Kan extensions, descent theory and localization of ∞-categories

Kan extensions:

$\mathcal{H}$ = „homotopy cat. of spaces": obj. are CW-complexes, Hom: continuous comp.-open top.

equiv. of $\mathcal{H}$-enriched cats. between their homotopy categories ($h\mathcal{C}$ [5])

Def [Lur09, Def 4.3.1.1] „relative colimits"

added terminal obj. by $K * \Delta^0$

$f: \mathcal{C} \to \mathcal{D}$; $\mathcal{C}_{\bar{p}/}$ = full subcat. of $\mathrm{Fun}(\mathcal{C}^{\triangleright}, \mathcal{D})$ s.t. $\mathcal{C} \hookrightarrow \mathcal{C}^{\triangleright} \to \mathcal{D}$ is f

Let $f: \mathcal{C} \to \mathcal{D}$ be an inner fibration (RLP for Λ^n_i, $0<i<n$) of simplicial sets, let $\underbrace{\bar{p}: K^{\triangleright} \to \mathcal{C}}_{\text{cocone}}$ be a diagram and $p = \bar{p}|_K$. (here $\Leftrightarrow$ categorical equiv.)

We will say that $\bar{p}$ is an f-colimit of p if the map $\mathcal{C}_{\bar{p}/} \to \mathcal{C}_{p/} \times_{\mathcal{D}_{fp/}} \mathcal{D}_{f\bar{p}/}$ is a trivial fibration (RLP for $\partial\Delta^n$, „trivial Kan fibration")

of simplicial sets. In this case, we will also say that $\bar{p}$ is an f-colimit diagram.

$$\Leftrightarrow \begin{tikzcd} \mathcal{C}_{\bar{p}/} \arrow[r] \arrow[d] & \mathcal{D}_{f\bar{p}/} \arrow[d] \\ \mathcal{C}_{p/} \arrow[r] & \mathcal{D}_{fp/} \end{tikzcd} \text{ is homotopy cartesian}$$

$\mathcal{D} = \Delta^0 = *$ (terminal obj)

Example [Lur09, Ex. 4.3.1.3] „relative colimits generalize colimits"

Let $\mathcal{C}$ be an ∞-cat. and $f: \mathcal{C} \to *$ the projection of $\mathcal{C}$ to a point. Then a diagram $\bar{p}: K^{\triangleright} \to \mathcal{C}$ is an f-colimit if and only if it is a colimit in the previous sense ([Lur09, Def. 1.2.13.4])

$p(\infty)$ is initial in $\mathcal{C}_{p/}$, $0 \in \mathcal{C}$ initial $\Leftrightarrow$ $\mathrm{Map}_{\mathcal{C}}(0, x)$ contractible $\forall x \in \mathcal{C}$ (final in $\mathcal{C}^{op}$) $\hookrightarrow$ weakly contractible if not CW-complexes

$K = \Delta^0 = *$, $K^{\triangleright} = \Delta^1$

Example [Lur09, Ex. 4.3.1.4] „f-coCartesian edges as f-colimits"

Let $f: \mathcal{C} \to \mathcal{D}$ be an inner fibration of simplicial sets and $e: \Delta^1 = (\Delta^0)^{\triangleright} \to \mathcal{C}$ be an edge of $\mathcal{C}$.

Then e is an f-colimit if and only if it is f-coCartesian.

(the description above is exactly the def) or this one

$$\Leftrightarrow \begin{tikzcd} \Delta^1 \cong \Delta^{\{0,1\}} \arrow[d, hook] \arrow[dr, "e"] & \\ \Lambda^n_0 \arrow[r, "e_0"] \arrow[d, hook] & \mathcal{C} \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

only for inclusions $\mathcal{C}^0 \subseteq \mathcal{C}$ of full subcat.

