Cuspidal representation 2

Throughout our discussion, we fix a non-Archimedean local field F with valuation ring $\mathcal{O}_F$ and maximal ideal $\mathfrak{p}$. We set $k = \mathcal{O}_F/\mathfrak{p}$ and we write $p = \operatorname{char} k$, $q = \# k$. Moreover, $U_F := \mathcal{O}_F^\times$ and $U_F^n = 1 + \mathfrak{p}^n$ for $n \geq 1$

§1 Lattice chains

We wish to define a particular family of compact open subgroups of $G := GL_2(F)$, which will be useful in the study of cuspidal smooth representations of G.

For this section, let us set $V = F \oplus F$ and $A = \operatorname{End}_F(V)$, so that $G = GL_F(V) = A^\times$.

Def: An "$\mathcal{O}_F$-lattice chain" is a non-empty family $\mathcal{L} = \{L_i\}_{i \in \mathbb{Z}}$ of $\mathcal{O}_F$-lattices L_i of V such that $L_{i+1} \subsetneq L_i$ for all $i \in \mathbb{Z}$ and $\mathcal{L}$ is stable under multiplication by $F^\times$.

Remark: The latter property implies that for any $\mathcal{O}_F$-lattice chain $\mathcal{L}$ there is a positive integer $e = e_{\mathcal{L}}$ satisfying $xL_i = L_{i+e\, v_F(x)}$ for all $i \in \mathbb{Z}$ and $x \in F^\times$

For example, the families of $\mathcal{O}_F$-lattices $\mathcal{L}^{(1)} = \{L_i^{(1)}\}_{i \in \mathbb{Z}}$ and $\mathcal{L}^{(2)} = \{L_i^{(2)}\}_{i \in \mathbb{Z}}$ given by $L_i^{(1)} := \mathfrak{p}^i \oplus \mathfrak{p}^i$ and $L_{2i}^{(2)} := \mathfrak{p}^i \oplus \mathfrak{p}^i$, $L_{2i+1}^{(2)} := \mathfrak{p}^i \oplus \mathfrak{p}^{i+1}$ are $\mathcal{O}_F$-lattice chains.

Up to automorphisms of V, these are essentially the only examples of $\mathcal{O}_F$-lattice chains, as shown by the following result.

Proposition 1: Let $\mathcal{L} = \{L_i\}_{i \in \mathbb{Z}}$ be an $\mathcal{O}_F$-lattice chain. Then one and only one of the following holds:

1) $e = e_{\mathcal{L}} = 1$ and there is $g \in G = GL_F(V)$ such that $gL_i = L_i^{(1)}$ for all $i \in \mathbb{Z}$;
2) $e = e_{\mathcal{L}} = 2$ and there is $g \in G = GL_F(V)$ such that $gL_i = L_i^{(2)}$ for all $i \in \mathbb{Z}$.

Proof: Since $\mathfrak{p}L_i = L_{i+e}$ for all $i \in \mathbb{Z}$, we have $L_e = \mathfrak{p}L_0$. Thus $L_0/L_e = L_0/\mathfrak{p}L_0$ is a 2-dimensional k-vector space. The $\mathcal{O}_F$-lattices L_i for $i = 0, \dots, e$ make up a flag of subspace in L_0/L_e, which cannot be therefore longer than 2, whence $e = 1$ or $e = 2$. The lattice L_0 is the $\mathcal{O}_F$-span of a F-basis for V, so we may find $g \in G$ such that $gL_0 = \mathcal{O}_F \oplus \mathcal{O}_F$, and thus $gL_{ei} = \mathfrak{p}^i \oplus \mathfrak{p}^i$ for any $i \in \mathbb{Z}$. If $e = 1$, this concludes the first case. Suppose $e = 2$, so that $\mathfrak{p} \oplus \mathfrak{p} \subsetneq gL_1 \subsetneq \mathcal{O}_F \oplus \mathcal{O}_F$. Since $\mathcal{O}_F \oplus \mathcal{O}_F/(\mathfrak{p} \oplus \mathfrak{p}) = k \oplus k$ has dimension 2, $gL_1/(\mathfrak{p} \oplus \mathfrak{p})$ must be one-dimensional, so we can find $\bar{h} \in GL_2(k)$ such that $\bar{h}(gL_1/(\mathfrak{p} \oplus \mathfrak{p})) = k \oplus 0$. By lifting $\bar{h}$ to $h \in GL_2(\mathcal{O}_F)$, it follows that $hgL_1 = \mathcal{O}_F \oplus \mathfrak{p}$. $\square$

Given a lattice chain $\mathcal{L}$, we set
$\mathfrak{A}_{\mathcal{L}} := \{x \in A = \mathrm{End}_F(V) \mid x L_i \subseteq L_i \ \forall i \in \mathbb{Z}\} = \bigcap_{i \in \mathbb{Z}} \mathrm{End}_{\mathcal{O}_F}(L_i) = \bigcap_{0 \le i \le e-1} \mathrm{End}_{\mathcal{O}_F}(L_i)$
It is clear that $\mathfrak{A}_{\mathcal{L}}$ is an $\mathcal{O}_F$-order of A, that is, it is both a $\mathcal{O}_F$-lattice and a ring (non-commutative, but with 1).
We may translate the previous Proposition in terms of $\mathfrak{A}_{\mathcal{L}}$:
__Corollary 2:__ Let $\mathcal{L}$ be an $\mathcal{O}_F$-lattice chain in V. Then there is $g \in G$ such that

1) if $e_{\mathcal{L}} = 1$, $g \mathfrak{A}_{\mathcal{L}} g^{-1} = \mathfrak{M}$, where $\mathfrak{M} := \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathcal{O}_F & \mathcal{O}_F \end{pmatrix}$;
2) if $e_{\mathcal{L}} = 2$, $g \mathfrak{A}_{\mathcal{L}} g^{-1} = \mathfrak{J}$, where $\mathfrak{J} = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathfrak{p} & \mathcal{O}_F \end{pmatrix}$.

An $\mathcal{O}_F$-lattice in V is said to be an $\mathfrak{A}_{\mathcal{L}}$-lattice if it is also an $\mathfrak{A}_{\mathcal{L}}$-module.
One may recover the $\mathcal{O}_F$-lattices in $\mathcal{L}$ starting from $\mathfrak{A}_{\mathcal{L}}$ as follows.
__Proposition 3:__ If L is an $\mathfrak{A}_{\mathcal{L}}$-lattice, then $L \in \mathcal{L}$.
__Proof:__ It is enough to deal with the cases $\mathfrak{A}_{\mathcal{L}} = \mathfrak{M}$ and $\mathfrak{A}_{\mathcal{L}} = \mathfrak{J}$.
· If $\mathfrak{A}_{\mathcal{L}} = \mathfrak{J}$, observe that $\mathfrak{J}$ contains the idempotents $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, so $L \supseteq e_1 L + e_2 L$, hence $L = e_1 L \oplus e_2 L$. This means that $L = \mathfrak{p}^a \oplus \mathfrak{p}^b$ for certain integers a, b. Since L is invariant under upper triangular unipotent matrices in $\mathfrak{J}$ we must have $a \le b$. By arguing with lower triangular matrices, it also follows $b \le a+1$. We conclude that either $L = \mathfrak{p}^a \oplus \mathfrak{p}^a$ or $L = \mathfrak{p}^a \oplus \mathfrak{p}^{a+1}$. Similarly for $\mathfrak{A}_{\mathcal{L}} = \mathfrak{M}$ $\square$

