
)

PROP.: There is an isomorphism of $\mathbb{C}$-algebras

$$\mathcal{H}(G,\rho) \xrightarrow{\Theta} \mathrm{End}_G\left(\text{c-Ind}_K^G \rho\right)$$

$$\phi \longmapsto \left[f \mapsto \phi * f : g \longmapsto \int_{G/Z} \phi(z) f(z^{-1}g)\, d\mu(z)\right]$$

proof:

$$\mathrm{End}_G\left(\text{c-Ind}_K^G \rho\right) \xrightarrow{(-)\circ \phi^\circ} \mathrm{Hom}_K\left(\rho, \text{c-Ind}_K^G \rho\right)$$

$$\mathrm{id} \longmapsto \phi^\circ : \rho \longrightarrow \text{c-Ind}_K^G \rho$$

$$w \longmapsto \phi^\circ_w$$

$\mathrm{Supp}\, \phi^\circ_w = K$ and $\phi^\circ_w(k) = \rho(k) w$

$$\mathcal{H}(G,\rho) \longrightarrow \mathrm{End}_G\left(\text{c-Ind}_K^G \rho\right) \xrightarrow{\sim} \mathrm{Hom}_K\left(\rho, \text{c-Ind}_K^G \rho\right)$$

$$\phi : \rho \longrightarrow \text{c-Ind}_K^G \rho$$

$\mu(K/Z)^{-1} \cdot \Phi : G \longrightarrow \mathrm{End}_{\mathbb{C}}(W)$, $\Phi(g)(w) := \phi_w(g)$

$\Phi(kg)(w) = \phi_w(kg) = \rho(k)\phi_w(g) = \rho_k \Phi(g)(w)$

$\Phi(gk)(w) = \phi_w(g^k) = \phi_{\rho(k)w}(g)$

$\rightsquigarrow \Phi \in$

$\square$

THEOREM: $Z \leq K \leq G = GL_2(F)$ open compact mod Z

$(\rho, W) \in \mathrm{Rep}(K)$ irred.

Suppose that $g \in G$ intertwines $\rho \iff g \in K$.

Then $\text{c-Ind}_K^G \rho$ is irreducible and cuspidal.

proof: (1) $\exists \gamma \in C(\pi)$ compactly supported modulo Z

$$\mathrm{End}_G(\text{c-Ind}_K^G \rho) \cong \mathrm{Hom}_K(\rho, \text{c-Ind}_K^G \rho)$$

$$\mathrm{id} \longmapsto \phi^\circ : w \mapsto \phi^\circ_w : g \mapsto \begin{cases} \rho(g)w, & g \in K \\ 0 & \text{else.} \end{cases}$$

$$\mathrm{im}\left(\phi^\circ : \rho \longrightarrow \text{c-Ind}_K^G \rho\right) = \left\{ f \in \text{c-Ind}_K^G \rho \mid \mathrm{supp} f \subseteq K \right\}$$

$$(\text{c-Ind}_K^G \rho)^\vee \cong \mathrm{Ind}_K^G \rho^\vee$$

$$\mathrm{im} = \{ f \mid \mathrm{supp} \subseteq K \}$$

$$\rho^\vee \subseteq \text{c-Ind}_K^G \rho^\vee$$

pick $0 \neq w \in \rho$ and $0 \neq \ell \in \rho^\vee \hookrightarrow (\text{c-Ind}_K^G \rho)^\vee$

$\rightsquigarrow$ Claim: $\mathrm{Supp}(\gamma_{\ell \otimes w}) \subseteq K$

More precisely:
$$\rho^\vee \hookrightarrow \mathrm{Ind}_K^G \rho^\vee \xrightarrow{\sim} (\text{c-Ind}_K^G \rho)^\vee$$
$$\ell \longmapsto [\varphi \mapsto \ell(\varphi(1))]$$

$$\rho \hookrightarrow \text{c-Ind}_K^G \rho$$
$$w \longmapsto \left[g \mapsto \begin{cases} \rho(g)w, & g \in K \\ 0 \end{cases}\right]$$

So $$\gamma_{\ell \otimes w}(g) = \begin{cases} \ell(\rho(g)), & g \in K \\ 0 & \text{else} \end{cases}$$
$\neq 0$

$\Rightarrow$ compact supp. mod Center.

(2) irreducible

$Z \curvearrowright \text{c-Ind}_K^G \rho$ acts via ω_ρ: $(zf)(g) = f(gz) = f(zg) = \rho(z)f(g) = \omega_\rho(z)f(g)$

$\rightsquigarrow \text{c-Ind}_K^G \rho \cong \bigoplus_{\tilde{\rho} \in \hat{K}} (\text{c-Ind}_K^G \rho)^{\tilde{\rho}}$

$$\text{Hom}_K(\rho, \text{c-Ind}_K^G \rho) \cong \text{Hom}_K(\rho, (\text{c-Ind}_K^G \rho)^\rho)$$

$$\cong$$

$$\text{End}_G(\text{c-Ind}_K^G \rho) \cong \mathcal{H}(G, \rho) \to \text{1-dim}^l$$

$g \in G$ intertwines ρ $\Leftrightarrow$ $g \in K$

$\Updownarrow$

$\exists f \in \mathcal{H}(G,\rho)$ supp. $= KgK$

$\Rightarrow \dim_{\mathbb{C}} \text{Hom}_K(\rho, (\text{c-Ind}_K^G \rho)^\rho) = 1 \Rightarrow \rho \cong (\text{c-Ind}_K^G \rho)^\rho$

Let $Y \subseteq \text{c-Ind}_K^G \rho$, $Y \neq 0$, G-subspace

Claim: $Y = \text{c-Ind}_K^G \rho$

$\ulcorner$

$$0 \neq \text{Hom}_G(Y, \text{c-Ind}_K^G \rho) \subseteq \text{Hom}_G(Y, \text{Ind}_K^G \rho) \cong \text{Hom}_K(Y, \rho)$$

Y semisimple $/K \Rightarrow Y^\rho \neq 0$

$\Longrightarrow Y \supseteq (\text{c-Ind}_K^G \rho)^\rho$ ← generates $\text{c-Ind}_K^G \rho$ over G

$\Longrightarrow Y = \text{c-Ind}_K^G \rho$

$\Rightarrow$ $c\text{-}\mathrm{Ind}_K^G \rho$ is irreducible

$\square$ Claim

$\llcorner$

1 + 2

$\Rightarrow$ $c\text{-}\mathrm{Ind}_K^G \rho$ is p-cuspidal. $\Rightarrow$ $c\text{-}\mathrm{Ind}_K^G \rho$ is cuspidal

$\square$

$\llcorner$