Def [Lur09, Def. 4.3.2.2] „p-left Kan extensions"

$$\begin{tikzcd} \mathcal{C}^0 \arrow[r, "F_0"] \arrow[d] & \mathcal{D} \\ \mathcal{C} \arrow[ur, "F"'] & \end{tikzcd}$$ is left Kan-extension" of F_0 if $F(C) = \mathrm{colim}_{C' \in \mathcal{C}^0_{/C}} F(C')$ $\forall C \in \mathcal{C}$

Suppose we are given a comm. diagram of ∞-cat. $\begin{tikzcd} \mathcal{C}^0 \arrow[r, "F_0"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\ \mathcal{C} \arrow[r] \arrow[ur, "F"] & \mathcal{D}' \end{tikzcd}$, where p is an inner fibration

and the left vertical map is the inclusion of a full subcat. $\mathcal{C}^0 \subseteq \mathcal{C}$.

We will say that F is a p-left Kan extension of F_0 at $C \in \mathcal{C}$ if the induced diagram

$$\begin{tikzcd} \mathcal{C}^0_{/C} \arrow[r, "F_C"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\ (\mathcal{C}^0_{/C})^{\triangleright} \arrow[r] \arrow[ur] & \mathcal{D}' \end{tikzcd}$$ exhibits $F(C)$ as a p-colimit of F_C.

For $C \in \mathcal{C}^0$ there is nothing to check: id_C is final obj. of $\mathcal{C}^0_{/C}$ $\Rightarrow$ $\{\mathrm{id}_C\} \hookrightarrow \mathcal{C}^0_{/C}$ is cofinal. Reduced to $F(\mathrm{id}_C): \Delta^1 \to \mathcal{D}$ is colimit of its restr. to $\{0\}$ ✓

We will say that F is a p-left Kan extension of F_0 if it is a p-left Kan extension of F_0 at C for every object $C \in \mathcal{C}$.

In the case where $\mathcal{D}' = \Delta^0$, we will omit mention of p and simply say that F is a left Kan extension of F_0 if the above condition is satisfied.

special inclusion $\mathcal{C} \subseteq \mathcal{C}^{\triangleright}$

Example [Lur09, Ex. 4.3.2.4] „p-left Kan extensions as p-colimits"

Consider a diagram $\begin{tikzcd} \mathcal{C} \arrow[r, "q"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\ \mathcal{C}^{\triangleright} \arrow[r] \arrow[ur, "\bar{q}"] & \mathcal{D}' \end{tikzcd}$. The map $\bar{q}$ is a p-left Kan extension of q if and only if it is a p-colimit of q.

„easy, using cofinal, id_C final obj. of $\mathcal{C}'_{/C}$ $\Rightarrow$ $\mathcal{C}^0_{/C} \times \{\mathrm{id}_C\} \subseteq \mathcal{C}^0_{/C} \times \mathcal{C}'_{/C}$ is cofinal"

Proposition [Lur09, Prop. 4.3.2.9] „checking p-left Kan extensions objectwise in one component"

Let $F: \mathcal{C} \times \mathcal{C}' \to \mathcal{D}$ denote a functor between ∞-cat., $p: \mathcal{D} \to \mathcal{D}'$ a categorical fibration (slightly stronger than „inner fibration"; RLP w.r.t. maps that are cofibrations (monomorph. in Set_Δ) and cat. equiv (see above)) of ∞-cat., and $\mathcal{C}^0 \subseteq \mathcal{C}$ a full subcat. The following are equivalent:

1) The functor F is a p-left Kan extension of $F|_{\mathcal{C}^0 \times \mathcal{C}'}$

2) For each object $C' \in \mathcal{C}'$ the induced functor $F_{C'}: \mathcal{C} \times \{C'\} \to \mathcal{D}$ is a p-left Kan extension of $F_{C'}|_{\mathcal{C}^0 \times \{C'\}}$

<u>Proposition</u> [Lur09, Cor. 4.3.2.16] „functorial association of limits" „morphism between directed systems gives morph. between colimits"

↳ Follows from [Lur09, Prop. 4.3.2.15] which follows from [Lur09, Lem. 4.3.2.13] and [Lur09, Cor. 4.3.1.11].

