Talk 8: Automorphic L-functions 1

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These are notes for a talk given in the PhD-seminar on *Local Langlands for* GL_2 in the summer term 2023 in Essen. The main reference for these notes is [BH06]. In this talk, we will introduce two invariants, L-functions and ε-factors, associated with representations of $\mathrm{GL}_2(F)$, and we will see some properties of them in the non-cuspidal case.

In the following let F be a valued field, let $\mathfrak{o}$ be its local ring, and let q be the number of elements of the residue field. Denote the units of $\mathfrak{o}$ by U_F. Recall that $C_c^\infty(F)$ is the $\mathbb{C}$-algebra of compactly supported, locally constant functions $F \to \mathbb{C}$. On this, we have defined Haar measures which are F-translation-invariant measures.

1 Functional Equations for $GL(1)$

This section is based on [BH06, § 23]. Before introducing L-functions and ε-factors in the two-dimensional case, we will introduce them in the one-dimensional case. Apart from the situation here being much more explicit, we will also need this case to prove facts over non-cuspidal representations in the two-dimensional case. We will also see functional equations relating different L-functions and ε-factors.

Let μ^* be a Haar measure on $F^\times$. Let χ be a character of $F^\times$, let $\Phi \in C_c^\infty(F)$, and choose a prime element ϖ of F. For $m \in \mathbb{Z}$, the set $\varpi^m U_F = \mathfrak{p}^m \setminus \mathfrak{p}^{m+1}$ is compact and open. Therefore for any $\Phi \in C_c^\infty(F)$, the function $\Phi \cdot \mathbb{I}_{\varpi^m U_F}$ is again locally constant and compactly supported on $F^\times$ because it is supported on $\varpi^m U_F$, which is compact and open in $F^\times$. Thus $\Phi \cdot \mathbb{I}_{\varphi^m U_F} \in C_c^\infty(F^\times)$ and we can integrate

$$z_m := z_m(\Phi, \chi) := \int_{\varpi^m U_F} \Phi(x)\chi(x)d\mu^*(x)$$

for $m \in \mathbb{Z}$. (The function $\Phi\chi$ is again locally constant because we have seen in Talk 2 [BH06, § 1.6, Prop.] that every character of F is locally constant.) Because $\Phi \cdot \mathbb{I}_{\varpi^m U_F}$ is identically zero for $m \ll 0$, one can make the following definition:

Definition 1.1. We define the formal Laurent series

$$Z(\Phi, \chi, X) := \sum_{m\in\mathbb{Z}} z_m X^m \in \mathbb{C}((X)).$$

This assembles into a linear map $C_c^\infty(F) \to \mathbb{C}((X)); \Phi \mapsto Z(\Phi, \chi, X)$. We denote its image by

$$\mathcal{Z}(\chi) := \mathcal{Z}(\chi, X) := \{Z(\Phi, \chi, X) \mid \Phi \in C_c^\infty(F)\}.$$ ⌟

For $a \in F^\times$, we denote $x \mapsto \Phi(a^{-1}x)$ by $a\Phi$. Then we have (using that we integrate over a Haar measure)

$$Z(a\Phi, \chi, X) = \chi(a)X^{v_F(a)} Z(\Phi, \chi, X),$$

where the $X^{v_F(a)}$ comes from the shift induced by multiplication with a^{-1} on the $\varpi^m U_F$. This shows that $\mathcal{Z}(\chi)$ is closed under multiplication by X, X^{-1}, and, therefore, $\mathcal{Z}(\chi)$ is a module over the ring $\mathbb{C}[X, X^{-1}]$ of Laurent polynomials.

This will be useful later because $\mathbb{C}[X, X^{-1}]$ is a principal ideal domain whose unit group consists of the monomials aX^b with $a \in \mathbb{C}^\times$ and $b \in \mathbb{Z}$.

We can describe $\mathcal{Z}(\chi, X)$ explicitly:

Proposition 1.2 ([BH06, § 23.2])**.** *Let χ be a character of $F^\times$; then*

$$\mathcal{Z}(\chi, X) = P_\chi(X)^{-1}\mathbb{C}[X, X^{-1}] \subset \mathbb{C}((X)),$$

where

$$P_\chi(X) = \begin{cases} 1 - \chi(\varpi)X & \text{if } \chi \text{ is unramified,} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. We first suppose that $\Phi(0) = 0$. Thus $\Phi|_{F^\times}$ lies in $C_c^\infty(F^\times)$ and $Z(\Phi, \chi, X)$ has only finitely many non-zero coefficients (at some point you're integrating over an area where Φ is constant). That is $\mathbb{Z}(\Phi, \chi, X) \in \mathbb{C}[X, X^{-1}]$. Furthermore, if Φ is the characteristic function of a sufficiently small neighbourhood of 1 (namely it needs to be small enough for χ to be a constant on it and the intersection with $\varpi^m U_F$ should be 0 for $m \neq 0$), then $Z(\Phi, \chi, X)$ is a positive constant. Thus $1 \in \mathcal{Z}(\chi)$ and

$$\{Z(\Phi, \chi, X) \mid \Phi \in C_c^\infty(F^\times)\} = \mathbb{C}[X, X^{-1}].$$

We can write every element of $C_c^\infty(F)$ as the sum of a $\mathbb{C}$-multiple of Φ_0 and an element of $C_c^\infty(F^\times)$, where Φ_0 is the characteristic function of $\mathfrak{o}$. Then, we have by definition and the substitution rule that

$$Z(\Phi_0, \chi, X) = \sum_{m \geq 0} \chi(\varpi^m) X^m \int_{U_F} \chi(x) d\mu^*(x). \tag{$*$}$$

