SIX LECTURES ON MOTIVES

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Contents

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Introduction

The notes are a hastily prepared revision of the slides I used to give a series of six lectures on motives as part of the ICTP Workshop on K-theory and Motives, May 14-25, 2007. I have also included notes of a second lecture on pure motives, originally intended as the second lecture in the series, which was omitted due to time constraints.

Because these notes are essentially the slides of my lectures, I have not followed the usual style of numbering the various theorems, propositions, definitions, etc., and all the internal references are accomplished by giving names (e.g., the PST theorem) to the key results. Also, I did not use internal citations of the literature, but rather have included a copy of the list of reference works that I thought would be helpful for the workshop participants.

Finally, I would like to express my heartfelt thanks to the organizers, Eric Friedlander, 'Remi Kuku and Claudio Pedrini, for putting together a very useful and interesting workshop, as well as the staff at the ICTP for their help in making the workshop a very enjoyable experience.

Lecture 1. Pure motives

Outline:

- Pre-history of motives
- Adequate equivalence relations
- Weil cohomology
- Grothendieck motives

1. Pre-history of motives

1.1. **Part I: Algebraic cycles.** X: a scheme of finite type over a field k.

Definition. An *algebraic cycle* on X is $Z = \sum_{i=1}^m n_i Z_i$, $n_i \in \mathbb{Z}$, $Z_i \subset X$ integral closed subschemes.

$\mathcal{Z}(X) :=$ the group of algebraic cycles on X.

$\mathcal{Z}(X) = \mathcal{Z}_*(X) := \oplus_{r\geq 0}\mathcal{Z}_r(X)$ graded by dimension.

$\mathcal{Z}(X) = \mathcal{Z}^*(X) := \oplus_{r\geq 0}\mathcal{Z}^r(X)$ graded by codimension (for X equi-dimensional).

1.2. **Functoriality.** $X \mapsto \mathcal{Z}_*(X)$ is a covariant functor for proper maps $f : X \to Y$:

$$f_*(Z) := \begin{cases} 0 & \text{if } \dim_k f(Z) < \dim_k Z \\ [k(Z) : k(f(Z))] \cdot f(Z) & \text{if } \dim_k f(Z) = \dim_k Z. \end{cases}$$

For $p : X \to \operatorname{Spec} k$ projective over k, have $\deg : \mathcal{Z}_0(X) \to \mathbb{Z}$ by

$$\deg(z) := p_*(z) \in \mathcal{Z}_0(\operatorname{Spec} k) = \mathbb{Z} \cdot [\operatorname{Spec} k] \cong \mathbb{Z}.$$

$X \mapsto \mathcal{Z}^*(X)$ is a contravariant functor for flat maps $f : Y \to X$:

$$f^*(Z) := \operatorname{cyc}(f^{-1}(Z)) := \sum_{T \subset f^{-1}(Z)} \ell_{\mathcal{O}_{Y,T}}(\mathcal{O}_{Z,T}) \cdot T;$$

sum over irreducible components T of $f^{-1}(Z)$.

1.3. **Intersection theory.** Take X smooth, $Z, W \subset X$ irreducible.

Z and W *intersect properly* on X: each irreducible component T of $Z \cap W$ has

$$\operatorname{codim}_X T = \operatorname{codim}_X Z + \operatorname{codim}_X W.$$

The *intersection product* is

$$Z \cdot_X W := \sum_T m(T; Z \cdot_X W) \cdot T.$$

$m(T; Z \cdot_X W)$ is *Serre's intersection multiplicity*:

$$m(T; Z \cdot_X W) := \sum_i (-1)^i \ell_{\mathcal{O}_{X,T}}(\operatorname{Tor}_i^{\mathcal{O}_{X,T}}(\mathcal{O}_{Z,T}, \mathcal{O}_{W,T})).$$

Extend to cycles $Z = \sum_i n_i Z_i$, $W = \sum_j m_j W_j$ of pure codimension by linearity.

1.4. **Contravariant functoriality.** Intersection theory extends flat pull-back to a partially defined pull-back for $f : Y \to X$ in $\mathbf{Sm}/k$:

$$f^*(Z) := p_{1*}(\Gamma_f \cdot p_2^*(Z))$$

$\Gamma_f \subset Y \times X$ the graph of f, $p_1 : \Gamma_f \to Y$, $p_2 : Y \times X \to X$ the projections.

And: a partially defined associative, commutative, unital graded ring structure on $\mathcal{Z}^*(X)$ with (when defined)

$$f^*(a \cdot b) = f^*(a) \cdot f^*(b)$$

and (the *projection formula*)

$$f_*(f^*(a) \cdot b) = a \cdot f_*(b)$$

for f projective.

1.5. **Example: the zeta-function.** X: smooth projective over $\mathbb{F}_q$.

$$Z_X(t) := \exp\Big(\sum_{n \geq 1} \frac{\#X(\mathbb{F}_{q^n})}{n} \cdot t^n\Big).$$

Note that

$$\#X(\mathbb{F}_{q^n}) = \deg(\Delta_X \cdot \Gamma_{Fr_X^n})$$

$\Delta_X \subset X \times X$ the diagonal, Fr_X the *Frobenius*

$$Fr_X^*(h) := h^q.$$

1.6. **Part II: cohomology.** Weil: the singular cohomology of varieties over $\mathbb{C}$ should admit a purely algebraic version, suitable also for varieties over $\mathbb{F}_q$.

Grothendieck *et al.*: *étale cohomology* with $\mathbb{Q}_\ell$ coefficients ($\ell \neq$ char(k), $k = \bar{k}$) works.

Example. The Lefschetz trace formula $\Longrightarrow$

$$\deg(\Delta_X \cdot \Gamma_{Fr_X^n}) = \sum_{i=0}^{2d_X} (-1)^i Tr(Fr_X^{n*}|_{H^i(\bar{X},\mathbb{Q}_\ell)})$$

$$Z_X(t) = \frac{\det(1 - tFr_X^*|_{H^-(\bar{X},\mathbb{Q}_\ell)})}{\det(1 - tFr_X^*|_{H^+(\bar{X},\mathbb{Q}_\ell)})}$$

Thus: $Z_X(t)$ is a *rational function* with $\mathbb{Q}$-coefficients.

In fact, by the Weil conjectures, the characteristic polymomial $\det(1-tFr_X^*|_{H^i(\bar{X},\mathbb{Q}_\ell)})$ has $\mathbb{Q}$ (in fact $\mathbb{Z}$) coefficients, *independent of* ℓ.

However: Serre's example of an elliptic curve E over $\mathbb{F}_{p^2}$ with $\mathrm{End}(E)_\mathbb{Q}$ a quaternion algebra shows: there is no "good" cohomology over $\bar{\mathbb{F}}_p$ with $\mathbb{Q}$-coefficients.

Grothendieck suggested: there is a $\mathbb{Q}$-linear category of "motives" over k which has the properties of a universal cohomology theory for smooth projective varieties over k.

This category would explain why the étale cohomology $H^*(-, \mathbb{Q}_\ell)$ for different ℓ all yield the same data.

Grothendieck's idea: make a cohomology theory purely out of algebraic cycles.

2. Adequate equivalence relations

To make cycles into cohomology, we need to make the pull-back and intersection product everywhere defined.

Consider an equivalence relation $\sim$ on $\mathcal{Z}^*$ for smooth projective varieties: for each $X \in \mathbf{SmProj}/k$ a graded quotient $\mathcal{Z}^*(X) \twoheadrightarrow \mathcal{Z}^*_\sim(X)$.

Definition. $\sim$ is an *adequate equivalence relation* if, for all $X, Y \in \mathbf{SmProj}/k$:

1. Given $a, b \in \mathcal{Z}^*(X)$ there is $a' \sim a$ such that a' and b intersect properly on X
2. Given $a \in \mathcal{Z}^*(X)$, $b \in \mathcal{Z}^*(X \times Y)$ such that $p_1^*(a)$ intersects b properly. Then

$$a \sim 0 \Longrightarrow p_{2*}(p_1^*(a) \cdot b) \sim 0.$$

For a field F (usually $\mathbb{Q}$) make the same definition with $\mathcal{Z}^*(X)_F$ replacing $\mathcal{Z}^*(X)$.

2.1. **Functoriality.** (1) and (2) imply:

• The partially defined intersection product on $\mathcal{Z}^*(X)$ descend to a well-defined product on $\mathcal{Z}^*_\sim(X)$.

• Push-forward for projective $f : Y \to X$ descends to $f_* : \mathcal{Z}_\sim(Y) \to \mathcal{Z}_\sim(X)$

• Partially defined pull-back for $f : Y \to X$ descends a well-defined $f^* : \mathcal{Z}^*_\sim(X) \to \mathcal{Z}^*_\sim(Y)$.

Order adequate equivalence relations by $\sim_1 \succ \sim_2$ if $Z \sim_1 0 \Longrightarrow Z \sim_2 0$: $\sim_1$ is *finer than* $\sim_2$.

2.2. **Geometric examples.** Take $Z \in \mathcal{Z}^n(X)$.

1. $Z \sim_{\mathrm{rat}} 0$ if there is a $W \in \mathcal{Z}^*(X \times \mathbb{P}^1)$ with

$$p_{1*}[(X \times 0 - X \times \infty) \cdot W] = Z.$$

2. $Z \sim_{\mathrm{alg}} 0$ if there is a smooth projective curve C with k-points c, c' and $W \in \mathcal{Z}^*(X \times C)$ with

$$p_{1*}[(X \times c - X \times c') \cdot W] = Z.$$

3. $Z \sim_{\mathrm{num}} 0$ if for $W \in \mathcal{Z}^{d_X - n}(X)$ with $W \cdot_X Z$ defined,

$$\deg(W \cdot_X Z) = 0.$$

$$\sim_{\mathrm{rat}} \succ \sim_{\mathrm{alg}} \succ \sim_{\mathrm{num}}$$

Write $\mathrm{CH}^*(X) := \mathcal{Z}^*_{\sim_{\mathrm{rat}}}(X) = \mathcal{Z}^*_{\mathrm{rat}}(X)$: the *Chow ring* of X. Write $\mathcal{Z}_{\mathrm{num}} := \mathcal{Z}_{\sim_{\mathrm{num}}}$, etc.

Remark. $\sim_{\mathrm{rat}}$ is the *finest* adequate equivalence relation $\sim$:

i. $[0] \sim \sum_i n_i[t_i]$ with $t_i \neq 0$ all i by (1).

ii. Let $f(x) = 1 - \prod_i f_{t_i}(x)$,
$f_{t_i}(x) \in k[x/(x-1)]$ minimal polynomial of t_i, normalized by $f_{t_i}(0) = 1$.
Then $f(0) = 0$ $f(t_i) = 1$, so

$$f_*([0] - \sum_i n_i[t_i]) = [0] - (\sum_i n_i')[1] \sim 0,$$

by (2), where $n_i' = [k(t_i) : k]n_i$.

iii. Send $x \mapsto 1/x$, get $[\infty] - (\sum_i n_i')[1] \sim 0$, so $[0] \sim [\infty]$ by (2).

iv. $\sim_{\mathrm{rat}} \succ \sim$ follows from (2).

Remark. $\sim_{\mathrm{num}}$ is the coarsest non-zero adequate equivalence relation $\sim$ (with fixed coefficient field $F \supset \mathbb{Q}$).

If $\sim \neq 0$, then $F = \mathcal{Z}^0(\operatorname{Spec} k)_F \to \mathcal{Z}^0_\sim(\operatorname{Spec} k)_F$ is an isomorphism: if not, $\mathcal{Z}^0_\sim(\operatorname{Spec} k)_F = 0$ so

$$[X]_\sim = p_X^*([\operatorname{Spec} k]_\sim) = 0$$

for all $X \in \mathbf{SmProj}/k$. But $? \cdot [X]_\sim$ acts as id on $\mathcal{Z}_\sim(X)_F$.

If $Z \sim 0$, $Z \in \mathrm{CH}^r(X)_F$ and W is in $\mathrm{CH}^{d_X - r}(X)$ then $Z \cdot W \sim 0$ so

$$0 = p_{X*}(Z \cdot W) \in \mathcal{Z}^0_\sim(\operatorname{Spec} k)_F = \mathcal{Z}^0_{\mathrm{num}}(\operatorname{Spec} k)_F$$

i.e. $Z \sim_{\mathrm{num}} 0$.

3. Weil cohomology

$\mathbf{SmProj}/k :=$ smooth projective varieties over k.

$\mathbf{SmProj}/k$ is a symmetric monoidal category with product $= \times_k$ and symmetry the exchange of factors $t : X \times_k Y \to Y \times_k X$.

$\mathrm{Gr}^{\geq 0}\mathrm{Vec}_K$ is the tensor category of graded finite dimensional K vector spaces $V = \oplus_{r \geq 0} V^r$.

$\mathrm{Gr}^{\geq 0}\mathrm{Vec}_K$ has tensor $\otimes_K$ and symmetry

$$\tau(v \otimes w) := (-1)^{\deg v \deg w} w \otimes v$$

for homogeneous elements v, w.

Definition. A *Weil cohomology theory* over k is a symmetric monoidal functor

$$H^* : \mathbf{SmProj}/k^{\mathrm{op}} \to \mathrm{Gr}^{\geq 0}\mathrm{Vec}_K,$$

K is a field of characteristic 0, satisfying some axioms.

Note: H^* monoidal means: $H^*(X \times Y) = H^*(X) \otimes H^*(Y)$. Using

$$\delta_X^* : H^*(X \times X) \to H^*(X)$$

makes H^* a functor to graded-commutative K-algebras.

3.1. **The axioms.** 1. $\dim_K H^2(\mathbb{P}^1) = 1$. Write $V(r)$ for $V \otimes_F H^2(\mathbb{P}^1)^{\otimes -r}$, $r \in \mathbb{Z}$.

2. If X has dimension d_X, then there is an isomorphism

$$Tr_X : H^{2d_X}(X)(d_X) \to K$$

such that $Tr_{X\times Y} = Tr_X \otimes Tr_Y$ and the pairing

$$H^i(X) \otimes H^{2d_X - i}(X)(d_X) \xrightarrow{\cup_X} H^{2d_X}(X)(d_X) \xrightarrow{Tr_X} K$$

is a perfect duality.

3. There is for $X \in \mathbf{SmProj}/k$ a *cycle class homomorphism*

$$\gamma_X^r : \mathrm{CH}^r(X) \to H^{2r}(X)(r)$$

compatible with f^*, $\cdot_X$ and with $Tr_X \circ \gamma_X^{d_X} = \deg$.

Remarks. By (2), $H^i(X) = 0$ for $i > 2d_X$. Also, $H^0(\operatorname{Spec} k) = K$ with $1 = \gamma([\operatorname{Spec} k])$. $\gamma_X([X])$ is the unit in $H^*(X)$.

Using Poincaré duality (2), we have $f_* : H^*(X)(d_X) \to H^{*+2c}(Y)(d_Y)$ for $f : X \to Y$ projective, $c = 2d_Y - 2d_X$ defined as the dual of f^*. $Tr_X = p_{X*}$

By (3), the cycle class maps γ_X are natural with respect to f_*.

3.2. **Correspondences.** For $a \in \mathrm{CH}^{\dim X + r}(X \times Y)$ define:

$$a_* : H^*(X) \to H^{*+2r}(Y)(r)$$
$$a_*(x) := p_{2*}(p_{1*}(x) \cup \gamma(a))).$$

Example. $a = {}^t\Gamma_f$ for $f : Y \to X$ $(r = 0)$. $a_* = f^*$.

$a = \Gamma_g$ for $g : X \to Y$ $(r = \dim Y - \dim X)$. $a_* = f_*$.

3.3. **Composition law.** Given $a \in \mathrm{CH}^{\dim X + r}(X \times Y)$, $b \in \mathrm{CH}^{\dim Y + s}(Y \times Z)$ set

$$b \circ a := p_{13*}(p_{12}^*(a) \cdot p_{23}^*(b)) \in \mathrm{CH}^{\dim X + r + s}(X \times Z).$$

Then

$$(b \circ a)_* = b_* \circ a_*.$$

Lemma. $H^1(\mathbb{P}^1) = 0$.

Proof. Set $i := i_0 : \operatorname{Spec} k \to \mathbb{P}^1$, $p : \mathbb{P}^1 \to \operatorname{Spec} k$.

$\Gamma_{\mathrm{id}_{\mathbb{P}^1}} = \Delta_{\mathbb{P}^1} \sim_{\mathrm{rat}} 0 \times \mathbb{P}^1 + \mathbb{P}^1 \times 0 \Longrightarrow$

$$\begin{aligned}\mathrm{id}_{H^1(\mathbb{P}^1)} &= \Delta_{\mathbb{P}^1 *} \\ &= (0 \times \mathbb{P}^1)_* + (\mathbb{P}^1 \times 0)_* \\ &= p^* i^* + i_* p_*.\end{aligned}$$

But $H^n(\operatorname{Spec} k) = 0$ for $n \neq 0$, so

$$i^* : H^1(\mathbb{P}^1) \to H^1(\operatorname{Spec} k);\ p_* : H^1(\mathbb{P}^1) \to H^{-1}(\operatorname{Spec} k)(-1)$$

are zero.

A Weil cohomology H yields an adequate equivalence relation: $\sim_H$ by

$$Z \sim_H 0 \Longleftrightarrow \gamma(Z) = 0$$

Note: $\sim_{\mathrm{rat}} \succ \sim_H \succ \sim_{\mathrm{num}}$.

Lemma. $\sim_{\mathrm{alg}} \succ \sim_H$.

Take $x, y \in C(k)$. $p : C \to \operatorname{Spec} k$. Then $p_* = Tr_C : H^2(C)(1) \to H^0(\operatorname{Spec} k) = K$ is an isomorphism and

$$Tr_C(\gamma_C(x-y)) = \gamma_{\operatorname{Spec} k}(p_*(x-y)) = 0$$

so $\gamma_C(x-y) = 0$. Promote to $\sim_{\rm alg}$ by naturality of γ.

Conjecture: $\sim_H$ is independent of the choice of Weil cohomology H.

We write $\sim_H$ as $\sim_{\rm hom}$.

3.4. **Lefschetz trace formula.** $V = \oplus_r V_r$: a graded K-vector space with dual $V^\vee = \oplus_r V^\vee_{-r}$ and duality pairing

$$<,>_V : V \otimes V^\vee \to K.$$

Identify $(V^\vee)^\vee = V$ by $<v^\vee, v>_{V^\vee} := (-1)^{\deg v}<v, v^\vee>$.

$\operatorname{Hom}_{\rm GrVec}(V, V) \cong \oplus_r V^\vee_{-r} \otimes V_r$ and for $f = v^\vee \otimes v : V \to V$ the graded trace is

$$Tr_V f = <v^\vee, v> = (-1)^{\deg v} v^\vee(v).$$

The graded trace is $(-1)^r$ times the usual trace on V_r.

If $W = \oplus_s W_s$ is another graded K vector space, identify $(V^\vee \otimes W)^\vee = V \otimes W^\vee$ by the pairing

$$<v^\vee \otimes w, v \otimes w^\vee> := (-1)^{\deg w \deg v}<v^\vee, v><w, w^\vee>$$

Given

$$\phi \in \operatorname{Hom}_{\rm GrVec}(V, W) \subset V^\vee \otimes W$$
$$\psi \in \operatorname{Hom}_{\rm GrVec}(W, V) \subset W^\vee \otimes V$$

get $\phi \circ \psi : W \to W$.

Let $c : W^\vee \otimes V \to V \otimes W^\vee$ be the exchange isomorphism, giving

$$c(\psi) \in V \otimes W^\vee = (V^\vee \otimes W)^\vee.$$

Checking on decomposable tensors gives the LTF:

$$Tr_W(\phi \circ \psi) = <\phi, c(\psi)>_{V^\vee \otimes W}.$$

Apply the LTF to $V = W = H^*(X)$. We have

$$V^\vee = H^*(X)(d_X)$$
$$\oplus_r V_r \otimes V^\vee_{-r} = \oplus_r H^r(X) \otimes H^{2d_X - r}(X)(d_X) = H^{2d_X}(X \times X)(d_X)$$
$$<,>_V = Tr_X \circ \delta^*_X : H^{2d_X}(X \times X)(d_X) \to K$$

Theorem (Lefschetz trace formula)**.** *Let $a, b \in \mathcal{Z}^{d_X}(X \times X)$ be correspondences. Then*

$$\deg(a \cdot {}^t b) = \sum_{i=0}^{2d_X} (-1)^i Tr(a_* \circ b_*)_{|H^i(X)}.$$

Just apply the LTF to $\phi = a_* = H^*(a)$, $\psi = b_* = H^*(b)$ and note: H^* intertwines ^{t} and c and $\deg(a \cdot {}^t b) = <H^*(a), H^*({}^t b)>_{H^*(X)}$.

Taking $b = \Delta_X$ gives the Lefschetz fixed point formula.

3.5. **Classical Weil cohomology.** 1. Betti cohomology ($K=\mathbb{Q}$): $\sigma : k \to \mathbb{C} \rightsquigarrow H^*_{\mathfrak{B},\sigma}$

$$H^*_{\mathfrak{B},\sigma}(X) := H^*(X_\sigma(\mathbb{C}), \mathbb{Q})$$

2. De Rham cohomology ($K = k$, for $\operatorname{char} k = 0$):

$$H^*_{dR}(X) := \mathbb{H}^*_{\mathrm{Zar}}(X, \Omega^*_{X/k})$$

3. Étale cohomology ($K = \mathbb{Q}_\ell$, $\ell \neq \operatorname{char} k$):

$$H^*_{\text{ét}}(X)_\ell := H^*_{\text{ét}}(X \times_k k^{sep}, \mathbb{Q}_\ell)$$

In particular: for each k, there exists a Weil cohomology theory on $\mathbf{SmProj}/k$.

3.6. **An application.**

Proposition. *Let F be a field of characteristic zero. $X \in \mathbf{SmProj}/k$. Then the intersection pairing*

$$\cdot_X : \mathcal{Z}^r_{\mathrm{num}}(X)_F \otimes_F \mathcal{Z}^{d_X - r}_{\mathrm{num}}(X)_F \to F$$

is a perfect pairing for all r.

Proof. May assume $F =$ the coefficient field of a Weil cohomology H^* for k.

$$H^{2r}(X)(r) \hookleftarrow \mathcal{Z}^r_{\mathrm{hom}}(X)_F \twoheadrightarrow \mathcal{Z}^r_{\mathrm{num}}(X)_F$$

so $\dim_F \mathcal{Z}^r_{\mathrm{num}}(X)_F < \infty$.

By definition of $\sim_{\mathrm{num}}$, $\cdot_X$ is non-degenerate; since the factors are finite dimensional, $\cdot_X$ is perfect.

3.7. **Matsusaka's theorem (weak form).**

Proposition. $\mathcal{Z}^1_{\mathrm{alg}}(X)_\mathbb{Q} = \mathcal{Z}^1_H(X)_\mathbb{Q} = \mathcal{Z}^1_{\mathrm{num}}(X)_\mathbb{Q}$.

Proof. Matsusaka's theorem is $\mathcal{Z}^1_{\mathrm{alg}\mathbb{Q}} = \mathcal{Z}^1_{\mathrm{num}\mathbb{Q}}$.

But $\sim_{\mathrm{alg}} \succ \sim_H \succ \sim_{\mathrm{num}}$.

4. Grothendieck motives

How to construct the category of motives for an adequate equivalence relation $\sim$.

4.1. **Pseudo-abelian categories.** An additive category $\mathcal{C}$ is *abelian* if every morphism $f : A \to B$ has a (categorical) kernel and cokernel, and the canonical map $\operatorname{coker}(\ker f) \to \ker(\operatorname{coker} f)$ is always an isomorphism.

An additive category $\mathcal{C}$ is *pseudo-abelian* if every idempotent endomorphism $p : A \to A$ has a kernel:

$$A \cong \ker p \oplus \ker 1 - p.$$

4.2. **The pseudo-abelian hull.** For an additive category $\mathcal{C}$, there is a universal additive functor to a pseudo-abelian category $\psi : \mathcal{C} \to \mathcal{C}^\natural$.

$\mathcal{C}^\natural$ has objects (A, p) with $p : A \to A$ an idempotent endomorphism,

$$\mathrm{Hom}_{\mathcal{C}^\natural}((A,p),(B,q)) = q\mathrm{Hom}_{\mathcal{C}}(A,B)p.$$

and $\psi(A) := (A, \mathrm{id})$, $\psi(f) = f$.

If $\mathcal{C}, \otimes$ is a tensor category, so is $\mathcal{C}^\natural$ with

$$(A,p) \otimes (B,q) := (A \otimes B, p \otimes q).$$

4.3. **Correspondences again.** The category $\mathrm{Cor}_\sim(k)$ has the same objects as $\mathbf{SmProj}/k$. Morphisms (for X irreducible) are

$$\mathrm{Hom}_{\mathrm{Cor}_\sim}(X,Y) := \mathcal{Z}^{d_X}_\sim(X\times Y)_\mathbb{Q}$$

with composition the composition of correspondences.

In general, take the direct sum over the components of X.

Write X (as an object of $\mathrm{Cor}_\sim(k)$) $= h_\sim(X)$ or just $h(X)$. For $f : Y \to X$, set $h(f) := {}^t\Gamma_f$. This gives a functor

$$h_\sim : \mathbf{SmProj}/k^{\mathrm{op}} \to \mathrm{Cor}_\sim(k).$$

1. $\mathrm{Cor}_\sim(k)$ is an additive category with $h(X)\oplus h(Y) = h(X \amalg Y)$.

2. $\mathrm{Cor}_\sim(k)$ is a tensor category with $h(X)\otimes h(Y) = h(X\times Y)$. For $a \in \mathcal{Z}^{d_X}_\sim(X\times Y)_\mathbb{Q}$, $b\in \mathcal{Z}^{d_{X'}}_\sim(X'\times Y')_\mathbb{Q}$

$$a\otimes b := t^*(a\times b)$$

with $t : (X\times X')\times(Y\times Y') \to (X\times Y)\times(X'\times Y')$ the exchange.

$h_\sim$ is a symmetric monoidal functor.

4.4. **Effective pure motives.**

Definition. $M^{\mathrm{eff}}_\sim(k) := \mathrm{Cor}_\sim(k)^\natural$. For a field $F \supset \mathbb{Q}$, set

$$M^{\mathrm{eff}}_\sim(k)_F := [\mathrm{Cor}(k)_F]^\natural$$

Explicitly, $M^{\mathrm{eff}}_\sim(k)$ has objects (X,α) with $X\in \mathbf{SmProj}/k$ and $\alpha \in \mathcal{Z}^{d_X}_\sim(X\times X)_\mathbb{Q}$ with $\alpha^2=\alpha$ (as correspondence mod $\sim$).

$M^{\mathrm{eff}}_\sim(k)$ is a tensor category with unit $\mathbb{1} = (\mathrm{Spec}\,k, [\mathrm{Spec}\,k])$.

Set $\mathfrak{h}_\sim(X) := (X,\Delta_X)$, for $f : Y\to X$, $\mathfrak{h}_\sim(f) := {}^t\Gamma_f$.

This gives the symmetric monoidal functor

$$\mathfrak{h}_\sim : \mathbf{SmProj}(k)^{\mathrm{op}} \to M^{\mathrm{eff}}_\sim(k).$$

4.5. **Universal property.**

Theorem. *Let H be a Weil cohomology on $\mathbf{SmProj}/k$. Then the functor H^* : $\mathbf{SmProj}/k^{\mathrm{op}} \to \mathrm{Gr}^{\geq 0}\mathrm{Vec}_K$ extends to a tensor functor $H^* : M^{\mathrm{eff}}_{\mathrm{hom}}(k) \to \mathrm{Gr}^{\geq 0}\mathrm{Vec}_K$ making*

$$\begin{tikzcd}
\mathbf{SmProj}/k^{\mathrm{op}} \arrow[d, "\mathfrak{h}"'] \arrow[dr, "H^*"] & \\
M^{\mathrm{eff}}_{\mathrm{hom}}(k) \arrow[r, "H^*"'] & \mathrm{Gr}^{\geq 0}\mathrm{Vec}_K.
\end{tikzcd}$$

commute.