§2 Chain orders

__Def:__ A "chain order" in $A = \mathrm{End}_F(V)$ is a ring of the form $\mathfrak{A} = \mathfrak{A}_{\mathcal{L}}$ for some lattice chain $\mathcal{L}$
We shall write $e_{\mathfrak{A}} := e_{\mathcal{L}}$.
__Proposition 4:__ Let $\mathfrak{A}$ be a chain order in $\mathfrak{A}$ and set $\mathfrak{P} := \mathrm{rad}\, \mathfrak{A}$, $e := e_{\mathfrak{A}}$.
Then $\mathfrak{P}^e = \mathfrak{p} \mathfrak{A}$ and there is $\Pi \in G$ such that $\mathfrak{P} = \Pi \mathfrak{A} = \mathfrak{A} \Pi$.
__Proof:__ It is enough to assume $\mathfrak{A} = \mathfrak{M}$ or $\mathfrak{A} = \mathfrak{J}$. If π is a uniformizer of F, then we may take $\Pi = \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}$ or $\Pi = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$ respectively $\square$
According to this Proposition, we have $\mathfrak{P}^n = \Pi^n \mathfrak{A} = \mathfrak{A} \Pi^n$ for all $n \ge 0$. One similarly defines $\mathfrak{P}^n$ for $n < 0$. By definition of lattice chain, if $\mathfrak{A} = \mathfrak{A}_{\mathcal{L}}$ for $\mathcal{L} = \{L_i\}_{i \in \mathbb{Z}}$, then $\Pi L_i = L_{i+1}$ and $\mathfrak{P}^n = \bigcap_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{O}_F}(L_i, L_{i+n}) = \{x \in A \mid x L_i \subseteq L_{i+n} \text{ for all } i \in \mathbb{Z}\}$.

Let $\mathfrak{A}$ be a chain order in A, $\mathfrak{P} = \operatorname{rad} \mathfrak{A}$. We define:

$U^0_{\mathfrak{A}} = U_{\mathfrak{A}} = \mathfrak{A}^\times$, $\quad U^n_{\mathfrak{A}} = 1 + \mathfrak{P}^n$ for $n \geq 1$.

For all $n \geq 0$, $U^n_{\mathfrak{A}}$ is a compact open subgroup of G, and $U^n_{\mathfrak{A}}$ is normal in $U_{\mathfrak{A}}$.

If $1 \leq m < n \leq 2m$, there is an isomorphism $\mathfrak{P}^m/\mathfrak{P}^n \xrightarrow{\sim} U^m_{\mathfrak{A}}/U^n_{\mathfrak{A}}$, $x \mapsto 1+x$

We also define $K_{\mathfrak{A}} := \{ g \in G \mid g\mathfrak{A}g^{-1} = \mathfrak{A} \}$.

Notice that, if $\mathfrak{A} = \mathfrak{A}_{\mathcal{L}}$, $\mathcal{L} = \{L_i\}_{i \in \mathbb{Z}}$, then $K_{\mathfrak{A}} = \operatorname{Aut}_{\mathcal{O}_F}(\mathcal{L}) = \{ g \in G \mid gL \in \mathcal{L} \ \forall L \in \mathcal{L} \}$.

It is easily checked that $K_{\mathfrak{A}} = \langle \Pi \rangle \ltimes U_{\mathfrak{A}}$, so $K_{\mathfrak{A}}$ is open in G, it contains the center Z of G and $K_{\mathfrak{A}}/Z$ is compact.

The following result will be useful when dealing with admissible pairs in the next talk. We are not going to use it throughout this talk

<u>Proposition 5</u>: Let E be an F subalgebra of A such that E/F is a quadratic field extension. Then:

1) The set of $\mathcal{O}_E$-lattices in V forms an $\mathcal{O}_F$-lattice chain in V, with the property that $e_{\mathcal{L}} = e(E/F)$. Moreover, $\mathcal{L}$ is the unique lattice chain in V which is stable under translation by $E^\times$.
2) The order $\mathfrak{A} = \mathfrak{A}_{\mathcal{L}}$ is the unique chain order in A such that $E^\times \subseteq K_{\mathfrak{A}}$
3) If $\mathfrak{P} = \operatorname{rad} \mathfrak{A}$, then $x\mathfrak{A} = \mathfrak{P}^{v_E(x)}$ for $x \in E^\times$ and $K_{\mathfrak{A}} = E^\times U_{\mathfrak{A}}$.

<u>Proof</u>: Points (2) and (3) easily follow from (1).

We start by observing that V is a one-dimensional E-vector space (if $E = F(\phi)$ for some $\phi \in E \setminus F$, then for all $v \in V$, $v \neq 0$ we have $E.v = Fv \oplus F\phi(v)$, which has dimension 2 over F just like V). Thus, the $\mathcal{O}_E$-lattices in V are precisely $\mathcal{L} = \{ \mathfrak{p}_E^j v \mid j \in \mathbb{Z} \}$ for any fixed $v \in V$, $v \neq 0$, and $\mathcal{L}$ does not depend on v. It is clear that $\mathcal{L}$ is an $\mathcal{O}_F$-lattice chain stable under multiplication by $E^\times$.

If $\mathcal{L}'$ is a lattice chain stable under multiplication by $E^\times$, each $L \in \mathcal{L}'$ is stable under the action of U_E. Since $\mathcal{O}_E = \mathcal{O}_F[U_E]$, L is an $\mathcal{O}_E$-lattice and therefore $\mathcal{L}' \subseteq \mathcal{L}$. But for any $L \in \mathcal{L}'$ we have $\mathcal{L} = E^\times L \subseteq \mathcal{L}'$, so $\mathcal{L} = \mathcal{L}'$. $\square$

§3 The normalized level of a representation

We shall now pass to the study of characters on the groups $U^n_{\mathfrak{A}}/U^{n+1}_{\mathfrak{A}}$.

Fix a character $\psi \in \hat{F}$, $\psi \neq 1$ (thus, ψ is a non-trivial group homomorphism $F \to \mathbb{C}^\times$ with open kernel). Let $\hat{A}$ be the group of characters of A, and define $\psi_A : A \to \mathbb{C}^\times$, $x \mapsto \psi(\mathrm{tr}_A x)$, where $\mathrm{tr}_A : A \to F$ is the usual trace map.

If $a \in A$, we also define $a\psi_A : A \to \mathbb{C}^\times$, $x \mapsto a\psi_A(x) = \psi_A(xa)$ $(= \psi_A(ax))$.

All characters of A arise in this way:

Lemma 6: The map $A \to \hat{A}$, $a \mapsto a\psi_A$ is an isomorphism.