The inner integral evaluates to $\mu^*(U_F)$ if χ is unramified, since then $\chi(x) = 1$, and to zero otherwise, since an integral over a non-trivial character is zero. (If $\chi(h) \neq 1$, then $\int_{U_F} \chi(g) d\mu^*(g) = \int_{U_F} \chi(hg) d\mu^*(g) = \chi(h) \cdot \int_{U_F} \chi(hg) d\mu^*(g)$, which can only occur if the integral is zero.) Thus

$$\mu^*(U_F)^{-1} Z(\Phi_0, \chi, X) = \begin{cases} (1 - \chi(\varpi)X)^{-1} & \text{if } \chi \text{ is unramified,} \\ 0 & \text{otherwise.} \end{cases}$$

Because Φ_0 and $C_c^\infty(F^\times)$ span $C_c^\infty(F)$, we get the desired result. ∎

We now want to see that a version of $Z(\Phi, \chi, X)$ converges in $\mathbb{C}$ such that we get a map to $\mathbb{C}$. For this we will use the above proposition. Furthermore, we will prove a transformation formula.

In order to express this formula, we need to introduce the Fourier transform.

Definition 1.3. Fix $\psi \in \hat{F}$, $\psi \neq 1$, and a Haar measure μ on F. For $\Phi \in C_c^\infty(F)$, we define the *Fourier transform* $\hat{\Phi}$ of Φ (relative to μ and ψ) by

$$\hat{\Phi}(x) = \int_F \Phi(y)\psi(xy) d\mu(y)$$

for $x \in F$. ⌟

Note that the integrand in the above definition is locally constant since Φ and all characters are so by Talk 1, and that it is compactly supported because Φ is. The Fourier transform has the following properties:

Proposition 1.4 ([BH06, § 23.1])**.** *(a) For $\Phi \in \mathbb{C}_c^\infty(F)$, the function $\hat{\Phi}$ lies in $C_c^\infty(F)$.*

(b) There is a positive real number $c = c(\psi, \mu)$ such that

$$\hat{\hat{\Phi}}(x) = c\Phi(-x)$$

for all $\Phi \in C_c^\infty(F)$ and all $x \in F$.

(c) For a given ψ, there is a unique Haar measure μ_ψ for which $c(\psi, \mu_\psi) = 1$. This measure satisfies $\mu_\psi(\mathfrak{o}) = q^{l/2}$ where l is the level of ψ.

(d) For $a \in F^\times$, we have $\mu_{a\psi} = ||a||^{\frac{1}{2}}\mu_\psi$.

Proof. We proof this result in the same fashion as many other measure theory results are proven: We consider generators of $C_c^\infty(F)$, for which we can check the statements by hand and then use the fact that they generate $C_c^\infty(F)$.

Let l be the level of ψ (i.e. the smallest l such that $\mathfrak{p}^l \subset \ker\psi$). We now consider $\Phi_j = \mathbb{I}_{\mathfrak{p}^j}$, the characteristic function of $\mathfrak{p}^j$. Now for $a \in F$, the character $a\psi|_{\mathfrak{p}^j}$ is trivial if and only if $a \in \mathfrak{p}^{l-j}$ (since $a\psi(x) = \psi(ax)$). The support of $\hat{\Phi}_j$ is therefore $\mathfrak{p}^{l-j}$. Indeed, we have that ψ only assumes finitely many values on $\mathfrak{p}^j$. Let $a \notin \mathfrak{p}^{l-j}$. Then $\hat{\Phi}(a) = \int_{\mathfrak{p}^j} \psi(ax)d\mu(x) = 0$, as an integral over a non-trivial character is zero. For $x \in \mathfrak{p}^{l-j}$, we have

$$\hat{\Phi}_j(x) = \int_{\mathfrak{p}^j} 1 d\mu(y) = \mu(\mathfrak{p}^j) = \mu(\mathfrak{o}) \cdot q^{-j}.$$

Since $l - (l - j) = j$, this shows that the assertions (a) and (b) hold for the Φ_j, where the constant is $\mu(\mathfrak{o})^2 q^{-l}$.

In order to get a generating set from these, we need to also consider shifts of the Φ_j. Therefore let $a \in F$ and let Ψ denote the function $x \mapsto \Phi(x - a)$, where $\Phi \in C_c^\infty(F)$. We then have

$$\hat{\Psi}(x) = \int_F \Phi(y-a)\psi(xy)d\mu(y) = \psi(xa)\int_F \Phi(y-a)\psi(x(y-a))d\mu(y) = \psi(xa)\hat{\Phi}(x) = a\psi(x)\hat{\Phi}(x).$$

The function $a\psi$ is locally constant so $\hat{\Psi} \in C_c^\infty(F)$ since Φ lies there. Calculating the Fourier transform again (which is essentially the same calculation again), we get

$$\hat{\hat{\Psi}}(x) = \hat{\hat{\Phi}}(a + x).$$

Therefore the assertions (a) and (b) hold for the shifts of the Φ_j, with the constant $\mu(\mathfrak{o})^2 q^{-l}$. As these generate $C_c^\infty(F)$, we get both assertions in general.