Proof. Extend H^* to $\mathrm{Cor}_{\mathrm{hom}}(k)$ by $H^*(a) = a_*$ for each correspondence a. Since $\mathrm{Gr}^{\geq 0}\mathrm{Vec}_K$ is pseudo-abelian, H^* extends to $M^{\mathrm{eff}}_{\mathrm{hom}}(k) = \mathrm{Cor}_{\mathrm{hom}}(k)^\natural$.

Examples. 1. $\Delta_{\mathbb{P}^1} \sim \mathbb{P}^1\otimes 0 + 0\otimes \mathbb{P}^1$ gives

$$\mathfrak{h}(P^1) = (\mathbb{P}^1, \mathbb{P}^1\otimes 0)\oplus(\mathbb{P}^1, 0\times\mathbb{P}^1).$$

The maps $i_0 : \mathrm{Spec}\,k \to \mathbb{P}^1$, $p : \mathbb{P}^1 \to \mathrm{Spec}\,k$, give

$$p^* : \mathfrak{h}(\mathrm{Spec}\,k) \to \mathfrak{h}(\mathbb{P}^1)$$
$$i_0^* : \mathfrak{h}(\mathbb{P}^1) \to \mathfrak{h}(\mathrm{Spec}\,k)$$

and define an isomorphism

$$\mathbb{1} \cong (\mathbb{P}^1, 0 \times \mathbb{P}^1).$$

The remaining factor $(\mathbb{P}^1, \mathbb{P}^1 \otimes 0)$ is the *Lefschetz motive* $\mathbb{L}$.

2. $\Delta_{\mathbb{P}^n} \sim \sum_{i=0}^n \mathbb{P}^i \times \mathbb{P}^{n-i}$. The $\mathbb{P}^i \times \mathbb{P}^{n-i}$ are orthogonal idempotents so

$$\mathfrak{h}(\mathbb{P}^n) = \oplus_{i=0}^n (\mathbb{P}^n, \mathbb{P}^i \times \mathbb{P}^{n-i}).$$

In fact $(\mathbb{P}^n, \mathbb{P}^i \times \mathbb{P}^{n-i}) \cong \mathbb{L}^{\otimes i}$ so

$$\mathfrak{h}(\mathbb{P}^n) \cong \oplus_{i=0}^n \mathbb{L}^i.$$

3. Let C be a smooth projective curve with a k-point 0. $0 \times C$ and $C \times 0$ are orthogonal idempotents in $\mathrm{Cor}(C, C)$. Let $\alpha := \Delta_C - 0 \times C - C \times 0$ so

$$\mathfrak{h}(C) = (C, 0 \times C) + (C, \alpha) + (C, C \times 0) \cong \mathbb{1} \oplus (C, \alpha) \oplus \mathbb{L}$$

Each decomposition of $\mathfrak{h}(X)$ in $M_{\mathrm{hom}}^{\mathrm{eff}}(k)$ gives a corresponding decomposition of $H^*(X)$ by using the action of correspondences on H^*.

1. The decomposition $\mathfrak{h}(\mathbb{P}^1) = \mathbb{1} \oplus \mathbb{L}$ decomposes $H^*(\mathbb{P}^1)$ as $H^0(\mathbb{P}^1) \oplus H^2(\mathbb{P}^1)$, with $\mathbb{1} \leftrightarrow H^0(\mathbb{P}^1) = K$ and $\mathbb{L} \leftrightarrow H^2(\mathbb{P}^1) = K(-1)$. Set

$$\mathfrak{h}_\sim^0(\mathbb{P}^1) := (\mathbb{P}^1, 0 \times \mathbb{P}^1), \mathfrak{h}_\sim^2(\mathbb{P}^2) := (\mathbb{P}^1, \mathbb{P}^1 \times 0)$$

so $\mathfrak{h}_\sim(\mathbb{P}^1) = \mathfrak{h}_\sim^0(\mathbb{P}^1) \oplus \mathfrak{h}_\sim^2(\mathbb{P}^1)$ and

$$H^*(\mathfrak{h}_{\mathrm{hom}}^i(\mathbb{P}^1)) = H^i(\mathbb{P}^1)$$

2. The factor $(\mathbb{P}^n, \mathbb{P}^{n-i} \times \mathbb{P}^i)$ of $[\mathbb{P}^n]$ acts by

$$(\mathbb{P}^i \times \mathbb{P}^{n-i})_* : H^*(\mathbb{P}^n) \to H^*(\mathbb{P}^n)$$

which is projection onto the summand $H^{2i}(\mathbb{P}^n)$. Since $(\mathbb{P}^n, \mathbb{P}^i \times \mathbb{P}^{n-i}) \cong \mathbb{L}^{\otimes i}$ this gives

$$H^{2i}(\mathbb{P}^n) \cong K(-i) \cong H^2(\mathbb{P}^1)^{\otimes i}.$$

Setting $\mathfrak{h}_\sim^{2i}(\mathbb{P}^n) := (\mathbb{P}^n, \mathbb{P}^i \times \mathbb{P}^{n-i})$ gives

$$\mathfrak{h}_\sim(\mathbb{P}^n) = \oplus_{i=0}^n \mathfrak{h}_\sim^{2i}(\mathbb{P}^n),$$

with $H^*(\mathfrak{h}_{\mathrm{hom}}^r(\mathbb{P}^n)) = H^r(\mathbb{P}^n)$.

3. The decomposition $\mathfrak{h}_\sim(C) = \mathbb{1} \oplus (C, \alpha) \oplus \mathbb{L}$ gives

$$H^*(C) = H^0(C) \oplus H^1(C) \oplus H^2(C) = K \oplus H^1(C) \oplus K(-1).$$

Thus we write $\mathfrak{h}^1(C) := (C, \alpha)$, $\mathfrak{h}_\sim^0(C) := (C, 0 \times C)$, $\mathfrak{h}_\sim^2(C) := (C, C \times 0)$ and

$$\mathfrak{h}_\sim(C) \cong \mathfrak{h}_\sim^0(C) \oplus \mathfrak{h}_\sim^1(C) \oplus \mathfrak{h}_\sim^2(C).$$

with $H^*(\mathfrak{h}_{\mathrm{hom}}^r(C)) = H^r(C)$.

Remark. $\mathfrak{h}_\sim^1(C) \neq 0$ iff $g(C) \geq 1$. It suffices to take $\sim =$ num. Since $\dim C \times C = 2$, it suffices to show $\mathfrak{h}_{\mathrm{hom}}^1(C) \neq 0$ for some classical Weil cohomology. But then $H^1(C) \cong K^{2g}$.

The decompositions in (1) and (2) are canonical. In (3), this depends (for e.g $\sim =$ rat, but not for $\sim =$ hom, num) on the choice of $0 \in C(k)$ (or degree 1 cycle $0 \in \mathrm{CH}_0(C)_{\mathbb{Q}}$).

4.6. **Grothendieck motives.**

Definition. 1. $\mathrm{Cor}^*_\sim(k)$ has objects $h(X)(r)$, $r \in \mathbb{Z}$ with

$$\mathrm{Hom}_{\mathrm{Cor}^*_\sim(k)}(h(X)(r), h(Y)(s)) := \mathcal{Z}^{d_X+s-r}_\sim(X \times Y)$$

with composition as correspondences.

2. $M_\sim(k) := \mathrm{Cor}^*_\sim(k)^\natural$. For a field $F \supset \mathbb{Q}$, set

$$M_\sim(k)_F := [\mathrm{Cor}^*(k)_F]^\natural$$

Sending X to $\mathfrak{h}(X) := h(X)(0)$, $f : Y \to X$ to ${}^t\Gamma_f$ defines the functor

$$\mathfrak{h}_\sim : \mathbf{SmProj}/k^{\mathrm{op}} \to M_\sim(k).$$

Examples. 1. $0 \in \mathcal{Z}^1(\mathbb{P}^1)$ gives a map $i_0 : \mathbb{1}(-1) \to \mathfrak{h}(\mathbb{P}^1)$, identifying

$$\mathbb{1}(-1) \cong \mathbb{L}.$$

2. $\mathbb{1}(-r) \cong \mathbb{L}^{\otimes r}$, so $\mathfrak{h}(\mathbb{P}^n) \cong \oplus_{r=0}^n \mathbb{1}(-r)$ and $\mathfrak{h}^{2r}(\mathbb{P}^n) = \mathbb{1}(-r)$
3. For C a curve, $\mathfrak{h}^0(C) = \mathbb{1}$, $\mathfrak{h}^2(C) = \mathbb{1}(-1)$.
4. The objects $\mathfrak{h}(X)(r)$ are *not* in $M^{\mathrm{eff}}_\sim(k)$ for $r > 0$.
For $r < 0$ $\mathfrak{h}(X)(r) \cong \mathfrak{h}(X) \otimes \mathbb{L}^{\otimes r}$.

4.7. **Inverting $\mathbb{L}$.** Sending $(X, \alpha) \in M^{\mathrm{eff}}_\sim(k)$ to $(X, 0, \alpha) \in M_\sim(k)$ defines a full embeding

$$i : M^{\mathrm{eff}}_\sim(k) \hookrightarrow M_\sim(k).$$

Since $i(\mathbb{L}) \cong \mathbb{1}(-1)$, the functor $\otimes\mathbb{L}$ on $M^{\mathrm{eff}}_\sim(k)$ has inverse $\otimes\mathbb{1}(1)$ on $M_\sim(k)$.
$(X, r, \alpha) = (X, 0, \alpha) \otimes \mathbb{1}(r) \cong i(X, \alpha) \otimes \mathbb{L}^{\otimes -r}$.
Thus $M_\sim(k) \cong M^{\mathrm{eff}}_\sim(k)[(-\otimes \mathbb{L})^{-1}]$.

4.8. **Universal property.** Let GrVec_K be the tensor category of finite dimensional graded K vector spaces.

Theorem. *Let H be a Weil cohomology on* $\mathbf{SmProj}/k$*. Then the functor* H^* : $\mathbf{SmProj}/k^{\mathrm{op}} \to \mathrm{Gr}^{\geq 0}\mathrm{Vec}_K$ *extends to a tensor functor* $H^* : M_{\mathrm{hom}}(k) \to \mathrm{GrVec}_K$ *making*

$$\begin{tikzcd}
\mathbf{SmProj}/k^{\mathrm{op}} \arrow[d, "\mathfrak{h}"'] \arrow[dr, "H^*"] & \\
M_{\mathrm{hom}}(k) \arrow[r, "H^*"'] & \mathrm{GrVec}_K.
\end{tikzcd}$$

commute.

Proof. Extend H^* to $H^* : \mathrm{Cor}^*_{\mathrm{hom}}(k) \to$ by

$$H^n(X, r) := H^n(X)(r),\ H^*(a) = a_*$$

for each correspondence a. Since GrVec_K is pseudo-abelian, H^* extends to $M_{\mathrm{hom}}(k) = \mathrm{Cor}^*_{\mathrm{hom}}(k)^\natural$.

4.9. **Duality.** Why extend $M^{\text{eff}}(k)$ to $M(k)$? In $M(k)$, each object has a dual:

$$(X, r, \alpha)^\vee := (X, d_X - r, {}^t\alpha)$$

The diagonal Δ_X yields

$$\delta_X : \mathbb{1} \to \mathfrak{h}(X \times X)(d_X) = \mathfrak{h}(X)(r) \otimes \mathfrak{h}(X)(r)^\vee$$
$$\epsilon_X : \mathfrak{h}(X)(r)^\vee \otimes \mathfrak{h}(X)(r) = \mathfrak{h}(X \times X)(d_X) \to \mathbb{1}$$

with composition

$$\mathfrak{h}(X)(r) = \mathbb{1} \otimes \mathfrak{h}(X) \xrightarrow{\delta\otimes\mathrm{id}} \mathfrak{h}(X)(r) \otimes \mathfrak{h}(X)(r)^\vee \otimes \mathfrak{h}(X)(r)$$
$$\xrightarrow{\mathrm{id}\otimes\epsilon} \mathfrak{h}(X)(r) \otimes \mathbb{1} = \mathfrak{h}(X)$$

the identity.

This yields a natural isomorphism

$$\mathrm{Hom}(A \otimes \mathfrak{h}(X)(r), B) \cong \mathrm{Hom}(A, B \otimes \mathfrak{h}(X)(r)^\vee)$$

by sending $f : A \otimes \mathfrak{h}(X)(r) \to B$ to

$$A = A \otimes \mathbb{1} \xrightarrow{\delta} A \otimes \mathfrak{h}(X)(r) \otimes \mathfrak{h}(X)(r)^\vee \xrightarrow{f\otimes\mathrm{id}} B \otimes \mathfrak{h}(X)(r)^\vee$$

The inverse is similar, using ϵ.

This extends to objects (X, r, α) by projecting. $A \to (A^\vee)^\vee = A$ is the identity.

Theorem. *$M_\sim(k)$ is a* rigid *tensor category. For $\sim=$* hom*, the functor H^* is compatible with duals.*

4.10. **Chow motives and numerical motives.** If $\sim\succ\approx$, the surjection $\mathcal{Z}_\sim \to \mathcal{Z}_\approx$ yields functors $\mathrm{Cor}_\sim(k) \to \mathrm{Cor}_\approx(k)$, $\mathrm{Cor}^*_\sim(k) \to \mathrm{Cor}^*_\approx(k)$ and thus

$$M^{\text{eff}}_\sim(k) \to M^{\text{eff}}_\approx(k);\ M_\sim(k) \to M_\approx(k).$$

Thus the category of pure motives with the most information is for the finest equivalence relation $\sim=$ rat. Write

$$CHM(k)_F := M_{\text{rat}}(k)_F$$

For example $\mathrm{Hom}_{CHM(k)}(\mathbb{1}, \mathfrak{h}(X)(r)) = \mathrm{CH}^r(X)$.

The coarsest equivalence is $\sim_{\text{num}}$, so $M_{\text{num}}(k)$ should be the most simple category of motives.

Set $NM(k) := M_{\text{num}}(k)$, $NM(k)_F := M_{\text{num}}(k)_F$.

4.11. **Jannsen's semi-simplicity theorem.**

Theorem (Jannsen)**.** *Fix F a field, char $F = 0$. $NM(k)_F$ is a semi-simple abelian category. If $M_\sim(k)_F$ is semi-simple abelian, then $\sim=\sim_{\text{num}}$.*

Proof. $d := d_X$. We show $\mathrm{End}_{NM(k)_F}(\mathfrak{h}(X)) = \mathcal{Z}^d_{\text{num}}(X^2)_F$ is a finite dimensional semi-simple F-algebra for all $X \in$ **SmProj**$/k$. We may extend F, so can assume $F = K$ is the coefficient field for a Weil cohomology on **SmProj**$/k$.

Consider the surjection $\pi : \mathcal{Z}^d_{\text{hom}}(X^2)_F \to \mathcal{Z}^d_{\text{num}}(X^2)_F$. $\mathcal{Z}^d_{\text{hom}}(X^2)_F$ is finite dimensional, so $\mathcal{Z}^d_{\text{num}}(X^2)_F$ is finite dimensional.

Also, the radical $\mathcal{N}$ of $\mathcal{Z}^d_{\text{hom}}(X^2)_F$ is nilpotent and it suffices to show that $\pi(\mathcal{N}) = 0$.

Take $f \in \mathcal{N}$. Then $f \circ {}^t g$ is in $\mathcal{N}$ for all $g \in \mathcal{Z}^d_{\mathrm{hom}}(X^2)_F$, and thus $f \circ {}^t g$ is nilpotent. Therefore

$$Tr(H^+(f \circ {}^t g)) = Tr(H^-(f \circ {}^t g)) = 0.$$

By the LTF

$$\deg(f \cdot g) = Tr(H^+(f \circ {}^t g)) - Tr(H^-(f \circ {}^t g)) = 0$$

hence $f \sim_{\mathrm{num}} 0$, i.e., $\pi(f) = 0$.

4.12. **Chow motives.** $CHM(k)_F$ has a nice universal property extending the one we have already described:

Theorem. *Giving a Weil cohomology theory H^* on* **SmProj**$/k$ *with coefficient field $K \supset F$ is equivalent to giving a tensor functor*

$$H^* : CHM(k)_F \to \mathrm{GrVec}_K$$

with $H^i(\mathbb{1}(-1)) = 0$ for $i \neq 2$.

"Weil cohomology" $\rightsquigarrow H^*$ because $\sim_{\mathrm{rat}} \succ \sim_H$.

$H^* \rightsquigarrow$ Weil cohomology: $\mathbb{1}(-1)$ is invertible and $H^i(\mathbb{1}(-1)) = 0$ for $i \neq 2 \Longrightarrow H^2(\mathbb{P}^1) \cong K$.

$\mathfrak{h}(X)^\vee = \mathfrak{h}(X)(d_X) \rightsquigarrow H^*(\mathfrak{h}(X))$ is supported in degrees $[0, 2d_X]$

Rigidity of $CHM(k)_F \rightsquigarrow$ Poincaré duality.

4.13. **Adequate equivalence relations revisited.**

Definition. Let $\mathcal{C}$ be an additive category. The *Kelly radical* $\mathcal{R}$ is the collection

$$\mathcal{R}(X, Y) := \{f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \mid \forall g \in \mathrm{Hom}_{\mathcal{C}}(Y, X), 1 - gf \text{ is invertible}\}$$

$\mathcal{R}$ forms an *ideal* in $\mathcal{C}$ (subgroups of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ closed under $\circ g$, $g \circ$).

Lemma. *$\mathcal{C} \to \mathcal{C}/\mathcal{R}$ is conservative, and $\mathcal{R}$ is the largest such ideal.*

Remark. If $\mathcal{I} \subset \mathcal{C}$ is an ideal such that $\mathcal{I}(X, X)$ is a nil-ideal in $\mathrm{End}(X)$ for all X, then $\mathcal{I} \subset \mathcal{R}$.

Definition. $(\mathcal{C}, \otimes)$ a tensor category. A ideal $\mathcal{I}$ in $\mathcal{C}$ is a $\otimes$ *ideal* if $f \in \mathcal{I}$, $g \in \mathcal{C} \Rightarrow f \otimes g \in \mathcal{I}$.

$\mathcal{C} \to \mathcal{C}/\mathcal{I}$ is a tensor functor iff $\mathcal{I}$ is a tensor ideal. $\mathcal{R}$ is *not* in general a $\otimes$ ideal.

Theorem. *There is a 1-1 correspondence between adequate equivalence relations on* **SmProj**$/k$ *and proper $\otimes$ ideals in $CHM(k)_F$: $M_\sim(k)_F := (CHM(k)_F/\mathcal{I}_\sim)^\natural$.*

In particular: Let $\mathcal{N} \subset CHM(k)_{\mathbb{Q}}$ be the tensor ideal defined by numerical equivalence. Then $\mathcal{N}$ is the largest proper $\otimes$ ideal in $CHM(k)_{\mathbb{Q}}$.

Lecture 2. Mixed motives: conjectures and constructions

Outline:

- Mixed motives
- Triangulated categories
- Geometric motives

1. Mixed motives

1.1. **Why mixed motives?** Pure motives describe the cohomology of smooth projective varieties

Mixed motives should describe the cohomology of *arbitrary* varieties.

Weil cohomology is replaced by *Bloch-Ogus* cohomology: Mayer-Vietoris for open covers and a purity isomorphism for cohomology with supports.

1.2. **An analog: Hodge structures.** The cohomology H^n of a smooth projective variety over $\mathbb{C}$ has a natural pure Hodge structure.

Deligne gave the cohomology of an arbitrary variety over $\mathbb{C}$ a natural *mixed* Hodge structure.

The category of (polarizable) pure Hodge structures is a semi-simple abelian rigid tensor category. The category of (polarizable) mixed Hodge structures is an abelian rigid tensor category, but has non-trivial extensions. The semi-simple objects in MHS are the pure Hodge structures.

MHS has a natural exact finite *weight filtration* on W_*M on each object M, with graded pieces $\mathrm{gr}_n^W M$ pure Hodge structures.

There is a functor

$$RHdg : \mathbf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to D^b(MHS)$$

with $R^nHdg(X) = H^n(X)$ with its MHS, lifting the singular cochain complex functor

$$C^*(-,\mathbb{Z}) : \mathbf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to D^b(\mathbf{Ab}).$$

In addition, all natural maps involving the cohomology of X: pull-back, proper pushforward, boundary maps in local cohomology or Mayer-Vietoris sequences, are maps of MHS.

1.3. **Beilinson's conjectures.** Beilinson conjectured that the semi-simple abelian category of pure motives $M_{\mathrm{hom}}(k)_{\mathbb{Q}}$ should admit a full embedding as the semi-simple objects in an abelian rigid tensor category of *mixed motives* $MM(k)$.

$MM(k)$ should have the following structures and properties:

• a natural finite exact weight filtration W_*M on each M with graded pieces $\mathrm{gr}_n^W M$ pure motives.

• For $\sigma : k \to \mathbb{C}$ a *realization functor* $\Re_\sigma : MM(k) \to MHS$ compatible with all the structures.

• A functor $R\mathfrak{h} : \mathbf{Sch}_k^{\mathrm{op}} \to D^b(MM(k))$ such that $\Re_\sigma(R^n\mathfrak{h}(X))$ is $H^n(X)$ as a MHS.

• A natural isomorphism ($\mathbb{Q}(n)[2n] \cong \mathbb{1}(n)$)

$$\mathrm{Hom}_{D^b(MM(k))}(\mathbb{Q}, R\mathfrak{h}(X)(q)[p]) \cong K_{2q-p}^{(q)}(X),$$

in particular $\mathrm{Ext}^p_{MM(k)}(\mathbb{Q},\mathbb{Q}(q)) \cong K_{2q-p}^{(q)}(k)$.

• All "universal properties" of the cohomology of algebraic varieties should be reflected by identities in $D^b(MM(k))$ of the objects $R\mathfrak{h}(X)$.

1.4. **Motivic sheaves.** In fact, Beilinson views the above picture as only the story over the generic point $\mathrm{Spec}\, k$.

He conjectured further that there should be a system of categories of "motivic sheaves"

$$S \mapsto MM(S)$$

together with functors Rf_*, f^*, $f^!$ and $Rf_!$, as well as $\mathcal{H}om$ and $\otimes$, all satisfying the yoga of Grothendieck's six operations for the categories of sheaves for the étale topology.

1.5. **A partial success.** The categories $MM(k)$, $MM(S)$ have not been constructed.

However, there are now a number of (equivalent) constructions of *triangulated tensor categories* that satisfy all the structural properties expected of the derived categories $D^b(MM(k))$ and $D^b(MM(S))$, except those which exhibit these as a derived category of an abelian category (t-structure).

There are at present various attempts to extend this to the triangulated version of Beilinson's vision of motivic sheaves over a base S.

We give a discussion of the construction of various versions of triangulated categories of mixed motives over k due to Voevodsky.

2. Triangulated categories

2.1. **Translations and triangles.** A *translation* on an additive category $\mathcal{A}$ is an equivalence $T : \mathcal{A} \to \mathcal{A}$. We write $X[1] := T(X)$.

Let $\mathcal{A}$ be an additive category with translation. A *triangle* (X, Y, Z, a, b, c) in $\mathcal{A}$ is the sequence of maps

$$X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} X[1].$$

A morphism of triangles

$$(f, g, h) : (X, Y, Z, a, b, c) \to (X', Y', Z', a', b', c')$$

is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[r, "b"] \arrow[d, "g"'] & Z \arrow[r, "c"] \arrow[d, "h"'] & X[1] \arrow[d, "{f[1]}"'] \\ X' \arrow[r, "a'"'] & Y' \arrow[r, "b'"'] & Z' \arrow[r, "c'"'] & X'[1]. \end{tikzcd}$$

Verdier has defined a *triangulated category* as an additive category $\mathcal{A}$ with translation, together with a collection $\mathcal{E}$ of triangles, called the *distinguished triangles* of $\mathcal{A}$, which satisfy:

TR1

$\mathcal{E}$ is closed under isomorphism of triangles.

$A \xrightarrow{\mathrm{id}} A \to 0 \to A[1]$ is distinguished.

Each $X \xrightarrow{u} Y$ extends to a distinguished triangle

$$X \xrightarrow{u} Y \to Z \to X[1]$$

TR2

$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished

$\Leftrightarrow Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is distinguished

TR3
Given a commutative diagram with distinguished rows

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & & & \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & X'[1] \end{array}$$

there exists a morphism $h : Z \to Z'$ such that (f, g, h) is a morphism of triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & X'[1] \end{array}$$

TR4
If we have three distinguished triangles $(X, Y, Z', u, i, *)$, $(Y, Z, X', v, *, j)$, and $(X, Z, Y', w, *, *)$, with $w = v \circ u$, then there are morphisms $f : Z' \to Y'$, $g : Y' \to X'$ such that

- (id_X, v, f) is a morphism of triangles
- (u, id_Z, g) is a morphism of triangles
- $(Z', Y', X', f, g, i[1] \circ j)$ is a distinguished triangle.

A graded functor $F : \mathcal{A} \to \mathcal{B}$ of triangulated categories is called *exact* if F takes distinguished triangles in $\mathcal{A}$ to distinguished triangles in $\mathcal{B}$.

Remark. Suppose $(\mathcal{A}, T, \mathcal{E})$ satisfies (TR1), (TR2) and (TR3). If (X, Y, Z, a, b, c) is in $\mathcal{E}$, and A is an object of $\mathcal{A}$, then the sequences

$$\ldots \xrightarrow{c[-1]_*} \mathrm{Hom}_{\mathcal{A}}(A, X) \xrightarrow{a_*} \mathrm{Hom}_{\mathcal{A}}(A, Y) \xrightarrow{b_*} \mathrm{Hom}_{\mathcal{A}}(A, Z) \xrightarrow{c_*} \mathrm{Hom}_{\mathcal{A}}(A, X[1]) \xrightarrow{a[1]_*} \ldots$$

and

$$\ldots \xrightarrow{a[1]^*} \mathrm{Hom}_{\mathcal{A}}(X[1], A) \xrightarrow{c^*} \mathrm{Hom}_{\mathcal{A}}(Z, A) \xrightarrow{b^*} \mathrm{Hom}_{\mathcal{A}}(Y, A) \xrightarrow{a^*} \mathrm{Hom}_{\mathcal{A}}(X, A) \xrightarrow{c[-1]^*} \ldots$$

are exact.

This yields:

- (five-lemma): If (f, g, h) is a morphism of triangles in $\mathcal{E}$, and if two of f, g, h are isomorphisms, then so is the third.
- If (X, Y, Z, a, b, c) and (X, Y, Z', a, b', c') are two triangles in $\mathcal{E}$, there is an isomorphism $h : Z \to Z'$ such that

$$(\mathrm{id}_X, \mathrm{id}_Y, h) : (X, Y, Z, a, b, c) \to (X, Y, Z', a, b', c')$$

is an isomorphism of triangles.

If (TR4) holds as well, then $\mathcal{E}$ is closed under taking finite direct sums.

2.2. **The main point.** A triangulated category is a machine for generating natural long exact sequences.

2.3. **An example.** Let $\mathcal{A}$ be an additive category, $C^?(\mathcal{A})$ the category of cohomological complexes (with boundedness condition $? = \emptyset, +, -, b$), and $K^?(\mathcal{A})$ the homotopy category.

For a complex (A, d_A), let $A[1]$ be the complex

$$A[1]^n := A^{n+1}; \quad d^n_{A[1]} := -d^{n+1}_A.$$

For a map of complexes $f : A \to B$, we have the *cone sequence*

$$A \xrightarrow{f} B \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} A[1]$$

where $\mathrm{Cone}(f) := A^{n+1} \oplus B^n$ with differential

$$d(a, b) := (-d_A(a), f(a) + d_B(b))$$

i and p are the evident inclusions and projections.