Suppose we are given a diagram of ∞-cat. $\mathcal{C} \to \mathcal{D}' \xleftarrow{p} \mathcal{D}$, where p is a categorical fibration. Dual: right Kan extension and limits

Let $\mathcal{C}^0$ be a full subcat. of $\mathcal{C}$. Suppose further that, for every functor $F_0 \in \mathrm{Map}_{\mathcal{D}'}(\mathcal{C}^0, \mathcal{D})$, there exists a functor $F \in \mathrm{Map}_{\mathcal{D}'}(\mathcal{C}, \mathcal{D})$ which is a p-left Kan extension of F_0.

special case $\mathcal{C}^{(0)} = K$, $\mathcal{C} = K^{\triangleright}$ ⇝ left Kan extension = colimit on obj $i_!$: functor $f: K \to \mathcal{D}$ to its colimit diagram $\bar{f}: K^{\triangleright} \to \mathcal{D}$

Then the restriction map $i^*: \mathrm{Map}_{\mathcal{D}'}(\mathcal{C}, \mathcal{D}) \to \mathrm{Map}_{\mathcal{D}'}(\mathcal{C}^0, \mathcal{D})$ admits a section $i_!$ whose essential image consists of precisely those functors F which are p-left Kan extensions of $F|_{\mathcal{C}^0}$.

called „left Kan extension functor" is unique up to homotopy (collection param. by contractible Kan complex)

<u>Lemma</u>: [Lur09, Lem 5.5.2.3] „limits commute" „Fubini for limits" Proof uses [Lur09, Prop. 4.3.2.9]

show that different $p|(X^{\triangleright}, Y^{(\triangleright)})^{(\triangleright)}$ are q-left Kan extensions

Let X, Y be simplicial sets, let $q: \mathcal{C} \to \mathcal{D}$ be a categorical fibration of ∞-cat. and let $p: X^{\triangleright} \times Y^{\triangleright} \to \mathcal{C}$ be a diagram. Suppose that:

$x \in X$: „$\operatorname{colim}_{y} H(x,y)$ exists"

$x = \infty \in X^{\triangleright}$: „$\operatorname{colim}_{x} \operatorname{colim}_{y} H(x,y)$ exists"

1) For every vertex x of $X^{\triangleright}$, the associated map $p_x: Y^{\triangleright} \to \mathcal{C}$ is a q-colimit diagram

„$\operatorname{colim}_{x} H(x,y)$ exists"

2) For every vertex y of Y, the associated map $p_y: X^{\triangleright} \to \mathcal{C}$ is a q-colimit diagram

Let „∞" denote the cone point of $Y^{\triangleright}$. Then the restriction $p_\infty: X^{\triangleright} \to \mathcal{C}$ is a q-colimit diagram

⇝ „$\operatorname{colim}_{y} \operatorname{colim}_{x} H(x,y)$ exists"

and „$\operatorname{colim}_{y} \operatorname{colim}_{x} H(x,y) \cong \operatorname{colim}_{x} \operatorname{colim}_{y} H(x,y)$"

<u>Descent theory</u>

<u>Def</u> [Lur18b, Def. A.3.1.1] „finitary Grothendieck topologies"

⌈sieve ≙ right ideal under precomp., i.e. closed under precomp. with arbi. morph.

↳ $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$ full subcat with $C_0 \to C_1$, $\varphi_0: C_0 \to C$, $\varphi_1: C_1 \to C$ comm.: $\varphi_1 \in \mathcal{C}^{(0)}_{/C} \Rightarrow \varphi_0 \in \mathcal{C}^{(0)}_{/C}$

Let $\mathcal{C}$ be an ∞-cat. which admits pullbacks. We say that a <u>Grothendieck topology on $\mathcal{C}$</u> is finitary if it satisfies the following condition:

for each obj $C \in \mathcal{C}$ a special class of sieves on C (→ covering sieves)