For $b > 0$ note that we have $c(\psi, b\mu) = b^2 c(\psi, \mu)$. To achieve $c(\psi, \mu) = 1$, we must have $\mu(\mathfrak{o})^2 q^{-l} = 1$, which is achievable by scaling. Therefore we get (c). Part (d) follows directly from solving the above equation. ■

The measure μ_ψ is called *the self-dual Haar measure on F, relative to ψ.* Using μ_ψ to compute the Fourier transform

$$\hat{\Phi}(x) = \int_F \Phi(y)\psi(xy)d\mu_\psi(y)$$

gives the *Fourier inversion formula*

$$\hat{\hat{\Phi}}(x) = \Phi(-x)$$

for $\Phi \in C_c^\infty(F)$ and $x \in F$.

Theorem 1.5 ([BH06, § 23.3])**.** *Let χ be a character of $F^\times$. There is a unique rational function $c(\chi,\psi,X) \in \mathbb{C}(X)$ such that*

$$Z(\hat{\Phi},\check{\chi},\frac{1}{qX}) = c(\chi,\psi,X)Z(\Phi,\chi,X)$$

for all $\Phi \in C_c^\infty(F)$.

Proof. Consider the space Λ of linear maps $\lambda\colon C_c^\infty(F) \to \mathbb{C}(X)$ satisfying the scaling property of the $Z(\Phi,\chi,X)$, i.e.

$$\lambda(a\Phi) = \chi(a)X^{v_F(a)}\lambda(\Phi)$$

for $\Phi \in C_c^\infty(F)$ and $a \in F^\times$. Surely Λ is a $\mathbb{C}(X)$-vector space and it contains the map $\lambda_0\colon \Phi \mapsto Z(\Phi,\chi,X)$, which is non-zero by Proposition 1.2. For $\Phi \in C_c^\infty(F)$ and $a \in F^\times$, we get using a substitution rule for the integral

$$\widehat{a\Phi} = ||a|| \cdot a^{-1}\hat{\Phi}.$$

Therefore, the map

$$\lambda_1\colon \Phi \mapsto Z(\hat{\Phi},\check{\chi},1/qX)$$

is also in Λ, when one uses that $\check{\chi}(a) = \chi(a^{-1})$. Therefore this theorem is a direct consequence of the following Lemma. ■

Lemma 1.6 ([BH06, § 23.3])**.** *The space Λ has dimension one over $\mathbb{C}(X)$.*

Proof. Choose $n \geq 1$ such that $U_F^n \subset \ker\chi$. For $k \geq 1$, let $\Phi_k = \mathbb{I}_{U_F^k}$ be the characteristic function of U_F^k. We consider the map

$$\Lambda \to \mathbb{C}(X); \lambda \mapsto \lambda(\Phi_n).$$

We will show that this map is injective, which then proves the lemma because $\mathbb{C}(X)$ is a principal ideal domain.

Suppose that $\lambda(\Phi_n) = 0$. The defining condition on λ yields for all $k \geq n$ and all $a \in U_F^n$ that $\lambda(a\Phi_k) = \chi(a)X^{v_F(a)}\lambda(\Phi_k) = \lambda(\Phi_k)$ (because a is a unit and thus $v_F(a) = 0$). Therefore we get

$$\lambda(\Phi_k) = q^{n-k}\lambda(\Phi_n) = 0$$

for $k \geq n$ using that we can cover U_F^n disjointly by U_F^k for $k \geq n$. Any $\Phi \in C_c^\infty(F^\times)$ is a finite linear combination of $F^\times$-translates of functions Φ_k. Thus $\lambda(\Phi_K) = 0$ implies $\lambda(\Phi) = 0$ for all $\Phi \in C_c^\infty(F^\times)$. Therefore the value of $\lambda(\Phi)$ for $\Phi \in C_c^\infty(F)$ only depends on $\Phi(0)$. Therefore $\lambda(a\Phi) = \lambda(\Phi)$ for all $a \in F^\times$ and the transformation formula that λ satisfies now implies $\lambda(\Phi) = 0$, because otherwise $\lambda(\Phi)$ and $\lambda(a\Phi)$ cannot be the same whenever $a \notin \ker\chi$. ■

We shall now introduce a more traditional notation for the constructions that we've seen above. We set

$$\begin{aligned}\zeta(\Phi,\chi,s) &= Z(\Phi,\chi,q^{-s}),\\ L(\chi,s) &= P_\chi(q^{-s})^{-1},\\ \gamma(\chi,s,\Psi) &= c(\chi,\Psi,q^{-s}).\end{aligned}$$

In particular, we have

$$\zeta(\Phi,\chi,s) = \int_{F^\times}\Phi(x)\chi(x)||x||^s d\mu^*(x)$$

in the following sense: We cannot necessarily directly integrate $\Phi(x)\chi(x)||x||^s$ on $F^\times$ in the language of Talk 3 because it might not be compactly supported. However $(*)$ shows that this integral can be written as a limit of something that converges, absolutely and uniformly in vertical strips, in some half-plane $\operatorname{Re} s > s_0$. It there then represents a rational function in q^{-s} by Proposition 1.2 and therefore possesses an analytic continuation to a meromorphic function on the whole s-plane, i.e. $\mathbb{C}$ with the parameter being s.

The two languages (of Z, P, and c; and of ζ, L, and γ) are equivalent, and the relation between them is transparent. Therefore we will use the two languages interchangeably to facilitate the more useful one at any time.