We make $K^?(\mathcal{A})$ a triangulated category by declaring a triangle to be exact if it is isomorphic to the image of a cone sequence.

2.4. **Tensor structure.**

Definition. Suppose $\mathcal{A}$ is both a triangulated category and a tensor category (with tensor operation $\otimes$) such that $(X \otimes Y)[1] = X[1] \otimes Y$.

Suppose that, for each distinguished triangle (X, Y, Z, a, b, c), and each $W \in \mathcal{A}$, the sequence

$$X \otimes W \xrightarrow{a \otimes \mathrm{id}_W} Y \otimes W \xrightarrow{b \otimes \mathrm{id}_W} Z \otimes W \xrightarrow{c \otimes \mathrm{id}_W} X[1] \otimes W = (X \otimes W)[1]$$

is a distinguished triangle. Then $\mathcal{A}$ is a *triangulated tensor category.*

Example. If $\mathcal{A}$ is a tensor category, then $K^?(\mathcal{A})$ inherits a tensor structure, by the usual tensor product of complexes, and becomes a triangulated tensor category. (For $? = \emptyset$, $\mathcal{A}$ must admit infinite direct sums).

2.5. **Thick subcategories.**

Definition. A full triangulated subcategory $\mathcal{B}$ of a triangulated category $\mathcal{A}$ is *thick* if $\mathcal{B}$ is closed under taking direct summands.

If $\mathcal{B}$ is a thick subcategory of $\mathcal{A}$, the set of morphisms $s : X \to Y$ in $\mathcal{A}$ which fit into a distinguished triangle $X \xrightarrow{s} Y \to Z \to X[1]$ with Z in $\mathcal{B}$ forms a *saturated multiplicative system* of morphisms.

The intersection of thick subcategories of $\mathcal{A}$ is a thick subcategory of $\mathcal{A}$, So, for each set $\mathcal{T}$ of objects of $\mathcal{A}$, there is a smallest thick subcategory $\mathcal{B}$ containing $\mathcal{T}$, called the thick subcategory *generated* by $\mathcal{T}$.

Remark. The original definition (Verdier) of a thick subcategory had the condition:

Let $X \xrightarrow{f} Y \to Z \to X[1]$ be a distinguished triangle in $\mathcal{A}$, with Z in $\mathcal{B}$. If f factors as $X \xrightarrow{f_1} B' \xrightarrow{f_2} Y$ with B' in $\mathcal{B}$, then X and Y are in $\mathcal{B}$.

This is equivalent to the condition given above, that $\mathcal{B}$ is closed under direct summands in $\mathcal{A}$ (*cf.* Rickard).

2.6. **Localization of triangulated categories.** Let $\mathcal{B}$ be a thick subcategory of a triangulated category $\mathcal{A}$. Let $\mathcal{S}$ be the saturated multiplicative system of map $A \xrightarrow{s} B$ with "cone" in $\mathcal{B}$.

Form the category $\mathcal{A}[\mathcal{S}^{-1}] = \mathcal{A}/\mathcal{B}$ with the same objects as $\mathcal{A}$, with

$$\mathrm{Hom}_{\mathcal{A}[\mathcal{S}^{-1}]}(X, Y) = \varinjlim_{s:X'\to X\in\mathcal{S}} \mathrm{Hom}_{\mathcal{A}}(X', Y).$$

Composition of diagrams

$$\begin{tikzcd}
 & Y' \arrow[r, "g"] \arrow[d, "t"] & Z \\
X' \arrow[r, "f"'] \arrow[d, "s"'] & Y & \\
X & &
\end{tikzcd}$$

is defined by filling in the middle

$$\begin{tikzcd}
X'' \arrow[r, "f'"] \arrow[d, "s'"'] & Y' \arrow[r, "g"] \arrow[d, "t"] & Z \\
X' \arrow[r, "f"'] \arrow[d, "s"'] & Y & \\
X. & &
\end{tikzcd}$$

One can describe $\mathrm{Hom}_{\mathcal{A}[\mathcal{S}^{-1}]}(X, Y)$ by a calculus of left fractions as well, i.e.,

$$\mathrm{Hom}_{\mathcal{A}[\mathcal{S}^{-1}]}(X, Y) = \varinjlim_{s:Y\to Y'\in\mathcal{S}} \mathrm{Hom}_{\mathcal{A}}(X, Y').$$

Let $Q_{\mathcal{B}} : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ be the canonical functor.

Theorem (Verdier)**.** *(i) $\mathcal{A}/\mathcal{B}$ is a triangulated category, where a triangle T in $\mathcal{A}/\mathcal{B}$ is distinguished if T is isomorphic to the image under $Q_{\mathcal{B}}$ of a distinguished triangle in $\mathcal{A}$.*

(ii) The functor $Q_{\mathcal{B}}$ is universal for exact functors $F : \mathcal{A} \to \mathcal{C}$ such that $F(B)$ is isomorphic to 0 for all B in $\mathcal{B}$.

(iii) $\mathcal{S}$ is equal to the collection of maps in $\mathcal{A}$ which become isomorphisms in $\mathcal{A}/\mathcal{B}$ and $\mathcal{B}$ is the subcategory of objects of $\mathcal{A}$ which becomes isomorphic to zero in $\mathcal{A}/\mathcal{B}$.

Remark. If $\mathcal{A}$ admits some infinite direct sums, it is sometimes better to preserve this property. A subscategory $\mathcal{B}$ of $\mathcal{A}$ is called *localizing* if $\mathcal{B}$ is thick and is closed under direct sums which exist in $\mathcal{A}$.

For instance, if $\mathcal{A}$ admits arbitrary direct sums and $\mathcal{B}$ is a localizing subcategory, then $\mathcal{A}/\mathcal{B}$ also admits arbitrary direct sums.

Localization with respect to localizing subcategories has been studied by Thomason and by Ne'eman.

2.7. **Localization of triangulated tensor categories.** If $\mathcal{A}$ is a triangulated tensor category, and $\mathcal{B}$ a thick subcategory, call $\mathcal{B}$ a *thick tensor subcategory* if A in $\mathcal{A}$ and B in $\mathcal{B}$ implies that $A \otimes B$ and $B \otimes A$ are in $\mathcal{B}$.

The quotient $Q_{\mathcal{B}} : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ of $\mathcal{A}$ by a thick tensor subcategory inherits the tensor structure, and the distinguished triangles are preserved by tensor product with an object.

Example. The classical example is the *derived category* $D^?(\mathcal{A})$ of an abelian category $\mathcal{A}$. $\mathcal{D}^?(\mathcal{A})$ is the localization of $K^?(\mathcal{A})$ with respect to the multiplicative system of *quasi-isomorphisms* $f : A \to B$, i.e., f which induce isomorphisms $H^n(f) : H^n(A) \to H^n(B)$ for all n.

If $\mathcal{A}$ is an abelian tensor category, then $D^-(\mathcal{A})$ inherits a tensor structure $\otimes^L$ if each object A of $\mathcal{A}$ admits a surjection $P \to A$ where P is *flat*, i.e. $M \mapsto M \otimes P$ is an exact functor on $\mathcal{A}$. If each A admits a finite flat (right) resolution, then $D^b(\mathcal{A})$ has a tensor structure $\otimes^L$ as well. The tensor structure $\otimes^L$ is given by forming for each $A \in K^?(\mathcal{A})$ a quasi-isomorphism $P \to A$ with P a complex of flat objects in $\mathcal{A}$, and defining

$$A \otimes^L B := \mathrm{Tot}(P \otimes B).$$

3. Geometric motives

Voevodsky constructs a number of categories: the category of geometric motives $DM_{\mathrm{gm}}(k)$ with its effective subcategory $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$, as well as a sheaf-theoretic construction DM_-^{eff}, containing $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ as a full dense subcategory. In contrast to many earlier constructions, these are based on *homology* rather than cohomology as the starting point, in particular, the motives functor from $\mathbf{Sm}/k$ to these categories is covariant.

3.1. **Finite correspondences.** To solve the problem of the partially defined composition of correspondences, Voevodsky introduces the notion of *finite* correspondences, for which all compositions are defined.

Definition. Let X and Y be in $\mathbf{Sch}_k$. The group $c(X, Y)$ is the subgroup of $z(X \times_k Y)$ generated by integral closed subschemes $W \subset X \times_k Y$ such that

(1) the projection $p_1 : W \to X$ is finite
(2) the image $p_1(W) \subset X$ is an irreducible component of X.

The elements of $c(X, Y)$ are called the *finite* correspondences from X to Y.

The following basic lemma is easy to prove:

Lemma. *Let X, Y and Z be in $\mathbf{Sch}_k$, $W \in c(X, Y)$, $W' \in c(Y, Z)$. Suppose that X and Y are irreducible. Then each irreducible component C of $|W| \times Z \cap X \times |W'|$ is finite over X and $p_X(C) = X$.*

Thus: for $W \in c(X, Y)$, $W' \in c(Y, Z)$, we have the composition:

$$W' \circ W := p_{XZ*}(p_{XY}^*(W) \cdot p_{YZ}^*(W')),$$

This operation yields an associative bilinear *composition law*

$$\circ : c(Y, Z) \times c(X, Y) \to c(X, Z).$$

3.2. **The category of finite correspondences.**

Definition. The category $\mathrm{Cor}_{\mathrm{fin}}(k)$ is the category with the same objects as $\mathbf{Sm}/k$, with

$$\mathrm{Hom}_{\mathrm{Cor}_{\mathrm{fin}}(k)}(X,Y) := c(X,Y),$$

and with the composition as defined above.

Remarks. (1) We have the functor $\mathbf{Sm}/k \to \mathrm{Cor}_{\mathrm{fin}}(k)$ sending a morphism $f : X \to Y$ in $\mathbf{Sm}/k$ to the graph $\Gamma_f \subset X \times_k Y$.

(2) We write the morphism corresponding to Γ_f as f_*, and the object corresonding to $X \in \mathbf{Sm}/k$ as $[X]$.

(3) The operation $\times_k$ (on smooth k-schemes and on cycles) makes $\mathrm{Cor}_{\mathrm{fin}}(k)$ a tensor category. Thus, the bounded homotopy category $K^b(\mathrm{Cor}_{\mathrm{fin}}(k))$ is a triangulated tensor category.

3.3. **The category of effective geometric motives.**

Definition. The category $\widehat{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ is the localization of $K^b(\mathrm{Cor}_{\mathrm{fin}}(k))$, as a triangulated tensor category, by

- *Homotopy.* For $X \in \mathbf{Sm}/k$, invert $p_* : [X \times \mathbb{A}^1] \to [X]$
- *Mayer-Vietoris.* Let X be in $\mathbf{Sm}/k$. Write X as a union of Zariski open subschemes U, V: $X = U \cup V$.

We have the canonical map

$$\mathrm{Cone}([U \cap V] \xrightarrow{(j_{U,U\cap V*}, -j_{V,U\cap V*})} [U] \oplus [V]) \xrightarrow{(j_{U*}+j_{V*})} [X]$$

since $(j_{U*} + j_{V*}) \circ (j_{U,U\cap V*}, -j_{V,U\cap V*}) = 0$. Invert this map.

The category $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ of *effective geometric motives* is the pseudo-abelian hull of $\widehat{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ (Balmer-Schlichting).

3.4. **The motive of a smooth variety.** Let $M_{\mathrm{gm}}(X)$ be the image of $[X]$ in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$. Sending $f : X \to Y$ to $M_{\mathrm{gm}}(f) := [\Gamma_f] = f_*$ defines

$$M^{\mathrm{eff}}_{\mathrm{gm}} : \mathbf{Sm}/k \to DM^{\mathrm{eff}}_{\mathrm{gm}}(k).$$

Remark. $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ is modeled on *homology*, so $M^{\mathrm{eff}}_{\mathrm{gm}}$ is a covariant functor. In fact, $M^{\mathrm{eff}}_{\mathrm{gm}}$ is a symmetric monoidal functor.

3.5. **The category of geometric motives.** To define the category of geometric motives we invert the Lefschetz motive.

For $X \in \mathbf{Sm}_k$, the *reduced motive* (in $C^b(\mathrm{Cor}_{\mathrm{fin}}(k))$) is

$$\widetilde{[X]} := \mathrm{Cone}(p_* : [X] \to [\mathrm{Spec}\, k])[-1].$$

If X has a k-point $0 \in X(k)$, then $p_* i_{0*} = \mathrm{id}_{[\mathrm{Spec}\, k]}$ so

$$[X] = \widetilde{[X]} \oplus [\mathrm{Spec}\, k]$$

and

$$\widetilde{[X]} \cong \mathrm{Cone}(i_{0*} : [\mathrm{Spec}\, k] \to [X])$$

in $K^b(\mathrm{Cor}_{\mathrm{fin}}(k))$.

Write $\widetilde{M^{\mathrm{eff}}_{\mathrm{gm}}(X)}$ for the image of $\widetilde{[X]}$ in $M^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

Set $\mathbb{Z}(1) := \widetilde{M^{\mathrm{eff}}_{\mathrm{gm}}(\mathbb{P}^1)}[-2]$, and set $\mathbb{Z}(n) := \mathbb{Z}(1)^{\otimes n}$ for $n \geq 0$.

Thus $M^{\text{eff}}_{\text{gm}}(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$, $\mathbb{Z} := M_{\text{gm}}(\operatorname{Spec} k) = \mathbb{Z}(0)$. So $\mathbb{Z}(1)[2]$ is like the Lefschetz motive.

Definition. The category of *geometric motives*, $DM_{\text{gm}}(k)$, is defined by inverting the functor $\otimes\mathbb{Z}(1)$ on $DM^{\text{eff}}_{\text{gm}}(k)$, i.e., for $r, s \in \mathbb{Z}$, $M, N \in DM^{\text{eff}}_{\text{gm}}(k)$,

$$\operatorname{Hom}_{DM_{\text{gm}}(k)}(M(r), N(s)) := \varinjlim_n \operatorname{Hom}_{DM^{\text{eff}}_{\text{gm}}(k)}(M \otimes \mathbb{Z}(n+r), N \otimes \mathbb{Z}(n+s)).$$

Remark. In order that $DM_{\text{gm}}(k)$ be again a triangulated category, it suffices that the commutativity involution $\mathbb{Z}(1) \otimes \mathbb{Z}(1) \to \mathbb{Z}(1) \otimes \mathbb{Z}(1)$ be the identity, which is in fact the case.

Of course, there arises the question of the behavior of the evident functor $DM^{\text{eff}}_{\text{gm}}(k) \to DM_{\text{gm}}(k)$. Here we have

Theorem (Cancellation)**.** *The functor $i : DM^{\text{eff}}_{\text{gm}}(k) \to DM_{\text{gm}}(k)$ is a fully faithful embedding.*

Let $M_{\text{gm}} : \mathbf{Sm}/k \to DM_{\text{gm}}(k)$ be $i \circ M^{\text{eff}}_{\text{gm}}$. $M_{\text{gm}}(X)$ is the *motive of* X.

Remark. We will see later that, just as for $M_\sim(k)$, $DM_{\text{gm}}(k)$ is a *rigid* tensor (triangulated) category: we invert $\mathbb{Z}(1)$ so that every object has a dual.

4. Elementary constructions in $DM^{\text{eff}}_{\text{gm}}(k)$

4.1. Motivic cohomology.

Definition. For $X \in \mathbf{Sm}/k$, $q \in \mathbb{Z}$, set

$$H^p(X, \mathbb{Z}(q)) := \operatorname{Hom}_{DM_{\text{gm}}(k)}(M_{\text{gm}}(X), \mathbb{Z}(q)[p]).$$

Compare with $\mathrm{CH}^r(X) = \operatorname{Hom}_{CHM(k)}(\mathbb{1}(-r), \mathfrak{h}(X))$. In fact, for all $X \in \mathbf{Sm}/k$, there is a natural isomorphism

$$\begin{aligned}\mathrm{CH}^r(X) &= \operatorname{Hom}_{CHM(k)}(\mathbb{1}(-r), \mathfrak{h}(X)) \\ &= \operatorname{Hom}_{DM_{\text{gm}}(k)}(M_{\text{gm}}(X), \mathbb{Z}(r)[2r]) = H^{2r}(X, \mathbb{Z}(r)).\end{aligned}$$

In particular, sending $\mathfrak{h}(X)$ to $M_{\text{gm}}(X)$ for $X \in \mathbf{SmProj}/k$ gives a full embedding

$$CHM(k)^{\text{op}} \hookrightarrow DM_{\text{gm}}(k)$$

4.2. Products.

Define the cup product

$$H^p(X, \mathbb{Z}(q)) \otimes H^{p'}(X, \mathbb{Z}(q')) \to H^{p+p'}(X, \mathbb{Z}(q+q'))$$

by sending $a \otimes b$ to

$$M_{\text{gm}}(X) \xrightarrow{\delta} M_{\text{gm}}(X) \otimes M_{\text{gm}}(X) \xrightarrow{a \otimes b} \mathbb{Z}(q)[p] \otimes \mathbb{Z}(q')[p'] \cong \mathbb{Z}(q+q')[p+p'].$$

This makes $\oplus_{p,q} H^p(X, \mathbb{Z}(q))$ a graded commutative ring with unit 1 the map $M_{\text{gm}}(X) \to \mathbb{Z}$ induced by $p_X : X \to \operatorname{Spec} k$.

4.3. Homotopy property.

Applying $\operatorname{Hom}_{DM_{\text{gm}}}(-, \mathbb{Z}(q)[p])$ to the isomorphism $p_* : M_{\text{gm}}(X \times \mathbb{A}^1) \to M_{\text{gm}}(X)$ gives the *homotopy property* for $H^*(-, \mathbb{Z}(*))$:

$$p^* : H^p(X, \mathbb{Z}(q)) \xrightarrow{\sim} H^p(X \times \mathbb{A}^1, \mathbb{Z}(q)).$$

4.4. **Mayer-Vietoris.** For $U, V \subset X$ open subschemes we can apply $\mathrm{Hom}_{DM_{\mathrm{gm}}}(-, \mathbb{Z}(q)[p])$ to the distinguished triangle

$$M_{\mathrm{gm}}(U \cap V) \to M_{\mathrm{gm}}(U) \oplus M_{\mathrm{gm}}(V) \to M_{\mathrm{gm}}(U \cup V) \to M_{\mathrm{gm}}(U \cap V)[1].$$

This gives the *Mayer-Vietoris exact sequence* for $H^*(-, \mathbb{Z}(*))$:

$$\ldots \to H^{p-1}(U \cup V, \mathbb{Z}(q)) \to H^p(U \cap V, \mathbb{Z}(q)) \\ \to H^p(U, \mathbb{Z}(q)) \oplus H^p(V, \mathbb{Z}(q)) \to H^p(U \cup V, \mathbb{Z}(q)) \to \ldots$$

4.5. **Chern classes of line bundles.** For each n, let Γ_n be the cycle on $\mathbb{P}^n \times \mathbb{P}^1$ defined by the bi-homogeneous equation

$$\sum_{i=0}^{n} X_i T_1^{n-i} T_0^i$$

where $X_0, \ldots, X_n$ are homogeneous coordinates on $\mathbb{P}^n$ and T_0, T_1 are homogeneous coordinates on $\mathbb{P}^1$.

Let $\gamma_n := \Gamma_n - n \cdot \mathbb{P}^n \times \infty$.

Then γ_n is finite over $\mathbb{P}^n$, so defines an element

$$\gamma_n \in \mathrm{Hom}_{\mathrm{Cor}_{\mathrm{fin}}(k)}(\mathbb{P}^n, \mathbb{P}^1).$$

In fact $p_* \circ \gamma_n = 0$, $p : \mathbb{P}^1 \to \mathrm{Spec}\, k$ the projection, so we have

$$[\gamma_n] \in \mathrm{Hom}_{\mathrm{Cor}_{\mathrm{fin}}(k)}(\mathbb{P}^n, \mathbb{Z}(1)[2]).$$

Definition. Let $L \to X$ be a line bundle on $X \in \mathbf{Sm}_k$. Suppose that L is generated by $n+1$ global sections $f_0, \ldots, f_n$; let $f := (f_0 : \ldots : f_n) : X \to \mathbb{P}^n$ be the resulting morphism.

We let

$$c_1(L) \in H^2(X, \mathbb{Z}(1)) := \mathrm{Hom}_{DM_{\mathrm{gm}}^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(X), \mathbb{Z}(1)[2]$$

be the element $[\gamma_n] \circ f_*$

Proposition. *For $X \in \mathbf{Sm}/k$, sending L to $c_1(L)$ extends to a well-defined homomorphism*

$$c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}(1)).$$

c_1 is natural: $c_1(L) \circ g_* = c_1(g^*L)$ for $g : Y \to X$ in $\mathbf{Sm}/k$

There are of course a number of things to check, mainly that $c_1(L)$ is independent of the choice of generating sections $f_0, \ldots, f_n$, and that $c_1(L \otimes M) = c_1(L) + c_1(M)$ when L and M are globally generated. These follow by explicit $\mathbb{A}^1$-homotopies relating the representing cycles.

4.6. **Weighted spheres.** Before we compute the motive of $\mathbb{P}^n$, we need:

Lemma. *There is a canonical isomorphism*
$M_{\mathrm{gm}}(\mathbb{A}^n \setminus 0) \to \mathbb{Z}(n)[2n-1] \oplus \mathbb{Z}$.

Proof. For $n = 1$, we have $M_{\mathrm{gm}}(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$, by definition of $\mathbb{Z}(1)$. The Mayer-Vietoris distinguished triangle

$$M_{\mathrm{gm}}(\mathbb{A}^1 \setminus 0) \to M_{\mathrm{gm}}(\mathbb{A}^1) \oplus M_{\mathrm{gm}}(\mathbb{A}^1) \to M_{\mathrm{gm}}(\mathbb{P}^1) \to M_{\mathrm{gm}}(\mathbb{A}^1 \setminus 0)[1]$$

defines an isomorphism $t : M_{\mathrm{gm}}(\mathbb{A}^1 \setminus 0) \to \mathbb{Z}(1)[1] \oplus \mathbb{Z}$.

For general n, write $\mathbb{A}^n \setminus 0 = \mathbb{A}^n \setminus \mathbb{A}^{n-1} \cup \mathbb{A}^n \setminus \mathbb{A}^1$. By induction, Mayer-Vietoris and homotopy invariance, this gives the distinguished triangle

$$(\mathbb{Z}(1)[1] \oplus \mathbb{Z}) \otimes (\mathbb{Z}(n-1)[2n-3] \oplus \mathbb{Z})$$

$$\to (\mathbb{Z}(1)[1] \oplus \mathbb{Z}) \oplus (\mathbb{Z}(n-1)[2n-3] \oplus \mathbb{Z})$$

$$\to M_{\mathrm{gm}}(\mathbb{A}^n \setminus 0)$$

$$\to (\mathbb{Z}(1)[1] \oplus \mathbb{Z}) \otimes (\mathbb{Z}(n-1)[2n-3] \oplus \mathbb{Z})[1]$$

yielding the result.

4.7. **Projective bundle formula.** Let $E \to X$ be a rank $n+1$ vector bundle over $X \in \mathbf{Sm}/k$, $q : \mathbb{P}(E) \to X$ the resulting $\mathbb{P}^{n-1}$ bundle, $\mathcal{O}(1)$ the tautological quotient bundle.

Define $\alpha_j : M_{\mathrm{gm}}(\mathbb{P}(E)) \to M_{\mathrm{gm}}(X)(j)[2j]$ by

$$M_{\mathrm{gm}}(\mathbb{P}(E)) \xrightarrow{\delta} M_{\mathrm{gm}}(\mathbb{P}(E)) \otimes M_{\mathrm{gm}}(\mathbb{P}(E)) \xrightarrow{q \otimes c_1(\mathcal{O}(1))^j} M_{\mathrm{gm}}(X)(j)[2j]$$

Theorem. $\oplus_{j=0}^n \alpha_j : M_{\mathrm{gm}}(\mathbb{P}(E)) \to \oplus_{j=0}^n M_{\mathrm{gm}}(X)(j)[2j]$ *is an isomorphism.*

Proof. The map is natural in X. Mayer-Vietoris reduces to the case of a trivial bundle, then to the case $X = \operatorname{Spec} k$, so we need to prove:

Lemma. $\oplus_{j=0}^n \alpha_j : M_{\mathrm{gm}}(\mathbb{P}^n) \to \oplus_{j=0}^n \mathbb{Z}(j)[2j]$ *is an isomorphism.*

Proof. Write $\mathbb{P}^n = \mathbb{A}^n \cup (\mathbb{P}^n \setminus 0)$. $M_{\mathrm{gm}}(\mathbb{A}^n) = \mathbb{Z}$. $\mathbb{P}^n \setminus 0$ is an $\mathbb{A}^1$ bundle over $\mathbb{P}^{n-1}$, so induction gives

$$M_{\mathrm{gm}}(\mathbb{P}^n \setminus 0) = \oplus_{j=0}^{n-1} \mathbb{Z}(j)[2j].$$

Also $M_{\mathrm{gm}}(\mathbb{A}^n \setminus 0) = \mathbb{Z}(n)[2n-1] \oplus \mathbb{Z}$.

The Mayer-Vietoris distinguished triangle

$$M_{\mathrm{gm}}(\mathbb{A}^n \setminus 0) \to M_{\mathrm{gm}}(\mathbb{A}^n) \oplus M_{\mathrm{gm}}(\mathbb{P}^n \setminus 0) \to M_{\mathrm{gm}}(\mathbb{P}^n) \to M_{\mathrm{gm}}(\mathbb{A}^n \setminus 0)[1]$$

gives the result.

4.8. **The end of the road?** It is difficult to go much further using only the techniques of geometry and homological algebra.

One would like to have:

- A Gysin isomorphism
- A computation of the morphisms in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ as algebraic cycles.

Voevodsky achieves this by viewing $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ as a subcategory of a derived category of “Nisnevich sheaves with transfer”.

Lecture 3. Motivic sheaves

Outline:

- Sites and sheaves
- Categories of motivic complexes
- The Suslin complex
- The main results: the localization and embedding theorems

1. Sites and sheaves

We give a quick review of the theory of sheaves on a Grothendieck site.

1.1. **Presheaves.**

A *presheaf* P on a small category $\mathcal{C}$ with values in a category $\mathcal{A}$ is a functor

$$P : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}.$$

Morphisms of presheaves are natural transformations of functors. This defines the category of $\mathcal{A}$-valued presheaves on $\mathcal{C}$, $PreShv^{\mathcal{A}}(\mathcal{C})$.

Remark. We require $\mathcal{C}$ to be small so that the collection of natural transformations $\vartheta : F \to G$, for presheaves F, G, form a set. It would suffice that $\mathcal{C}$ be essentially small (the collection of isomorphism classes of objects form a set).

1.2. **Structural results.**

Theorem. *(1) If $\mathcal{A}$ is an abelian category, then so is $PreShv^{\mathcal{A}}(\mathcal{C})$, with kernel and cokernel defined objectwise: For $f : F \to G$,*

$$\begin{aligned} ker(f)(x) &= \ker(f(x) : F(x) \to G(x)); \\ \mathrm{coker}(f)(x) &= \mathrm{coker}(f(x) : F(x) \to G(x)). \end{aligned}$$

(2) For $\mathcal{A} = \mathbf{Ab}$, $PreShv^{\mathbf{Ab}}(\mathcal{C})$ has enough injectives.

The second part is proved by using a result of Grothendieck, noting that $PreShv^{\mathbf{Ab}}(\mathcal{C})$ has the set of generators $\{\mathbb{Z}_X \mid X \in \mathcal{C}\}$, where $\mathbb{Z}_X(Y)$ is the free abelian group on $\mathrm{Hom}_{\mathcal{C}}(Y, X)$.

