A SHORT COURSE IN K-THEORY
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Contents

Part 1. K_0 and the Chow ring

In this first part, we give the definition of algebraic K_0 and the Chow ring of a smooth variety over a field k. We sketch the basic properties of these two objects: homotopy, left-exact short localization sequence, projective bundle formula, and describe how to relate K_0 and CH via Chern classes and the Grothendieck-Riemann-Roch theorem.

1. *Basic definitions*

1.1. **Algebraic K_0.** First, let R be a noetherian commutative ring. Recall that an R module P is called *projective* if there is an R-module Q with $P \oplus Q$ a free R module. If P is finitely generated, then one can choose Q finitely generated, giving

$$P \oplus Q \cong R^n$$

for some n. We let $\mathcal{M}_R$ denote the category of finitely generated R-modules, and $\mathcal{P}_R$ the full subcategory of finitely generated projective R-modules.

Remark 1.1. If

$$0 \to M \to N \to P \to 0$$

is an exact sequence with P projective, then there is an isomorphism $N \cong M \oplus P$ so that the sequence becomes the evident split sequence.

Definition 1.2. The Grothendieck group $K_0(R)$ is the free abelian group on the isomorphism classes of finitely generated projective R-modules, modulo the relation $[P] = [P'] + [P'']$ if there is a short exact sequence $0 \to P' \to P \to P'' \to 0$.

By remark 1.1, one can just as well impose the relations $[P' \oplus P''] = [P'] + [P'']$, but in more general situations, this won't work.

Alternatively, one can define $K_0(R)$ as the set of isomorphism classes of projectives, modulo the relation of *stable equivalence*: $P \sim P'$ if there is a projective module (or even a free module) P'' with $P \oplus P'' \cong P' \oplus P''$.

Tensor product of R-modules makes $K_0(R)$ into a commtutative ring.

We can also make a construction with all finitely generated R-modules:

Definition 1.3. $G_0(R)$ is the free abelian group on the isomorphism classes of finitely generated R-modules, modulo the relations $[M] = [M'] + [M'']$ if there exists a short exact sequence

$$0 \to M' \to M \to M'' \to 0.$$

Notice that it now makes a difference if we use short exact sequences instead of direct sums, because not every short exact sequence of R-modules splits.

Remark 1.4. Let $\mathcal{A}$ be an abelian category with a set of isomorphism classes of objects (*essentially small*). Define $K_0(\mathcal{A})$ to be the free abelian group on the isomorphism classes of objects of $\mathcal{A}$, modulo the relations $[M] = [M'] + [M'']$ if there exists a short exact sequence

$$0 \to M' \to M \to M'' \to 0.$$

$K_0(\mathcal{A})$ is called the *Grothendieck group* of $\mathcal{A}$. Clearly $G_0(X) = K_0(\mathcal{M}_X)$.

As tensor product with a projective module preserves exactness, $G_0(R)$ is a $K_0(R)$-module. Also, considering a projective R-module as an R-module defines a homomorphism $K_0(R) \to G_0(R)$ which is sometimes, but not usually, an isomorphism. More about this later.

Now suppose X is a (noetherian) scheme. Replace $\mathcal{M}_R$ with $\mathcal{M}_X$, the category of coherent sheaves, and $\mathcal{P}_R$ with $\mathcal{P}_X$, the category of locally free sheaves, and we have the commutative ring $K_0(X)$, the $K_0(X)$-module $G_0(X)$, and the homomorphism $K_0(X) \to G_0(X)$. If $X = \operatorname{Spec} R$ is affine, we recover $K_0(R)$ and $G_0(R)$, since we have the equivalence of categories (preserving exact sequences)