In the more classical language, we can write

$$L(\chi, s) = \begin{cases} (1 - \chi(\varpi)q^{-s})^{-1} & \text{if } \chi \text{ is unramified,} \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, the L-function $L(\chi, s)$ carries no information about χ if χ is ramified. If χ is unramified, it is completely determined by its L-function.

Corollary 1.7 ([BH06, § 23.4, Cor. 1])**.** *Let χ_1 and χ_2 be unramified characters of $F^\times$. The following are equivalent*

(a) the meromorphic functions $L(\chi_1, s)$ and $L(\chi_2, s)$ have a pole in common;

(b) the meromorphic functions $L(\chi_1, s)$ and $L(\chi_2, s)$ have the same set of poles;

(c) $\chi_1 = \chi_2$.

Proof. We know the structure of $L(\chi, s)$ as the inverse of $1 - \chi(\varpi)q^{-s}$ which in any case determines $\chi(\varpi)$. Since an unramified character is determined by the image of ϖ, we get the equivalence. ■

We are now going to examine the structure of $\gamma(\chi, s, \psi)$ further. This will be helpful (and necessary) when studying cuspidal representations in the next talk. We define a rational function $\varepsilon(\chi, s, \psi) \in \mathbb{C}(q^{-s})$ by

$$\varepsilon(\chi, s, \psi) = \gamma(\chi, s, \psi)\frac{L(\chi, s)}{L(\check{\chi}, 1-s)}.$$

$\varepsilon(\chi, s, \psi)$ is called an *ε-factor*. These factors satisfy the following functional equation.

Corollary 1.8 ([BH06, § 23.4, Cor. 2])**.** *The function $\varepsilon(\chi, s, \psi)$ satisfies the functional equation*

$$\varepsilon(\chi, s, \psi)\varepsilon(\check{\chi}, 1-s, \psi) = \chi(-1).$$

It is of the form

$$\varepsilon(\chi, s\psi) = q^{(\frac{1}{2}-s)n(\chi,\psi)}\varepsilon(\chi, \frac{1}{2}, \psi),$$

for some $n(\chi, \psi) \in \mathbb{Z}$.

Proof. In the classical language, Theorem 1.5 reads as

$$\zeta(\hat{\Phi}, \check{\chi}, 1-s) = \gamma(\chi, s, \psi)\zeta(\Phi, \chi, s). \tag{†}$$

If we apply this twice, we get

$$\zeta(\hat{\hat{\Phi}}, \chi, s) = \gamma(\check{\chi}, 1 - s\psi)\gamma(\chi, s, \psi)\zeta(\Phi, \chi, s).$$

Fourier inversion gives $\hat{\hat{\Phi}} = (-1)\Phi$ and therefore the transformation formula yields $\zeta(\hat{\hat{\Phi}}, \chi, s) = \chi(-1)\zeta(\Phi, \chi, s)$, and thus

$$\gamma(\check{\chi}, 1-s, \psi)\gamma(\chi, s, \psi) = \varepsilon(\check{1} - s, \psi)\varepsilon(\chi, s, \psi) = \chi(-1),$$

which is the first formula of this corollary.

To get the second formula, we rewrite (†) in the form

$$\frac{\zeta(\hat{\check{\Phi}}, \check{\chi}, 1-s)}{L(\check{\chi}, 1-s)} = \varepsilon(\chi, s, \psi)\frac{\zeta(\Phi, \chi, s)}{L(\chi, s)}$$

where the fraction on either side is lying in $\mathbb{C}[q^s, q^{-s}]$. As in the proof of Proposition 1.2, we can choose Φ with $\zeta(\Phi, \chi, s) = L(\chi, s)$. Therefore $\varepsilon(\chi, s, \psi) \in \mathbb{C}[q^s, q^{-s}]$. The functional equation from the first part of this corollary implies that $\varepsilon(\chi, s, \psi)$ is a unit of $\mathbb{C}[q^s, q^{-s}]$, and hence equal to aq^{ms} for some $a \in C^\times$ and $m \in \mathbb{Z}$. This can be rewritten in the way stated in the corollary. ■

Remark. The relation in the previous corollary is generally referred to as *Tate's (local) functional equation.* The function $\varepsilon(\chi, s, \psi)$ is the *Tate local constant* of χ (relative to ψ). ⌟

2 Functional Equation for $\mathrm{GL}(2)$

This section is based on [BH06, § 24]. We now want to generalise the constructions from the GL(1)-case to GL(2). This leads us to a functional equation and local constant by Godement-Jacquet. For this, we will first do the constructions in general and state their relations similar to the relations in the one-dimensional case. Then we will see the proof of these in the case of non-cuspidal representations. The cuspidal case will be deferred to the next talk.

Let $\mathfrak{M} := M_2(\mathfrak{o}) \subset M_2(F) =: A$, and let $G := \mathrm{GL}_2(F)$. Then the space $C_c^\infty(A)$ is spanned by characteristic functions of the sets $a + \mathfrak{p}^j\mathfrak{M}$ for $a \in A, j \in \mathbb{Z}$.