1.3. **Pre-topologies.**

Definition. Let $\mathcal{C}$ be a category. A *Grothendieck pre-topology* τ on $\mathcal{C}$ is given by defining, for $X \in \mathcal{C}$, a collection $\mathrm{Cov}_\tau(X)$ of *covering families* of X: a covering family of X is a set of morphisms $\{f_\alpha : U_\alpha \to X\}$ in $\mathcal{C}$. These satisfy:

A1. $\{\mathrm{id}_X\}$ is in $\mathrm{Cov}_\tau(X)$ for each $X \in \mathcal{C}$.

A2. For $\{f_\alpha : U_\alpha \to X\} \in \mathrm{Cov}_\tau(X)$ and $g : Y \to X$ a morphism in $\mathcal{C}$, the fiber products $U_\alpha \times_X Y$ all exist and $\{p_2 : U_\alpha \times_X Y \to Y\}$ is in $\mathrm{Cov}_\tau(Y)$.

A3. If $\{f_\alpha : U_\alpha \to X\}$ is in $\mathrm{Cov}_\tau(X)$ and if $\{g_{\alpha\beta} : V_{\alpha\beta} \to U_\alpha\}$ is in $\mathrm{Cov}_\tau(U_\alpha)$ for each α, then $\{f_\alpha \circ g_{\alpha\beta} : V_{\alpha\beta} \to X\}$ is in $\mathrm{Cov}_\tau(X)$.

A category with a (pre) topology is a *site*

1.4. **Sheaves on a site.**

For S presheaf of abelian groups on $\mathcal{C}$ and $\{f_\alpha : U_\alpha \to X\} \in \mathrm{Cov}_\tau(X)$ for some $X \in \mathcal{C}$, we have the "restriction" morphisms

$$\begin{aligned} f_\alpha^* &: S(X) \to S(U_\alpha) \\ p_{1,\alpha,\beta}^* &: S(U_\alpha) \to S(U_\alpha \times_X U_\beta) \\ p_{2,\alpha,\beta}^* &: S(U_\beta) \to S(U_\alpha \times_X U_\beta). \end{aligned}$$

Taking products, we have the sequence of abelian groups

$$0 \to S(X) \xrightarrow{\prod f_\alpha^*} \prod_\alpha S(U_\alpha) \xrightarrow{\prod p_{1,\alpha,\beta}^* - \prod p_{2,\alpha,\beta}^*} \prod_{\alpha,\beta} S(U_\alpha \times_X U_\beta). \tag{1}$$

Definition. A presheaf S is a *sheaf* for τ if for each covering family $\{f_\alpha : U_\alpha \to X\} \in \mathrm{Cov}_\tau$, the sequence (1) is exact. The category $Shv_\tau^{\mathbf{Ab}}(\mathcal{C})$ of sheaves of abelian groups on $\mathcal{C}$ for τ is the full subcategory of $PreShv^{\mathbf{Ab}}(\mathcal{C})$ with objects the sheaves.

Proposition. *(1) The inclusion* $i : Shv_\tau^{\mathbf{Ab}}(\mathcal{C}) \to PreShv_\tau^{\mathbf{Ab}}(\mathcal{C})$ *admits a left adjoint: "sheafification".*

(2) $Shv_\tau^{\mathbf{Ab}}(\mathcal{C})$ *is an abelian category: For* $f : F \to G$, $\ker(f)$ *is the presheaf kernel.* $\operatorname{coker}(f)$ *is the sheafification of the presheaf cokernel.*

(3) $Shv_\tau^{\mathbf{Ab}}(\mathcal{C})$ *has enough injectives.*

2. Categories of motivic complexes

Nisnevich sheaves The sheaf-theoretic construction of mixed motives is based on the notion of a *Nisnevich sheaf with transfer.*

Definition. Let X be a k-scheme of finite type. A *Nisnevich cover* $\mathcal{U} \to X$ is an étale morphism of finite type such that, for each finitely generated field extension F of k, the map on F-valued points $\mathcal{U}(F) \to X(F)$ is surjective.

Using Nisnevich covers as covering families gives us the *small Nisnevich site on* X, X_{Nis}. The *big Nisnevich site over* k, with underlying category $\mathbf{Sm}/k$, is defined similarly.

Notation $\mathrm{Sh}^{\mathrm{Nis}}(X) :=$ Nisnevich sheaves of abelian groups on X,
$\mathrm{Sh}^{\mathrm{Nis}}(k) :=$ Nisnevich sheaves of abelian groups on $\mathbf{Sm}/k$

For a presheaf $\mathcal{F}$ on $\mathbf{Sm}/k$ or X_{Nis}, we let $\mathcal{F}_{\mathrm{Nis}}$ denote the associated sheaf.

For a category $\mathcal{C}$, we have the category of presheaves of abelian groups on $\mathcal{C}$, i.e., the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$.

Definition. (1) The category $\mathrm{PST}(k)$ of presheaves with transfer is the category of presheaves of abelian groups on $\mathrm{Cor}_{\mathrm{fin}}(k)$.

(2) The category of Nisnevich sheaves with transfer on $\mathbf{Sm}/k$, $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$, is the full subcategory of $\mathrm{PST}(k)$ with objects those F such that, for each $X \in \mathbf{Sm}/k$, the restriction of F to X_{Nis} is a sheaf.

Remark. A PST F is a presheaf on $\mathbf{Sm}/k$ together with *transfer maps*

$$\mathrm{Tr}(a) : F(Y) \to F(X)$$

for every finite correspondence $a \in \mathrm{Cor}_{\mathrm{fin}}(X, Y)$, with:
$\mathrm{Tr}(\Gamma_f) = f^*$, $\mathrm{Tr}(a \circ b) = \mathrm{Tr}(b) \circ \mathrm{Tr}(a)$, $\mathrm{Tr}(a \pm b) = \mathrm{Tr}(a) \pm \mathrm{Tr}(b)$.

Definition. Let F be a presheaf of abelian groups on $\mathbf{Sm}/k$. We call F *homotopy invariant* if for all $X \in \mathbf{Sm}/k$, the map

$$p^* : F(X) \to F(X \times \mathbb{A}^1)$$

is an isomorphism.

We call F *strictly homotopy invariant* if for all $q \geq 0$, the cohomology presheaf $X \mapsto H^q(X_{\mathrm{Nis}}, F_{\mathrm{Nis}})$ is homotopy invariant.

Theorem (PST)**.** *Let F be a homotopy invariant PST on* $\mathbf{Sm}/k$. *Then*

(1) The cohomology presheaves $X \mapsto H^q(X_{\mathrm{Nis}}, F_{\mathrm{Nis}})$ *are PST's*

(2) F_{Nis} *is strictly homotopy invariant.*

(3) $F_{\mathrm{Zar}} = F_{\mathrm{Nis}}$ *and* $H^q(X_{\mathrm{Zar}}, F_{\mathrm{Zar}}) = H^q(X_{\mathrm{Nis}}, F_{\mathrm{Nis}})$.

Remarks. (1) uses the fact that for finite map $Z \to X$ with X Hensel local and Z irreducible, Z is also Hensel local. (2) and (3) rely on Voevodsky's generalization of Quillen's proof of Gersten's conjecture, viewed as a "moving lemma using transfers". For example:

Lemma (Voevodsky's moving lemma). *Let X be in $\mathbf{Sm}/k$, S a finite set of points of X, $j_U : U \to X$ an open subscheme. Then there is an open neighborhood $j_V : V \to X$ of S in X and a finite correspondence $a \in c(V,U)$ such that, for all homotopy invariant PST's F, the diagram*

$$\begin{tikzcd} F(X) \arrow[r, "j_U^*"] \arrow[d, "j_V^*"'] & F(U) \arrow[dl, "Tr(a)"] \\ F(V) & \end{tikzcd}$$

commutes.

One consequence of the lemma is

(1) If X is semi-local, then $F(X) \to F(U)$ is a split injection.

Variations on this construction prove:

(2) If X is semi-local and smooth then $F(X) = F_{\mathrm{Zar}}(X)$ and $H^n(X_{\mathrm{Zar}}, F_{\mathrm{Zar}}) = 0$ for $n > 0$.

(3) If U is an open subset of $\mathbb{A}^1_k$, then $F_{\mathrm{Zar}}(U) = F(U)$ and $H^n(U, F_{\mathrm{Zar}}) = 0$ for $n > 0$.

(4) If $j : U \to X$ has complement a smooth k-scheme $i : Z \to X$, then $coker F(X_{\mathrm{Zar}}) \to j_*F(U_{\mathrm{Zar}})$ (as a sheaf on Z_{Zar}) depends only on the Nisnevich neighborhood of Z in X.

(1)-(4) together with some cohomological techniques prove the theorem.

2.1. **The category of motivic complexes.**

Definition. Inside the derived category $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$, we have the full subcategory $DM_-^{\mathrm{eff}}(k)$ consisting of complexes whose cohomology sheaves are homotopy invariant.

Proposition. *$DM_-^{\mathrm{eff}}(k)$ is a triangulated subcategory of $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$.*

This follows from

Lemma. *Let $HI(k) \subset \mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$ be the full subcategory of homotopy invariant sheaves. Then $HI(k)$ is an abelian subcategory of $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$, closed under extensions in $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$.*

Proof of the lemma. Given $f : F \to G$ in $HI(k)$, $ker(f)$ is the presheaf kernel, hence in $HI(k)$. The presheaf $coker(f)$ is homotopy invariant, so by the PST theorem $coker(f)_{\mathrm{Nis}}$ is homotopy invariant.

To show $HI(k)$ is closed under extensions: Given $0 \to A \to E \to B \to 0$ exact in $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$ with $A, B \in HI(k)$, consider $p : X \times \mathbb{A}^1 \to X$. The PST theorem implies $R^1p_*A = 0$, so

$$0 \to p_*A \to p_*E \to p_*B \to 0$$

is exact as sheaves on X. Thus $p_*E = E$, so E is homotopy invariant. $\square$

3. The Suslin complex

3.1. **The Suslin complex.** Let $\Delta^n := \operatorname{Spec} k[t_0, \dots, t_n]/\sum_{i=0}^n t_i - 1$.
$n \mapsto \Delta^n$ defines the cosimplicial k-scheme Δ^*.

Definition. Let F be a presheaf on $\mathrm{Cor}_{\mathrm{fin}}(k)$. Define the presheaf $C_n(F)$ by

$$C_n(F)(X) := F(X \times \Delta^n)$$

The *Suslin complex* $C_*(F)$ is the complex with differential

$$d_n := \sum_i (-1)^i \delta_i^* : C_n(F) \to C_{n-1}(F).$$

For $X \in \mathbf{Sm}/k$, let $C_*(X)$ be the complex of sheaves $C_n(X)(U) := \mathrm{Cor}_{\mathrm{fin}}(U \times \Delta^n, X)$.

Remarks. (1) If F is a sheaf with transfers on $\mathbf{Sm}/k$, then $C_*(F)$ is a complex of sheaves with transfers.

(2) The homology presheaves $h_i(F) := \mathcal{H}^{-i}(C_*(F))$ are homotopy invariant. Thus, by Voevodsky's PST theorem, the associated Nisnevich sheaves $h_i^{\mathrm{Nis}}(F)$ are strictly homotopy invariant. We thus have the functor

$$C_* : \mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)) \to DM_-^{\mathrm{eff}}(k).$$

(3) For X in $\mathbf{Sch}_k$, we have the sheaf with transfers $L(X)(Y) = \mathrm{Cor}_{\mathrm{fin}}(Y, X)$ for $Y \in \mathbf{Sm}/k$.

For $X \in \mathbf{Sm}/k$, $L(X)$ is the free sheaf with transfers generated by the representable sheaf of sets $\mathrm{Hom}(-, X)$.

We have the canonical isomorphisms $\mathrm{Hom}(L(X), F) = F(X)$ and $C_*(X) = C_*(L(X))$.

In fact: For $F \in \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$ there is a canonical isomorphism

$$\mathrm{Ext}^n_{\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))}(L(X), F) \cong H^n(X_{\mathrm{Nis}}, F)$$

4. Statement of main results

4.1. **The localization theorem.**

Theorem. *The functor C_* extends to an exact functor*

$$\mathbf{R}C_* : D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))) \to DM_-^{\mathrm{eff}}(k),$$

left adjoint to the inclusion $DM_-^{\mathrm{eff}}(k) \to D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$.

$\mathbf{R}C_*$ identifies $DM_-^{\mathrm{eff}}(k)$ with the localization $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))/\mathcal{A}$, where $\mathcal{A}$ is the localizing subcategory of $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$ generated by complexes

$$L(X \times \mathbb{A}^1) \xrightarrow{L(p_1)} L(X); \quad X \in \mathbf{Sm}/k.$$

4.2. **The tensor structure.** We define a tensor structure on $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$:

Set $L(X) \otimes L(Y) := L(X \times Y)$.

For a general F, we have the canonical surjection

$$\oplus_{(X, s \in F(X))} L(X) \to F.$$

Iterating gives the canonical left resolution $\mathcal{L}(F) \to F$. Define

$$F \otimes G := H_0^{\mathrm{Nis}}(\mathcal{L}(F) \otimes \mathcal{L}(G)).$$

The unit for $\otimes$ is $L(\operatorname{Spec} k)$.

There is an *internal Hom* in $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$:

$$\mathcal{H}om(L(X), G)(U) = G(U \times X);$$
$$\mathcal{H}om(F, G) := H^0_{\mathrm{Nis}}(\mathcal{H}om(\mathcal{L}(F), G)).$$

4.3. **Tensor structure in DM^{eff}_-.** The tensor structure on $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$ induces a tensor structure $\otimes^L$ on $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$.

We make $DM^{\mathrm{eff}}_-(k)$ a tensor triangulated category via the localization theorem:

$$M \otimes N := \mathbf{R}C_*(i(M) \otimes^L i(N)).$$

4.4. **The embedding theorem.**

Theorem. *There is a commutative diagram of exact tensor functors*

$$\begin{array}{ccc}
\mathcal{H}^b(\mathrm{Cor}_{\mathrm{fin}}(k)) & \xrightarrow{L} & D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))) \\
\downarrow & & \downarrow{\scriptstyle \mathbf{R}C_*} \\
DM^{\mathrm{eff}}_{\mathrm{gm}}(k) & \xrightarrow[i]{} & DM^{\mathrm{eff}}_-(k)
\end{array}$$

such that
1. i is a full embedding with dense image.
2. $\mathbf{R}C_*(L(X)) \cong C_*(X)$.

Corollary. *For X and $Y \in \mathbf{Sm}/k$,* $\mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(Y), M_{\mathrm{gm}}(X)[n]) \cong \mathbb{H}^n(Y_{\mathrm{Nis}}, C_*(X)) \cong \mathbb{H}^n(Y_{\mathrm{Zar}}, C_*(X))$.

Lecture 4. Consequences and computations

Outline:
- Consequences of the localization and embedding theorems
- Computations in $DM^{\mathrm{eff}}_{\mathrm{gm}}$
- The Gysin distinguished triangle

1. Consequences of the localization and embedding theorems

1.1. **Morphisms as hypercohomology.**

Corollary. *For X and $Y \in \mathbf{Sm}/k$,*

$$\mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(Y), M_{\mathrm{gm}}(X)[n]) \cong \mathbb{H}^n(Y_{\mathrm{Nis}}, C_*(X)) \cong \mathbb{H}^n(Y_{\mathrm{Zar}}, C_*(X)).$$

Proof. For a sheaf F, and $Y \in \mathbf{Sm}/k$,

$$\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}})}(L(Y), F) = F(Y)$$

Thus the Hom in the derived category, for F a complex of sheaves, is:

$$\mathrm{Hom}_{D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}))}(L(Y), F[n]) = \mathbb{H}^n(Y_{\mathrm{Nis}}, F).$$

Thus (using the embedding theorem and localization theorem)

$$\begin{aligned}
&\mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(Y), M_{\mathrm{gm}}(X)[n]) \\
&\quad = \mathrm{Hom}_{DM^{\mathrm{eff}}_-(k)}(C_*(Y), C_*(X)[n]) \\
&\quad = \mathrm{Hom}_{D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}))}(L(Y), C_*(X)[n]) \\
&\quad = \mathbb{H}^n(Y_{\mathrm{Nis}}, C_*(X)).
\end{aligned}$$

PST theorem $\Longrightarrow \mathbb{H}^n(Y_{\mathrm{Zar}}, C_*(X)) = \mathbb{H}^n(Y_{\mathrm{Nis}}, C_*(X))$.

1.2. Mayer-Vietoris for Suslin homology.

Definition. For $X \in \mathbf{Sm}/k$, define the *Suslin homology* of X as

$$H_i^{\mathrm{Sus}}(X) := H_i(C_*(X)(\operatorname{Spec} k)).$$

Theorem. *Let U, V be open subschemes of $X \in \mathbf{Sm}/k$. Then there is a long exact Mayer-Vietoris sequence*

$$\ldots \to H_{n+1}^{\mathrm{Sus}}(U \cup V) \to H_n^{\mathrm{Sus}}(U \cap V) \\ \to H_n^{\mathrm{Sus}}(U) \oplus H_n^{\mathrm{Sus}}(V) \to H_n^{\mathrm{Sus}}(U \cup V) \to \ldots$$

Proof. By the embedding theorem $[U \cap V] \to [U] \oplus [V] \to [U \cup V]$ maps to the distinguished triangle in $DM_-^{\mathrm{eff}}(k)$:

$$C_*(U \cap V)_{\mathrm{Nis}} \to C_*(U)_{\mathrm{Nis}} \oplus C_*(V)_{\mathrm{Nis}} \to C_*(U \cup V)_{\mathrm{Nis}} \to$$

This yields a long exact sequence upon applying $\operatorname{Hom}_{DM^{\mathrm{eff}}}(M_{\mathrm{gm}}(Y), -)$ for any $Y \in \mathbf{Sm}/k$.

By the corollary to the embedding theorem, this gives the long exact sequence

$$\ldots \to \mathbb{H}^{-n}(Y_{\mathrm{Nis}}, C_*(U \cap V)) \to \mathbb{H}^{-n}(Y_{\mathrm{Nis}}, C_*(U)) \oplus \mathbb{H}^{-n}(Y_{\mathrm{Nis}}, C_*(V)) \\ \to \mathbb{H}^{-n}(Y_{\mathrm{Nis}}, C_*(U \cap V) \to \mathbb{H}^{-n+1}(Y_{\mathrm{Nis}}, C_*(U \cap V)) \to \ldots$$

Now just take $Y = \operatorname{Spec} k$, since

$$\mathbb{H}^{-n}(\operatorname{Spec} k_{\mathrm{Nis}}, C_*(X)) = H_n(C_*(X)(\operatorname{Spec} k)) = H_n^{\mathrm{Sus}}(X).$$

Remark. In fact, the embedding theorem implies that for all $Y \in \mathbf{Sm}/k$, the homology sheaves $h_n^{\mathrm{Zar}}(Y)$ associated to the presheaf $U \mapsto H_n(C_*(Y)(U))$ are the same as the sheaves associated to the presheaf

$$U \mapsto \operatorname{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(U), M_{\mathrm{gm}}(Y)[-n]).$$

Thus
- $h_n^{\mathrm{Zar}}(Y \times \mathbb{A}^1) \to h_n^{\mathrm{Zar}}(Y)$ is an isomorphism
- for $Y = U \cup V$, have a long exact Mayer-Vietoris sequence

$$\ldots \to h_p^{\mathrm{Zar}}(U \cap V) \to h_p^{\mathrm{Zar}}(U) \oplus h_p^{\mathrm{Zar}}(V) \to h_p^{\mathrm{Zar}}(Y) \to h_{p-1}^{\mathrm{Zar}}(U \cap V) \to \ldots$$

2. Fundamental constructions in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$

2.1. Weight one motivic cohomology.

$\mathbb{Z}(1)[2]$ is the reduced motive of $\mathbb{P}^1$, and $M_{\mathrm{gm}}(\mathbb{P}^1)$ is represented in DM_-^{eff} by the Suslin complex $C_*(\mathbb{P}^1)$. The homology sheaves of $C_*(\mathbb{P}^1)$ and $C_*(\operatorname{Spec} k)$ are given by:

Lemma. *$h_0^{\mathrm{Zar}}(\mathbb{P}^1) = \mathbb{Z}$, $h_1^{\mathrm{Zar}}(\mathbb{P}^1) = \mathbb{G}_m$ and $h_n^{\mathrm{Zar}}(\mathbb{P}^1) = 0$ for $n \geq 2$. $h_0^{\mathrm{Zar}}(\operatorname{Spec} k) = \mathbb{Z}$, $h_n^{\mathrm{Zar}}(\operatorname{Spec} k) = 0$ for $n \geq 1$.*

Sketch of proof: $C_n(\operatorname{Spec} k)(Y) = c(Y \times \Delta^n, \operatorname{Spec} k) = H^0(Y_{\mathrm{Zar}}, \mathbb{Z})$. Thus

$$h_p(C_*(\operatorname{Spec} k)(Y)) = \begin{cases} H^0(Y_{\mathrm{Zar}}, \mathbb{Z}) & \text{for } p = 0 \\ 0 & \text{for } p \neq 0 \end{cases}$$

We have $h_p^{\mathrm{Zar}}(\mathbb{A}^1) = h_p^{\mathrm{Zar}}(\operatorname{Spec} k)$ and we have a Meyer-Vietoris sequence

$$\ldots \to h_p^{\mathrm{Zar}}(\mathbb{A}^1) \oplus h_p^{\mathrm{Zar}}(\mathbb{A}^1) \to h_p^{\mathrm{Zar}}(\mathbb{P}^1) \to h_{p-1}^{\mathrm{Zar}}(\mathbb{A}^1 \setminus 0) \to \ldots$$

giving

$$h_p^{\mathrm{Zar}}(\mathbb{P}^1) = h_p^{\mathrm{Zar}}(\operatorname{Spec} k) \oplus h_{p-1}^{\mathrm{Zar}}(\mathbb{G}_m).$$

where $h_{p-1}^{\mathrm{Zar}}(\mathbb{G}_m) := h_{p-1}^{\mathrm{Zar}}(\mathbb{A}^1 \setminus 0)/h_{p-1}^{\mathrm{Zar}}(1)$.

So we need to see that

$$h_p^{\mathrm{Zar}}(\mathbb{A}^1 \setminus 0) = \begin{cases} \mathbb{G}_m \oplus \mathbb{Z} \cdot [1] & \text{for } p = 0 \\ 0 & \text{else.} \end{cases}$$

For this, let $Y = \operatorname{Spec} \mathcal{O}$ for $\mathcal{O} = \mathcal{O}_{X,x}$ some $x \in X \in \mathbf{Sm}/k$

$$h_p^{\mathrm{Zar}}(\mathbb{A}^1 \setminus 0)_{X,x} = H_p(C_*(\mathbb{A}^1 \setminus 0)(Y)).$$

For $W \subset Y \times \Delta^n \times (\mathbb{A}^1 \setminus 0)$ finite and surjective over $Y \times \Delta^n$, W has a monic defining equation

$$F_W(y,t,x) = x^N + \sum_{i=1}^{N-1} F_i(y,t)x^i + F_0(y,t)$$

with $F_0(y,t)$ a unit in $\mathcal{O}[t_0,\ldots,t_n]/\sum_i t_i - 1$.

Map $h_0^{\mathrm{Zar}} \to \mathbb{Z}$ by $W \mapsto \deg_Y W$.

Define $\mathrm{cl}_Y : \mathbb{G}_m(Y) \to H_0(C_*(\mathbb{A}^1 \setminus 0)(Y))_{\deg 0}$ by

$$\mathrm{cl}_Y(u) := [\Gamma_u - \Gamma_1],$$

$\Gamma_u \subset Y \times \mathbb{A}^1 \setminus 0$ the graph of $u : Y \to \mathbb{A}^1 \setminus 0$.

One shows cl_Y is a group homomorphism by using the cycle T on $Y \times \Delta^1 \times \mathbb{A}^1 \setminus 0$ defined by

$$t(x-uv)(x-1) + (1-t)(x-u)(x-v),$$

$$dT = (\Gamma_{uv} - \Gamma_1) - (\Gamma_u - \Gamma_1) - (\Gamma_v - \Gamma_1).$$

To show cl_Y is surjective: If $W \subset Y \times \mathbb{A}^1 \setminus 0$ is finite over Y, we have the unit $u := (-1)^N F_W(y,0)$ with $F_W(y,x)$ the monic defining equation for W, $N = \deg_Y W$. The function

$$F(y,x,t) := tF_W(y,x) + (1-t)(x-u)(x-1)^{N-1}$$

defines a finite cycle T on $Y \times \Delta^1 \times \mathbb{A}^1 \setminus 0$ with

$$dT = W - \Gamma_u - (N-1)\Gamma_1 = (W - \deg_Y W \cdot \Gamma_1) - (\Gamma_u - \Gamma_1).$$

To show that cl_Y is injective: show sending W to $(-1)^N F_W(y,0)$ passes to H_0. This can be done by noting that there are no non-constant maps $f : \mathbb{A}^1 \to \mathbb{A}^1 \setminus 0$.

The proof that $h_p^{\mathrm{Zar}}(Y) = 0$ for $p > 0$ is similar.

This computation implies that

$$\mathbb{Z}(1) \cong \mathbb{G}_m[-1]$$

in $DM_-^{\mathrm{eff}}(k)$. Indeed:

$$\begin{aligned} \mathbb{Z}(1)[2] &\cong \operatorname{Cone}(C_*(\mathbb{P}^1) \to C_*(\operatorname{Spec} k))[-1] \\ &\cong h_1^{\mathrm{Zar}}(\mathbb{P}^1)[1] = \mathbb{G}_m[1] \end{aligned}$$

This yields:

Proposition. *For $X \in \mathbf{Sm}/k$, we have*

$$H^n(X, \mathbb{Z}(1)) = \begin{cases} H^0_{\mathrm{Zar}}(X, \mathcal{O}_X^*) & \text{for } n = 1 \\ \mathrm{Pic}(X) := H^1_{\mathrm{Zar}}(X, \mathcal{O}_X^*) & \text{for } n = 2 \\ 0 & \text{else.} \end{cases}$$

Proof.. Since $\mathbb{Z}(1) \cong \mathbb{G}_m[-1]$ in $DM_-^{\mathrm{eff}}(k)$, the corollary to the embedding theorem gives:

$$\begin{aligned} \mathrm{Hom}_{DM_{\mathrm{gm}}^{\mathrm{eff}}}(M_{\mathrm{gm}}(X), \mathbb{Z}(1)[n]) &\cong \mathbb{H}^n_{\mathrm{Nis}}(X, \mathbb{Z}(1)) \\ &\cong \mathbb{H}^n_{\mathrm{Zar}}(X, \mathbb{Z}(1)) \cong H^{n-1}_{\mathrm{Zar}}(X, \mathbb{G}_m). \end{aligned}$$

Remark. The isomorphism $H^2(X, \mathbb{Z}(1)) \cong \mathrm{Pic}(X)$ gives another way of associating an element of $H^2(X, \mathbb{Z}(1))$ to a line bundle L on X; one can show that this agrees with the 1st Chern class $c_1(L)$ defined in Lecture 3.

2.2. **Gysin isomorphism.**

Definition. For $i : Z \to X$ a closed subset, let $M_{\mathrm{gm}}(X/X \setminus Z) \in DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ be the image in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ of the complex $[X \setminus Z] \xrightarrow{j} [X]$, with $[X]$ in degree 0.

Remark. The Mayer-Vietoris property for $M_{\mathrm{gm}}(-)$ yields a Zariski excision property: If Z is closed in U, an open in X, then $M_{\mathrm{gm}}(U/U \setminus Z) \to M_{\mathrm{gm}}(X/X \setminus Z)$ is an isomorphism.

In fact, Voevodsky's moving lemma shows that $M_{\mathrm{gm}}(X/X \setminus Z)$ depends only on the Nisnevich neighborhood of Z in X: this is the *Nisnevich excision* property.