Proof: Since $F \xrightarrow{\sim} \hat{F}$ via $a \mapsto (x \mapsto \psi(ax))$, we have $\hat{A} = (\hat{F})^4 \cong F^{\oplus 4}$. A character of A is therefore of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \psi(\alpha a)\psi(\beta b)\psi(\gamma c)\psi(\delta d)$ for uniquely determined $\alpha, \beta, \gamma, \delta \in F$. This expression coincides with $\psi_A\left(\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left(\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}\psi_A\right)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ □

From now forth, let us assume that ψ has level 1 (that is, $\mathfrak{p} \subseteq \ker\psi$ but $\mathcal{O}_F \not\subseteq \ker\psi$)

Let L be an $\mathcal{O}_F$-lattice in A. We define $L^* \subseteq A$ as

$$L^* = \{ x \in A \mid \psi_A(xy) = 1 \text{ for all } y \in L \}$$

This is also a $\mathcal{O}_F$-lattice in A, and $(L^*)^* = L$. Notice that for any pair L_1, L_2 of $\mathcal{O}_F$-lattices in A we have $(L_1 + L_2)^* = L_1^* \cap L_2^*$ and $(L_1 \cap L_2)^* = L_1^* + L_2^*$.

Via the isomorphism $A \cong \hat{A}$, we may identify L^* with

$$L^* = \{ x \in A \mid (x\psi_A)(y) = 1 \text{ for all } y \in L \} \cong \{ \varphi \in \hat{A} \mid \varphi|_L \equiv 1 \}.$$

Also, if $L_1 \subseteq L_2$, then $L_2^* \subseteq L_1^*$ and L_1^*/L_2^* corresponds to the characters of L_2 which are trivial on L_1.

Proposition 7: Let $\mathfrak{A}$ be a chain order in A with radical $\mathfrak{P} = \mathrm{rad}\,\mathfrak{A}$, $\psi \in \hat{F}$ of level one.

1) For $n \in \mathbb{Z}$, we have $(\mathfrak{P}^n)^* = \mathfrak{P}^{1-n}$.

2) Let m, n be integers such that $0 < m < n \leq 2m+1$. Then the map

$$\mathfrak{P}^{-n}/\mathfrak{P}^{-m} \longrightarrow \left(U^{m+1}_{\mathfrak{A}}/U^{n+1}_{\mathfrak{A}}\right)^\wedge, \quad \alpha + \mathfrak{P}^{-m} \mapsto \psi_{A,a}\big|_{U^{m+1}_{\mathfrak{A}}}$$

is an isomorphism (here, $\psi_{A,a}(x) := \psi_A(a(x-1))$ for all $x \in A$)

Proof: For (1), one can check that $\mathfrak{A}^* = \mathfrak{P}$ by reducing to the case of $\mathfrak{A} = \mathfrak{M}$ or $\mathfrak{A} = \mathfrak{I}$.

If L is an $\mathcal{O}_F$-lattice in A and $g \in G$, then $(gL)^* = L^* g^{-1}$. So, if $\mathfrak{P} = \Pi\mathfrak{A}$, then $(\mathfrak{P}^n)^* = (\Pi^n \mathfrak{A})^* = \mathfrak{A}^* \Pi^{-n} = \mathfrak{P}^{1-n}$.

For (2), we have $\mathfrak{P}^{-n}/\mathfrak{P}^{-m} = (\mathfrak{P}^{-n}/\mathfrak{P}^{-m})^{**} \cong \left((\mathfrak{P}^{-m})^*/(\mathfrak{P}^{-n})^*\right)^* \cong \left(\mathfrak{P}^{m+1}/\mathfrak{P}^{n+1}\right)^* \cong \left(U^{m+1}_{\mathfrak{A}}/U^{n+1}_{\mathfrak{A}}\right)^\wedge$. □

Let us now consider an irreducible smooth representation (ϖ, V) of G.
We denote by $S(\varpi)$ the set of pairs $(\mathcal{A}, n)$ where $\mathcal{A}$ is a chain order in A and $n \geq 0$ is an integer such that ϖ contains the trivial character of $U_{\mathcal{A}}^{n+1}$.
Clearly if $(\mathcal{A}, n) \in S(\varpi)$ then $(\mathcal{A}, m) \in S(\varpi)$ for all $m \geq n$.
Def: The "normalized level" of ϖ is $\ell(\varpi) = \min\{\frac{n}{e_{\mathcal{A}}} \mid (\mathcal{A}, n) \in S(\varpi)\} \in \frac{1}{2}\mathbb{N}$.
Notice that ϖ contains the trivial character on $U_{\mathcal{A}}^{n+1}$ if and only if it contains the trivial character of $U_{g\mathcal{A}g^{-1}}^{n+1} = g\, U_{\mathcal{A}}^{n+1} g^{-1}$ for $g \in G$. As a result, $\ell(\varpi)$ can be computed by looking at pairs of the form $(\mathfrak{M}, n)$ or $(\mathcal{J}, n)$.
Also notice that $\ell(\varpi) < \infty$ because ϖ is smooth.
The fact that $U_{\mathfrak{M}}^1 \subseteq U_{\mathcal{J}}^1$ implies the following
Proposition 8: $\ell(\varpi) = 0$ if and only if ϖ contains the trivial character of $U_{\mathfrak{M}}^1$.
Proof: This follows from the last example of Talk 5.

§4 Strata

Fix a character $\psi \in \hat{F}$ of level 1, thus an isomorphism $A \cong \hat{A}$
Def: A "stratum" in A is a triple $(\mathcal{A}, n, a)$ where $\mathcal{A}$ is a chain order in A, $n \in \mathbb{Z}$ and $a \in \mathfrak{P}^{-n}$. Two strata $(\mathcal{A}, n, a_1), (\mathcal{A}, n, a_2)$ are equivalent if $a_1 - a_2 \in \mathfrak{P}^{1-n}$.
For $n \geq 1$, we can associate with a stratum $(\mathcal{A}, n, a)$ the character ψ_a of $U_{\mathcal{A}}^n$, which is trivial on $U_{\mathcal{A}}^{n+1}$ and only depends on the equivalence class of $(\mathcal{A}, n, a)$.
Proposition 9: Let $(\mathcal{A}_1, n_1, a_1), (\mathcal{A}_2, n_2, a_2)$ be strata, $\mathfrak{P}_1 = \operatorname{rad} \mathcal{A}_1$, $\mathfrak{P}_2 = \operatorname{rad} \mathcal{A}_2$ and let $g \in G$. Suppose that $n_1, n_2 \geq 1$. The following are equivalent:
1) g intertwines the character ψ_{a_1} of $U_{\mathcal{A}_1}^{n_1}$ with the character ψ_{a_2} of $U_{\mathcal{A}_2}^{n_2}$.
2) The intersection $g^{-1}(a_1 + \mathfrak{P}^{1-n_1})g \cap (a_2 + \mathfrak{P}^{1-n_2})$ is not empty.

Proof: Consider $\mathcal{A}_3 := g^{-1}\mathcal{A}_1 g$, with radical $\mathfrak{P}_3 = g^{-1}\mathfrak{P}_1 g$. The character $(\psi_{a_1})^g$ is associated with the stratum $(\mathcal{A}_3, n_1, g^{-1}a_1 g)$ and we may therefore reduce to the case $g = 1$.
$2) \Rightarrow 1)$ Pick $a \in (a_1 + \mathfrak{P}^{1-n_1}) \cap (a_2 + \mathfrak{P}^{1-n_2})$. Then $\psi_a = \psi_{a_i}$ on $U_{\mathcal{A}_i}^{n_i}$, so $\psi_{a_1} = \psi_{a_2} = \psi_a$ on $U_{\mathcal{A}_1}^{n_1} \cap U_{\mathcal{A}_2}^{n_2}$. This implies that $\operatorname{Hom}_{U_{\mathcal{A}_1}^{n_1} \cap U_{\mathcal{A}_2}^{n_2}}(\psi_{a_1}, \psi_{a_2}) \neq 0$.