Let (π, V) be an irreducible smooth representation of G. The one-dimensional ζ-function has three inputs: An element $\Phi \in C_c^\infty(F)$, a character of $F^\times$, i.e. an irreducible (one-dim) representation of $F^\times$, and a complex number $s \in \mathbb{C}$. To get to the two-dimensional one, we can replace Φ by its two-dimensional analogue $\Phi \in C_c^\infty(A)$, and we can keep s. For the irreducible representation (π, V), we want to pick something which carries more information than the character of π: We pick the coefficients of π, $C(\pi)$, which is the group (see [BH06, § 10.1] for details)

$$C(\pi) = \left\{ f\colon G \to \mathbb{C} \mid \exists(\check{v}_1 \otimes v_1, \ldots, \check{v}_n \otimes v_n) \in \check{V} \otimes V, \lambda_1, \ldots, \lambda_n \in K\colon f(g) = \sum_{i=1}^n \lambda_i \check{v}_i(\pi(g)v) \forall g \in G \right\}.$$

The elements $f \in C(\pi)$ are locally constant in g because π is smooth. The z_m in the previous definition were precisely the integrals over the norm $(1/q)^m$-part of the units of F and therefore, we can now analogously define

$$\zeta(\Phi, f, s) := \int_G \Phi(x) f(x) ||\det x||^s d\mu^*(x) \tag{‡}$$

where μ^* is a Haar measure on G. In the following, we will often abbreviate $d\mu^*(x)$ with d^*x. The integrand is locally constant and compactly supported on A. We do, however, not know whether it is also compactly supported on G. (This is probably not the case.) But we can, as in analysis, approximate the integral by truncations to compact sets and see whether this converges.

Theorem 2.1 ([BH06, Thm. 1, § 24.2])**.** *Let (π, V) be an irreducible smooth representation of G.*

(a) There exists $s_0 \in \mathbb{R}$ such that the integral (‡) converges absolutely and uniformly in vertical strips in the region $\operatorname{Re} s > s_0$, for all Φ and f. The integral represents a rational function in q^{-s}.

(b) Define

$$\mathcal{Z}(\pi) = \{\zeta(\Phi, f, s + \frac{1}{2}) \mid \Phi \in C_c^\infty(A), f \in C(\pi)\}.$$

Then there is a unique polynomial $P_\pi(X) \in \mathbb{C}[X]$, satisfying $P_\pi(0) = 1$ and

$$\mathcal{Z}(\pi) = P_\pi(q^{-s})^{-1}\mathbb{C}[q^s, q^{-s}].$$

Again, we set

$$L(\pi, s) = P_\pi(q^{-s})^{-1}.$$

One can show that this definition is independent of the choice of Haar measure μ^*.

In order to state the functional equation, we again need a Fourier transform. As in the one-dimensional case, fix a character $\psi \in \hat{F}$, $\psi \neq 1$, and set $\psi_A = \psi \circ \mathrm{trace}_A$. We now define the *Fourier transform* $\hat{\Phi}$ of $\Phi \in C_c^\infty(A)$ analogously to the one-dimensional case via

$$\hat{\Phi}(x) = \int_A \Phi(y)\psi_A(xy)d\mu(y)$$

relative to a Haar measure μ on A. Exactly as in Proposition 1.4, one can show that $\hat{\Phi} \in C_c^\infty(A)$ and that there is a unique Haar measure μ_ψ^A on A for which the Fourier inversion formula

$$\hat{\hat{\Phi}}(x) = \Phi(-x)$$

holds. This is the *self-dual Haar measure on A, relative to ψ*. We again get the relations

$$\mu_\psi^A(\mathfrak{M}) = q^{2l} \quad \text{and} \quad \mu_{a\psi}^A = ||a||^2\mu_\psi^A$$

where $a \in F^\times$ and l is the level of ψ.

In the following we will fix a non-trivial character $\psi \in \hat{F}$ and denote by $\hat{\Phi}$ the Fourier transform of $\Phi \in C_c^\infty(A)$ with respect to ψ and the self-dual Haar measure on A relative to ψ. Furthermore, as in the one-dimensional case, we need a dual notion of $f \in C(\pi)$ for a representation (π, V). For this, we note that $g \mapsto f(g^{-1})$ is in the coefficients for the dual representation $C(\check{\pi})$. The map $f \mapsto \check{f}$ yields a linear isomorphism $C(\pi) \cong C(\check{\pi})$. With this, we can state the functional equation.

Theorem 2.2 (Godement-Jacquet functional equation, [BH06, Thm. 2, § 24.2])**.** *Let (π, V) be an irreducible smooth representation of G. There is a unique rational funciton $\gamma(\pi, s, \psi) \in \mathbb{C}(q^{-s})$ such that*

$$\zeta(\hat{\Phi}, \check{f}, \frac{3}{2} - s) = \gamma(\pi, s, \psi)\zeta(\Phi, f, \frac{1}{2} + s)$$

for all $\Phi \in C_c^\infty(A), f \in C(\pi)$.

Corollary 2.3 (Godement-Jacquet local constant, [BH06, § 24.2])**.** *Define*

$$\varepsilon(\pi, s, \psi) = \gamma(\pi, s, \psi)\frac{L(\pi, s)}{L(\check{\pi}, 1 - s)}.$$

The function $\varepsilon(\pi, s, \psi)$ satisfies the functional equation

$$\varepsilon(\pi, s, \psi)\varepsilon(\check{\pi}, 1 - s, \psi) = \omega_\pi(-1).$$

Moreover, there exists $a \in \mathbb{C}^\times$ and $b \in \mathbb{Z}$ such that $\varepsilon(\pi, s, \psi) = aq^{bs}$.