2.3. **Motivic cohomology with support.** Let $Z \subset X$ be a closed subset, $U := X \setminus Z$. Setting

$$H^p_Z(X, \mathbb{Z}(q)) := \mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(X/U), \mathbb{Z}(q)[p])$$

gives the long exact sequence for cohomology with support:

$$\ldots \to H^p_Z(X, \mathbb{Z}(q)) \xrightarrow{i_*} H^p(X, \mathbb{Z}(q)) \xrightarrow{j^*} H^p(U, \mathbb{Z}(q)) \to H^{p+1}_Z(X, \mathbb{Z}(q)) \to$$

Theorem (Gysin isomorphism)**.** *Let $i : Z \to X$ be a closed embedding in $\mathbf{Sm}/k$ of codimension n, $U = X \setminus Z$. Then there is a natural isomorphism in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$*

$$M_{\mathrm{gm}}(X/U) \cong M_{\mathrm{gm}}(Z)(n)[2n].$$

In particular:

$$\begin{aligned} H^p_Z(X, \mathbb{Z}(q)) &= \mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(X/U), \mathbb{Z}(q)[p]) \\ &= \mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(Z)(n)[2n], \mathbb{Z}(q)[p]) \\ &= \mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(Z), \mathbb{Z}(q-n)[p-2n]) \\ &= H^{p-2n}(Z, \mathbb{Z}(q-n)). \end{aligned}$$

2.4. **Gysin distinguished triangle.**

Theorem. *Let $i : Z \to X$ be a codimension n closed immersion in $\mathbf{Sm}/k$ with open complement $j : U \to X$. There is a canonical distinguished triangle in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$:*

$$M_{\mathrm{gm}}(U) \xrightarrow{j_*} M_{\mathrm{gm}}(X) \to M_{\mathrm{gm}}(Z)(n)[2n] \to M_{\mathrm{gm}}(U)[1]$$

Proof. By definition of $M_{\rm gm}(X/U)$, we have the canonical distinguished triangle in $DM^{\rm eff}_{\rm gm}(k)$:

$$M_{\rm gm}(U) \xrightarrow{j_*} M_{\rm gm}(X) \to M_{\rm gm}(X/U) \to M_{\rm gm}(U)[1]$$

then insert the Gysin isomorphism $M_{\rm gm}(X/U) \cong M_{\rm gm}(Z)(n)[2n]$.

Applying $\mathrm{Hom}(-, \mathbb{Z}(q)[p])$ to the Gysin distinguished triangle gives the long exact Gysin sequence

$$\ldots \to H^{p-2n}(Z, \mathbb{Z}(q-n)) \xrightarrow{i_*} H^p(X, \mathbb{Z}(q)) \\ \xrightarrow{j^*} H^p(U, \mathbb{Z}(q)) \xrightarrow{\partial} H^{p-2n+1}(Z, \mathbb{Z}(q-n)) \to$$

which is the same as the sequence for cohomology with supports, using the Gysin isomorphism

$$H^{p-2n}(Z, \mathbb{Z}(q-n)) \cong H^p_Z(X, \mathbb{Z}(q)).$$

Now for the proof of the Gysin isomorphism theorem:

We first prove a special case:

Lemma. *Let $E \to Z$ be a vector bundle of rank n with zero section s. Then $M_{\rm gm}(E/E \setminus s(Z)) \cong M_{\rm gm}(Z)(n)[2n]$.*

Proof. Since $M_{\rm gm}(E) \to M_{\rm gm}(Z)$ is an isomorphism by homotopy, we need to show

$$M_{\rm gm}(E \setminus s(Z)) \cong M_{\rm gm}(Z) \oplus M_{\rm gm}(Z)(n)[2n-1].$$

Let $\mathbb{P} := \mathbb{P}(E \oplus \mathcal{O}_Z)$, and write $\mathbb{P} = E \cup (\mathbb{P} \setminus s(Z))$. Mayer-Vietoris gives the distinguished triangle

$$M_{\rm gm}(E \setminus s(Z)) \to M_{\rm gm}(E) \oplus M_{\rm gm}(\mathbb{P} \setminus s(Z)) \to M_{\rm gm}(\mathbb{P}) \to M_{\rm gm}(E \setminus s(Z))[1]$$

Since $\mathbb{P} \setminus s(Z) \to \mathbb{P}(E)$ is an $\mathbb{A}^1$ bundle, the projective bundle formula gives the isomorphism we wanted.

2.5. **Deformation to the normal bundle.** For $i : Z \to X$ a closed immersion in $\mathbf{Sm}/k$, let

$$p : (X \times \mathbb{A}^1)_{Z \times 0} \to X \times \mathbb{A}^1$$

be the blow-up of $X \times \mathbb{A}^1$ along $Z \times 0$. Set

$$Def(i) := (X \times \mathbb{A}^1)_{Z \times 0} \setminus p^{-1}[X \times 0].$$

We have $\tilde{i} : Z \times \mathbb{A}^1 \to Def(i)$, $q : Def(i) \to \mathbb{A}^1$.

The fiber $\tilde{i}_1$ is $i : Z \to X$, the fiber $\tilde{i}_0$ is $s : Z \to N_{Z/X}$.

Lemma. *The maps*

$$M_{\rm gm}(N_{Z/X}/N_{Z/X} \setminus s(Z)) \to M_{\rm gm}(Def(i)/[Def(i) \setminus Z \times \mathbb{A}^1]) \\ M_{\rm gm}(X/X \setminus Z) \to M_{\rm gm}(Def(i)/[Def(i) \setminus Z \times \mathbb{A}^1])$$

are isomorphisms.

Proof. By Nisnevich excision, we reduce to the case $Z \times 0 \to Z \times \mathbb{A}^n$. In this case, $Z \times \mathbb{A}^1 \to Def(i)$ is just $(Z \times 0 \to Z \times \mathbb{A}^n) \times \mathbb{A}^1$, whence the result.

Proof of the theorem.

$$M_{\rm gm}(X/X \setminus Z) \cong M_{\rm gm}(N_{Z/X}/N_{Z/X} \setminus s(Z)) \cong M_{\rm gm}(Z)(n)[2n]$$

Lecture 5. **Mixed motives and cycle complexes, I**

0.6. **Outline:**

- Proofs of the localization and embedding theorems
- Cycle complexes
- Bivariant cycle cohomology

1. Proofs of the localization and embedding theorems

1.1. **Statement of main results.** We recall the statements of the results we are to prove:

Theorem (The localization theorem)**.** *The functor* $C_* : \mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)) \to DM_-^{\mathrm{eff}}(k)$ *extends to an exact functor*

$$\mathbf{R}C_* : D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))) \to DM_-^{\mathrm{eff}}(k),$$

left adjoint to the inclusion $DM_-^{\mathrm{eff}}(k) \to D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$*.*

$\mathbf{R}C_*$ *identifies* $DM_-^{\mathrm{eff}}(k)$ *with the localization* $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))/\mathcal{A}$*, where* $\mathcal{A}$ *is the localizing subcategory of* $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$ *generated by complexes*

$$L(X \times \mathbb{A}^1) \xrightarrow{L(p_1)} L(X); \quad X \in \mathbf{Sm}/k.$$

Theorem (The embedding theorem)**.** *There is a commutative diagram of exact tensor functors*

$$\begin{array}{ccc} K^b(\mathrm{Cor}_{\mathrm{fin}}(k)) & \xrightarrow{L} & D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \mathbf{R}C_*} \\ DM_{\mathrm{gm}}^{\mathrm{eff}}(k) & \xrightarrow[i]{} & DM_-^{\mathrm{eff}}(k) \end{array}$$

such that
1. i is a full embedding with dense image.
2. $\mathbf{R}C_*(L(X)) \cong C_*(X)$*.*

Now to work. We will use the result from Lecture 3:

Theorem (Global PST)**.** *Let F^* be a complex of PSTs on* $\mathbf{Sm}/k$*:* $F \in C^-(PST)$*. Suppose that the cohomology presheaves $h^i(F)$ are homotopy invariant. Then*
(1) For $Y \in \mathbf{Sm}/k$*,* $\mathbb{H}^i(Y_{\mathrm{Nis}}, F^*_{\mathrm{Nis}}) \cong \mathbb{H}^i(Y_{\mathrm{Zar}}, F^*_{\mathrm{Zar}})$
(2) The presheaf $Y \mapsto \mathbb{H}^i(Y_{\mathrm{Nis}}, F^*_{\mathrm{Nis}})$ *is homotopy invariant*

(1) and (2) follows from the PST theorem using the spectral sequence:

$$E_2^{p,q} = H^p(Y_\tau, h^q(F)_\tau) \Longrightarrow \mathbb{H}^{p+q}(Y_\tau, F_\tau), \tau = \mathrm{Nis}, \mathrm{Zar}.$$

1.2. $\mathbb{A}^1$**-homotopy.** The inclusions $i_0, i_1 : \mathrm{Spec}\, k \to \mathbb{A}^1$ give maps of PST's $i_0, i_1 : 1 \to L(\mathbb{A}^1)$.

Definition. Two maps of PST's $f, g : F \to G$ are $\mathbb{A}^1$-*homotopic* if there is a map

$$h : F \otimes L(\mathbb{A}^1) \to F$$

with $f = h \circ (\mathrm{id} \otimes i_0)$, $g = h \circ (\mathrm{id} \otimes i_1)$.

The usual definition gives the notion of $\mathbb{A}^1$-*homotopy equivalence*.

These notions extend to complexes by allowing chain homotopies.

Example. $p^*: F \to C_n(F)$ is an $\mathbb{A}^1$-homotopy equivalence:

$n = 1$ is the crucial case since $C_1(C_{n-1}(F)) = C_n(F)$.

We have the homotopy inverse $i_0^*: C_1(F) \to F$.

To define a homotopy $h: C_1(F) \otimes L(\mathbb{A}^1) \to C_1(F)$ between $p^* i_0^*$ and id:

$$\begin{aligned}\mathrm{Hom}(&C_1(F) \otimes L(\mathbb{A}^1), C_1(F))\\ &= \mathrm{Hom}(\mathcal{H}om(L(\mathbb{A}^1), F), \mathcal{H}om(L(\mathbb{A}^1) \otimes L(\mathbb{A}^1), F))\end{aligned}$$

so we need a map $\mu: \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$.

Taking $\mu(x, y) = xy$ works.

Lemma. *The inclusion $F = C_0(F) \to C_*(F)$ is an $\mathbb{A}^1$-homotopy equivalence.*

Proof. Let F_* be the "constant" complex, $F_n := F$, $d_n = 0, \mathrm{id}$.

$F \to F_*$ is a chain homotopy equivalence.

$F_* \to C_*(F)$ is an $\mathbb{A}^1$-homotopy equivalence by the Example.

1.3. $\mathbb{A}^1$-homotopy and $\mathrm{Ext}_{\mathrm{Nis}}$.

Lemma. *Let F, G be in $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$, with G homotopy invariant. Then $\mathrm{id} \otimes p_*: F \otimes L(\mathbb{A}^1) \to F$ induces an isomorphism*

$$\mathrm{Ext}^n(F, G) \to \mathrm{Ext}^n(F \otimes L(\mathbb{A}^1), G).$$

Here Ext *is in* $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$.

Proof. For $F = L(X)$, we have

$$\mathrm{Ext}^n(L(X), G) \cong H^n(X_{\mathrm{Nis}}, G),$$

so the statement translates to:

$$p^*: H^n(X_{\mathrm{Nis}}, G) \to H^n(X \times \mathbb{A}^1_{\mathrm{Nis}}, G)$$

is an isomorphism. This follows from: G strictly homotopy invariant and the Leray spectral sequence.

In general: use the left resolution $\mathcal{L}(F) \to F$.

Proposition. *Let $f: F_* \to F'_*$ be an $\mathbb{A}^1$-homotopy equivalence in $C^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$. Then*

$$\mathrm{Hom}_{D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))}(F_*, G[n]) \xrightarrow{f^*} \mathrm{Hom}_{D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))}(F'_*, G[n])$$

is an isomorphism for all $G \in HI(k)$.

Theorem ($\mathbb{A}^1$-resolution)**.** *For $G \in HI(k)$, F a PST, we have*

$$\mathrm{Ext}^n(F_{\mathrm{Nis}}, G) \cong \mathrm{Hom}_{D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))}(C_*(F)_{\mathrm{Nis}}, G[n])$$

for all n. Hence:

$\mathrm{Ext}^i(F_{\mathrm{Nis}},, G) = 0$ *for* $0 \le i \le n$ *and all* $G \in HI(k) \Leftrightarrow h_i^{\mathrm{Nis}}(F) = 0$ *for* $0 \le i \le n$.

Proof. The $\mathbb{A}^1$-homotopy equivalence $F \to C_*(F)$ induces an $\mathbb{A}^1$-homotopy equivalence $F_{\mathrm{Nis}} \to C_*(F)_{\mathrm{Nis}}$.

1.4. **Nisnevich acyclicity theorem.** A very important consequence of the $\mathbb{A}^1$-resolution theorem is

Theorem. *Let F be a PST such that $F_{\mathrm{Nis}} = 0$. Then $C_*(F)_{\mathrm{Nis}}$ and $C_*(F)_{\mathrm{Zar}}$ are acyclic complexes of sheaves.*

Proof. We need to show that

$$h_i^{\mathrm{Nis}}(F) = 0 = h_i^{\mathrm{Zar}}(F)$$

for all i. The vanishing of the $h_i^{\mathrm{Nis}}(F)$ follows from the $\mathbb{A}^1$-resolution theorem.

Since $h_i(F)$ is a homotopy invariant PST, it follows from the PST theorem that

$$h_i^{\mathrm{Zar}}(F) = h_i^{\mathrm{Nis}}(F)$$

hence $h_i^{\mathrm{Zar}}(F) = 0$.

1.5. **The localization theorem.**

Theorem. *The functor C_* extends to an exact functor*

$$\mathbf{R}C_* : D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))) \to DM_-^{\mathrm{eff}}(k),$$

left adjoint to the inclusion $DM_-^{\mathrm{eff}}(k) \to D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$.

$\mathbf{R}C_*$ identifies $DM_-^{\mathrm{eff}}(k)$ with the localization $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))/\mathcal{A}$, where $\mathcal{A}$ is the localizing subcategory of $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$ generated by complexes

$$L(X \times \mathbb{A}^1) \xrightarrow{L(p_1)} L(X); \quad X \in \mathbf{Sm}/k.$$

Proof. It suffices to prove

1. For each $F \in \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$, $F \to C_*(F)$ is an isomorphism in $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))/\mathcal{A}$.

2. For each $T \in DM_-^{\mathrm{eff}}(k)$, $B \in \mathcal{A}$, $\mathrm{Hom}(B, T) = 0$.

Indeed: (1) implies $DM_-^{\mathrm{eff}}(k) \to D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))/\mathcal{A}$ is surjective on isomorphism classes.

(2) implies $DM_-^{\mathrm{eff}}(k) \to D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))/\mathcal{A}$ is fully faithful, hence an equivalence.

(1) again implies the composition

$$D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))) \to D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))/\mathcal{A} \to DM_-^{\mathrm{eff}}(k)$$

sends F to $C_*(F)$.

To prove: 2. For each $T \in DM_-^{\mathrm{eff}}(k)$, $B \in \mathcal{A}$, $\mathrm{Hom}(B, T) = 0$.

$\mathcal{A}$ is generated by complexes $I(X) := L(X \times \mathbb{A}^1) \xrightarrow{L(p_1)} L(X)$.

But $\mathrm{Hom}(L(Y), T) \cong \mathbb{H}^0(Y_{\mathrm{Nis}}, T)$ for $T \in D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$ and

$$\mathbb{H}^*(X, T) \cong \mathbb{H}^*(X \times \mathbb{A}^1, T)$$

since T is in $DM_-^{\mathrm{eff}}(k)$, so $\mathrm{Hom}(I(X), T) = 0$.

To prove: 1. For each $F \in \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$, $F \to C_*(F)$ is an isomorphism in $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))/\mathcal{A}$.

First: $\mathcal{A}$ is a $\otimes$-ideal: $A \in \mathcal{A}$, $B \in D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))) \Longrightarrow A \otimes B \in \mathcal{A}$.

$\mathcal{A}$ is localizing, so can take $A = I(X)$, $B = L(Y)$. But then $A \otimes B = I(X \times Y)$.

Second: $F_* \to C_*(F)$ is a term-wise $\mathbb{A}^1$-homotopy equivalence and $F \to F_*$ is an iso in $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$, so it suffices to show:

For each $F \in \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$, $\mathrm{id} \otimes i_0 = \mathrm{id} \otimes i_1 : F \to F \otimes \mathbb{A}^1$ in $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))/\mathcal{A}$.

To show: For each $F \in \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))$, $\mathrm{id} \otimes i_0 = \mathrm{id} \otimes i_1 : F \to F \otimes \mathbb{A}^1$ in $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))/\mathcal{A}$.

For this: $i_0 - i_1 : L(\mathrm{Spec}\, k) \to L(\mathbb{A}^1)$ goes to 0 after composition with $L(\mathbb{A}^1) \to L(\mathrm{Spec}\, k)$, so lifts to a map $\phi : L(\mathrm{Spec}\, k) \to I(\mathbb{A}^1)$.

Thus $\mathrm{id} \otimes i_0 - \mathrm{id} \otimes i_1 : F \to F \otimes L(\mathbb{A}^1)$ lifts to $\mathrm{id} \otimes \phi : F \to F \otimes I(\mathbb{A}^1) \in \mathcal{A}$.

1.6. **The embedding theorem.**

Theorem. *There is a commutative diagram of exact tensor functors*

$$\begin{array}{ccc} K^b(\mathrm{Cor}_{\mathrm{fin}}(k)) & \xrightarrow{L} & D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k))) \\ \downarrow & & \downarrow{\mathbf{R}C_*} \\ DM^{\mathrm{eff}}_{\mathrm{gm}}(k) & \xrightarrow[i]{} & DM^{\mathrm{eff}}_-(k) \end{array}$$

such that
1. *i is a full embedding with dense image.*
2. *$\mathbf{R}C_*(L(X)) \cong C_*(X)$.*

Proof of the embedding theorem.
We already know that $\mathbf{R}C_*(L(X)) \cong C_*(L(X)) = C_*(X)$.
To show that $i : DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to DM^{\mathrm{eff}}_-(k)$ exists:
$DM^{\mathrm{eff}}_-(k)$ is already pseudo-abelian. Using the localization theorem, we need to show that the two types of complexes we inverted in $K^b(\mathrm{Cor}_{\mathrm{fin}}(k))$ are already inverted in $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))/\mathcal{A}$.

Type 1. $[X \times \mathbb{A}^1] \to [X]$. This goes to $L(X \times \mathbb{A}^1) \to L(X)$, which is a generator in $\mathcal{A}$.

Type 2. $([U \cap V] \to [U] \oplus [V]) \to [U \cup V]$. The sequence

$$0 \to L(U \cap V) \to L(U) \oplus L(V) \to L(U \cup V) \to 0$$

is exact as Nisnevich sheaves (N.B. *not* as Zariski sheaves), hence the map is inverted in $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{\mathrm{fin}}(k)))$.

To show that i is a full embedding:
We need show show that $L^{-1}(\mathcal{A})$ is the thick subcategory generated by cones of maps of Type 1 and Type 2.

The proof uses results of Neeman on compact objects in triangulated categories.

To show that i has dense image: This uses the canonical left resolution $\mathcal{L}(F) \to F$.

2. Cycle complexes

We introduce various cycle complexes and describe their main properties.

Our goal is to describe the morphisms in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ using algebraic cycles, more precisely, as the homology of a cycle complex.

2.1. **Bloch's cycle complex.** A *face* of $\Delta^n := \mathrm{Spec}\, k[t_0, \ldots, t_n]/\sum_i t_i - 1$ is a closed subset defined by $t_{i_1} = \ldots = t_{i_s} = 0$.

Definition. $X \in \mathbf{Sch}_k$. $z_r(X, n) \subset z_{r+n}(X \times \Delta^n)$ is the subgroup generated by the closed irreducible $W \subset X \times \Delta^n$ such that

$$\dim W \cap X \times F \leq r + \dim F$$

for all faces $F \subset \Delta^n$.

If X is equi-dimensional over k of dimension d, set

$$z^q(X,n) := z_{d-q}(X,n).$$

Let $\delta_i^n : \Delta^n \to \Delta^{n+1}$ be the inclusion to the face $t_i = 0$.
The cycle pull-back δ_i^{n*} is a well-defined map

$$\delta_i^{n*} : z_r(X,n+1) \to z_r(X,n)$$

Definition. Bloch's cycle complex $z_r(X,*)$ is $z_r(X,n)$ in degree n, with differential

$$d_n := \sum_{i=0}^{n+1} (-1)^i \delta_i^{n*} : z_r(X,n+1) \to z_r(X,n)$$

Bloch's *higher Chow groups* are

$$\mathrm{CH}_r(X,n) := H_n(z_r(X,*)).$$

For X locally equi-dimensional over k, we have the complex $z^q(X,*)$ and the higher Chow groups $\mathrm{CH}^q(X,n)$.

2.2. **A problem with functoriality.** Even for $X \in \mathbf{Sm}/k$, the complex $z^q(X,*)$ is only functorial for *flat* maps, and covariantly functorial for proper maps (with a shift in q). This complex is NOT a complex of PST's.

This is corrected by a version of the the classical *Chow's moving lemma* for cycles modulo rational equivalence.

2.3. **Products.** There is an external product $z^q(X,*) \otimes z^{q'}(Y,*) \to z^{q+q'}(X \times_k Y,*)$, induced by taking products of cycles. For X smooth, this induces a cup product, using δ_X^*.

2.4. **Properties of the higher Chow groups.** (1) **Homotopy.** $p^* : z_r(X,*) \to z_{r+1}(X \times \mathbb{A}^1,*)$ is a quasi-isomorphism for $X \in \mathbf{Sch}_k$.

(2) **Localization amd Mayer-Vietoris.** For $X \in \mathbf{Sch}_k$, let $i : W \to X$ be a closed subset with complement $j : U \to X$. Then

$$z_r(W,*) \xrightarrow{i_*} z_r(X,*) \xrightarrow{j^*} z_r(U,*)$$

canonically extends to a distinguished triangle in $D^-(\mathbf{Ab})$. Similarly, if $X = U \cup V$, U, V open in X, the sequence

$$z_r(X,*) \to z_r(U,*) \oplus z_r(V,*) \to z_r(U \cap V,*)$$

canonically extends to a distinguished triangle in $D^-(\mathbf{Ab})$.

(3) **K-theory.** For X regular, there is a functorial *Chern character isomorphism*

$$ch : K_n(X)_{\mathbb{Q}} \to \oplus_q \mathrm{CH}^q(X,n)_{\mathbb{Q}}$$

identifying $\mathrm{CH}^q(X,n)_{\mathbb{Q}}$ with the weight q eigenspace $K_n(X)^{(q)}$ for the Adams operations.

(4) **Classical Chow groups.** $\mathrm{CH}^n(X,0) = \mathrm{CH}^n(X)$.

(5) **Weight one.** For $X \in \mathbf{Sm}/k$, $\mathrm{CH}^1(X,1) = H^0(X,\mathcal{O}_X^*)$, $\mathrm{CH}^1(X,0) = H^1(X,\mathcal{O}_X^*) = \mathrm{Pic}(X)$, $\mathrm{CH}^1(X,n) = 0$ for $n > 1$.

The proof of the localization property uses a different type of moving lemma (Bloch's moving by blowing up faces).

2.5. **Equi-dimensional cycles.**

Definition. Fix $X \in \mathbf{Sch}_k$. For $U \in \mathbf{Sm}/k$ let $z_r^{\mathrm{equi}}(X)(U) \subset z(X \times U)$ be the subgroup generated by the closed irreducible $W \subset X \times U$ such that $W \to U$ is equi-dimensional with fibers of dimension r (or empty).

Remark. The standard formula for composition of correspondences makes $z_r^{\mathrm{equi}}(X)$ a PST; in fact $z_r^{\mathrm{equi}}(X)$ is a Nisnevich sheaf with transfers.

Definition. The complex of *equi-dimensional cycles* is

$$z_r^{\mathrm{equi}}(X, *) := C_*(z_r^{\mathrm{equi}}(X))(\mathrm{Spec}\, k).$$

Explicitly: $z_r^{\mathrm{equi}}(X, n)$ is the subgroup of $z_{r+n}(X \times \Delta^n)$ generated by irreducible W such that $W \to \Delta^n$ is equi-dimensional with fiber dimension r. Thus:

There is a natural inclusion

$$z_r^{\mathrm{equi}}(X, *) \to z_r(X, *).$$

Remark. $z_0^{\mathrm{equi}}(X)(Y) \subset \mathcal{Z}(Y \times X)$ is the subgroup generated by integral closed subschemes $W \subset Y \times X$ just that $W \to Y$ is *quasi-finite* and dominant over some component of Y.

Write $C_*^c(X)$ for $C_*(z_0^{\mathrm{equi}}(X))$.

Since $z_r^{\mathrm{equi}}(X)$ is a Nisnevich sheaf with transfers, $C_*^c(X)$ defines an object $M_{\mathrm{gm}}^c(X)$ of $DM_-^{\mathrm{eff}}(k)$.

$X \mapsto M_{\mathrm{gm}}^c(X)$ is covariantly functorial for *proper* maps and contravariantly functorial for *flat maps of relative dimension 0* (e.g. open immersions).

Similarly, we can define the PST $L(X)$ for $X \in \mathbf{Sch}_k$ by $L(X)(Y) =$ the cycles on $X \times Y$, finite over X. This gives the object

$$M_{\mathrm{gm}}(X) := C_*(X) := C_*(L(X))$$

of $DM_-^{\mathrm{eff}}(k)$, covariantly functorial in X, extending the definition of M_{gm} from $\mathbf{Sm}/k$ to $\mathbf{Sch}_k$.

3. Bivariant cycle cohomology

3.1. **The** cdh **topology.**

Definition. The cdh site is given by the pre-topology on $\mathbf{Sch}_k$ with covering families generated by

1. Nisnevich covers
2. $p \amalg i : Y \amalg F \to X$, where $i : F \to X$ is a closed immersion, $p : Y \to X$ is proper, and

$$p : Y \setminus p^{-1}F \to X \setminus F$$

is an isomorphism (abstract blow-up).

Remark. If k admits resolution of singularities (for finite type k-schemes and for abstract blow-ups to smooth k-schemes), then each cdh cover admits a refinement consisting of *smooth* k-schemes.

Definition. Take $X, Y \in \mathbf{Sch}_k$. The *bivariant cycle cohomology* of Y with coefficients in cycles on X are

$$A_{r,i}(Y, X) := \mathbb{H}^{-i}(Y_{\mathrm{cdh}}, C_*(z_r^{\mathrm{equi}}(X))_{\mathrm{cdh}}).$$

$A_{r,i}(Y,X)$ is contravariant in Y and covariant in X (for proper maps).

We have the natural map

$$h_i(z_r^{\mathrm{equi}}(X))(Y) := H_i(C_*(z_r^{\mathrm{equi}}(X))(Y)) \to A_{r,i}(Y,X).$$

3.2. **Mayer-Vietoris and blow-up sequences.** Since Zariski open covers and abstract blow-ups are covering families in the cdh topology, we have a Mayer-Vietoris sequence for $U, V \subset Y$:

$$\ldots \to A_{r,i}(U \cup V, X) \to A_{r,i}(U,X) \oplus A_{r,i}(V,X) \\ \to A_{r,i}(U \cap V, X) \to A_{r,i-1}(U \cup V, X) \to \ldots$$

and for $p \amalg i : Y' \amalg F \to Y$:

$$\ldots \to A_{r,i}(Y,X) \to A_{r,i}(Y',X) \oplus A_{r,i}(F,X) \\ \to A_{r,i}(p^{-1}(F),X) \to A_{r,i-1}(Y,X) \to \ldots$$

Additional properties of $A_{r,i}$, to be discussed in the next lecture, require some fundamental results on the behavior of homotopy invariant PST's with respect to cdh-sheafification. Additionally, we will need some essentially algebro-geometric results comparing different cycle complexes. These two types of results are:

1. Acyclicity theorems. We have already seen the Nisnevich acyclicity theorem:

Theorem. *Let F be a PST F with $F_{\mathrm{Nis}} = 0$. Then the Suslin complex $C_*(F)_{\mathrm{Zar}}$ is acyclic.*

We will also need the cdh version:

Theorem (cdh-acyclity)**.** *Assume that k admits resolution of singularities. For F a PST with $F_{\mathrm{cdh}} = 0$, the Suslin complex $C_*(F)_{\mathrm{Zar}}$ is acyclic.*

This result transforms sequences of PST's which become short exact after cdh-sheafification, into distinguished triangles after applying $C_*(-)_{\mathrm{Zar}}$.