covering sieve
- $\mathcal{C}_{/C}$ is a covering sieve on C for all $C \in \mathcal{C}$
- $\varphi: C \to D$ morph in $\mathcal{C}$, $\mathcal{C}^{(0)}_{/D}$ cov. sieve on D ⇒ $\varphi^* \mathcal{C}^{(0)}_{/D}$ cov. sieve on C
- $\mathcal{C}^{(0)}_{/C}$ cov. sieve on C, $\mathcal{C}^{(1)}_{/D}$ arb. sieve on D: $\varphi^* \mathcal{C}^{(1)}_{/D}$ cov. sieve on C $\forall \varphi \in \mathcal{C}^{(0)}_{/C}$, $\varphi: C \to D$ ⇒ $\mathcal{C}^{(1)}_{/D}$ cov. sieve on D
- $\varphi: C \to D$, $\mathcal{C}^{(0)}_{/D}$ sieve, $\varphi^* \mathcal{C}^{(0)}_{/D} \subseteq \mathcal{C}_{/C}$ full subcat spanned by $\psi: C_0 \to C$ s.t. $\varphi \circ \psi$ belongs to $\mathcal{C}^{(0)}_{/D}$

For every object $C \in \mathcal{C}$ and every covering sieve $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$, there exists a finite collection of morphisms $\{C_i \to C\}_{1 \leq i \leq n}$ in $\mathcal{C}^{(0)}_{/C}$ which generate a covering of C.

⌈conditions I) „isom. are coverings"

II) „cov. of set gives cov. of subset"

III) „refined cov. is a cov."

(i.e. the smallest sieve $\mathcal{C}^{(1)}_{/C}$ containing each C_i is also a covering sieve on C)

<u>Proposition</u> [Lur18b, Prop. A.3.2.1] „construction of finitary Grothendieck topologies"

There are three conditions for the collection described to be a covering sieve

a) ⇒ I)

b)+d) ⇒ II)

a)+c) ⇒ III)

Let $\mathcal{C}$ be an ∞-cat. and let S be a collection of morphisms in $\mathcal{C}$. Assume that:

a) The collection S contains all equivalences and is stable under composition

b) $\mathcal{C}$ admits pullbacks. Moreover S is stable under pullbacks: $\begin{array}{ccc} C' & \to & C \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ D' & \to & D \end{array}$ pullback: $f \in S \Rightarrow f' \in S$

c) $\mathcal{C}$ admits finite coproducts. Moreover S is stable under fin. coproducts: fin. collect. $f_i: C_i \to D_i$ in $S \Rightarrow \coprod C_i \to \coprod D_i$ in S.

d) Finite coproducts are universal: $\coprod_{1 \leq i \leq n} C_i \to D \leftarrow D' \Rightarrow$ canon. map $\coprod_{1 \leq i \leq n} (C_i \times_D D') \to (\coprod_{1 \leq i \leq n} C_i) \times_D D'$ is equiv. in $\mathcal{C}$.

Then there exists a Grothendieck topology on $\mathcal{C}$ which can be described as follows:

A sieve $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$ on an object $C \in \mathcal{C}$ is a covering if and only if it contains a finite collection of morphisms $\{C_i \to C\}_{1 \leq i \leq n}$ such that the induced map $\coprod_{i=1}^{n} C_i \to C$ belongs to S.

long proof
reduce to $\mathcal{D}=\mathcal{S}$
by looking at $h^D \circ \mathcal{F}: \mathcal{C}^{op} \to \mathcal{S}$
where h^D corepres. D and
do that $\forall D \in \mathcal{D}$
∞-cat. of spaces

Proposition [Lur18b, A.3.3.1] „sheaf condition by descent along Čech nerves"

Let $\mathcal{C}$ be an ∞-cat. and S a collection of morphisms in $\mathcal{C}$. Assume that $\mathcal{C}$ and S satisfy the previous hypothesis a)-d), together with

initial obj

$$\begin{array}{ccc} * & \to & C \\ \downarrow & & \downarrow \\ C' & \to & C' \amalg C \end{array}$$

or is cartesian (it is already cocartesian)

e) Coproducts in $\mathcal{C}$ are disjoint. That means C, C' obj. in $\mathcal{C}$ then $C \times_{C \amalg C'} C'$ is initial in $\mathcal{C}$.

$\Delta_+^{\leq n} \subseteq \Delta_+$ full subcat spanned by obj $\{[k]\}_{-1 \leq k \leq n}$

Let $\mathcal{D}$ be an arbitrary ∞-cat., and let $\mathcal{F}: \mathcal{C}^{op} \to \mathcal{D}$ be a functor.