Proof. This is again a rather straightforward computation. The difference to Corollary 1.8 is that we need to work a bit harder to insert the L-functions into the picture. If we apply Theorem 2.2 twice, we get

$$\zeta(\hat{\hat{\Phi}}, f, \frac{1}{2}+s) = \gamma(\check{\pi}, 1-s, \psi)\gamma(\pi, s, \psi)\zeta(\Phi, f, \frac{1}{2}+s).$$

From the Fourier transform formula, we get $\zeta(\hat{\hat{\Phi}}, f, s) = \omega_\pi(-1)\zeta(\Phi, f, s)$, which yields the functional equation for ε.

For the second part, note that, by definition, we can find $\Phi_i \in C_c^\infty(A)$ and $f_i \in C(\pi)$, $i = 1, \dots, r$ with

$$\sum_{i=1}^{r} \zeta(\Phi_i, f_i, s+\frac{1}{2}) = L(\pi, s).$$

Now Theorem 2.2 yields

$$L(\check{\pi}, 1-s)^{-1} \sum_{i=1}^{r} \zeta(\hat{\Phi}_i, \check{f}_i, \frac{3}{2}-s) = \varepsilon(\pi, s, \psi).$$

By definition, the left hand side is in $\mathbb{C}[q^s, q^{-s}]$, so $\varepsilon(\pi, s, \psi) \in \mathbb{C}[q^s, q^{-s}]$. Similarly, $\varepsilon(\check{\pi}, 1-s, \psi)$ is in $\mathbb{C}[q^s, q^{-s}]$. Now the functional equation of ε tells us that $\varepsilon(\pi, s, \psi)$ is a unit in $\mathbb{C}[q^s, q^{-s}]$ and therefore of the desired form. ■

3 L-functions and ε-factors for non-cuspidal irreducible representations of $\mathrm{GL}_2(F)$

This section is based on [BH06, § 26]. We are now going to prove the two theorems from the last section in the case where our representation is non-cuspidal. We will see a sketch of the proof in the cuspidal case in the next Talk. In order to do this, we again need some subgroups of $G = \mathrm{GL}_2(F)$:

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in G \right\}, \quad N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in G \right\}, \quad T = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G \right\}, \quad Z = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in G \right\}.$$

We encountered these groups already in Talk 3. B is the standard Borel subgroup of G, N the unipotend radical of B, T the standard split maximal torus in G, and Z the centre of G. Recall that a representation is non-cuspidal if and only if it is a G-composition factor of $\iota_B^G \chi$ for $\chi = \chi_1 \otimes \chi_2$ a character of T, which is a quotient of B. Furthermore recall that $\iota_B^G \chi = \mathrm{Ind}_B^G \chi \otimes \delta^{-1/2}$ is the normalized smooth induction where $\delta(x) = ||x||$. The proof of the above theorems is closely related to the proof of the following theorem, which is interesting in its own right and which we will use later to prove the converse theorem. We will therefore see some parts of its proof as well.

Theorem 3.1 ([BH06, § 26.1])**.** *Let $\chi = \chi_1 \otimes \chi_2$ be a character of the group T, and let π be a G-composition factor of $\iota_B^G \chi$. For any $\psi \in \hat{F}, \psi \neq 1$, we have*

$$L(\pi, s) = L(\chi_1, s)L(\chi_2, s),$$
$$\varepsilon(\chi, s, \psi) = \varepsilon(\chi_1, s, \psi)\varepsilon(\chi_2, s, \psi),$$

except when $\pi \cong \phi \cdot St_G$, for an unramified character ϕ of $F^\times$. In this exceptional case, we have

$$L(\pi, s) = L(\phi, s+\frac{1}{2}), \quad \varepsilon(\pi, s, \psi) = -\varepsilon(\phi, s, \psi).$$

The proof of Theorem 2.1 and Theorem 2.2 in the non-cuspidal case works, essentially, by first proving it in the case where $\pi = \iota_B^G \chi$. This representation need not be irreducible, so we need to adjust some statements such that they also work when this π is reducible. We start with Theorem 2.1 in this case.

Proposition 3.2 ([BH06, § 26.2])**.** *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T, and put $(\pi, V) = \iota_B^G \chi$.*

(a) There exists $s_0 \in \mathbb{R}$ depending only on χ, such that $\zeta(\Phi, f, s)$ converges absolutely and uniformly in vertical strips, in the region $\operatorname{Re} s > s_0$.

(b) The integral $\zeta(\Phi, f, s)$ represents a rational function in q^{-s} and

$$\mathscr{Z}(\pi, q^{-s}) = \mathscr{Z}(\chi_1, q^{-s})\mathscr{Z}(\chi_2, q^{-s}).$$

Sketch of proof. The idea of this proof is to transfer everything back to $T = F^\times \times F^\times$. For this let D be the algebra of diagonal matrices in A, i.e. $T = D^\times$ and the space $C_c^\infty(D)$ is canonically isomorphic to $C_c^\infty(F) \otimes C_c^\infty(F)$. Let $\theta \in V, \tau \in \check{V}$, i.e. by definition θ and τ are certain functions $G \to \mathbb{C}$ with an induction condition. With this, we can define a coefficient

$$f(g) = \langle \tau, \pi(g)\theta \rangle = \int_{B \backslash G} \tau(x)\theta(xg) d\dot{x} = \int_K \tau(k)\theta(kg) dk$$

where $d\dot{x}$ is a positive semi-invariant measure on $B \setminus G$. We can rewrite this as above with $K = \mathrm{GL}_2(\mathfrak{o})$ and dk a good Haar measure on K. With this, we can write