Indeed, given a linear map $f:\mathbb{C}\to\mathbb{C}$, the condition that it lies in $\mathrm{Hom}_{U^{m_1}_{\mathfrak{A}_1}\cap U^{m_2}_{\mathfrak{A}_2}}(\psi_{a_1},\psi_{a_2})$ reads: $\forall\, h\in U^{m_1}_{\mathfrak{A}_1}\cap U^{m_2}_{\mathfrak{A}_2}$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "f"] \arrow[d, "\psi_{a_1}(h)\cdot"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{C} \arrow[d, "\psi_{a_2}(h)\cdot"] \\ \mathbb{C} \arrow[r, "f"'] & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} z \arrow[r, mapsto] \arrow[dd, mapsto] & f(z) \arrow[d, mapsto] \\ & \psi_{a_2}(h)\cdot f(z) \arrow[d, phantom, "\parallel"] \\ \psi_{a_1}(h)z \arrow[r, mapsto] & \psi_{a_1}(h)\cdot f(z) \end{tikzcd}$$

1) $\Rightarrow$ 2) Suppose that ψ_{a_i} agree on $U^{m_1}_{\mathfrak{A}_1}\cap U^{m_2}_{\mathfrak{A}_2}$. This means that $\psi_A(a_1x)=\psi_A(a_2x)$ for all $x\in\mathfrak{P}^{m_1}\cap\mathfrak{P}^{m_2}$. By taking duals:
$(\mathfrak{P}_1^{m_1}\cap\mathfrak{P}_2^{m_2})^* = \mathfrak{P}_1^* + \mathfrak{P}_2^* = \mathfrak{P}_1^{1-m_1}+\mathfrak{P}_2^{1-m_2}$, so $a_1\equiv a_2 \pmod{\mathfrak{P}_1^{1-m_1}+\mathfrak{P}_2^{1-m_2}}$.
Let $x_1\in\mathfrak{P}^{1-m_1}$ and $x_2\in\mathfrak{P}^{1-m_2}$ be such that $a_1=a_2+x_1+x_2$. Then
$a_1-x_1=a_2+x_2\in(a_1+\mathfrak{P}_1^{1-m_1})\cap(a_2+\mathfrak{P}_2^{1-m_2})$. □

An element $g\in G$ that satisfies one of the two equivalent conditions of the previous Proposition is said to "intertwine $(\mathfrak{A}_1,m_1,a_1)$ with $(\mathfrak{A}_2,m_2,a_2)$".

Def: A stratum $(\mathfrak{A},n,a)$ in A is called "fundamental" if the coset $a+\mathfrak{P}^{1-n}$ contains no nilpotent element of A.

It is readily checked that this definition only depends on the equivalence class of a stratum. A useful characterization is the following:

Proposition 10: Let $(\mathfrak{A},n,a)$ be a stratum in A, $\mathfrak{P}=\mathrm{rad}\,\mathfrak{A}$. The following are equivalent:

1) The coset $a+\mathfrak{P}^{1-n}$ contains a nilpotent element;
2) There is $r\in\mathbb{Z}$, $r\geq 1$ such that $a^r\in\mathfrak{P}^{1-nr}$.

Proof: Observe that (2) is equivalent to say that there is $r\in\mathbb{Z}$, $r\geq 1$ such that for any $x_1,\dots,x_r\in a+\mathfrak{P}^{1-n}$ we have $x_1\cdots x_r\in\mathfrak{P}^{1-nr}$. Thus, (2) only depends on the coset $a+\mathfrak{P}^{1-n}$; it can be rephrased as: there are $x\in a+\mathfrak{P}^{1-n}$ and $r\in\mathbb{Z}$, $r\geq 1$ such that $x^r\in\mathfrak{P}^{1-nr}$. The implication 1) $\Rightarrow$ 2) becomes then clear.
For 2) $\Rightarrow$ 1), it is enough to assume that $\mathfrak{A}=\mathfrak{M}$ or $\mathfrak{A}=\mathfrak{J}$. The claim is also invariant under replacing $(\mathfrak{A},n,a)$ by $(\mathfrak{A},n-e_{\mathfrak{A}},\pi a)$ for a uniformizer $\pi\in F$.
We may therefore assume $(\mathfrak{A},n)=(\mathfrak{M},0),(\mathfrak{J},0)$ or $(\mathfrak{J},-1)$.
If $(\mathfrak{A},n,a)=(\mathfrak{M},0,a)$, then $a\in\mathfrak{M}$. Thus, (2) is equivalent to $\bar a^r\equiv 0$ for some $r\geq 1$, where $\bar a\in M_2(k)$ is the image of a under $M_2(\mathcal{O}_F)\to M_2(k)$. This means that either $\bar a=0$ or $\bar a$ is $GL_2(k)$-conjugate to $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\in M_2(k)$. This results into either $a\in\mathfrak{P}$

or $a+\mathfrak{P}$ containing a $GL_2(\mathcal{O}_F)$-conjugate of $\begin{pmatrix}0&1\\0&0\end{pmatrix}$, so (1) holds.
If $\mathfrak{P}=\operatorname{rad}\mathfrak{J}$, then for $m=0,1$ we have $a+\mathfrak{P}^{m+1}=\begin{pmatrix}0&1\\\pi&0\end{pmatrix}^m\begin{pmatrix}a_1&0\\0&a_2\end{pmatrix}+\mathfrak{P}^{m+1}$ for $a_i\in\mathcal{O}_F$.
If $m=0$, then (2) is equivalent to $a_1\equiv a_2\equiv 0 \pmod{\pi}$, which implies (1). If $m=1$, then (2) is equivalent to $a_1a_2\equiv 0 \pmod{\pi}$, which also implies (1) $\square$

<u>Corollary 11</u>: Let $\pi\in F$ be a uniformizer, $(\mathfrak{A},n,a)$ a non-trivial stratum in A (in the sense that $a\notin\mathfrak{P}^{1-n}$, with $\mathfrak{P}=\operatorname{rad}\mathfrak{A}$). If $(\mathfrak{A},n,a)$ is not fundamental, then it is equivalent to a G-conjugate of either $(\mathfrak{M},n,\pi^{-n}a)$ for $a=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ or $(\mathfrak{J},2n-1,\pi^{-n}\alpha)$ for $\alpha=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ for some $n\in\mathbb{Z}$.
This gives a classification of all non-trivial non-fundamental strata, up to equivalence.

