Tuesday, January 19, 2021 4:01 PM

Khan's Virtual Classes

Question: What are (virtual) Fundamental Classes.

Déglise-Jin-Khan: Fundamental classes are a special instance of the notion of orientation in the setting of bivariant theories
(Inspired by Fulton)

Let's Recall: $\mathcal{F} \in SH(S)$, $\mathcal{F}$ is oriented, and multiplicative (i.e. ring object) $\quad \mathcal{F}(X/S, v) \simeq \mathcal{F}(X/S, \mathcal{O}^r)$

For any $n, r \in \mathbb{Z}$, $f: X \to S$

$$H_n^{BM}(X/S, \mathcal{F}(r)) := \pi_0 \operatorname{Maps}_{SH(S)}(1_S(r)[n], f_* f^! \mathcal{F})$$

Remark: $\pi_n \mathcal{F}(X/S, r) = H^{BM}_{2r+n}(X/S, \mathcal{F}(r))$

Orientation: Is a pair (η_f, r) for some $r \in \mathbb{Z}$ Such that $\eta_f \in H^{BM}_{2r}(X/S, \mathcal{F}(r))$

Example: Suppose $f: X \to S$ is smooth of rel. d.
then purity: $\gamma: \Sigma^{T_f} f^* \xrightarrow{\simeq} f^!$

$$[X/S]: 1_S \to \mathcal{F} \xrightarrow{\text{adjunction}} f_* f^* \mathcal{F} \simeq f_* \Sigma^{-T_f} f^! \mathcal{F}$$

$$\in \pi_0 \mathcal{F}(X/S, T_f) \simeq H^{BM}_{2d}(X/S, \mathcal{F}(d))$$

DJK: Let $\mathcal{C}$ = Class of Morphisms.

$\rightsquigarrow$ System of Fundamental classes for $\mathcal{C}$.

(i) For each $f \in \mathcal{C} \rightsquigarrow$ orientation. (η_f, r)

normalized (ii)
assoc. (iii)
transverse Base change. (iv)

$\mathcal{C} = \{$ smoothable l.c.i maps $\}$. admits a system of fundamental classes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "\text{closed reg. embedding}"'] & Y \\ P \arrow[ur, "\text{smooth}"'] & \end{tikzcd}$$

Let's recall the first half of the construction.
i.e. focus on the case of closed reg. embedd.

Two important things we'll need:

① Localization: $U \overset{j}{\underset{\text{open}}{\hookrightarrow}} X \overset{i}{\underset{\text{closed}}{\hookleftarrow}} Z$

then there is long exact sequence in BM-homology.

① Localization: $U \overset{u}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z$ (open, closed)

then there is a long exact sequence in BM-homology

$$\cdots \xrightarrow{\partial} H^{BM}_{n+1}(Z/S, \mathcal{F}(r)) \xrightarrow{i_*} H^{BM}_{n+1}(X/S, \mathcal{F}(r)) \xrightarrow{j^!} H^{BM}_{n+1}(U/S, \mathcal{F}(r)) \xrightarrow{\partial} \cdots$$

For every $r \in \mathbb{Z}$.

② Homotopy Invariance

$V \xrightarrow{\pi} X$ vector bundle of rk d

$$\Rightarrow \pi^! : H^{BM}_n(X/S, \mathcal{F}(r)) \xrightarrow{\sim} H^{BM}_{n+2d}(V/S, \mathcal{F}(r+d)).$$

for all $r, n \in \mathbb{Z}$.

suppose we have a fixed closed reg. embedding

$$i : X \hookrightarrow Y$$

① $\mathbb{G}_m \times Y \underset{\text{open}}{\hookrightarrow} \mathbb{A}^1 \times Y \underset{\text{closed}}{\hookleftarrow} \{0\} \times Y$

② Deformation to the normal cone:

$$\mathbb{G}_m \times Y \underset{\text{open}}{\overset{h}{\hookrightarrow}} D_X Y \underset{\text{closed}}{\overset{k}{\hookleftarrow}} N_X Y$$

①: $0 \to H^{BM}_{n+1}(\mathbb{A}^1 \times Y/S, \mathcal{F}(r)) \to H^{BM}_{n+1}(\mathbb{G}_m \times Y/S, \mathcal{F}(r)) \to H^{BM}_n(Y/S, \mathcal{F}(r)) \to 0$

γ_t

DJK 3.2.2.

