Babyseminar SS2023 - Local Langlands for GL_2

Talk 1 : Introduction & motivation

In January 1967, R. Langlands sent a handwritten letter to A. Weil, outlining what is today known as Langlands Program.

In this letter Langlands posed two main questions / conjectures that go under the name of

(i) Langlands reciprocity (or correspondence)

(ii) Langlands functoriality

and can be formulated in a local or global setting.

§ A global motivation

Fix a positive integer $N \geq 5$ and let $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \bmod N \right\}$.

The moduli problem

$$\begin{aligned} Alg_{\mathbb{Q}} &\longrightarrow Sets \\ R &\longmapsto \left\{ (E, H) \;\middle|\; \begin{array}{l} E \text{ elliptic curve over } Spec(R) \\ H \subseteq E \text{ cyclic subgroup scheme} \\ \text{of order } N \end{array} \right\} \Big/ \cong \end{aligned}$$

admits a coarse moduli space, the so-called "open" modular curve $Y_0(N)$, a smooth affine curve over $\mathbb{Q}$.

$\mathcal{H} = \{ \tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0 \}$

One can show that $Y_0(N)(\mathbb{C}) \cong \Gamma_0(N) \backslash \mathcal{H}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \dfrac{a\tau + b}{c\tau + d}$ $\quad \tau \in \mathcal{H}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

$Y_0(N)$ can be "compactified" adding the so-called cusps.

One obtains a smooth projective curve $X_0(N)$ over $\mathbb{Q}$ such that

$X := X_0(N)(\mathbb{C}) \cong \Gamma_0(N) \backslash \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ has a natural structure of compact Riemann surface

One can define $S_2(\Gamma_0(N)) := H^0(X, \Omega_X)$ $\quad \Omega_X$ sheaf of holomorphic differentials

$S_2(\Gamma_0(N))$ is known as the $\mathbb{C}$-vector space of cuspidal modular forms of weight 2 and level $\Gamma_0(N)$.

Note that $\dim_{\mathbb{C}} S_2(\Gamma_0(N)) = \text{genus of } X_0(N) =: g$

One can form the Jacobian $J_0(N)$ of $X_0(N)$ over $\mathbb{Q}$. It is an abelian variety over $\mathbb{Q}$ of dimension g such that:

$$\mathrm{Jac}(X) = \frac{H^0(X, \Omega_X)^\vee}{H_1(X, \mathbb{Z})} \cong J_0(N)(\mathbb{C})$$

For every prime ℓ we can form the ℓ-adic Tate module

$$Ta_\ell(J_0(N)) := \varprojlim_n J_0(N)[\ell^n] \cong \mathbb{Z}_\ell^{2g}$$

and one gets an ℓ-adic Galois representation of $G_\mathbb{Q} = Gal(\bar{\mathbb{Q}}/\mathbb{Q})$

$$\rho_\ell : G_\mathbb{Q} \to GL(Ta_\ell(J_0(N)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \cong GL_{2g}(\mathbb{Q}_\ell)$$

FACT: ρ_ℓ is unramified at primes NOT dividing $N \cdot \ell$

There is a so-called Hecke algebra acting on $S_2(\Gamma_0(N))$, $\mathcal{H} = \mathbb{Z}[T_p, p \text{ prime}]$
$\mathcal{H} \subseteq \mathrm{End}_\mathbb{C}(S_2(\Gamma_0(N)))$, where T_p is induced by a correspondence on $X_0(N)$

$$\begin{tikzcd}
 & (E,H) \quad X_0(Np) \arrow[dl] \arrow[dr] & \\
(E/H_p, H/H_p) \quad X_0(N) & & X_0(N)
\end{tikzcd}$$

$(E,H) \mapsto (E,H_N)$

$H_p \leq H$ unique cyclic subgroup of order p

$H_N \subseteq H$ unique cyclic subgroup of order N

FACT: $\mathcal{H}$ is a commutative and finite $\mathbb{Z}$-algebra

Assume from now on for simplicity that N is a prime. This implies that $\mathcal{H} \otimes \mathbb{Q}$ is semisimple $\Rightarrow$ $S_2(\Gamma_0(N))$ has a basis of eigenforms for all the Hecke operators; moreover the Hecke eigenvalues are rational numbers (actually integers).

$$\left\{ \begin{array}{c} \text{normalized eigenforms} \\ \text{in } S_2(\Gamma_0(N)) \end{array} \right\} \overset{1:1}{\longleftrightarrow} \mathrm{Hom}_{\mathbb{Z}\text{-alg}}(\mathcal{H}, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Q}\text{-alg}}(\mathcal{H} \otimes \mathbb{Q}, \mathbb{Q})$$

$$f \longmapsto \lambda_f \qquad \lambda_f(T) f = Tf$$

By construction $\mathcal{H} \otimes \mathbb{Q} \subseteq \mathrm{End}(J_0(N)) \otimes \mathbb{Q}$. Since the Hecke action is linear, one can check that $\mathcal{H}$ acts on $Ta_\ell(J_0(N))$ $\forall \ell$. Moreover the Hecke action is defined over $\mathbb{Q}$, so that it commutes with the $G_\mathbb{Q}$-action on $Ta_\ell(J_0(N))$.