Let us now go back to representations. Consider an irreducible smooth representation (ϖ,V) of G. We say that ϖ contains the stratum $(\mathfrak{A},n,a)$ if $n\geq 1$ and ϖ contains the character ψ_a of $U^n_{\mathfrak{A}}$. By definition, if this is the case then $\ell(\varpi)\leq \frac{n}{e_{\mathfrak{A}}}$.
<u>Theorem 12</u>: Let $(\mathfrak{A},n,a)$ be a stratum in A contained in ϖ. The following are equivalent
1) $(\mathfrak{A},n,a)$ is fundamental;
2) $\ell(\varpi)=\frac{n}{e_{\mathfrak{A}}}$.
In particular, ϖ contains a fundamental stratum if and only if $\ell(\varpi)>0$.
<u>Sketch of the proof</u>:
<u>Step 1</u>: One shows that, if $(\mathfrak{A},n,a)$ is not fundamental, then there is a chain order $\mathfrak{A}_1$ in A with $\mathfrak{P}_1:=\operatorname{rad}\mathfrak{A}_1$ such that $a+\mathfrak{P}^{1-n}=\mathfrak{P}_1^{-n_1}$ for some integer n_1 such that $\frac{n_1}{e_{\mathfrak{A}_1}}<\frac{n}{e_{\mathfrak{A}}}$. One can achieve this by reducing to the special cases of Corollary 11. It follows that $\ell(\varpi)\leq\frac{n_1}{e_{\mathfrak{A}_1}}<\frac{n}{e_{\mathfrak{A}}}$.
<u>Step 2</u>: If $\ell(\varpi)>0$, by definition ϖ contains a stratum $(\mathfrak{A},n,a)$ with $\ell(\varpi)=\frac{n}{e_{\mathfrak{A}}}$. By Step 1, this stratum is fundamental. Suppose that $(\mathfrak{B},m,b)$ is another stratum contained in ϖ, so it must intertwine $(\mathfrak{A},n,a)$. One then shows that $\frac{m}{e_{\mathfrak{B}}}\geq\frac{n}{e_{\mathfrak{A}}}$, with equality if and only if $(\mathfrak{B},m,b)$ is fundamental.
<u>Step 3</u>: It remains to deal with $\ell(\varpi)=0$. In this case, ϖ contains a stratum $(\mathfrak{J},1,a)$ with $a\equiv\begin{pmatrix}0&0\\a_0&0\end{pmatrix}\pmod{\mathfrak{J}}$ for some $a_0\in\mathcal{O}_F$, which is not fundamental. Conclude with Step 2. $\square$

§5 Classification of fundamental strata.

So far, we have seen that an irreducible smooth representation (ϖ, V) contains a fundamental stratum if and only if $\ell(\varpi) > 0$. We shall now classify all fundamental strata up to G-conjugacy, thereby reducing to the case of the two chain order $\mathfrak{M}$ and $\mathfrak{I}$.

Def: A "ramified simple stratum" is a fundamental stratum $(\mathfrak{A}, n, a)$ such that $e_{\mathfrak{A}} = 2$ and n is odd.

Thus, an irreducible smooth representation (ϖ, V) factors through a ramified simple stratum if and only if $\ell(\varpi) \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.

Proposition 13: Let ϖ be an irreducible smooth representation for G. If $0 < \ell(\varpi) = n \in \mathbb{Z}$, then ϖ contains a fundamental stratum of the form $(\mathfrak{M}, n, a)$

Proof: This follows from Theorem 12 together with $U_{\mathfrak{M}}^{n+1} \subseteq U_{\mathfrak{I}}^{2n+1} \subseteq U_{\mathfrak{I}}^{2n} \subseteq U_{\mathfrak{M}}^{n}$. □

Thus, every irreducible smooth representations contains a fundamental stratum of the form $(\mathfrak{A}, n, a)$ where $\mathfrak{A} = \mathfrak{M}$ or $\mathfrak{I}$ and $\gcd(n, e_{\mathfrak{A}}) = 1$. This means that we can leave out strata of the form $(\mathfrak{I}, 2n, a)$

Let us now focus on the case $e_{\mathfrak{A}} = 1$; pick a stratum $(\mathfrak{A}, n, \alpha)$ with $e_{\mathfrak{A}} = 1$. We may write $\alpha = \pi^{-n}\alpha_0$ for some $\alpha_0 \in \mathfrak{A}$. We denote by $f_\alpha(t) \in \mathcal{O}_F[t]$ the characteristic polynomial of α_0 and by $\tilde{f}_\alpha(t) \in k[t]$ its reduction modulo $\mathfrak{p}$.

It is readily checked that, up to fixing the uniformizer π of F, the polynomial $\tilde{f}_\alpha(t)$ only depends on the equivalence class of the stratum $(\mathfrak{A}, n, \alpha)$. Moreover, $\tilde{f}_\alpha(t) \neq t^2$ if and only if α is not nilpotent, and since we may replace α with any element of $\alpha + \mathfrak{P}^{1-n}$, this is equivalent to $(\mathfrak{A}, n, \alpha)$ being fundamental.

Def: Let $(\mathfrak{A}, n, \alpha)$ be a fundamental stratum with $e_{\mathfrak{A}} = 1$. $(\mathfrak{A}, n, \alpha)$ is called:

1) "unramified simple" if $\tilde{f}_\alpha(t)$ is irreducible in $k[t]$.
2) "split" if $\tilde{f}_\alpha(t)$ has distinct roots in k
3) "essentially scalar" if $\tilde{f}_\alpha(t)$ has a repeated root in $k^\times$.

Proposition 14: 1) A ramified simple stratum cannot intertwine any fundamental stratum of the form $(\mathfrak{M}, n, \alpha)$

2) If $(\mathfrak{M}, n, \alpha), (\mathfrak{M}, n, \beta)$ are fundamental and intertwine, then $\tilde{f}_\alpha(t) = \tilde{f}_\beta(t)$.

Proof: 1) follows from "Step 2" in the proof of Theorem 12.
2) Take $g \in G$ and $\beta' \in \beta + \mathfrak{P}_{\mathfrak{M}}^{1-n}$ such that $g^{-1}\beta' g \in \alpha + \mathfrak{P}_{\mathfrak{M}}^{1-n}$. $\pi^n\beta'$ and $\pi^n g^{-1}\beta' g$ have the same characteristic polynomials, so $\tilde{f}_\beta(t) = \tilde{f}_\alpha(t)$ reducing modulo $\mathfrak{p}$ (exploiting the fact that $\tilde{f}_\alpha(t) = \tilde{f}_{g^{-1}\beta' g}(t)$).

We may now translate these classification to representations:

Theorem 15: Let ϖ be an irreducible smooth representation of G with $\ell(\varpi) > 0$. Then the following are equivalent:

1) ϖ contains an essentially scalar stratum $(\mathfrak{M}, n, \alpha)$
2) There is a character χ of $F^\times$ such that $\ell(\chi\varpi) < \ell(\varpi)$.

Recall that $\chi\varpi$ is the representation given by $g \mapsto \chi(\det g)\,\varpi(g)$

Proof: Omitted.

Corollary 16: Let ϖ be an irreducible smooth representation of G such that $0 < \ell(\varpi) \leq \ell(\varpi\chi)$ for all characters χ of $F^\times$. Then one and only one of the following holds:

1) ϖ contains a split fundamental stratum;
2) ϖ contains a ramified simple stratum;
3) ϖ contains an unramified simple stratum.

It is perhaps a good moment to pause our exposition and summarize what we have achieved so far.