②: we get a map

$$\partial : H^{BM}_{n+1}(\mathbb{G}_m \times Y/S, \mathcal{F}(r)) \longrightarrow H^{BM}_n(N_X Y/S, \mathcal{F}(r))$$

$$sp_{X/Y} : H^{BM}_n(Y/S, \mathcal{F}(r)) \to H^{BM}_{n+1}(\mathbb{G}_m \times Y/S, \mathcal{F}(r)) \to H^{BM}_n(N_X Y/S, \mathcal{F}(r))$$

Homotopy invariance:

$$i^! : H^{BM}_n(Y/S, \mathcal{F}(r)) \xrightarrow{sp_{X/Y}} H^{BM}_n(N_X Y/S, \mathcal{F}(r)) \xrightarrow{\simeq} H^{BM}_{n-2d}(X/S, \mathcal{F}(n-d))$$

$S = Y$, $n=0$, $r=0$

$$i^! : H^{BM}_0(Y/Y, \mathcal{F}) \longrightarrow H^{BM}_{-2d}(X/S, \mathcal{F}(-d))$$

$$1 \longmapsto i^!(1) =: [X/Y]$$

$$X \underset{\text{closed reg}}{\hookrightarrow} P \underset{\text{smooth}}{\longrightarrow} Y, \quad f : X \to Y$$

$$X \underset{\substack{\text{closed}\\ \text{reg}}}{\hookrightarrow} P \xrightarrow{\text{smooth}} Y$$

<u>Remark</u>: If you could define $SH(-)$ on stacks you don't have to do the second half of DJK.

f smoothable $\Rightarrow$ $N_X Y = \operatorname{Spec}\operatorname{Sym}(L_{X/Y}[-1])$ perfect $[1,0]$

is a vector bundle stack on X.

homotopy inv. VB stacks. + Def. to the normal sheaf. (KResch) + localization $\Rightarrow$ Gives the orientation.

- Suppose further $f: X \to Y$ had a perfect obs. theory.

<u>Remark</u>: If $f: X \to Y$ is a quasi-smooth morphism of derived schemes. $\Rightarrow$ $L_{X/Y}$ is perfect in $[1,0]$.

A bit about quasi-smooth morphisms.

quasi-smooth schemes $\rightsquigarrow$ perfect obstr. theories.

More precisely: If $f: X \to pt$ is quasi-smooth.

$\Rightarrow i^* L_X \to L_{X^{cl}}$ is a perfect obstr. theory on X^{cl}.

(<u>Recall</u>: The adjunction $Sch \underset{i}{\overset{\pi}{\rightleftarrows}} DerSch$ gives $X^{cl} := i\pi_0 X \underset{i}{\hookrightarrow} X$ closed embedding)

<u>Remark</u>: The notion of p.o.t. is more general than quasi-smoothness.

<u>Lemma</u>: X/pt is quasi-smooth $\Leftrightarrow$ Zariski locally we can describe X as

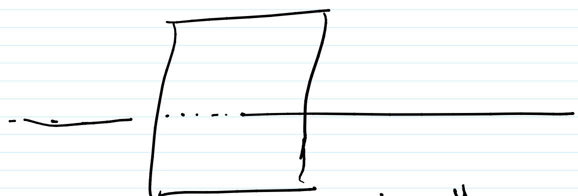
Lemma: X/pt is quasi-smooth $\Longleftrightarrow$ Zariski locally we can describe X as

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathbb{A}^n \arrow[d, "{(f_1,\dots,f_m)}"] \\ \{0\} \arrow[r, hook] & \mathbb{A}^m \end{tikzcd}$$

Example: Consider

$$\begin{tikzcd} X = \operatorname{Spec}\left(\mathbb{C}[x,y,z]/(xz,yz)\right) \arrow[r, hook] \arrow[d] & \mathbb{A}^3 \arrow[d, "{(xy,\, yz)}"] \\ 0 \arrow[r, hook] & \mathbb{A}^2 \end{tikzcd}$$

Pullback in ordinary schemes.



But:

$$\begin{tikzcd} X_{der} \arrow[r, hook, "\text{quasi-smooth closed embedded}"] \arrow[d] & \mathbb{A}^3 \arrow[d, "{(xz,\, yz)}"] \\ \{0\} \arrow[r, hook] & \mathbb{A}^2 \end{tikzcd}$$

then X_{der} quasi-smooth

$X_{der}^{cl} = X$.

For a general description of an arbitrary quasi-smooth map.
Arinkin-Gaitsgory: Sing. Supp. Corollary 2.1.11.

Adeel Khan: Let's SH to derived schemes and stacks to obtain a virtual class.