$$\zeta(\Phi, f, s) = \int_K \int_K \int_B \Phi(k^{-1}bk')\tau(k)\theta(bk') db dk' dk$$

for a Haar measure db on B. Because the integrad is locally constant, we find a $K_1 \subset K$ open such that we can rewrite

$$\zeta(\Phi, f, s) = \mu(K_1)^2 \sum_{i,j \in K/K_1} \int_B \Phi^{ij}(b)\theta(bk_i)\tau(k_j) ||\det b||^s db$$

where k_i and k_j range independently over K/K_1 and $\Phi^{ij}(x) = \Phi(k_i^{-1}xk_j)$. We can now split up $db = dtdn$ for Haar measures dt on T and dn on N, and rewrite a typical summand of the above sum as

$$\theta(k_i)\tau(k_j) \int_T \Phi_T^{ij}(t)\chi(t) ||\det t||^{s-\frac{1}{2}} dt.$$

The Φ_T^{ij} is a construction from the next lemma, where we reduce functions to D. We get the convergence of the right hand side from the one-dimensional case. With this, we also get that $\zeta(\Phi, f, s + \frac{1}{2}) \in \mathbb{Z}(\chi_1)\mathbb{Z}(\chi_2)$, which also proves $\mathbb{Z}(\pi) \subset \mathbb{Z}(\chi_1)\mathbb{Z}(\chi_2)$.

In order to complete the proof, one still needs to show $\mathbb{Z}(\chi_1)\mathbb{Z}(\chi_2) \subset \mathbb{Z}(\pi)$. This can be done via a lengthy computation, in which you have to compute Φ. ■

Lemma 3.3 ([BH06, § 26.2])**.** *Let $\Phi \in C_c^\infty(A)$. There exists a unique function $\Phi_T \in C_c^\infty(D)$ such that*

$$\Phi_T(t) = ||t_1|| \int_N \Phi(tn) dn, \quad \text{where} \quad t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in T.$$

The map $\Phi \mapsto \Phi_T$ is a linear surjection $C_c^\infty(A) \to C_c^\infty(D)$.

Proof. The well-definedness is clear, i.e. the integral converges and the map is linear. The surjectivity can be shown on characteristic functions. ■

After proving an analogue of Theorem 2.1 in the case $\pi = \iota_B^G\chi$, we will now do the same for Theorem 2.2.

Proposition 3.4 ([BH06, § 26.3]). *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T and set $(\pi, V) = \iota_B^G\chi$. Let $\psi \in \hat{F}$, $\psi \neq 1$. There is a unique rational function $\gamma(\pi, s, \psi) \in \mathbb{C}(q^{-s})$ such that*

$$\zeta(\hat{\Phi}, \check{f}, \frac{3}{2} - s) = \gamma(\pi, s, \psi)\zeta(\Phi, f, s + \frac{1}{2}),$$

for all $\Phi \in C_c^\infty(A)$ and all $f \in C(\pi)$. Moreover,

$$\gamma(\pi, s, \psi) = \gamma(\chi_1, s, \psi)\gamma(\chi_2, s, \psi).$$

Sketch of proof. As in the previous proposition, we write

$$\zeta(\hat{\Phi}, \check{f}, s) = \mu(K_1)\sum_{i,j}\theta(k_i)\tau(k - J)\int_T \hat{\Phi}_T^{ji}(t)\chi(t)^{-1}||\det t||^{s-\frac{1}{2}}dt$$

where $\hat{\Phi}^{ji}(x)0\hat{\Phi}(k_j^{-1}xk_i)$. From the following Lemma we get $\hat{\Phi}ji_T = \widehat{\Phi_T^{ji}}$. With this, we get the desired equation. ■

Lemma 3.5 ([BH06, § 26.3]). *For $\Phi \in C_c^\infty(A)$, we have $(\hat{\Phi})_T = \widehat{\Phi_T}$.*

Proof. Straightforward computation after writing Φ as a matrix. ■

With this, we are done with the case of the theorems where the involved ι_B^G is already irreducible. In this case, we can also write down the involved L-functions and local constants, giving a first part of a proof of Theorem 3.1.

Proposition 3.6 ([BH06, § 26.4]). *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T such that $\pi = \iota_B^G\chi$ is irreducible. Then*

$$L(\pi, s) = L(\chi_1, s)L(\chi_2, s),$$
$$\varepsilon(\pi, s, \psi) = \varepsilon(\chi_1, s, \psi)\varepsilon(\chi_2, s, \psi)$$

for any $\psi \in \hat{F}, \psi \neq 1$.

Proof. The L-function relation reflects the equality $\mathbb{Z}(\pi) = \mathbb{Z}(\chi_1)\mathbb{Z}(\chi_2)$ from Proposition 3.2 and the ε-relation comes from the corresponding relation between the γ's in Proposition 3.4. ■

What remains in the proof of Theorem 2.1 and Theorem 2.2 in the non-cuspidal case is to deduce the case where π is a G-composition factor of $\iota_B^G\chi$ from the statements that we already know when $\iota_B^G\chi$ is reducible.

Corollary 3.7 ([BH06, § 26.5]). *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T and let π be a G-composition factor of $\sigma = \iota_B^G\chi$. We have*

$$\gamma(\pi, s, \psi) = \gamma(\iota_B^G\chi, s, \psi) = \gamma(\chi_1, s, \psi)\gamma(\chi_2, s, \psi)$$

and $P_\pi(t)$ divides $P_\Sigma(t) = P_{\chi_1}(t)P_{\chi_2}(t)$.