1) We have defined a stratum $(\mathcal{A}, n, a)$ (i.e., $\mathcal{A}$ is a chain order in A, $n \in \mathbb{Z}$ and $a \in \mathfrak{P}^{-n}$) and an equivalence relation on strata $((\mathcal{A}, n, a_1) \sim (\mathcal{A}, n, a_2) \Leftrightarrow a_1 - a_2 \in \mathfrak{P}^{1-n}$
   To each stratum $(\mathcal{A}, n, a)$ we have associated a character ψ_a of $U_{\mathcal{A}}^n$ which only depends on the equivalence class of $(\mathcal{A}, n, a)$.
2) Let ϖ be a smooth irreducible representation of G.
   $\leadsto$ ϖ contains a stratum $(\mathcal{A}, n, a)$ if it contains the character ψ_a of $U_{\mathcal{A}}^n$.
   (In particular, ψ_a contains the trivial character of $U_{\mathcal{A}}^{n+1}$)
   $\leadsto$ We have defined $\ell(\varpi) := \min\{ \frac{n}{e_{\mathcal{A}}} \mid \varpi$ contains a stratum $(\mathcal{A}, n, a)\}$.

Goal: Give a first classification of irreducible smooth representations of G according to the strata that they contain.

3) We have called a stratum $(\mathcal{A}, n, a)$ if $n \geq 1$ and $a + \mathfrak{P}^{1-n}$ contains no nilpotent elements. (Notice that $(\mathcal{A}, n, a)$ is then non-trivial, i.e. $a + \mathfrak{P}^{1-n} \neq \mathfrak{P}^{1-n}$)
Why are these strata important?
By Theorem 12: if ϖ (irreducible smooth) contains $(\mathcal{A}, n, a)$, then $(\mathcal{A}, n, a)$ is fundamental if and only if $\ell(\varpi) = \frac{n}{e_{\mathcal{A}}}$.
Thus, if $\ell(\varpi) > 0$, fundamental strata are precisely the ones whose invariants realize $\ell(\varpi)$, so $\ell(\varpi) = \frac{n}{e_{\mathcal{A}}}$ for some $(\mathcal{A}, n, e_{\mathcal{A}})$ fundamental.

4) By point (3), every irreducible smooth representation ϖ of G contains a fundamental stratum. How do we classify these?
   $\rightsquigarrow$ First, we begin by considering the cases when $\mathcal{A} = \mathfrak{M}$ or $\mathcal{A} = \mathfrak{J}$.
   $\rightsquigarrow$ By Proposition 13, we can ignore strata of the form $(\mathfrak{J}, 2n, a)$.
   We can therefore distinguish whether $\mathcal{A} = \mathfrak{M}$ or $\mathcal{A} = \mathfrak{J}$ according to whether $\ell(\varpi) \in \mathbb{Z}$ or $\ell(\varpi) \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.
   $\rightsquigarrow$ In order to classify fundamental strata of the form $(\mathfrak{M}, n, a)$, we can look at the characteristic polynomial of $\pi^n a$ modulo $\mathfrak{p}$.
   Since the latter is invariant under G-conjugation, this classification applies to any stratum $(\mathcal{A}, n, a)$ with $e_{\mathcal{A}} = 1$.
   $\rightsquigarrow$ We finally obtain Corollary 16.

5) As an extra, we have also classified non-trivial non-fundamental strata very explicitly in Corollary 11.

§6 Classification of simple strata

Def: An element $\alpha \in G \setminus Z$ is called "minimal" over F if the subalgebra $E = F[\alpha]$ of A is a field and, setting $n = -v_E(\alpha)$, one of the following holds:
1) E/F is totally ramified and n is odd;
2) E/F is unramified and, for a prime element π of F, the coset $\pi^n \alpha + \mathfrak{p}_E$ generates the field extension k_E/k

The assumption $\alpha \notin Z = Z(G)$ implies that $[E:F] = 2$. Also notice that $f(E/F)\, v_E(\alpha) = v_F(\det \alpha)$.

<u>Lemma 17</u>: Let $\alpha \in G$ be minimal over F; $E = F[\alpha]$, $n = -v_E(\alpha)$, π uniformizer of F.

Define: $\alpha_0 = \begin{cases} \pi^{\frac{n+1}{2}} \cdot \alpha & \text{if } E/F \text{ is ramified} \\ \pi^{n} \alpha & \text{if } E/F \text{ is unramified} \end{cases}$

Then $\mathcal{O}_E = \mathcal{O}_F[\alpha_0]$

<u>Proof</u>: α_0 is a uniformizer of $\mathcal{O}_E$ and $\mathcal{O}_F[\alpha_0]$ contains a k-basis of $\mathcal{O}_E/\pi\mathcal{O}_E$. $\square$

Minimal elements characterize simple strata, as the following two results show:

<u>Proposition 18</u>: Let $(\mathfrak{A}, n, \alpha)$ be a (ramified or unramified) simple stratum in $\mathfrak{A}$. Then:

1) α is minimal over F;
2) $F[\alpha]^\times \subseteq K_{\mathfrak{A}}$;
3) $e(F[\alpha]|F) = e_{\mathfrak{A}}$;
4) every $\alpha' \in \alpha + \mathfrak{P}^{1-n}$ is minimal over F.

<u>Proposition 19</u>: Let $\alpha \in G$ be minimal over F. Then there exists a unique chain order $\mathfrak{A}$ in A such that $\alpha \in K_{\mathfrak{A}}$. Moreover, $F[\alpha]^\times \subseteq K_{\mathfrak{A}}$ and $(\mathfrak{A}, n, \alpha)$ is a simple stratum, setting $n = -v_{F[\alpha]}(\alpha)$.

§7. The exhaustion theorem

We now wish to show that, given an irreducible smooth representation ϖ of G, we may distinguish between ϖ being cuspidal or not only by looking at a fundamental stratum contained in ϖ. Thus, cuspidality can be read at the level of strata. The main result is the following:

<u>Theorem 20</u>: (Exhaustion theorem)

Let ϖ be an irreducible smooth representation of G such that $\ell(\varpi) \leq \ell(\chi\varpi)$ for all characters χ of $F^\times$. The following are equivalent:

1) ϖ is cuspidal;
2) Either
   a) $\ell(\varpi) = 0$ and ϖ contains a representation of $U_{\mathfrak{M}} \cong GL_2(\mathcal{O}_F)$ inflated from an irreducible cuspidal representation of $GL_2(k)$, or
   b) $\ell(\varpi) > 0$ and ϖ contains a simple stratum.

Sketch of the proof:

First case: $\ell(\varpi)>0$.

1) $\Rightarrow$ 2) Suppose that ϖ does not contain a simple stratum. Since $\ell(\varpi)>0$, Corollary 16 ensures that ϖ contains a split fundamental stratum $(\mathfrak{M}, n, \alpha)$.

Now, the coset $\alpha+\mathfrak{P}^{1-n}$ can be replaced by a $U_{\mathfrak{M}}$-conjugate, so we may suppose that $\alpha=\pi^{-n}\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}$ where π is a uniformizer of F and $a,b\in \mathcal{O}_F\cap F^\times$, $a\not\equiv b \pmod{\mathfrak{p}}$.

This shows that $\alpha\in T$.