Extension to derived schemes: Khan's Thesis.

Roughly: $S \in \mathrm{DerSch}$.

$$\mathcal{H}(S) \underset{full}{\subseteq} \mathrm{PSh}_{\mathrm{Anima}}(\mathrm{Sm}/S)$$

, spanned by presheaves which are Nisnevich sheaves and $\mathbb{A}^1$-homotopy invariant.

$$\mathcal{H}(S)_\bullet := \mathcal{H}(S)_{S/} \quad \text{this is a symmetric monoid } \infty\text{-category} : \wedge$$

$$SH(S) = \mathcal{H}(S)_\bullet^\wedge [(\mathbb{P}^1,\infty)^{-1}]$$

<u>Khan</u>: Works even for derived algebraic spaces. and there is a six functor formalism.

<u>Important Property</u>: (Derived Invariance) $\quad S^{cl} \overset{i}{\hookrightarrow} S$

$$i^* : SH(S) \simeq SH(S^{cl}) : i_*$$

<u>Extension to Derived stacks</u>:

$$\begin{aligned} SH : (\text{Der Alg Sp})^{op} &\longrightarrow \infty\text{-cat} \\ X &\longmapsto SH(X) \\ f &\longmapsto f^* \end{aligned}$$

this is a Nisnevich sheaf.

$SH_{ét}(\ ) :=$ étale sheafification of SH.

Then take the right Kan extension to Derived Artin stacks.

<u>Explicitly</u>: For any derived Artin Stack $\mathcal{X}$ with smooth atlas.
$p : X \to \mathcal{X}$

$$SH_{ét}(\mathcal{X}) \simeq \varprojlim \left(SH_{ét}(X) \rightrightarrows SH_{ét}(X \times_{\mathcal{X}} X) \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots \right)$$

<u>Theorem</u>: (Khan) For every derived Artin stack there is a closed symmetric monoidal structure on $SH_{ét}(X)$ such that: All the six functors behave nicely.

Now redo everything from the beginning of the talk with $SH_{ét}$ instead of SH.

- Localization ✓

• Homotopy invariance: If $\mathcal{E}$ is perfect in $[0,-k]$ for $k \geq 0$ on a derived scheme/stack.

$V(\mathcal{E})$: $\mathrm{Spec}(A) \xrightarrow{e} X \longmapsto \mathrm{Maps}_{D(A)}(e^*\mathcal{E}, A)$

is a derived vector bundle stack

$d = \mathrm{rank}(\mathcal{E})$

$$\pi^! : H^{BM}_S(X/S, \mathcal{F}(r)) \xrightarrow{\simeq} H^{BM}_{S+2d}(V_X(\mathcal{E})/S, \mathcal{F}(r+d)).$$

• Deformation to the normal cone. (Khan - Rydh)

<u>Theorem</u>: (Khan-Rydh) Let $f: X \to Y$ be a quasi-smooth morphism of Artin stacks

(1) $\exists$ a quasi-smooth derived Artin stack : $D_X Y$ over $Y \times \mathbb{A}^1$ and a quasi-smooth map.

$$\begin{tikzcd} X \times \mathbb{A}^1 \arrow[rr] \arrow[dr] & & D_X Y \arrow[dl] \\ & Y \times \mathbb{A}^1 & \end{tikzcd}$$

the fiber over $\mathbb{G}_m$ is the quasi-smooth map $X \times \mathbb{G}_m \to Y \times \mathbb{G}_m$.
the fiber over 0 is the quasi-smooth closed immersion

$$0: X \longrightarrow \underline{\underline{N_X Y}} = \mathrm{Spec}\,\mathrm{Sym}(L_{X/Y}[-1]).$$

(2) The construction $D_X Y \longrightarrow Y$ is stable under arbitrary base change to Y.

Can take the Weil restriction of X along $Y \times \{0\} \hookrightarrow Y \times \mathbb{A}^1$.
(see SAG)

<u>Classical Things</u>: $D_X Y = \underset{\substack{U \xhookrightarrow{\text{quasismooth}} V \\ \downarrow \quad \downarrow \\ X \xrightarrow[\text{quasismooth}]{} Y}}{\mathrm{Colim}} D_U V \in \mathrm{Shv}_S(\mathrm{Aff})$

Some other nice things Khan proves:

Some other nice things Khan proves:

① Comparison with BeFa.

② Derived version of Bézout's Theorem.

③ Grothendieck - Riemann - Roch.

④ Absolute purity.