Using a hypercovering argument and Voevodsky's PST theorem, these results also show that cdh, Nis and Zar cohomology of a homotopy invariant PST all agree on smooth varieties:

Theorem (cdh-Nis-Zar)**.** *Assume that k admits resolution of singularities. For $U \in \mathbf{Sm}/k$, $F^* \in C^-(PST)$ such that the cohomology presheaves of F are homotopy invariant,*

$$\mathbb{H}^n(U_{\mathrm{Zar}}, F^*_{\mathrm{Zar}}) \cong \mathbb{H}^n(U_{\mathrm{Nis}}, F^*_{\mathrm{Nis}}) \cong \mathbb{H}^n(Y_{\mathrm{cdh}}, F^*_{\mathrm{cdh}})$$

We will derive the important consequences of the cdh acyclicity theorem for bivariant cohomology in the next lecture.

2. Moving lemmas. The bivariant cohomology $A_{r,i}$ is defined using cdh-hypercohomology of z_r^{equi}, so comparing z_r^{equi} with other complexes leads to identification of $A_{r,i}$ with cdh-hypercohomology of the other complexes. These comparisions of z_r^{equi} with other complexes is based partly on a number of very interesting geometric constructions, due to Friedlander-Lawson and Suslin. We will not discuss these results here, except to mention where they come in.

Lecture 6. **Mixed motives and cycle complexes, II**

Outline:
- Properties of bivariant cycle cohomology
- Morphisms and cycles
- Duality

1. Properties of bivariant cycle cohomology

To develop the properties of bivariant cycle cohomology, we need a number of tools.

We will use the global PST theorem:

Theorem (Global PST)**.** *Let F^* be a complex of PSTs on $\mathbf{Sm}/k$: $F \in C^-(PST)$. Suppose that the cohomology presheaves $h^i(F)$ are homotopy invariant. Then*

(1) For $Y \in \mathbf{Sm}/k$, $\mathbb{H}^i(Y_{\text{Nis}}, F^*_{\text{Nis}}) \cong \mathbb{H}^i(Y_{\text{Zar}}, F^*_{\text{Zar}})$
(2) The presheaf $Y \mapsto \mathbb{H}^i(Y_{\text{Nis}}, F^*_{\text{Nis}})$ is homotopy invariant

(1) and (2) follows from the PST theorem using the spectral sequence:

$$E_2^{p,q} = H^p(Y_\tau, h^q(F)_\tau) \Longrightarrow \mathbb{H}^{p+q}(Y_\tau, F_\tau), \tau = \text{Nis}, \text{Zar}.$$

Recall also:

Definition. Take $X, Y \in \mathbf{Sch}_k$. The *bivariant cycle cohomology* of Y with coefficients in cycles on X are

$$A_{r,i}(Y, X) := \mathbb{H}^{-i}(Y_{\text{cdh}}, C_*(z_r^{\text{equi}}(X))_{\text{cdh}}).$$

$A_{r,i}(Y, X)$ is contravariant in Y and covariant in X (for proper maps).

We have the natural map

$$h_i(z_r^{\text{equi}}(X))(Y) := H_i(C_*(z_r^{\text{equi}}(X))(Y)) \to A_{r,i}(Y, X).$$

The bivariant cycle cohomology $A_{r,i}(Y, X)$ has long exact Mayer-Vietoris sequence and a blow-up sequence with respect to Y.

We have already seen the Nisnevich acyclicity theorem:

Theorem. *Let F be a PST with $F_{\text{Nis}} = 0$. Then the Suslin complex $C_*(F)_{\text{Zar}}$ is acyclic.*

We will also need the cdh version:

Theorem (cdh-acyclity)**.** *Assume that k admits resolution of singularities. For F a PST with $F_{\text{cdh}} = 0$, the Suslin complex $C_*(F)_{\text{cdh}}$ is acyclic as on $\mathbf{Sch}_k$.*

Using a hypercovering argument again, and Voevodsky's PST theorem, these results lead to a proof that cdh, Nis and Zar cohomology of a homotopy invariant PST all agree on smooth varieties:

Theorem (cdh-Nis-Zar)**.** *Assume that k admits resolution of singularities. For $U \in \mathbf{Sm}/k$, $F^* \in C^-(PST)$ such that the cohomology presheaves of F are homotopy invariant,*

$$\mathbb{H}^n(U_{\text{Zar}}, F^*_{\text{Zar}}) \cong \mathbb{H}^n(U_{\text{Nis}}, F^*_{\text{Nis}}) \cong \mathbb{H}^n(Y_{\text{cdh}}, F^*_{\text{cdh}})$$

We will also use some essentially geometric *moving lemmas*, due to Suslin and Friedlander-Lawson. We will not discuss these results here, except to mention where they come in.

1.1. **Homotopy.** Bivariant cycle homolopy is homotopy invariant:

Proposition. *Suppose k admits resolution of singularities. Then the pull-back map*

$$p^* : A_{r,i}(Y,X) \to A_{r,i}(Y \times \mathbb{A}^1, X)$$

is an isomorphism.

Proof. Using hypercovers and resolution of singularities, we reduce to the case of smooth Y.

The cdh-Nis-Zar theorem changes the cdh hypercohomology defining $A_{r,i}$ to Nisnevich hypercohomology:

$$A_{r,i}(Y,X) = \mathbb{H}^i(Y_{\mathrm{Nis}}, C_*(z_r^{\mathrm{equi}}(X)_{\mathrm{Nis}}).$$

By the global PST theorem, the hypercohomology presheaves

$$Y \mapsto \mathbb{H}^i(Y_{\mathrm{Nis}}, C_*(z_r^{\mathrm{equi}}(X)_{\mathrm{Nis}}))$$

are homotopy invariant.

1.2. **The geometric comparison theorem.**

Theorem (Geometric comparison)**.** *Suppose k admits resolution of singularities. Take $X \in \mathbf{Sch}_k$. Then the natural map $z^{\mathrm{equi}}(X,*) \to z_r(X,*)$ is a quasi-isomorphism.*

This is based on *Suslin's moving lemma*, a purely algebro-geometric construction, in case X is affine. In addition, one needs to use the cdh techniques to prove a Meyer-Vietoris property for the complexes $z^{\mathrm{equi}}(X,*)$ (we'll see how this works a bit later).

1.3. **The geometric duality theorem.** Let $z_r^{\mathrm{equi}}(Z,X) := \mathcal{H}om(L(Z), z_r^{\mathrm{equi}}(X))$. Explicitly:

$$z_r^{\mathrm{equi}}(Z,X)(U) = z_r^{\mathrm{equi}}(X)(Z \times U).$$

We have the inclusion $z_r^{\mathrm{equi}}(Z,X) \to z_{r+\dim Z}^{\mathrm{equi}}(X \times Z)$.

Theorem (Geometric duality)**.** *Suppose k admits resolution of singularities. Take $X \in \mathbf{Sch}_k$, $U \in \mathbf{Sm}/k$, quasi-projective of dimension n. The inclusion $z_r^{\mathrm{equi}}(U,X) \to z_{r+n}^{\mathrm{equi}}(X \times U)$ induces a quasi-isomorphism of complexes on $\mathbf{Sm}/k_{\mathrm{Zar}}$:*

$$C_*(z_r^{\mathrm{equi}}(U,X))_{\mathrm{Zar}} \to C_*(z_{r+n}^{\mathrm{equi}}(X \times U))_{\mathrm{Zar}}$$

The proof for U and X smooth and projective uses the *Friedlander-Lawson moving lemma* for "moving cycles in a family". The extension to U smooth quasi-projective, and X general uses the cdh-acyclicity theorem.

1.4. **The** cdh **comparison and duality theorems.**

Theorem (cdh comparison)**.** *Suppose k admits resolution of singularities. Take $X \in \mathbf{Sch}_k$. Then for U smooth and quasi-projective, the natural map*

$$h_i(z_r^{\mathrm{equi}}(X))(U) \to A_{r,i}(U,X)$$

is an isomorphism.

Theorem (cdh duality)**.** *Suppose k admits resolution of singularities. Take $X, Y \in \mathbf{Sch}_k$, $U \in \mathbf{Sm}/k$ of dimension n. There is a canonical isomorphism*

$$A_{r,i}(Y \times U, X) \to A_{r+n,i}(Y, X \times U).$$

To prove the cdh comparison theorem, first use the cdh-Nis-Zar theorem to identify

$$\mathbb{H}^{-i}_{\mathrm{Zar}}(U, C_*(z_r^{\mathrm{equi}}(X))) \xrightarrow{\sim} \mathbb{H}^{-i}_{\mathrm{cdh}}(U, C_*(z_r^{\mathrm{equi}}(X))) =: A_{r,i}(U, X)$$

Next, if V_1, V_2 are Zariski open in U, use the geometric duality theorem to identify the Mayer-Vietoris sequence

$$C_*(z_r^{\mathrm{equi}}(X))(V_1 \cup V_2) \to C_*(z_r^{\mathrm{equi}}(X))(V_1) \oplus C_*(z_r^{\mathrm{equi}}(X))(V_2) \\ \to C_*(z_r^{\mathrm{equi}}(X))(V_1 \cap V_2)$$

with what you get by applying $C_*(-)(\mathrm{Spec}\, k)$ to

$$0 \to z_{r+d}^{\mathrm{equi}}(X \times (V_1 \cup V_2)) \to z_{r+d}^{\mathrm{equi}}(X \times V_1) \oplus z_{r+d}^{\mathrm{equi}}(X \times V_2) \\ \to z_{r+d}^{\mathrm{equi}}(X \times (V_1 \cap V_2))$$

$d = \dim U$.

But this presheaf sequence is exact, and $coker_{\mathrm{cdh}} = 0$. The cdh-acyclicity theorem thus gives us the distinguished triangle

$$C_*(z_{r+d}^{\mathrm{equi}}(X \times (V_1 \cup V_2)))_{\mathrm{Zar}} \\ \to C_*(z_{r+d}^{\mathrm{equi}}(X \times V_1))_{\mathrm{Zar}} \oplus C_*(z_{r+d}^{\mathrm{equi}}(X \times V_2))_{\mathrm{Zar}} \\ \to C_*(z_{r+d}^{\mathrm{equi}}(X \times (V_1 \cap V_2)))_{\mathrm{Zar}} \to$$

Evaluating at $\mathrm{Spec}\, k$, we find that our original Mayer-Vietoris sequence for $C_*(z_r^{\mathrm{equi}}(X))$ was in fact a distinguished triangle.

The Mayer-Vietoris property for $C_*(z_r^{\mathrm{equi}}(X))$ then formally implies that

$$h_i(C_*(z_r^{\mathrm{equi}}(X)))(U) \to \mathbb{H}^{-i}_{\mathrm{Zar}}(U, C_*(z_r^{\mathrm{equi}}(X))) = A_{r,i}(U, X)$$

is an isomorphism.

The proof of the cdh-duality theorem is similar, using the geometric duality theorem.

1.5. cdh-descent theorem.

Theorem (cdh-descent)**.** *Suppose k admits resolution of singularities. Take $Y \in$ $\mathbf{Sch}_k$.*
(1) Let $U \cup V = X$ be a Zariski open cover of $X \in \mathbf{Sch}_k$. There is a long exact sequence

$$\ldots \to A_{r,i}(Y, U \cap V) \to A_{r,i}(Y, U) \oplus A_{r,i}(Y, V) \\ \to A_{r,i}(Y, X) \to A_{r,i-1}(Y, U \cap V) \to \ldots$$

(2) Let $Z \subset X$ be a closed subset. There is a long exact sequence

$$\ldots \to A_{r,i}(Y, Z) \to A_{r,i}(Y, X) \to A_{r,i}(Y, X \setminus Z) \to A_{r,i-1}(Y, Z) \to \ldots$$

(3) Let $p \amalg i : X' \amalg F \to X$ be an abstract blow-up. There is a long exact sequence

$$\ldots \to A_{r,i}(Y, p^{-1}(F)) \to A_{r,i}(Y, X') \oplus A_{r,i}(Y, F) \\ \to A_{r,i}(Y, X) \to A_{r,i-1}(Y, p^{-1}(F)) \to \ldots$$

Proof. For (1) and (3), the analogous properties are obvious in the "first variable", so the theorem follows from duality.

For (2), the presheaf sequence

$$0 \to z_r^{\mathrm{equi}}(Z) \to z_r^{\mathrm{equi}}(X) \to z_r^{\mathrm{equi}}(X \setminus U)$$

is exact and $coker_{\mathrm{cdh}} = 0$. The cdh-acyclicity theorem says that applying $C_*(-)_{\mathrm{cdh}}$ to the above sequence yields a distinguished triangle.

1.6. **Localization for M^c_{gm}.** Continuing the argument for (2), the cdh-Nis-Zar theorem shows that the sequence

$$0 \to C_*(z_r^{\mathrm{equi}}(Z))_{\mathrm{Nis}} \to C_*(z_r^{\mathrm{equi}}(X))_{\mathrm{Nis}} \to C_*(z_r^{\mathrm{equi}}(X \setminus U))_{\mathrm{Nis}}$$

canonically defines a distinguished triangle in $DM_-^{\mathrm{eff}}(k)$. Taking $r = 0$ gives

Theorem (Localization)**.** *Suppose k admits resolution of singularities. Let $i : Z \to X$ be a closed immersion in $\mathbf{Sch}_k$ with complement $j : U \to X$. Then there is a canonical distinguished triangle in $DM_-^{\mathrm{eff}}(k)$*

$$M^c_{\mathrm{gm}}(Z) \xrightarrow{i_*} M^c_{\mathrm{gm}}(X) \xrightarrow{j^*} M^c_{\mathrm{gm}}(U) \to M^c_{\mathrm{gm}}(Z)[1]$$

Corollary. *Suppose k admits resolution of singularities. For each $X \in \mathbf{Sch}_k$, $M^c_{\mathrm{gm}}(X)$ is in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \subset DM^{\mathrm{eff}}_-(k)$.*

Proof. We proceed by induction on $\dim X$. First assume $X \in \mathbf{Sm}/k$. By resolution of singularities, we can find a smooth projective $\bar{X}$ containing X as a dense open subscheme. Since the complement $D := \bar{X} \setminus X$ has $\dim D <$ dim, $M^c_{\mathrm{gm}}(D)$ is in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

$M^c_{\mathrm{gm}}(\bar{X}) = M_{\mathrm{gm}}(\bar{X})$ since $\bar{X}$ is for proper. The localization distinguished triangle shows $M^c_{\mathrm{gm}}(X)$ is in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

For arbitrary X, take a stratification X_* of X by closed subschemes with $X_i \setminus X_{i-1}$. The localization triangle and the case of smooth X gives the result.

1.7. **A computation.**

Proposition. $M^c_{\mathrm{gm}}(\mathbb{A}^n) \cong \mathbb{Z}(n)[2n]$

Proof. For Z projective $M^c_{\mathrm{gm}}(Z) = M_{\mathrm{gm}}(Z)$. The localization sequence gives the distinguished triangle

$$M_{\mathrm{gm}}(\mathbb{P}^{n-1}) \to M_{\mathrm{gm}}(\mathbb{P}^n) \to M^c_{\mathrm{gm}}(\mathbb{A}^n) \to M_{\mathrm{gm}}(\mathbb{P}^{n-1})[1]$$

Then use the projective bundle formula:

$$M_{\mathrm{gm}}(\mathbb{P}^n) = \oplus_{i=0}^n \mathbb{Z}(i)[2i]$$
$$M_{\mathrm{gm}}(\mathbb{P}^{n-1}) = \oplus_{i=0}^{n-1} \mathbb{Z}(i)[2i].$$

Corollary (Duality)**.** *For $X, Y \in \mathbf{Sch}_k$, $n = \dim Y$ we have a canonical isomorphism*

$$\mathrm{CH}_{r+n}(X \times Y, i) \cong A_{r,i}(Y, X)$$

Proof. For $U \in \mathbf{Sm}/k$, quasi-projective, we have the quasi-isomorphisms

$$C_*(z_{r+n}^{\mathrm{equi}}(X \times U))(\mathrm{Spec}\, k) = z_{r+n}^{\mathrm{equi}}(X \times U, *) \to z_{r+n}(X \times U, *)$$

$$C_*(z_r^{\mathrm{equi}}(U, X))(\mathrm{Spec}\, k) \to C_*(z_{r+n}^{\mathrm{equi}}(X \times U))(\mathrm{Spec}\, k)$$

and the isomorphisms

$$A_{r,i}(U,X) \to A_{r+n,i}(\operatorname{Spec} k, X\times U) \leftarrow h_i(z^{\mathrm{equi}}_{r+n}(X\times U))(\operatorname{Spec} k)$$

This gives the isomorphism

$$\mathrm{CH}_{r+n}(X\times U, i) \to A_{r,i}(U,X).$$

One checks this map is natural with respect to the localization sequences for $\mathrm{CH}_{r+n}(X\times -, i)$ and $A_{r,i}(-,X)$.

Given $Y \in \mathbf{Sch}_k$, there is a filtration by closed subsets

$$\emptyset = Y_{-1} \subset Y_0 \subset \ldots \subset Y_m = Y$$

with $Y_i \setminus Y_{i-1} \in \mathbf{Sm}/k$ and quasi-projective (k is perfect), so this extends the result from $U \in \mathbf{Sm}/k$, quasi-projective, to $Y \in \mathbf{Sch}_k$.

Corollary. *Suppose k admits resolution of singularities. For $X, Y \in \mathbf{Sch}_k$ we have*
(1) (homotopy) *The projection $p : X \times \mathbb{A}^1 \to X$ induces an isomorphism $p^* : A_{r,i}(Y,X) \to A_{r+1,i}(Y, X\times\mathbb{A}^1)$.*

(2) (suspension) *The maps $i_0 : X \to X\times\mathbb{P}^1$, $p : X\times\mathbb{P}^1 \to X$ induce an isomorphism*

$$A_{r,i}(Y,X)\oplus A_{r-1,i}(Y,X) \xrightarrow{i_*+p^*} A_{r,i}(Y, X\times\mathbb{P}^1)$$

(3)(cosuspension) *There is a canonical isomorphism*

$$A_{r,i}(Y\times\mathbb{P}^1, X) \cong A_{r,i}(Y,X)\oplus A_{r+1,i}(Y,X)$$

(4) (localization) *Let $i : Z \to U$ be a codimension n closed embedding in $\mathbf{Sm}/k$. Then there is a long exact sequence*

$$\ldots \to A_{r+n,i}(Z,X) \to A_{r,i}(U,X) \xrightarrow{j^*} A_{r,i}(U\setminus Z, X) \to A_{r+n,i-1}(Z,X) \to \ldots$$

Proof. These all follow from the corresponding properties of $\mathrm{CH}^*(-,*)$ and the duality corollary:
(1) from homotopy
(2) and (3) from the projective bundle formula
(4) from the localization sequence.

2. Morphisms and cycles

We describe how morphisms in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ can be realized as algebraic cycles.

We assume throughout that k admits resolution of singularities.

2.1. Bivariant cycle cohomology reappears.

The cdh-acyclicity theorem relates the bivariant cycle cohomology (and hence higher Chow groups) with the morphisms in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

Theorem. *For $X, Y \in \mathbf{Sch}_k$ $r \geq 0$, $i \in \mathbb{Z}$, there is a canonical isomorphism*

$$\mathrm{Hom}_{DM^{\mathrm{eff}}_-(k)}(M_{\mathrm{gm}}(Y)(r)[2r+i], M^c_{\mathrm{gm}}(X)) \cong A_{r,i}(Y,X).$$

Proof. First use cdh hypercovers to reduce to $Y \in \mathbf{Sm}/k$.

For $r = 0$, the embedding theorem and localization theorem, together with the cdh-Nis-Zar theorem gives an isomorphism

$$\begin{aligned}\mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(C_*(Y)[i], C_*^c(X)) &\cong \mathbb{H}^{-i}(Y_{\mathrm{Nis}}, C_*(z_0^{\mathrm{equi}}(X))_{\mathrm{Nis}}) \\ &\cong \mathbb{H}^{-i}(Y_{\mathrm{cdh}}, C_*(z_0^{\mathrm{equi}}(X))_{\mathrm{cdh}}) = A_{0,i}(Y, X).\end{aligned}$$

To go to $r > 0$, use the case $r = 0$ for $Y \times (\mathbb{P}^1)^r$:

$$\mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(C_*(Y \times (\mathbb{P}^1)^r)[i], C_*^c(X)) \cong A_{0,i}(Y \times (\mathbb{P}^1)^r, X).$$

By the cosuspension isomorphism $A_{r,i}(Y, X)$ is a summand of $A_{0,i}(Y \times (\mathbb{P}^1)^r, X)$; by the definition of $\mathbb{Z}(1)$, $M_{\mathrm{gm}}(Y)(r)[2r]$ is a summand of $M_{\mathrm{gm}}(Y \times (\mathbb{P}^1)^r)$. One checks the two summands match up.

2.2. **Effective Chow motives.**

Corollary. *Sending a smooth projective variety X of dimension n to $M_{\mathrm{gm}}(X)$ extends to a full embedding $i : CHM^{\mathrm{eff}}(k)^{\mathrm{op}} \to DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$, $CHM^{\mathrm{eff}}(k) :=$ effective Chow motives,*

$$i(\mathfrak{h}(X)(-r)) = M_{\mathrm{gm}}(X)(r)$$

Proof. For X and Y smooth and projective

$$\begin{aligned}\mathrm{Hom}_{DM_{\mathrm{gm}}^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(Y), M_{\mathrm{gm}}(X)) &= A_{0,0}(Y, X) \\ &\cong A_{\dim Y,0}(\mathrm{Spec}\, k, Y \times X) \\ &\cong \mathrm{CH}_{\dim Y}(Y \times X) \\ &\cong \mathrm{CH}^{\dim X}(X \times Y) \\ &= \mathrm{Hom}_{CHM^{\mathrm{eff}}(k)}(X, Y).\end{aligned}$$

One checks that sending $a \in \mathrm{CH}^{\dim X}(X \times Y)$ to the corresponding map

$$[{}^t a] : M_{\mathrm{gm}}(Y) \to M_{\mathrm{gm}}(X)$$

satisfies $[{}^t(b \circ a)] = [{}^t a] \circ [{}^t b]$.

2.3. **The Chow ring reappears.**

Corollary. *For $Y \in \mathbf{Sch}_k$, equi-dimensional over k, $i \geq 0$, $j \in \mathbb{Z}$, $\mathrm{CH}^i(Y, j) \cong \mathrm{Hom}_{DM_{\mathrm{gm}}^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(Y), \mathbb{Z}(i)[2i - j])$. That is*

$$\mathrm{CH}^i(Y, j) \cong H^{2i-j}(Y, \mathbb{Z}(i)).$$

Take $i \geq 0$. Then $M_{\mathrm{gm}}^c(\mathbb{A}^i) \cong \mathbb{Z}(i)[2i]$ and

$$\begin{aligned}H^{2i-j}(Y, \mathbb{Z}(i)) &= \mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(Y)[j], M_{\mathrm{gm}}^c(\mathbb{A}^i)) \\ &\cong A_{0,j}(Y, \mathbb{A}^i) \\ &\cong A_{\dim Y,j}(\mathrm{Spec}\, k, Y \times \mathbb{A}^i) \\ &= \mathrm{CH}_{\dim Y}(Y \times \mathbb{A}^i, j) \\ &= \mathrm{CH}^i(Y \times \mathbb{A}^i, j) \\ &\cong \mathrm{CH}^i(Y, j)\end{aligned}$$

Remark. Combining the Chern character isomorphism

$$ch : K_j(Y)^{(i)} \cong \mathrm{CH}^i(Y, j)_{\mathbb{Q}}$$

(for $Y \in \mathbf{Sm}/k$) with our isomorphism $\mathrm{CH}^i(Y, j) \cong H^{2i-j}(Y, \mathbb{Z}(i))$ identifies rational motivic cohomology with weight-graded K-theory:

$$H^{2i-j}(Y, \mathbb{Q}(i)) \cong K_j(Y)^{(i)}.$$

Thus motivic cohomology gives an integral version of weight-graded K-theory, in accordance with conjectures of Beilinson on mixed motives.

Corollary (cancellation)**.** *For $A, B \in DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ the map*

$$- \otimes \mathrm{id} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A(1), B(1))$$

is an isomorphism. Thus

$$DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to DM_{\mathrm{gm}}(k)$$

is a full embedding.

Corollary. *For $Y \in \mathbf{Sch}_k$, $n, i \in \mathbb{Z}$, set*

$$H^n(Y, \mathbb{Z}(i)) := \mathrm{Hom}_{DM_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(Y), \mathbb{Z}(i)[n]).$$

Then $H^n(Y, \mathbb{Z}(i)) = 0$ for $i < 0$ and for $n > 2i$.

Corollary. *The full embedding $CHM^{\mathrm{eff}}(k)^{\mathrm{op}} \to DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ extends to a full embedding*

$$M_{\mathrm{gm}} : CHM(k)^{\mathrm{op}} \to DM_{\mathrm{gm}}(k).$$

Proof of the cancellation theorem. The Gysin distinguished triangle for for M_{gm} shows that $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ is generated by $M_{\mathrm{gm}}(X)$, X smooth and projective. So, we may assume $A = M_{\mathrm{gm}}(Y)[i]$, $B = M_{\mathrm{gm}}(X)$, X and Y smooth and projective, $i \in \mathbb{Z}$.

Then $M_{\mathrm{gm}}(X) = M^c_{\mathrm{gm}}(X)$ and $M_{\mathrm{gm}}(X)(1)[2] = M^c_{\mathrm{gm}}(X \times \mathbb{A}^1)$. Thus:

$$\begin{aligned}\mathrm{Hom}(M_{\mathrm{gm}}(Y)(1)[i], M_{\mathrm{gm}}(X)(1)) &\cong A_{1,i}(Y, X \times \mathbb{A}^1)\\ &\cong A_{0,i}(Y, X)\\ &\cong \mathrm{Hom}(M_{\mathrm{gm}}(Y)[i], M_{\mathrm{gm}}(X))\end{aligned}$$

For the second corollary, supposes $i < 0$. Cancellation implies

$$\begin{aligned}H^{2i-j}(Y, \mathbb{Z}(i)) &= \mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(Y)(-i)[j-2i], \mathbb{Z})\\ &= A_{-i,j}(Y, \mathrm{Spec}\, k)\\ &= A_{\dim Y - i, j}(\mathrm{Spec}\, k, Y)\\ &= H^{-j}(C_*(z^{\mathrm{equi}}_{\dim Y - i}(Y))(\mathrm{Spec}\, k)).\end{aligned}$$

Since $\dim Y - i > \dim Y$, $z^{\mathrm{equi}}_{\dim Y - i}(Y) = 0$.

If $i \geq 0$ but $n > 2i$, then $H^n(Y, \mathbb{Z}(i)) = \mathrm{CH}^i(Y, 2i - n) = 0$.

3. Duality

We describe the duality involution

$$^* : DM_{\mathrm{gm}}(k) \to DM_{\mathrm{gm}}(k)^{\mathrm{op}},$$

assuming k admits resolution of singularities.

3.1. **A reduction.**

Proposition. *Let $\mathcal{D}$ be a tensor triangulated category, $\mathcal{S}$ a subset of the objects of $\mathcal{D}$. Suppose*

1. Each $M \in \mathcal{S}$ has a dual M^*.

2. $\mathcal{D}$ is equal to the smallest full triangulated subcategory of $\mathcal{D}$ containing $\mathcal{S}$ and closed under isomorphisms in $\mathcal{D}$.