Set $\xi:=\psi_\alpha|_{U_{\mathfrak{M}}^n}$; we shall show that V^ξ has non-zero image in the Jacquet module V_N.

Suppose that this does not hold, so $V^\xi\subseteq V(N)$. For any $v\in V^\xi$ we may then write $v=\sum_{i=1}^{m} v_i-\varpi(n_i)v_i$ for suitable $v_i\in V$, $n_i\in N$. Since $N\cong F$ is the union of an ascending chain of compact open subgroups, we may find one of these subgroups, say $N(v)$, such that $n_i\in N(v)$ for all $i=1,\dots,n$. We infer that $\int_{N(v)}\varpi(n).v\, d\mu_N(n)=0$, because

$\int_{N(v)}\varpi(n).v\, d\mu_N(n)=\int_{N(v)}\varpi(n)\left(\sum_{i=1}^{m}v_i-\varpi(n_i)v_i\right)d\mu_N(n)=\sum_{i=1}^{m}\int_{N(v)}\varpi(n)v_i\, d\mu_N(n)-\sum_{i=1}^{m}\int_{N(v)}\varpi(nn_i).v\, d\mu_N(n)=$

$=\sum_{i=1}^{m}\int_{N(v)}\varpi(n)v_i\, d\mu_N(n)-\sum_{i=1}^{m}\int_{N(v)}\varpi(n).v\, d\mu_N(n)=0.$

Since smooth representations of G are admissible, V^ξ is finite dimensional and we may choose $N(v)$ large enough so that $\int_{N(v)}\varpi(n).v\, d\mu_N(n)=0$ holds for all $v\in V^\xi$.

Moreover, pick $j\in\mathbb{Z}$ such that $N(v)\subseteq N_j:=\begin{pmatrix} 1 & \mathfrak{p}^j \\ 0 & 1\end{pmatrix}$. We also have $\int_{N_j}\varpi(n).v\, d\mu_N(n)=0$.

Choose j maximal with this property, so there is $v_1\in V^\xi$ such that $\int_{N_{j+1}}\varpi(n).v_1\, d\mu_N(n)\neq 0$.

Now pick $t=\begin{pmatrix}\pi & 0\\ 0 & 1\end{pmatrix}$, which intertwines ξ. (This means that ξ and ξ^t agree on the subgroup $Y=U_{\mathfrak{M}}^n\cap t^{-1}U_{\mathfrak{M}}^n t=1+\begin{pmatrix}\mathfrak{p}^n & \mathfrak{p}^n\\ \mathfrak{p}^{n+1} & \mathfrak{p}^n\end{pmatrix}$)

We make the following remarks:

1) An irreducible representation (ρ, W) of $U_{\mathfrak{M}}^n$ containing $\xi|_Y$ is one-dimensional.

This follows from the fact that Y contains $U_{\mathfrak{M}}^{n+1}$

2) Let φ be a charachter of $U_{\mathfrak{M}}^n$ such that $\varphi|_Y=\xi|_Y$. Then there is $x\in N_0$ such that $\varphi^x=\xi$.

Indeed, $\varphi|_Y=\xi|_Y$ implies that $\varphi|_{U_{\mathfrak{M}}^{n+1}}$ is trivial, so φ is a character of $U_{\mathfrak{M}}^n/U_{\mathfrak{M}}^{n+1}$. Thus, $\varphi=\psi_\delta$ for some $\delta\in\mathfrak{P}^{-n}$. Imposing the condition $\psi_\delta|_Y=\psi_\alpha|_Y$ gives $\delta\equiv\pi^{-n}\begin{pmatrix} a & y\\ 0 & b\end{pmatrix} \pmod{\mathfrak{p}^{1-n}\mathfrak{M}}$ for some $y\in\mathcal{O}_F$. Since $a-b\not\equiv 0 \pmod{\mathfrak{p}}$, we have $x:=\frac{y}{a-b}$ lies in $\mathcal{O}_F$. It is then readily checked that $\delta^{\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}}\equiv\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix} \pmod{\mathfrak{p}\mathfrak{M}}$, so $\phi^x=\xi$.

Let us now consider the vector $v_2 := \varpi(t^{-1}).v_1$. By definition of j, we have

$$\int_{N_j} \varpi(u) v_2 \, d\mu_N(u) = \int_{N_j} \varpi(ut^{-1}).v_1 \, d\mu_N(u) = \varpi(t^{-1}) \int_{N_j} \varpi(tut^{-1}).v_1 \, d\mu_N(u) \underset{tN_jt^{-1} = N_{j+1}}{\underbrace{=}} c\,\varpi(t^{-1}) \int_{N_{j+1}} \varpi(u).v_1 \, d\mu_N(u) \neq 0,$$

where $c>0$ is a constant due to the change of variable $u \mapsto tut^{-1}$.

Let $\mathcal{Z}$ be the set of characters φ of $U_{\mathfrak{M}}^n$ such that $\varphi|_Y = \xi|_Y$.

Since $v_2 = \varpi(t^{-1}).v_1$ and $v_1 \in V^{\xi}$, we have $\varpi|_Y$ acts on v_2 as ξ^t, but $\xi^t|_Y = \xi|_Y$. Thus, $\varpi|_Y$ acts on v_2 as ξ. The representation $\varpi|_{U_{\mathfrak{M}}^n}$ can be decomposed into a direct sum of irreducible ones, since $U_{\mathfrak{M}}^n$ is compact; we may therefore write v_2 as a sum of vectors, one for each irreducible representation appearing in $\varpi|_{U_{\mathfrak{M}}^n}$. The only possibly non-zero such vectors will be the ones corresponding to irreducible representations of $U_{\mathfrak{M}}^n$ containing $\xi|_Y$. By Remark (1) above, these are 1-dimensional, and we may therefore write $v_2 = \sum_{\varphi \in \mathcal{Z}} v_\varphi$ for some $v_\varphi \in V^{\varphi}$.

Since $\int_{N_j} \varpi(u).v_2 \, d\mu_N(u) \neq 0$, we have $\int_{N_j} \varpi(u).v_\varphi \, d\mu_N(u) \neq 0$ for at least one $\varphi \in \mathcal{Z}$. However, by Remark (2) above $\varphi^x = \xi$, for some $x \in N_0$. It follows that the vector $v_3 = \varpi(x^{-1}).v_\varphi$ lies in V^{ξ} and satisfies $\int_{N_j} \pi(u) v_3 \, d\mu_N(u) \neq 0$, against the fact that such integral must vanish for all vectors of V^{ξ}.

2) $\Rightarrow$ 1) Conversely, suppose that ϖ is non-cuspidal. By the characterization of non-cuspidal representations, this means that ϖ is a G-subspace of $\Sigma = \operatorname{Ind}_B^G \chi$ for some character $\chi = \chi_1 \otimes \chi_2$ of T.

$\rightsquigarrow$ Suppose first that Σ is irreducible, so $\Sigma = \varpi$. If $\ell(\chi_i) = 0$ for $i = 1, 2$, then ϖ contains the trivial representation of $U_{\mathfrak{I}}^1$, whence $\ell(\varpi) = 0$, against our assumption. Thus, $\ell(\chi_i) \geq 1$ for some $i \in \{1,2\}$.