We have canonical identifications:

comparison isom. for étale cohomology

$$\mathrm{Hom}_{\mathbb{Z}_\ell}\big(Ta_\ell(J_0(N)),\mathbb{Z}_\ell\big) \cong H^1_{ét}\big(J_0(N)_{\bar{\mathbb{Q}}},\mathbb{Z}_\ell\big) \cong H^1_{sing}\big(Jac(X),\mathbb{Z}_\ell\big)$$

$$H^1_{ét}\big(J_0(N)_{\bar{\mathbb{Q}}},\mathbb{Z}_\ell\big) \cong \mathrm{Hom}_{\mathbb{Z}_\ell}\big(\pi_1^{ét}(J_0(N)),\mathbb{Z}_\ell\big) \cong \mathrm{Hom}_{\mathbb{Z}_\ell}\big(Ta_\ell(J_0(N)),\mathbb{Z}_\ell\big)$$

$$\pi_1^{ét}(J_0(N))\otimes\mathbb{Z}_\ell \cong Ta_\ell(J_0(N))$$

easy from the definition of $Jac(X)$

Hodge theory

and

$$H^1_{sing}(Jac(X),\mathbb{C}) \cong H^1_{sing}(X,\mathbb{C}) \cong S_2(\Gamma_0(N)) \oplus \overline{S_2(\Gamma_0(N))}$$

These identifications are all Hecke-equivariant, so given a normalised eigenform $f\in S_2(\Gamma_0(N))$ if we set

$$\big(Ta_\ell(J_0(N))\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell\big)[f] := \{v\in Ta_\ell(J_0(N))\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell \mid Tv=\lambda_f(T)\cdot v \quad \forall T\in\mathcal{H}\}$$

it is not surprising that we get a two-dimensional $\mathbb{Q}_\ell$-subspace of $Ta_\ell(J_0(N))\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ which is a $G_{\mathbb{Q}}$-submodule.

Fixing a basis we get $\rho_{f,\ell}: G_{\mathbb{Q}} \to GL_2(\mathbb{Q}_\ell)$

NOTE: $\rho_{f,\ell} \cong Ta_\ell(E_f)$ E_f elliptic curve over $\mathbb{Q}$ obtained as "$J_0(N)/I_f\cdot J_0(N)$" where $I_f=\{T\in\mathcal{H} \mid Tf=0\}$

Moreover the following "miracle" (Eichler-Shimura congruence relation) happens.
For p prime $p\nmid \ell N$ one finds that

$$\det(1-\mathrm{Frob}_p X \mid \rho_{f,\ell}) = 1-a_p(f)X+pX^2$$

Frob_p geometric Frobenius
$T_pf=a_p(f)f$

$\rightsquigarrow$ a statement of the form

$L(f,s)=L(E_f,s)=L((\rho_{f,\ell})_\ell,s)$ is now believable (and in fact true!)

§ Moral of the story

Starting from an analytic/"automorphic" object f for the reductive group $GL_2/\mathbb{Q}$ one can produce a compatible system of Galois representations valued in $GL_2(\mathbb{Q}_\ell)$ inside the cohomology of a suitable Shimura variety, in such a way that the corresponding L-functions match.

Langlands Program looks for a "good" framework where one can conjecture (and possibly realize) such correspondences for any reductive group G over any global field K. One would like these correspondences to be compatible with change of group and of ground field (this is the functoriality).

As usual when a global statement is too hard to prove (or it is even not clear what the correct statement should be), one can try to formulate suitable "local" analogues and attack those conjectures first (postulating a suitable local-global compatibility).

§ Our seminar

The goal of our seminar is to prove the local Langlands reciprocity for GL_2/F, F local field. In this setting one can produce a unique bijection

TALKS 8-9 (converse theorem)

TALKS 2-3 (general theory) TALKS 10-11

$$\left\{ \begin{array}{c} \text{isomorphism classes} \\ \text{of 2-dimensional Frob.} \\ \text{semisimple Weil-Deligne} \\ \text{representations} \end{array} \right\} \xleftrightarrow[\text{TALK 12}]{1:1} \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{irreducible smooth admissible} \\ \text{representations of } GL_2(F) \text{ in} \\ \mathbb{C}\text{- vector spaces} \end{array} \right\}$$

$$e \longmapsto \pi(e)$$

TALK 8 (autom. side), TALK 10 (Galois side)

such that: (i) L-functions and ε-factors are preserved

(ii) there is a compatibility with local class field theory (i.e. with twists by characters)

TALK 10

Under such a bijection we have that:

- (super) cuspidal representations correspond to irreducible WD repr.

  TALKS 5-7 TALKS 10-11
- non-cuspidal representations correspond to reducible WD repr.

  TALK 4

TALKS 13-14: Overview on orbit integrals & trace formulae: more advanced tools which are also ingredients for the proof of local Langlands reciprocity for GL_n $\forall n \geq 1$ (Harris-Taylor, Scholze)