Then each object in $\mathcal{D}$ has a dual, i.e. $\mathcal{D}$ is a rigid tensor triangulated category.

Idea of proof For $M \in \mathcal{S}$, we have the unit and trace

$$\delta_M : \mathbb{1} \to M^* \otimes M,\ \epsilon_M : M \otimes M^* \to \mathbb{1}$$

satisfying

$$(\epsilon \otimes \mathrm{id}_M) \circ (\mathrm{id}_M \otimes \delta) = \mathrm{id}_M,\ (\mathrm{id}_{M^*} \otimes \epsilon) \circ (\delta \otimes \mathrm{id}_{M^*}) = \mathrm{id}_{M^*}$$

Show that, if you have such δ, ϵ for M_1, M_2 in a distinguished triangle

$$M_1 \xrightarrow{a} M_2 \to M_3 \to M_1[1]$$

you can construct δ_3, ϵ_3 with M_3^* fitting in a distinguished triangle

$$M_3^* \to M_2^* \xrightarrow{a^*} M_1^* \to M_3^*[1]$$

3.2. **Duality for X projective.**

Proposition. *For $X \in \mathbf{SmProj}/k$, $r \in \mathbb{Z}$, $M_{\mathrm{gm}}(X)(r) \in DM_{\mathrm{gm}}(k)$ has a dual $(M_{\mathrm{gm}}(X)(r))^*$.*

We use the full embedding $CHM(k)^{\mathrm{op}} \hookrightarrow DM_{\mathrm{gm}}(k)$ sending $\mathfrak{h}(X)(-r)$ to $M_{\mathrm{gm}}(X)(r)$, and the fact that $\mathfrak{h}(X)(-r)$ has a dual in $CHM(k)$.

Proposition. *Suppose k admits resolution of singularities. Then $DM_{\mathrm{gm}}(k)$ is the smallest full triangulated subcategory of $DM_{\mathrm{gm}}(k)$ containing the $M_{\mathrm{gm}}(Y)(r)$ for $Y \in \mathbf{SmProj}/k$, $r \in \mathbb{Z}$ and closed under isomorphisms in $DM_{\mathrm{gm}}(k)$.*

Proof. Take $X \in \mathbf{Sm}/k$. By resolution of singularities, there is a smooth projective $\bar{X}$ containing X as a dense open subscheme, such that $D := \bar{X} - X$ is a strict normal crossing divisor:

$$D = \cup_{i=1}^m D_i$$

with each D_i smooth codimension one on $\bar{X}$ and each intersection: $I = \{i_1, \ldots, i_r\}$

$$D_I := D_{i_1} \cap \ldots \cap D_{i_r}$$

is smooth of codimension r.

Then $\bar{X}$ and each $D_{i_1} \cap \ldots \cap D_{i_r}$ is in $\mathbf{SmProj}/k$. So $M_{\mathrm{gm}} = M^c_{\mathrm{gm}}$ for all these.

The Gysin triangle for $W \subset Y$ both smooth, $n = \mathrm{codim}_Y W$,

$$M_{\mathrm{gm}}(Y \setminus W) \to M_{\mathrm{gm}}(Y) \to M_{\mathrm{gm}}(W)(n)[2n] \to M_{\mathrm{gm}}(Y \setminus W)[1],$$

and induction on $\dim X$ and descending induction on r shows that

$$M_{\mathrm{gm}}(\bar{X} \setminus \cup_{|I|=r} D_I)$$

is in the category generated by the $M_{\mathrm{gm}}(Y)(r)$, $Y \in \mathbf{SmProj}/k$, $r \in \mathbb{Z}$.

Theorem. *Suppose k admits resolution of singularities. Then $DM_{\mathrm{gm}}(k)$ is a rigid tensor triangulated category.*

Remark. In fact, one can show that (after embedding in $DM^{\mathrm{eff}}_-(k)$)

$$M_{\mathrm{gm}}(X)^* = M^c_{\mathrm{gm}}(X)(-d_X)[-2d_X]$$

Lecture 7. Pure motives, II

This was supposed to have been the second lecture in the series. However, time constraints made this impossible. so I am are just adding it at the end. It probably would have been nice to discuss how the theory of mixed motives fits in to all the conjectures discussed in this lecture, but unfortunately, I didn't have time to do this, maybe next time!

Outline:

- Standard conjectures
- Decompositions of the diagonal
- Filtrations on the Chow ring
- Nilpotence conjecture
- Finite dimensionality

1. The standard conjectures

We would like to think of our functor

$$\mathfrak{h} : \mathbf{SmProj}/k^{\mathrm{op}} \to M_{\mathrm{hom}}(k)$$

as the "universal Weil cohomology". What is lacking:

• We have the "total cohomology" $\mathfrak{h}(X)$, we would like the individual cohomologies $\mathfrak{h}^r(X)$.

• Other "higher level" properties of cohomology are missing, e.g., Lefschetz theorems.

• $\sim_{\mathrm{hom}}$ could depend on the choice of Weil cohomology.

• $M_{\mathrm{hom}}(k)$ is not a category of vector spaces, but it is at least pseudo-abelian. It would be nice if it were an abelian category.

1.1. **Künneth projectors.** Fix a Weil cohomology H^* and an $X \in \mathbf{SmProj}/k$. By the Künneth formula, we have

$$H^*(X \times X) = H^*(X) \otimes H^*(X)$$

so

$$H^{2d_X}(X \times X)(d_X) = \oplus_{n=0}^{2d_X} H^n(X) \otimes H^{2d_X-n}(X)(d_X)$$

By Poincaré duality, $H^{2d_X-n}(X)(d_X) = H^n(X)^\vee$, so

$$\begin{aligned} H^{2d_X}(X \times X)(d_X) &= \oplus_{n=0}^{2d_X} H^n(X) \otimes H^n(X)^\vee \\ &= \oplus_{n=0}^{2d_X} \mathrm{Hom}_K(H^n(X), H^n(X)). \end{aligned}$$

$$\begin{aligned} H^{2d_X}(X \times X)(d_X) &= \oplus_{n=0}^{2d_X} H^n(X) \otimes H^n(X)^\vee \\ &= \oplus_{n=0}^{2d_X} \mathrm{Hom}_K(H^n(X), H^n(X)). \end{aligned}$$

This identifies $H^{2d_X}(X \times X)(d_X)$ with the vector space of graded K-linear maps $f : H^*(X) \to H^*(X)$ and writes

$$\mathrm{id}_{H^*(X)} = \sum_{n=0}^{2d_X} \pi^n_{X,H}; \quad \pi^n_H \in H^n(X) \otimes H^n(X)^\vee.$$

The term

$$\pi^n_{X,H} : H^*(X) \to H^*(X)$$

is the projection on $H^n(X)$, called the *Künneth projector*

Since $\mathrm{id}_{\mathfrak{h}_{\mathrm{hom}}(X)}$ is represented by the diagonal $\Delta_X \in \mathcal{Z}^{d_X}(X \times X)$, we have

$$\gamma_{X,H}(\Delta_*) = \mathrm{id}_{H^*(X)} = \sum_n \pi^n_{X,H}$$

We can ask: are there correspondences $\pi^n_X \in \mathcal{Z}^{d_X}_{\mathrm{hom}}(X \times X)_{\mathbb{Q}}$ with

$$\gamma_{X,H}(\pi^n_X) = \pi^n_{X,H}.$$

Remarks. 1. The $\pi^n_{X,H}$ are idempotent endomorphisms $\implies (X, \pi^n_X)$ defines a summand $\mathfrak{h}^n(X)$ of $\mathfrak{h}(X)$ in $M^{\mathrm{eff}}_{\mathrm{hom}}(k)_{\mathbb{Q}}$.

2. If π^n_X exists, it is unique.

3. π^n_X exists iff $\mathfrak{h}_{\mathrm{hom}}(X) = \mathfrak{h}^n(X) \oplus \mathfrak{h}(X)'$ in $M^{\mathrm{eff}}(k)_{\mathbb{Q}}$ with $H^*(\mathfrak{h}^n(X)) \subset H^*(X)$ equal to $H^n(X)$.

If all the π^n_X exist:

$$\mathfrak{h}_{\mathrm{hom}}(X) = \oplus_{n=0}^{2d_X} \mathfrak{h}^n_{\mathrm{hom}}(X)$$

X has a *Künneth decomposition.*

Examples. 1. The decomposition

$$\mathfrak{h}(\mathbb{P}^n) = \oplus_{r=0}^n \mathfrak{h}^{2r}(\mathbb{P}^n)$$

in $CHM^{\mathrm{eff}}(k)$ maps to a Künneth decomposition of $\mathfrak{h}_{\mathrm{hom}}(\mathbb{P}^n)$.

2. For a curve C, the decomposition (depending on a choice of $0 \in C(k)$)

$$\mathfrak{h}(C) = \mathfrak{h}^0(X) \oplus \mathfrak{h}^1(C) \oplus \mathfrak{h}^2(C); \quad \mathfrak{h}^0(C) \cong \mathbb{1}, \mathfrak{h}^2(C) \cong \mathbb{1}(-1),$$

in $CHM^{\mathrm{eff}}(k)$ maps to a Künneth decomposition of $\mathfrak{h}_{\mathrm{hom}}(C)$.

3. For each $X \in \mathbf{SmProj}/k$, a choice of a k-point gives factors

$$\mathfrak{h}^0(X) := (X, 0 \times X) \cong \mathbb{1}$$
$$\mathfrak{h}^{2d_X}(X) := (X, X \times 0) \cong \mathbb{1}(-d_X).$$

of $\mathfrak{h}(X)$. Using the Picard and Albanese varieties of X, one can also define factors $\mathfrak{h}^1(X)$ and $\mathfrak{h}^{2d-1}(X)$, so

$$\mathfrak{h}(X) = \mathfrak{h}^0(X) \oplus \mathfrak{h}^1(X) \oplus \mathfrak{h}(X)' \oplus \mathfrak{h}^{2d_X-1}(X) \oplus \mathfrak{h}^{2d_X}(X)$$

which maps to a partial Künneth decomposition in $M^{\mathrm{eff}}_{\mathrm{hom}}(k)_{\mathbb{Q}}$. For $d_X = 2$, this gives a full Künneth decomposition (Murre).

1.2. **The Künneth conjecture.**

Conjecture (C(X))**.** *The Künneth projectors $\pi^n_{X,H}$ are algebraic for all n:*

$$\mathfrak{h}_{\mathrm{hom}}(X) = \oplus_{n=0}^{2d_X} \mathfrak{h}^n_{\mathrm{hom}}(X)$$

with $H^*(\mathfrak{h}^n_{\mathrm{hom}}(X)) = H^n(X) \subset H^*(X)$.

Consequence. *Let $a \in \mathcal{Z}^{d_X}(X \times X)_{\mathbb{Q}}$ be a correspondence.*

1. The characteristic polynomial of $H^n(a)$ on $H^n(X)$ has $\mathbb{Q}$-coefficients.

2. If $H^n(a) : H^n(X) \to H^n(X)$ is an automorphism, then $H^n(a)^{-1} = H^*(b)$ for some correspondence $b \in \mathcal{Z}^{d_X}(X \times X)_{\mathbb{Q}}$.

Proof. (1) The Lefschetz trace formula gives

$$Tr(a^m)_{|H^n(X)} = (-1)^n \deg({}^t a^m \cdot \pi_X^n) \in \mathbb{Q}.$$

But

$$\det(1 - ta_{|H^n(X)}) = \exp(-\sum_{m=1}^{\infty} \frac{1}{m} Tr(a^m_{H^n(X)})t^m).$$

(2) By Cayley-Hamilton and (1), there is a $Q_n(t) \in \mathbb{Q}[t]$ with

$$\begin{aligned} H^n(a)^{-1} &= Q_n(H^n(a)) \\ &= H^n(Q_n(a)) \\ &= H^*(Q_n(a)\pi_X^n) \end{aligned}$$

1.3. **Status:** $C(X)$ is known for "geometrically cellular" varieties ($\mathbb{P}^n$, Grassmannians, flag varieties, quadrics, etc.), curves, surfaces and abelian varieties: For an abelian variety A, one has

$$\mathfrak{h}^n_{\mathrm{hom}}(A) = \Lambda^n(\mathfrak{h}^1_{\mathrm{hom}}(A)).$$

$C(X)$ is true for all X if the base-field k is a finite field $\mathbb{F}_q$ and $H^* = H^*_{\text{ét}}(-, \mathbb{Q}_\ell)$:

Use the Weil conjectures to show that the characteristic polynomial $P_n(t)$ of Fr_X on $H^n(X, \mathbb{Q}_\ell)$ has $\mathbb{Q}$-coefficients and that $P_n(t)$ and $P_m(t)$ are relatively prime for $n \neq m$. Cayley-Hamilton and the Chinese remainder theorem yield polynomials $Q_n(t)$ with $\mathbb{Q}$-coefficients and

$$Q_n(Fr_X^*)_{|H^m(X)} = \delta_{n,m}\mathrm{id}_{H^m(X)}.$$

Then $\pi_X^n = Q_n({}^t\Gamma_{Fr_X})$.

1.4. **The sign conjecture** $C^+(X)$**.** This is a weak version of $C(X)$, saying that $\pi^+_{X,H} := \sum_{n=0}^{d_X} \pi^{2n}_{X,H}$ is algebraic. Equivalently, $\pi^-_{X,H} := \sum_{n=1}^{d_X} \pi^{2n-1}_{X,H}$ is algebraic.

$C^+(X)$ for all X/k says that we can impose a $\mathbb{Z}/2$-grading on $M_{\mathrm{hom}}(k)_{\mathbb{Q}}$:

$$\mathfrak{h}_{\mathrm{hom}}(X) = \mathfrak{h}^+_{\mathrm{hom}}(X) \oplus \mathfrak{h}^-_{\mathrm{hom}}(X)$$

so that $H^* : M_{\mathrm{hom}}(k)_{\mathbb{Q}} \to \mathrm{GrVec}_K$ defines

$$H^{\pm} : M_{\mathrm{hom}}(k)_{\mathbb{Q}} \to s\mathrm{Vec}_K$$

respecting the $\mathbb{Z}/2$ grading, where $s\mathrm{Vec}_K$ the tensor category of finite dimensional $\mathbb{Z}/2$-graded K vector spaces.

Consequence. *Suppose $C^+(X)$ for all $X \in$ **SmProj**$/k$. Then*

$$M_{\mathrm{hom}}(k)_{\mathbb{Q}} \to M_{\mathrm{num}}(k)_{\mathbb{Q}}$$

is conservative and essentially surjective.

This follows from:

Lemma. $C^+(X) \implies$ *the kernel of* $\mathcal{Z}^{d_X}_{\mathrm{hom}}(X \times X)_{\mathbb{Q}} \to \mathcal{Z}^{d_X}_{\mathrm{num}}(X \times X)_{\mathbb{Q}}$ *is a nil-ideal, hence* $\ker \subset \mathcal{R}$.

Proof. For $f \in ker$, $\deg(f^n \cdot \pi^+_X) = \deg(f^n \cdot \pi^-_X) = 0$. By Lefschetz

$$Tr(\gamma(f^n)_{|H^+(X)}) = Tr(\gamma(f^n)_{|H^-(X)}) = 0$$

Thus $\gamma(f)_{|H^*(X)}$ has characteristic polynomial t^N, $N = \dim H^*(X)$. $\square$

Remark. André and Kahn use the fact that the kernel of $M_{\mathrm{hom}}(k)_{\mathbb{Q}} \to M_{\mathrm{num}}(k)_{\mathbb{Q}}$ is a $\otimes$ nilpotent ideal to define a canonical $\otimes$ functor $M_{\mathrm{num}}(k)_{\mathbb{Q}} \to M_{\mathrm{hom}}(k)_{\mathbb{Q}}$. This allows one to define the "homological realization" for $M_{\mathrm{num}}(k)_{\mathbb{Q}}$.

1.5. **The Lefschetz theorem.** Take a smooth projective X over k with an embedding $X \subset \mathbb{P}^N$. Let $i : Y \hookrightarrow X$ be a smooth hyperplane section.

For a Weil cohomology H^*, this gives the operator

$$L : H^*(X) \to H^{*-2}(X)(-1)$$
$$L(x) := i_*(i^*(x)) = \gamma([Y]) \cup x.$$

L lifts to the correspondence $Y \times X \subset X \times X$.

The strong Leschetz theorem is

Theorem. *For H^* a "classical" Weil cohomology and $i \leq d_X$*

$$L^{d_X - i} : H^i(X) \to H^{2d_X - i}(X)(d_X - i)$$

is an isomorphism.

1.6. **The conjecture of Lefschetz type.** Let $*_{L,X}$ be the involution of $\oplus_{i,r} H^i(X)(r)$:

$$*_{L,X} \text{ on } H^i(X)(r) := \begin{cases} L^{d_X - i} & \text{for } 0 \leq i \leq d_X \\ (L^{i-d_X})^{-1} & \text{for } d_X < i \leq 2d_X. \end{cases}$$

Conjecture (B(X))**.** *The Lefschetz involution $*_{L,X}$ is algebraic: there is a correspondence $\alpha_{L,X} \in \mathcal{Z}^*(X \times X)_{\mathbb{Q}}$ with $\gamma(\alpha) = *_{L.X}$*

1.7. **Status.** $B(X)$ is known for curves, and for abelian varieties (Kleiman-Grothendieck). For abelian varieties Lieberman showed that the operator Λ (related to the inverse of L) is given by Pontryagin product (translation) with a rational multiple of $Y^{(d-1)}$.

1.8. **Homological and numerical equivalence.**

Conjecture (D(X))**.** $\mathcal{Z}^*_{\mathrm{hom}}(X)_{\mathbb{Q}} = \mathcal{Z}^*_{\mathrm{num}}(X)_{\mathbb{Q}}$

Proposition. *For $X \in \mathbf{SmProj}/k$, $D(X^2) \Longrightarrow \mathrm{End}_{M_{\mathrm{hom}}(k)_{\mathbb{Q}}}(\mathfrak{h}(X))$ is semi-simple.*

$D(X^2) \Longrightarrow \mathrm{End}_{M_{\mathrm{hom}}(k)_{\mathbb{Q}}}(\mathfrak{h}(X)) = \mathrm{End}_{M_{\mathrm{num}}(k)_{\mathbb{Q}}}(\mathfrak{h}(X))$, which is semi-simple by Jannsen's theorem.

Similarly, Jannsen's theorem shows:

Proposition. *If $D(X)$ is true for all $X \in \mathbf{SmProj}/k$, then $H^* : M_{\mathrm{hom}}(k)_F \to \mathrm{GrVec}_K$ is conservative and exact.*

In fact: $D(X^2) \Longrightarrow B(X) \Longrightarrow C(X)$.

Thus, if we know that hom = num (with $\mathbb{Q}$-coefficients) we have our universal cohomology of smooth projective varieties

$$\mathfrak{h} = \oplus_i \mathfrak{h}^i : \mathbf{SmProj}(k)^{\mathrm{op}} \to NM(k)_{\mathbb{Q}}$$

with values in the semi-simple abelian category $NM(k)_{\mathbb{Q}}$.

Also, for H^* = Betti cohomology, $B(X) \Longrightarrow D(X)$, so it would suffice to prove the conjecture of Lefschetz type.

$D(X)$ is known in codimension $0, d_X$ and for codimension 1 (Matsusaka's thm). In characteristic 0, also for codimension $2, d_X - 1$ and for abelian varieties (Lieberman).

2. Decompositions of the diagonal

We look at analogs of the Künneth projectors for $CHM(k)_{\mathbb{Q}}$.
First look at two basic properties of the Chow groups.

2.1. Localization.

Theorem. *Let $i : W \to X$ be a closed immersion, $j : U \to X$ the complement. Then*

$$\mathrm{CH}_r(W) \xrightarrow{i_*} \mathrm{CH}_r(X) \xrightarrow{j^*} \mathrm{CH}_r(U) \to 0$$

is exact.

Proof.

$$0 \to \mathcal{Z}_r(W) \xrightarrow{i_*} \mathcal{Z}_r(X) \xrightarrow{j^*} \mathcal{Z}_r(U) \to 0$$

is exact: Look at the basis given by subvarieties. At $\mathcal{Z}_r(U)$ take the closure to lift to $\mathcal{Z}_r(X)$. At $\mathcal{Z}_r(X)$ $j^{-1}(Z) = \emptyset$ means $Z \subset W$.

Do the same for $W \times \mathbb{P}^1 \subset X \times \mathbb{P}^1$ and use the snake lemma. □

2.2. Continuity.

Proposition. *Let $t : \mathrm{Spec}\,(L) \to T$ be a geometric generic point and take $X \in \mathbf{Sch}_k$ equi-dimensional. If $\eta \in \mathrm{CH}^r(X \times T)_{\mathbb{Q}} \mapsto 0 \in \mathrm{CH}^r(X_t)_{\mathbb{Q}}$, then there is a Zariski open subset of T containing the image of t such that $\eta \mapsto 0 \in \mathrm{CH}^r(X \times U)$.*

$\eta_t = 0 \Rightarrow \eta_K = 0$ for some $K/k(T)$ finite, Galois.
But $\mathrm{CH}^r(X_K)_{\mathbb{Q}}^{\mathrm{Gal}} = \mathrm{CH}^r(X_{k(X)})_{\mathbb{Q}} \Rightarrow \eta_{k(X)} = 0 \in \mathrm{CH}^r(X_{k(X)})_{\mathbb{Q}}$.
But $\mathrm{CH}^r(X_{k(X)}) = \lim_{\emptyset \neq U \subset T} \mathrm{CH}^r(X \times U)$.

Remark. This result is *false* for other $\sim$, e.g. $\sim_{\mathrm{hom}}$, $\sim_{\mathrm{alg}}$.

2.3. The first component.

Proposition (Bloch)**.** *$X \in \mathbf{SmProj}/k$. Suppose $\mathrm{CH}_0(X_{\bar{L}})_{\mathbb{Q}} = \mathbb{Q}$ (by degree) for all finitely generated field extensions $L \supset k$. Then*

$$\Delta_X \sim_{\mathrm{rat}} X \times 0 + \rho$$

with $\rho \in \mathcal{Z}^{d_X}(X \times X)$ supported in $D \times X$ for some divisor $D \subset X$ and $0 \in \mathrm{CH}_0(X)_{\mathbb{Q}}$ any degree 1 cycle.

Proof. Let $i : \eta \to X$ be a geometric generic point. Then $i^*(X \times 0)$ and $i^*(\Delta_X)$ are in $\mathrm{CH}_0(X_{k(\eta)})$ and both have degree 1. Thus

$$(i \times \mathrm{id})^*(X \times 0) = (i \times \mathrm{id})^*(\Delta_X) \text{ in } \mathrm{CH}_0(X_{k(\eta)})_{\mathbb{Q}}$$

By continuity, there is a dense open subscheme $j : U \hookrightarrow X$ with

$$(j \times \mathrm{id})^*(X \times 0) = (j \times \mathrm{id})^*(\Delta_X^*) \text{ in } \mathrm{CH}_0(U \times X)_{\mathbb{Q}}$$

By localization there is a $\tau \in \mathcal{Z}_{d_X}(D \times X)$ for $D = X \setminus U$ with

$$\Delta_X - X \times 0 = (i_{D*} \times \mathrm{id})_*(\tau) =: \rho.$$

2.4. **Mumford's theorem.** Take $k = \bar{k}$. Each X in $\mathbf{SmProj}/k$ has an associated *Albanese variety* $\mathrm{Alb}(X)$. A choice of $0 \in X(k)$ gives a morphism $\alpha_X : X \to \mathrm{Alb}(X)$ sending 0 to 0, which is universal for pointed morphisms to abelian varieties.

Extending by linearity and noting $\mathrm{Alb}(X \times \mathbb{P}^1) = \mathrm{Alb}(X)$ gives a canonical map

$$\alpha_X : \mathrm{CH}_0(X)_{\deg 0} \to \mathrm{Alb}(X)$$

Theorem (Mumford)**.** *X: smooth projective surface over $\mathbb{C}$. If $H^0(X, \Omega^2) \neq 0$, then the Albanese map $\alpha_X : \mathrm{CH}_0(X)_{\deg 0} \to \mathrm{Alb}(X)$ has "infinite dimensional" kernel.*

Here is Bloch's motivic proof (we simplify: assume $\mathrm{Alb}(X) = 0$, and show only that $\mathrm{CH}_0(X)_{\mathbb{Q}}$ is not $\mathbb{Q}$).

Since $\mathbb{C}$ has infinite transcendence degree over $\mathbb{Q}$, $\mathrm{CH}_0(X)_{\mathbb{Q}} = \mathbb{Q}$ implies $\mathrm{CH}_0(X_{\bar{L}})_{\mathbb{Q}} = \mathbb{Q}$ for all finitely generated fields $L/\mathbb{C}$.

Apply Bloch's decomposition theorem: $\Delta_X \sim_{\mathrm{rat}} X \times 0 + \rho$. Since

$$H^0(X, \Omega^2) = H^0(X \times \mathbb{P}^1, \Omega^2)$$

$\Delta_{X*} = (X \times 0)_* + \rho_*$ on 2-forms.

If $\omega \in H^0(X, \Omega^2)$ is a two form, then

$$\omega = \Delta_*(\omega) = (X \times 0)_*(\omega) + \rho_*(\omega) = 0 :$$

$(X \times 0)_*(\omega)$ is 0 on $X \setminus \{0\}$. $\rho_*(\omega)$ factors through the restriction $\omega_{|D}$. D is a curve, so $\omega_{|D} = 0$.

2.5. **Jannsen's surjectivity theorem.**

Theorem (Jannsen)**.** *Take $X \in \mathbf{SmProj}/\mathbb{C}$. Suppose the cycle-class map*

$$\gamma^r : \mathrm{CH}^r(X)_{\mathbb{Q}} \to H^{2r}(X(\mathbb{C}), \mathbb{Q})$$

is injective for all r. Then $\gamma^* : \mathrm{CH}^*(X) \to H^*(X, \mathbb{Q})$ is surjective, in particular $H^{odd}(X, \mathbb{Q}) = 0$.

Corollary. *If $\gamma^* : \mathrm{CH}^*(X)_{\mathbb{Q}} \to H^*(X(\mathbb{C}), \mathbb{Q})$ is injective, then the Hodge spaces $H^{p,q}(X)$ vanish for $p \neq q$.*

Compare with Mumford's theorem: if X is a surface and $\mathrm{CH}_0(X)_{\mathbb{Q}} = \mathbb{Q}$, then $H^{2,0}(X) = H^{0,2}(X) = 0$.

Note. The proof shows that the injectivity assumption yields a full decomposition of the diagonal

$$\Delta_X = \sum_{i=0}^{d_X} \sum_{j=1}^{n_i} a^{ij} \times b_{ij} \text{ in } \mathrm{CH}^{d_X}(X \times X)_{\mathbb{Q}}$$

with $a^{ij} \in \mathcal{Z}^i(X)_{\mathbb{Q}}$, $b_{ij} \in \mathcal{Z}_i(X)_{\mathbb{Q}}$. Applying Δ_{X*} to a cohomology class $\eta \in H^r(X, \mathbb{Q})$ gives

$$\eta = \Delta_{X*}(\eta) = \sum_{ij} Tr(\eta \cup \gamma(a^{ij})) \times \gamma(b_{ij})$$

This is 0 if r is odd, and is in the $\mathbb{Q}$-span of the $\gamma(b_{ij})$ for $r = 2d_X - 2i$.

Conversley, a decomposition of Δ_X as above yields

$$\mathfrak{h}(X)_{\mathbb{Q}} \cong \sum_{i=0}^{d_X} \mathbb{1}(-i)_{\mathbb{Q}}^{n_i} \text{ in } CHM(\mathbb{C})_{\mathbb{Q}}$$

which implies $\mathrm{CH}_i(X)_{\mathbb{Q}}$ is the $\mathbb{Q}$-span of the b_{ij} and that γ^* is an isomorphism.