Proposition 3.8 ([BH06, § 26.6]). *Let π be a composition factor of a representation $\iota_B^G\phi\delta_B^{\pm 1/2}$ and suppose that the character ϕ of $F^\times$ is not unramified. Then*

$$L(\pi, s) = 1, \quad \varepsilon(\pi, s, \psi) = \varepsilon(\phi, s - \frac{1}{2}, \psi)\varepsilon(\phi, s + \frac{1}{2}, \psi).$$

Proposition 3.9 ([BH06, § 26.7]). *Let ϕ be an unramified character of $F^\times$ and put $\pi = \phi \circ \det$. Then*

$$L(\pi, s) = L(\phi, s - \frac{1}{2})L(\phi, s + \frac{1}{2})$$
$$\varepsilon(\pi, s, \psi) = \varepsilon(\phi, s - \frac{1}{2}, \psi)\varepsilon(\phi, s + \frac{1}{2}, \psi).$$

In particular, if ψ has level one, then

$$\varepsilon(\phi \circ \det, s, \psi) = \phi(\varpi)^{-2} q^{2s-1}$$

for any prime element ϖ of F.

4 The Converse Theorem

This section is based on [BH06, § 27].

Theorem 4.1 (Converse Theorem, [BH06, § 27.1]). *Let $\psi \in \hat{F}, \psi \neq 1$. Let π_1, π_2 be irreducible, smooth representations of $G = \mathrm{GL}_2(F)$. Suppose that*

$$L(\chi\pi_1, s) = L(\chi\pi_2, s) \quad \text{and} \quad \varepsilon(\chi\pi_1, s, \psi) = \varepsilon(\chi\pi_2, s, \psi),$$

for all characters χ of $F^\times$. We then have $\pi_1 \cong \pi_2$.

The proof of this theorem can be split up into two parts: The case where both π_1 and π_2 are cuspidal, and the case where both are non-cuspidal. The reason for this is as follows:

Proposition 4.2. *An irreducible smooth representation π of G is cuspidal if and only if $L(\phi\pi, s) = 1$ for all characters ϕ of $F^\times$.*

Proof. If π is cuspidal, so is $\phi\pi$ for all $\phi \in \hat{F}^\times$. We will see next time [BH06, § 24.5, Cor.] that we have $L(\pi, s) = 1$ for a cuspidal representation.

If π is non-cuspidal, i.e. a composition factor of $\iota_B^G\chi$ for some character $\chi = \chi_1 \otimes \chi_2$ of T, then the representation $\phi\pi$ is a composition factor of $\iota_B^G\phi\chi$, and $\phi\chi = \phi\chi_1 \otimes \phi\chi_2$. If we choose $\phi = \chi_2^{-1}$, we get from Theorem 3.1 that $L(\phi\pi, s) \neq 0$ because $L(1, s) \neq 0$. ■

This proposition shows that the above theorem has two cases: The case where both π_1 and π_2 are non-cuspidal, and the case where both are cuspidal. We will see the latter case is the next talk. In this talk we are only going to sketch a proof of the theorem in the non-cuspidal case.

For this, we will show that a non-cuspidal representation is determined by the map $\phi \mapsto L(\phi\pi, s)$. By the proposition, we can assume that $L(\pi, s) \neq 1$. We will use Theorem 3.1 to reconstruct π from the $L(\phi\pi, s)$. By definition, since π is a non-cupsidal representation it is a G-composition factor of $\iota_B^G(\chi_1 \otimes \chi_2)$. By Proposition 1.2, we know $L(\chi, s)$. Therefore if $L(\pi, s)$ has degree two, we know that $L(\pi, s) = L(\chi_1, s)L(\chi_2, s)$ for two unramified characters χ_1 and χ_2 of $F^\times$. Now the character $\iota_B^G(\chi_1 \otimes \chi_2)$ is either irreducible (and therefore coincides with π) or it has twice the same G-composition factor. Therefore this determines π in the degree 2 case. (We get these cases because the representaion is irreducible if and only if $\chi_1\chi_2^{-1} \neq ||x||^{\pm 1}$. In the case where it is reducible, the latter equation shows that the two compositon factors of $\iota_B^G(\chi_1 \otimes \chi_2)$ are isomorphic.)

Therefore, we only have the case where $L(\pi, s)$ has degree 1 left. In that case Theorem 3.1 shows that $L(\pi, s) = L(\chi)$ for some unramified character χ of $F^\times$. This only happens in two instances by Proposition 1.2: Either $\pi \cong \chi \otimes \theta$ for a ramified character θ or $\pi \cong \chi' \cdot \mathrm{St}_G$ where $\chi'(x) = \chi(x)||x||^{-1/2}$.

We can distinguish these cases as follows: In the former case there is a ramified character ϕ such that $L(\phi\pi, s) \neq 1$. In the latter case there is not. To recover θ in the former case, we pick a ϕ so that $L(\phi\pi, s) = L(\chi', s)$. Then $\theta = \phi^{-1}\chi'$. (You get this by comparing the actions on both sides.)

References

[BH06] Colin J. Bushnell and Guy Henniart. *The local Langlands conjecture for* GL(2). Vol. 335. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2006, pp. xii+347. ISBN: 978-3-540-31486-8; 3-540-31486-5. DOI: `10.1007/3-540-31511-X`. URL: `https://doi.org/10.1007/3-540-31511-X`.