$\mathcal{F}: \mathcal{C}^{op} \to \mathcal{D}$ s.t. $\mathcal{F}$ exhibits $\mathcal{F}(C)$ as a limit of $\mathcal{F}|(\mathcal{C}^0_{/C})^{op}$ $\forall$ cov. sieve $\mathcal{C}^0_{/C} \subseteq \mathcal{C}_{/C}$ $\forall C$
canon. map into limit is equiv. in $\mathcal{D}$

Def [Lur09, Prop. 6.1.2.11]
Let $\mathcal{C}$ be ∞-cat. aug. simp. obj $U_\bullet^+ : N(\Delta_+)^{op} \to \mathcal{C}$ is a Čech-nerve if:
$U_\bullet^+$ is right Kan extension of $U_\bullet^+|_{N(\Delta_+^{\leq 0})^{op}}$
$\Leftrightarrow$ underlying simp obj $U_\bullet$ is groupoid obj of $\mathcal{C}$ and diag $U_\bullet^+|_{N(\Delta_+^{\leq 1})^{op}}$ is pullback square $\begin{array}{ccc} U_1 & \to & U_0 \\ \downarrow & & \downarrow \\ U_0 & \to & U_{-1} \end{array}$ in $\mathcal{C}$
$U_\bullet^+$ is determined (up to equiv) by $u: U_0 \to U_{-1}$
$\rightsquigarrow U_\bullet^+$ Čech-nerve of u

Then $\mathcal{F}$ is a $\mathcal{D}$-valued sheaf on $\mathcal{C}$ if and only if: use $\longrightarrow$

$U: N(\Delta)^{op} \to \mathcal{C}$ groupoid obj
$\forall n \geq 0$ $\forall$ part $[n] = S \cup S'$ and $\{s\} = S \cap S'$:
$$\begin{array}{ccc} U([n]) & \to & U(S) \\ \downarrow & & \downarrow \\ U(S') & \to & U(\{s\}) \end{array}$$ pullback

1) The functor $\mathcal{F}$ preserves finite products

2) Let $f: U_0 \to X$ be a morphism in $\mathcal{C}$ which belongs to S and let $U_\bullet$ be a Čech nerve of f, regarded as an augmented simplicial object of $\mathcal{C}$. Then the composite map
functor $N(\Delta_+)^{op} \to \mathcal{C}$

$$\Delta_+ \xrightarrow{U_\bullet^{op}} \mathcal{C}^{op} \xrightarrow{\mathcal{F}} \mathcal{D} \quad \text{is a limit diagram.}$$

(i.e. $\mathcal{F}$ exhibits $\mathcal{F}(X)$ as a totalization of the cosimplicial object $[n] \mapsto \mathcal{F}(U_n)$)
dual of geom. realization for simplicial objects
=end of $\cdot\,\Delta'$ instead of „coend of $\cdot \times \Delta$"
functor: $\Delta \to \mathcal{D}$

Def [Cho, Def. A.16.7][LZ17, Def. 3.1.1] „F-descent"

Let $\mathcal{C}$ be an ∞-cat. which admits pullbacks, $F: \mathcal{C}^{op} \to \mathcal{D}$ be a functor of ∞-cat. and $f: X_0^+ \to X_{-1}^+$ be a morphism in $\mathcal{C}$. Then f satisfies F-descent if:

$$F \circ (X_\bullet^+)^{op} : N(\Delta_+) \to \mathcal{D} \quad \text{is a limit diagram, where } X_\bullet^+ \text{ is the Čech nerve of } f.$$

Take iterated fiber products of the diagram to get a functor
$H: N(\Delta_+)^{op} \times N(\Delta_+)^{op} \to \mathcal{D}$
and the use „limits commute"

Lemma: [Cho, Lem. A.16.8] „Fubini for F-descent"

Let $\mathcal{D}$ be an ∞-cat. which admits products. Let
$$\begin{array}{ccc} D_3 & \xrightarrow{f_{32}} & D_2 \\ {\scriptstyle f_{31}}\downarrow & & \downarrow{\scriptstyle f_{20}} \\ D_1 & \xrightarrow{f_{10}} & D_0 \end{array}$$
be a comm. diagram in $\mathcal{D}$.