Proof. Show by induction that

$$\Delta_X = \sum_{i=0}^{r}\sum_{j=1}^{n_i} a^{ij}\times b_{ij} + \rho^r \text{ in } \mathrm{CH}^{d_X}(X\times X)_{\mathbb{Q}}$$

with $a^{ij}\in \mathcal{Z}^i(X)_{\mathbb{Q}}$, $b_{ij}\in\mathcal{Z}_i(X)_{\mathbb{Q}}$ and ρ^r supported on $Z^r\times X$, $Z^r\subset X$ a closed subset of codimension $r+1$.

The case $r=0$ is Bloch's decomposition theorem, since $H^{2d_X}(X,\mathbb{Q})=\mathbb{Q}$.

To go from r to $r+1$: ρ^r has dimension d_X. Think of $\rho^r\to Z^r$ as a familiy of codimension d_X-r-1 cycles on X, parametrized by Z^r (at least over some dense open subschemeof Z^r):

$$z\mapsto \rho^r(z)\in \mathrm{CH}^{d-r-1}(X)_{\mathbb{Q}} \xrightarrow{\gamma} H^{2d-2r-2}(X,\mathbb{Q})$$

For each component Z_i of Z, fix one point z_i. Then

$$\rho^r - \sum_i Z_i\times\rho^r(z_i)$$

goes to zero in $H^{2d-2r-2}(X,\mathbb{Q})$ at each geometric generic point of Z^r. Thus the cycle goes to zero in $\mathrm{CH}^{d-r-1}(X_{k(\eta_j)})$ for each generic point $\eta_j\in Z^r$.

By continuity, there is a dense open $U\subset Z^r$ with

$$(\rho^r - \sum_i Z_i\times\rho^r(z_i))\cap U\times X = 0 \text{ in } \mathrm{CH}^{d_X}(U\times X)_{\mathbb{Q}}$$

By localization

$$\rho^r = \sum_i Z_i\times\rho^r(z_i) + \rho^{r+1}\in \mathrm{CH}^*(Z^r\times X)_{\mathbb{Q}}$$

with ρ^{r+1} supported in $Z^{r+1}\times X$, $Z^{r+1}=X\setminus U$.

Combining with the identity for r gives

$$\Delta_X = \sum_{i=0}^{r+1}\sum_{j=1}^{n_i} a^{ij}\times b_{ij} + \rho^r \text{ in } \mathrm{CH}^{d_X}(X\times X)_{\mathbb{Q}}$$

2.6. **Esnault's theorem.**

Theorem (Esnault)**.** *Let X be a smooth Fano variety over a finite field $\mathbb{F}_q$. Then X has an $\mathbb{F}_q$-rational point.*

Recall: X is a Fano variety if $-K_X$ is ample.

Proof. Kollár shows that X Fano $\Longrightarrow X_{\bar{k}}$ is rationally connected (each two points are connected by a chain of rational curves).

Thus $\mathrm{CH}_0(X_L)_{\mathbb{Q}}=\mathbb{Q}$ for all $L\supset\bar{\mathbb{F}}_q$. Now use Bloch's decomposition (transposed):

$$\Delta_{\bar{X}} = 0\times\bar{X}+\rho$$

$0\in X(\bar{\mathbb{F}}_q)$, ρ supported on $\bar{X}\times D$.

Thus $H^n_{\text{ét}}(\bar{X},\mathbb{Q}_\ell)\to H^n_{\text{ét}}(\bar{X}\setminus D,\mathbb{Q}_\ell)$ is the zero map for all $n\geq 1$.

Purity of étale cohomology $\Longrightarrow$ EV of Fr_X on $H^n_{\text{ét}}(\bar{X},\mathbb{Q}_\ell)$ are divisible by q for $n\geq 1$.

Lefschetz fixed point formula $\Longrightarrow$

$$\#X(\mathbb{F}_q) = \sum_{n=0}^{2d_X}(-1)^n Tr(Fr_{X|H^n_{\text{ét}}(\bar{X},\mathbb{Q})}) \equiv 1 \mod q$$

2.7. **Bloch's conjecture.**

Conjecture. *Let X be a smooth projective surface over $\mathbb{C}$ with $H^0(X, \Omega^2) = 0$. Then the Albanese map*

$$\alpha_X : \mathrm{CH}_0(X) \to \mathrm{Alb}(X)$$

is an isomorphism.

This is known for surfaces *not* of general type (K_X ample) by Bloch-Kas-Lieberman, and for many examples of surfaces of general type.

Roitman has shown that α_X is an isomorphism on the torsion subgroups for arbitrary smooth projective X over $\mathbb{C}$.

2.8. **A motivic viewpoint.** Since X is a surface, we have Murre's decomposition of $\mathfrak{h}_{\mathrm{rat}}(X)_{\mathbb{Q}}$:

$$\mathfrak{h}(X)_{\mathbb{Q}} = \oplus_{i=0}^4 \mathfrak{h}^i(X)_{\mathbb{Q}} \cong \mathbb{1} \oplus \mathfrak{h}^1 \oplus \mathfrak{h}^2 \oplus \mathfrak{h}^1(-1) \oplus \mathbb{1}(-2).$$

Murre defined a filtration of $\mathrm{CH}^2(X)_{\mathbb{Q}}$:

$$F^0 := \mathrm{CH}^2(X)_{\mathbb{Q}} \subset F^1 = \mathrm{CH}^2(X)_{\mathbb{Q}\deg 0} \supset F^2 := \ker \alpha_X \supset F^3 = 0$$

and showed

$$F^2 = \mathrm{CH}^2(\mathfrak{h}^2(X)), \mathrm{gr}^1_F = \mathrm{CH}^2(\mathfrak{h}^3(X)), \mathrm{gr}^0_F = \mathrm{CH}^2(\mathfrak{h}^4(X))$$

$\mathrm{CH}^2(\mathfrak{h}^i(X)) = 0$ for $i = 0, 1$.

Suppose $p_g = 0$. Choose representatives $z_i \in \mathrm{CH}^1(X)$ for $\mathcal{Z}^1_{\mathrm{num}}(X)_{\mathbb{Q}} = H^2(X, \mathbb{Q})(1)$.

Since $\mathrm{CH}^1(X) = \mathrm{Hom}_{CHM}(\mathbb{1}(-1), \mathfrak{h}(X))$, we can use the z_i to lift $\mathfrak{h}^2_{\mathrm{num}}(X) = \mathbb{1}(-1)^\rho$ to a direct factor of $\mathfrak{h}^2(X)_{\mathbb{Q}}$:

$$\mathfrak{h}^2(X)_{\mathbb{Q}} = \mathbb{1}(-1)^\rho \oplus \mathfrak{t}^2$$

with $\mathfrak{t}^2_{\mathrm{num}}(X) = 0$.

$\mathrm{CH}^2(\mathbb{1}(-1)) := \mathrm{Hom}_{CHM(k)}(\mathbb{1}(-2), \mathbb{1}(-1)) = \mathrm{CH}^1(\mathrm{Spec}\, k) = 0$.

So Bloch's conjecture is:

$$\mathrm{CH}^2(\mathfrak{t}^2(X)) = 0.$$

3. Filtrations on the Chow ring

We have seen that a lifting of the Künneth decompostion in $\mathcal{Z}^*_{\mathrm{num}}(X^2)_{\mathbb{Q}}$ to a sum of products in $\mathrm{CH}^*(X^2)_{\mathbb{Q}}$ imposes strong restrictions on X. However, one can still ask for a lifting of the Künneth projectors π^n_X (assuming $C(X)$) to a mutually orthogonal decomposition of Δ_X in $\mathrm{CH}^*(X^2)_{\mathbb{Q}}$.

This leads to an interesting filtration on $\mathrm{CH}^*(X)_{\mathbb{Q}}$, generalizing the situation for dimension 2.

3.1. **Murre's conjecture.**

Conjecture (Murre)**.** *For all $X \in \mathbf{SmProj}/k$:*

1. the Künneth projectors π^n_X are algebraic.

2. There are lifts Π^n_X of π^n_X to $\mathrm{CH}^{d_X}(X^2)_{\mathbb{Q}}$ such that

i. the Π^n_X are mutually orthogonal idempotents with $\sum_n \Pi^n_X = 1$.

ii. Π^n_X acts by 0 on $\mathrm{CH}^r(X)_{\mathbb{Q}}$ for $n > 2r$

iii. the filtration

$$F^\nu \mathrm{CH}^r(X)_{\mathbb{Q}} := \cap_{n > 2r-\nu} \ker \Pi^n_X$$

is independent of the choice of lifting

iv. $F^1\mathrm{CH}^*(X)_{\mathbb{Q}} = \ker(\mathrm{CH}^*(X)_{\mathbb{Q}} \to \mathcal{Z}^r_{\mathrm{hom}}(X)_{\mathbb{Q}})$.

In terms of a motivic decomposition, this is the same as:
1. $\mathfrak{h}_{\mathrm{hom}}(X)$ has a Künneth decomposition in $M_{\mathrm{hom}}(k)_{\mathbb{Q}}$:

$$\mathfrak{h}_{\mathrm{hom}}(X) = \oplus_{n=0}^{2d_X} \mathfrak{h}^n_{\mathrm{num}}(X)$$

2. This decomposition lifts to a decomposition in $CHM(k)_{\mathbb{Q}}$:

$$\mathfrak{h}(X) = \oplus_{n=0}^{2d_X} \mathfrak{h}^n(X)$$

such that

ii. $\mathrm{CH}^r(\mathfrak{h}^n(X)) = 0$ for $n > 2r$
iii. the filtration

$$F^\nu \mathrm{CH}^r(X)_{\mathbb{Q}} = \sum_{n \leq 2r-\nu} \mathrm{CH}^r(\mathfrak{h}^n(X))$$

is independent of the lifting.
iv. $\mathrm{CH}^r(\mathfrak{h}^{2r}(X)) = \mathcal{Z}^r_{\mathrm{hom}}(X)_{\mathbb{Q}}$.

3.2. **The Bloch-Beilinson conjecture.**

Conjecture. *For all $X \in \mathbf{SmProj}/k$:*
1. *the Künneth projectors π^n_X are algebraic.*
2. *For each $r \geq 0$ there is a filtration $F^\nu \mathrm{CH}^r(X)_{\mathbb{Q}}$, $\nu \geq 0$ such that*

i. $F^0 = \mathrm{CH}^r$, $F^1 = \ker(\mathrm{CH}^r \to \mathcal{Z}^r_{\mathrm{hom}})$
ii. $F^\nu \cdot F^\mu \subset F^{\nu+\mu}$
iii. F^ν is stable under correspondences
iv. π^n_X acts by id *on $\mathrm{Gr}^\nu_F \mathrm{CH}^r$ for $n = 2r - \nu$, 0 otherwise*
v. $F^\nu \mathrm{CH}^r(X)_{\mathbb{Q}} = 0$ for $\nu >> 0$.

Murre's conjecture implies the BB conjecture by taking the filtration given in the statement of Murre's conjecture. In fact

Theorem (Jannsen)**.** *The two conjectures are equivalent, and give the same filtrations.*

Also: Assuming the Lefschetz-type conjectures $B(X)$ for all X, the condition (v) in BB is equivalent to $F^{r+1}\mathrm{CH}^r(X) = 0$ i.e.

$$\mathrm{CH}^r(\mathfrak{h}^n(X)) = 0 \text{ for } n < r.$$

3.3. **Saito's filtration.** Saito has defined a functorial filtration on the Chow groups, without requiring any conjectures. This is done inductively: $F^0\mathrm{CH}^r = \mathrm{CH}^r$, $F^1\mathrm{CH}^r := \ker(\mathrm{CH}^r \to \mathcal{Z}^r_{\mathrm{hom}})_{\mathbb{Q}}$ and

$$F^{\nu+1}\mathrm{CH}^r(X)_{\mathbb{Q}} := \sum_{Y,\rho,s} \mathrm{Im}(\rho_* : F^\nu \mathrm{CH}^{r-s}(Y)_{\mathbb{Q}} \to \mathrm{CH}^r(X)_{\mathbb{Q}})$$

with the sum over all $Y \in \mathbf{SmProj}/k$, $s \in \mathbb{Z}$ and $\rho \in \mathcal{Z}^{d_Y+s}(Y \times X)$ such that the map

$$\pi_X^{2r-\nu} \circ \rho_* : H^*(Y) \to H^{2r-\nu}(X)$$

is 0.

There is also a version with the Y restricted to lie in a subcategory $\mathcal{V}$ closed under products and disjoint union.

The only problem with Saito's filtration is the vanishing property: That $F^\nu \mathrm{CH}^r(X)$ should be 0 for $\nu >> 0$. The other properties for the filtration in the BB conjecture (2) are satisfied.

3.4. **Consequences of the BBM conjecture.** We assume the BBM conjectures are true for the $X \in \mathcal{V}$, some subset of **SmProj**$/k$ closed under products and disjoint union. Let $M_\sim(\mathcal{V})$ denote the full tensor pseudo-abelian subcategory of $M_\sim(k)$ generated by the $\mathfrak{h}(X)(r)$ for $X \in \mathcal{V}$, $r \in \mathbb{Z}$.

Lemma. *The kernel of $CHM(\mathcal{V})_\mathbb{Q} \to NM(\mathcal{V})_\mathbb{Q}$ is a nilpotent $\otimes$ ideal.*

The nilpotence comes from

1. $\ker(\mathrm{Hom}_{CHM}(\mathfrak{h}(X)(r), \mathfrak{h}(Y)(s)) \to \mathrm{Hom}_{NM}(\mathfrak{h}(X), \mathfrak{h}(Y)))$
$$= F^1\mathrm{CH}^{d_X - r + s}(X \times Y)$$
2. $F^\nu \cdot F^\mu \subset F^{\nu+\mu}$
3. $F^\nu \mathrm{CH}^r(X^2) = 0$ for $\nu >> 0$.

The $\otimes$ property is valid without using the filtration.

Proposition. *$CHM(\mathcal{V})_\mathbb{Q} \to NM(\mathcal{V})_\mathbb{Q}$ is conservative and essentially surjective.*

Indeed: $\ker \subset \mathcal{R}$

Proposition. *Let X be a surface over $\mathbb{C}$ with $p_g = 0$. The BBM conjectures for X^n (all n) imply Bloch's conjecture for X.*

Proof. Recall the decomposition $\mathfrak{h}(X) = \oplus_n \mathfrak{h}^n(X)$ and $\mathfrak{h}^2(X) = \mathbb{1}(-1)^\rho \oplus \mathfrak{t}^2(X)$, $\rho = \dim_\mathbb{Q} H^2(X, \mathbb{Q})$. We need to show that $\mathrm{CH}^2(\mathfrak{t}^2(X)) = 0$.

But $\mathfrak{h}^2_{\mathrm{hom}} = \mathfrak{h}^2_{\mathrm{num}} = \mathbb{1}(-1)^\rho$, so $\mathfrak{t}^2_{\mathrm{num}} = 0$. By the proposition $\mathfrak{t}^2 = 0$.

3.5. **Status.** The BBM conjectures are valid for X of dimension ≤ 2. For an abelian variety A, one can decompose $\mathrm{CH}^r(A)_\mathbb{Q}$ by the common eigenspaces for the multiplication maps $[m] : A \to A$ This gives

$$\mathrm{CH}^r(X)_\mathbb{Q} = \oplus_{i \geq 0} \mathrm{CH}^r_{(i)}(A)$$

with $[m]$ acting by $\times m^i$ on $\mathrm{CH}^r_{(i)}(A)$ for all m.

Beauville conjectures that $\mathrm{CH}^r_{(i)}(A) = 0$ for $i > 2r$, which would give a BBM filtration by

$$F^\nu \mathrm{CH}^r(A)_\mathbb{Q} = \oplus_{i=0}^{2r-\nu} \mathrm{CH}^r_{(i)}(A).$$

4. Nilpotence

We have seen how one can compare the categories of motives for $\sim \succ \approx$ if the kernel of $\mathcal{Z}^*_\sim(X^2) \to \mathcal{Z}^*_\approx(X^2)$ is nilpotent. Voevodsky has formalized this via the adequate equivalence relation $\sim_{\otimes \mathrm{nil}}$.

Definition. A correspondence $f \in \mathrm{CH}^*(X \times Y)_F$ is *smash nilpotent* if $f \times \ldots \times f \in \mathrm{CH}^*(X^n \times Y^n)$ is zero for some n.

Lemma. *The collection of smash nilpotent elements in $\mathrm{CH}^*(X \times Y)_F$ for $X, Y \in$* **SmProj**$/k$ *forms a tensor nil-ideal in $\mathrm{Cor}^*(k)_F$.*

Proof. For smash nilpotent f, and correspondences $g_0, \ldots, g_m$, the composition $g_0 \circ f \circ g_1 \circ \ldots \circ f \circ g_m$ is formed from $g_0 \times f \times \ldots \times f \times g_m$ by pulling back by diagonals and projecting. After permuting the factors, we see that $g_0 \times f \times \ldots \times f \times g_m = 0$ for $m >> 0$. $\square$

Remark. There is a 1-1 correspondence between tensor ideals in $\mathrm{Cor}_{\mathrm{rat}}(k)_F$ and adequate equivalence relations. Thus smash nilpotence defines an adequate equivalence relation $\sim_{\otimes \mathrm{nil}}$.

Corollary. *The functor $CHM(k)_F \to M_{\otimes \mathrm{nil}}(k)_F$ is conservative and a bijection on isomorphism classes.*

The kernel $\mathcal{I}_{\otimes \mathrm{nil}}$ of $CHM(k)_F \to M_{\otimes \mathrm{nil}}(k)_F$ is a nil-ideal, hence contained in $\mathcal{R}$.

Lemma. $\sim_{\otimes \mathrm{nil}} \succ \sim_{\mathrm{hom}}$

If a is in $H^*(X)$ then $a \times \ldots \times a \in H^*(X^r)$ is just $a^{\otimes r} \in (H^*(X))^{\otimes r}$, by the Künneth formula.

Conjecture (Voevodsky)**.** $\sim_{\otimes \mathrm{nil}} = \sim_{\mathrm{num}}$.

This conjecture thus implies the standard conjecture $\sim_{\mathrm{hom}} = \sim_{\mathrm{num}}$.

As some evidence, Voevodsky proves

Proposition. *If $f \sim_{\mathrm{alg}} 0$, then $f \sim_{\otimes \mathrm{nil}} 0$ (with $\mathbb{Q}$-coefs).*

By naturality, one reduces to showing $a^{\times n} = 0$ for $a \in \mathrm{CH}_0(C)_{\deg 0}$, $n >> 0$, C a curve.

Pick a point $0 \in C(k)$, giving the decomposition $\mathfrak{h}(C) = \mathbb{1} \oplus \tilde{\mathfrak{h}}(C)$. Since a has degree 0, this gives a map $a : \mathbb{1}(-1) \to \tilde{\mathfrak{h}}(C)$.

We view $a^{\times n}$ as a map $a^{\times n} : \mathbb{1}(-n) \to \tilde{\mathfrak{h}}(C)^{\otimes n}$, i.e. an element of $\mathrm{CH}^n(\tilde{\mathfrak{h}}(C)^{\otimes n})_{\mathbb{Q}}$.

$a^{\times n}$ is symmetric, so is in $\mathrm{CH}^n(\tilde{\mathfrak{h}}(C)^{\otimes n})_{\mathbb{Q}}^{S_n} \subset \mathrm{CH}^n(\tilde{\mathfrak{h}}(C)^{\otimes n})_{\mathbb{Q}}$

But
$$\mathrm{CH}^n(\tilde{\mathfrak{h}}(C))_{\mathbb{Q}}^{S_n} = \mathrm{CH}_0(\mathrm{Sym}^n C)_{\mathbb{Q}} / \mathrm{CH}_0(\mathrm{Sym}^{n-1} C)_{\mathbb{Q}}.$$
For $n > 2g - 1$ $\mathrm{Sym}^n C \to \mathrm{Jac}(C)$ and $\mathrm{Sym}^{n-1} C \to \mathrm{Jac}(C)$ are projective space bundles, so the inclusion $\mathrm{Sym}^{n-1} C \to \mathrm{Sym}^n C$ induces an iso on CH_0.

4.1. **Nilpotence and other conjectures.** For X a surface, the nilpotence conjecture for X^2 implies Bloch's conjecture for X: The nilpotence conjecture implies that $\mathfrak{t}^2_{\otimes \mathrm{nil}}(X) = 0$, but then $\mathfrak{t}^2(X) = 0$.

The BBM conjectures imply the nilpotence conjecture (O'Sullivan).

5. Finite dimensionality

Kimura and O'Sullivan have introduced a new notion for pure motives, that of finite dimensionality.

5.1. **Multi-linear algebra in tensor categories.** For vector spaces over a field F, one has the operations
$$V \mapsto \Lambda^n V,\ V \mapsto \mathrm{Sym}^n V$$
as well as the other Schur functors.

Define elements of $\mathbb{Q}[S_n]$ by
$$\lambda^n := \frac{1}{n!} \sum_{g \in S_n} \mathrm{sgn}(g) \cdot g$$
$$\mathrm{sym}^n := \frac{1}{n!} \sum_{g \in S_n} g$$

λ^n and sym^n are idempotents in $\mathbb{Q}[S_n]$.

Let S_n act on $V^{\otimes_F n}$ by permuting the tensor factors. This makes $V^{\otimes_F n}$ a $\mathbb{Q}[S_n]$ module (assume F has characteristic 0) and

$$\Lambda^n V = \lambda^n(V^{\otimes n}), \mathrm{Sym}^n V = \mathrm{sym}^n(V^{\otimes n}).$$

These operation extend to the abstract setting.

Let $(\mathcal{C}, \otimes, \tau)$ be a pseudo-abelian tensor category (over $\mathbb{Q}$). For each object V of $\mathcal{C}$, S_n acts on $V^{\otimes n}$ with simple transpositions acting by the symmetry isomorphisms τ.

Since $\mathcal{C}$ is pseudo-abelian, we can define

$$\Lambda^n V := \mathrm{Im}(\lambda^n : V^{\otimes n} \to V^{\otimes n})$$
$$\mathrm{Sym}^n V := \mathrm{Im}(\mathrm{sym}^n : V^{\otimes n} \to V^{\otimes n})$$

Remarks. 1. Let $\mathcal{C} = \mathrm{GrVec}_F$, and let $f : \mathrm{GrVec}_K \to \mathrm{Vec}_K$ be the functor "forget the grading". If V has purely odd degree, then

$$f(\mathrm{Sym}^n V) \cong \Lambda^n f(V), f(\Lambda^n V) = \mathrm{Sym}^n f(V)$$

If V has purely even degree, then

$$f(\mathrm{Sym}^n V) \cong \mathrm{Sym}^n f(V), f(\Lambda^n V) = \Lambda^n f(V).$$

2. Take $\mathcal{C} = \mathrm{Vec}_K^\infty$. Then $V \in \mathcal{C}$ is finite dimensional $\Leftrightarrow \Lambda^n V = 0$ for some n.

3. Take $\mathcal{C} = \mathrm{GrVec}_K^\infty$. Then $V \in \mathcal{C}$ is finite dimensional $\Leftrightarrow V = V^+ \oplus V^-$ with $\Lambda^n V^+ = 0$ and $\mathrm{Sym}^n V^- = 0$ for some n.

Definition. Let $\mathcal{C}$ be a pseudo-abelian tensor category over a field F of characteristic 0. Call $M \in \mathcal{C}$ *finite dimensional* if $M \cong M^+ \oplus M^-$ with

$$\Lambda^n M^+ = 0 = \mathrm{Sym}^m M^-$$

for some integers $n, m > 0$.

Proposition (Kimura, O'Sullivan)**.** *If M, N are finite dimensional, then so are $N \oplus M$ and $N \otimes M$.*

The proof uses the extension of the operations Λ^n, Sym^n to *all* Schur functors.

Theorem (Kimura,O'Sullivan)**.** *Let C be a smooth projective curve over k. Then $\mathfrak{h}(C) \in CHM(k)_\mathbb{Q}$ is finite dimensional.*

In fact
$\mathfrak{h}(C)^+ = \mathfrak{h}^0(C) \oplus \mathfrak{h}^2(C)$, $\mathfrak{h}(C)^- = \mathfrak{h}^1(C)$ and

$$\lambda^3(\mathfrak{h}^0(C) \oplus \mathfrak{h}^2(C)) = 0 = \mathrm{Sym}^{2g+1}\mathfrak{h}^1(C).$$

The proof that $\mathrm{Sym}^{2g+1}\mathfrak{h}^1(C) = 0$ is similar to the proof that the nilpotence conjecture holds for algebraic equivalence: One uses the structure of $\mathrm{Sym}^N C \to \mathrm{Jac}(C)$ as a projective space bundle.

Corollary. *Let M be in the pseudo-abelian tensor subcategory of $CHM(k)_\mathbb{Q}$ generated by the $\mathfrak{h}(C)$, as C runs over smooth projective curves over k. Then M is finite dimensional.*

For example $\mathfrak{h}(A)$ is finite dimensional if A is an abelian variety. $\mathfrak{h}(S)$ is finite dimensional if S is a Kummer surface. $\mathfrak{h}(C_1 \times \ldots \times C_r)$ is also finite dimensional.

It is not known if a general quartic surface $S \subset \mathbb{P}^3$ has finite dimensional motive.

5.2. Consequences.

Theorem. *Suppose M is a finite dimensional Chow motive. Then every $f \in \mathrm{Hom}_{CHM(k)_{\mathbb{Q}}}(M, M)$ with $H^*(f) = 0$ is nilpotent. In particular, if $H^*(M) = 0$ then $M = 0$.*

Corollary. *Suppose $\mathfrak{h}(X)$ is finite dimensional for a surface X. Then Bloch's conjecture holds for X.*

Indeed, $\mathfrak{h}(X)$ finite dimensional implies $\mathfrak{h}^2(X) = \mathbb{1}(-1)^\rho \oplus \mathfrak{t}^2(X)$ is evenly finite dimensional, so $\mathfrak{t}^2(X)$ is finite dimensional. But $\mathfrak{t}^2_{\mathrm{hom}}(X) = 0$.

Conjecture (Kimura, O'Sullivan)**.** *Each object of $CHM(k)_{\mathbb{Q}}$ is finite dimensional.*

Remark. The nilpotence conjecture implies the finite dimensionality conjecture.

In fact, let $\mathcal{I}_{\otimes\mathrm{nil}} \subset \mathcal{I}_{\mathrm{hom}} \subset \mathcal{I}_{\mathrm{num}}$ be the various ideals in $CHM(k)_{\mathbb{Q}}$.

Then $\mathcal{I}_{\otimes\mathrm{nil}} \subset \mathcal{R}$ (f smash nilpotent $\Rightarrow f$ nilpotent). So the nilpotence conjecture implies $\mathcal{R} = \mathcal{I}_{\mathrm{num}}$.

Thus $\phi : CHM(k)_{\mathbb{Q}} \to NM(k)_{\mathbb{Q}} = M_{\mathrm{hom}}(k)_{\mathbb{Q}}$ is conservative and essentially surjective.

Since $\sim_{\mathrm{hom}}=\sim_{\mathrm{num}}$, the Künneth projectors are algebraic: we can thus lift the decomposition $\mathfrak{h}_{\mathrm{hom}}(X) = \mathfrak{h}^+_{\mathrm{hom}}(X) \oplus \mathfrak{h}^-_{\mathrm{hom}}(X)$ to $CHM(k)$.

Since ϕ is conservative, $\mathfrak{h}(X) = \mathfrak{h}^+X(X) \oplus \mathfrak{h}^-(X)$ is finite dimensional:

$$\Lambda^{b^+(X)+1}(\mathfrak{h}^+(X)) = 0 = \mathrm{Sym}^{b^-(X)+1}(\mathfrak{h}^-(X)).$$

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