If $\chi_1\chi_2^{-1}|_{U_F^n} \neq 1$, then ϖ contains a split fundamental stratum. Indeed, choose $a_1, a_2 \in \mathfrak{p}^{-n}$ such that for $x \in \mathfrak{p}^n$ $\chi_i(1+x) = \psi(a_i x)$, where $n = \max(\ell(\chi_1), \ell(\chi_2))$. One can then check that ϖ contains the split stratum $(\mathfrak{M}, n, \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix})$, using the fact that $\varpi = \operatorname{Ind}_T^G \chi_1 \otimes \chi_2$.

If $\chi_1\chi_2^{-1}|_{U_F^n} = 1$, one can prove similarly that ϖ contains an essentially scalar stratum. The first case contradicts Corollary 16, the second Theorem 15.

⇝ Suppose that ϖ is not irreducible. By the classification of non-cuspidal representations we have $\varpi = \varphi \circ \det$ or $\varpi = \varphi \circ \mathrm{St}_G$ for some character φ of $F^\times$. Each of these cases leads to $\ell(\varpi) = \ell$, where ℓ is the level of φ. Since for every character χ of $F^\times$ we have $\ell(\varpi) \leq \ell(\chi\varpi)$, we deduce that $\ell = 0$, against $\ell(\varpi) > 0$.

Second case: $\ell(\varpi) = 0$.

By a result of the previous talk, we need only consider two cases:

⇝ ϖ contains a representation of $GL_2(\mathcal{O}_F)$ inflated from a representation of $GL_2(k)$. In this case, ϖ is cuspidal;

⇝ ϖ contains the trivial character of $U^1_{\mathcal{J}}$. Since $U_{\mathcal{J}}/U^1_{\mathcal{J}} \cong k^\times \times k^\times$, ϖ contains a character φ of I trivial on $U^1_{\mathcal{J}}$. One can then show with some work that the map $(\varpi, V) \to (\varpi_N, V_N)$ is injective on the isotropic space V^φ, so $(\varpi_N, V_N) \neq 0$ and ϖ is not cuspidal.

This concludes the proof of Theorem 20. □

§8. Cuspidal types

Our exhaustion theorem 20 ensures that cuspidality can be read at the level of fundamental strata. We now wish to refine our comprehension of simple fundamental strata in order to give a first classification of cuspidal representations.

We first need to define a special subgroup of G attached to a simple stratum. Thus, let $(\mathcal{A}, n, \alpha)$ be a simple stratum in A with $n \geq 1$; we know by Proposition 18 that α is minimal, $E := F[\alpha]$ is a field and $F[\alpha]^\times \subseteq K_{\mathcal{A}} = \{g \in G \mid g^{-1}\mathcal{A}g = \mathcal{A}\}$.

Def: We set $J_\alpha := E^\times U_{\mathcal{A}}^{\lfloor \frac{n+1}{2} \rfloor}$.

Notice that $J_\alpha \subseteq K_{\mathcal{A}}$, it is open in G, it contains and is compact modulo Z.

Our interest for the group J_α mainly comes from the following:

Proposition 21: Let Λ be an irreducible representation of J_α which contains the character ψ_α of $U_{\mathcal{A}}^{\lfloor \frac{n}{2} \rfloor + 1}$. Then:

1) The restriction of Λ to $U_{\mathcal{A}}^{\lfloor \frac{n}{2} \rfloor + 1}$ is a multiple of ψ_α;

2) The representation $\varpi_\Lambda := c\text{-}\mathrm{Ind}_{J_\alpha}^{G} \Lambda$ is irreducible and cuspidal.

This gives us a way to construct a cuspidal representation of G starting with a certain representation of J_α. Let us give a name to these representations.

<u>Def</u>: Let $(\mathfrak{A}, n, \alpha)$, $n \geq 1$, be a simple stratum. We denote by $C(\psi_\alpha, \mathfrak{A})$ the set of equivalence classes of irreducible representations Λ of the group $J_\alpha = F[\alpha]^\times U_{\mathfrak{A}}^{\lfloor \frac{n+1}{2} \rfloor}$ such that $\Lambda|_{U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor + 1}}$ is a multiple of ψ_α.

The construction of Proposition 21 has a useful uniqueness property.

<u>Proposition 22</u>: Let $(\mathfrak{A}_1, n_1, \alpha_1), (\mathfrak{A}_2, n_2, \alpha_2)$ be simple strata in A, $n_1, n_2 \geq 1$, and pick $\Lambda_1 \in C(\psi_{\alpha_1}, \mathfrak{A}_1)$, $\Lambda_2 \in C(\psi_{\alpha_2}, \mathfrak{A}_2)$. Suppose that π_{Λ_1} and π_{Λ_2} are equivalent. Then $n_1 = n_2$ and there is $g \in G$ such that $\mathfrak{A}_2 = g^{-1} \mathfrak{A}_1 g$, $J_{\alpha_1} = g^{-1} J_{\alpha_1} g$, $\Lambda_2 = \Lambda_1^g$.

We are now ready to describe the main objects that will allow us to classify cuspidal representations.

<u>Def.</u>: A "cuspidal type" in G is a triple $(\mathfrak{A}, J, \Lambda)$, where $\mathfrak{A}$ is a chain order in A, J is a subgroup of $K_{\mathfrak{A}}$ and Λ is an irreducible smooth representation of J, and these data must satisfy one of the following:

1) $\mathfrak{A} \cong \mathfrak{M}$, $J = Z U_{\mathfrak{A}}$ and $\Lambda|_{U_{\mathfrak{A}}}$ is the inflation of an irreducible cuspidal representation of the group $U_{\mathfrak{A}} / U_{\mathfrak{A}}^1 \cong GL_2(k)$.
2) There is a simple stratum $(\mathfrak{A}, n, \alpha)$, $n \geq 1$, such that $J = J_\alpha$ and $\Lambda \in C(\psi_\alpha, \mathfrak{A})$.
3) There is a triple $(\mathfrak{A}, J, \Lambda_0)$ satisfying (1) or (2), and a character χ of $F^\times$ such that $\Lambda \cong \Lambda_0 \oplus (\chi \circ \det)$.

Note that the set of cuspidal types is stable under G-conjugation. They key property is that, given a cuspidal type $(\mathfrak{A}, J, \Lambda)$, the representation $c\text{-}\mathrm{Ind}_J^G \Lambda$ is irreducible and cuspidal (either by Theorem 20 or by Proposition 21).

Here comes the big result!

<u>Theorem 23</u>: (Induction Theorem).

Let π be an irreducible cuspidal representation of G. Then there exists a cuspidal type $(\mathfrak{A}, J, \Lambda)$ in G such that $\pi \cong c\text{-}\mathrm{Ind}_J^G \Lambda$. Moreover, π determines $(\mathfrak{A}, J, \Lambda)$ uniquely up to conjugation in G.

Corollary 24: There is an isomorphism:

$$\left\{\begin{matrix}\text{Conjugacy classes of} \\ \text{cuspidal types}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Isomorphism classes of} \\ \text{irreducible cuspidal repr.}\end{matrix}\right\}$$

$$(\mathfrak{A}, J, \Lambda) \longmapsto c\text{-}\mathrm{Ind}_J^G \Lambda$$