Let $F: \mathcal{D}^{op} \to \mathcal{D}'$ be a functor of ∞-cat. Assume the following:

1) f_{32}, f_{31} satisfy F-descent „$\hat{=}$ componentwise limits exists"

2) f_{20} satisfies F-descent „$\hat{=}$ one of the iterated limits exists"

Then f_{10} satisfies F-descent. Also we have: „Fubini $\hat{=}$ the other one exists and they coincide"

$$\lim_{n \in \Delta^{op}} F(D_1^{\times n}) \xrightarrow{\cong} \lim_{n \in \Delta^{op}} F(D_1^{\times n} \times_{D_0} D_2^{\times n}) \xleftarrow{\cong} \lim_{n \in \Delta^{op}} F(D_2^{\times n})$$

Localization of ∞-categories

Def [Lan21, Def. 2.4.2] „Dwyer-Kan localization"

Let $\mathcal{C}$ be an ∞-cat. and let S be a collection of morphisms in $\mathcal{C}$. A functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$ is called a Dwyer-Kan localization of $\mathcal{C}$ along S, if for every ∞-cat. $\mathcal{D}$ the restriction functor $\mathrm{Fun}(\mathcal{C}[S^{-1}], \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is fully faithful and its essential image consists of those functors that send S to equivalences.
localization is unique up to categorical equivalence

([Lan21, Thm. 2.3.20] $\Leftrightarrow$ restriction functor factors through cat. equiv. $\mathrm{Fun}(\mathcal{C}[S^{-1}], \mathcal{D}) \to \mathrm{Fun}^S(\mathcal{C}, \mathcal{D})$)
(Lan21, Thm 2.3.20: fully faithful + ess. surj $\Rightarrow$ cat. equiv.)
functors $f: \mathcal{C} \to \mathcal{D}$ with $f(S) \subseteq \mathcal{D}^{\simeq}$, i.e. f maps morph of S to equiv. in $\mathcal{D}$

Lemma [Lan21, Lem. 2.4.5] „easiest example: $\bullet \longrightarrow \bullet$ becomes $\bullet \overset{\sim}{\underset{\sim}{\rightleftarrows}} \bullet$"

contractible groupoid with two objects $= N(\bullet \overset{\sim}{\rightleftarrows} \bullet)$

The map $\Delta^1 \to J$ is a localization at the unique morphism from 0 and 1.

Takes some work

res: $\mathrm{Fun}(J, D) \to \mathrm{Fun}^{\simeq}(\Delta^1, D) \subseteq \mathrm{Fun}(\Delta^1, D)$

show: this is cat. equiv.

Lemma [Lan21, Lem. 2.4.6] „localization at everything exists"

LLP w.r.t. fibrations

construct an anodyne map $\mathcal{C} \to X$ on ∞-groupoid

show: $\forall \infty$-cat D the res: $\mathrm{Fun}(X, D) \to \mathrm{Fun}(\mathcal{C}, D)$ factors through triv. fib. $\mathrm{Fun}(X, D) \to \mathrm{Fun}^{\simeq}(\mathcal{C}, D)$

For every ∞-cat. $\mathcal{C}$, there exists a localization along all morphisms of $\mathcal{C}$.

localizing along all morphisms is left adjoint to inclusion of ∞-groupoid into ∞-cat. (as ∞-functors between ∞-cat)

Proposition [Lan21, Prop. 2.4.8] „every localization exists"

For every collection S of morphisms in $\mathcal{C}$, there exists a localization of $\mathcal{C}$ along S.

consider smallest subcat. $\mathcal{C}_S \subseteq \mathcal{C}$ that contains S.

$\mathcal{C}[S] \hat{=} \mathcal{C}[\mathcal{C}_S]$

Localize $\mathcal{C}_S$ at all morphisms

pushout:

$$\begin{tikzcd}
\mathcal{C}_S \arrow[r] \arrow[d, "\text{as before}"'] & \mathcal{C} \arrow[d] \\
X \arrow[r] & W \arrow[r, "\simeq"] & \mathcal{C}[S]
\end{tikzcd}$$

inner-anodyne

LLP w.r.t inner-fibrations