$$\mathcal{P}_R \sim \mathcal{P}_X; \quad \mathcal{M}_R \sim \mathcal{M}_X.$$

1.2. **The Chow ring.** We now shift gears a bit. Let X be a variety over a field k. Suppose at first that X is non-singular in codimension one. Let $D \subset X$ be a subvariety of codimension one on X, and consider the local ring $\mathcal{O}_{X,D}$ of rational functions on X which are regular functions at the generic point of D. It is well-known that $\mathcal{O}_{X,D}$ has a unique maximal ideal m, which is principal. In fact m is just the set of all $f \in \mathcal{O}_{X,D}$ which vanish at the generic point of D. In any case, since $m = (t)$, and since the quotient field of $\mathcal{O}_{X,D}$ is the field of rational functions on X, $k(X)$, to each non-zero $f \in k(X)$, we can assign the integer $\operatorname{ord}_D(f)$ by writing

$$f = u \cdot t^n$$

with u a unit in $\mathcal{O}_{X,D}$ and then setting $\operatorname{ord}_D(f) := n$.

Since each non-zero rational function on X is regular and non-zero on some dense open subset of X, it follows that $\operatorname{ord}_D(f) = 0$ for all but finitely many D. The *divisor* of f is the formal $\mathbb{Z}$-linear combination

$$\operatorname{div}(f) := \sum_D \operatorname{ord}_D(f) \cdot D.$$

This sum takes place in the group of *divisors* on X, i.e., the free abelian group on the codimension one subvarieties of X, denoted $Z^1(X)$. We set $\mathrm{CH}^1(X) := Z^1(X)/\{\mathrm{div}(f) \mid f \in k(X)^*\}$.

If X is not smooth in codimension one, we replace X with its normalization $p : X^N \to X$. Since $p^* : k(X) \to k(X^N)$ is an isomorphism, we can, for $f \in k(X)^*$, take $\mathrm{div}_{X^N}(f)$, and then apply the operation $p_* : Z^1(X^N) \to Z^1(X)$ to get $\mathrm{div}(f)$, where $p_*(D) = [k(D) : k(p(D))] \cdot p(D)$ for a codimension one subvariety D of X^N.

The group $\mathrm{CH}^1(X)$ is well-known to algebraic geometers of the 19th century as the group of divisors modulo *linear equivalence*. For X a smooth projective curve over $\mathbb{C}$, Abel's theorem identifies $\mathrm{CH}^1(X)$ with the Jacobian variety $H^0(X, \Omega^1_X)^*/H_1(X, \mathbb{Z})$. We will return to this later.

We can generalize this construction as follows: Let $Z_p(X)$ be the free abelian group on the dimension p subvarieties of X: the *dimension p algebraic cycles on X*. Let $R_p(X) \subset Z_p(X)$ be the subgroup generated by cycles of the form $i_*(\mathrm{div}(f))$, where $i : W \to X$ is the inclusion of a dimension $p+1$ subvariety, and f is a non-zero rational function on W; $i_* : Z^1(W) \to Z^p(X)$ is the map sending $D \subset W$ to $i(D) \subset X$ and extending by linearity.

Definition 1.5. The *Chow group* of dimension p cycles on X, modulo rational equivalence, is the quotient group $\mathrm{CH}_p(X) := Z_p(X)/R_p(X)$.

Now assume that X is smooth over k. We usually label with codimension instead of dimension, writing this as a superscript, e.g., $\mathrm{CH}^p(X)$. There is a partially defined *intersection product* of cycles on X. For this, let Z and W be subvarieties of X of codimension p and q, respectively, and let T be an irreducible component of $Z \cap W$. Suppose that T has the "correct" codimension $p + q$ on X. One can define the positive integer $m(T; Z \cdot W)$ by

$$m(T; Z \cdot W) = \sum_{i=0}^{\dim_k X} (-1)^i \ell(\mathrm{Tor}_i^{\mathcal{O}_{X,T}}(\mathcal{O}_{X,T}/\mathcal{I}_Z, \mathcal{O}_{X,T}/\mathcal{I}_W)).$$

Here $\mathcal{I}_Z$ and $\mathcal{I}_W$ are the defining ideal sheaves of Z and W, and ℓ means length as an $\mathcal{O}_{X,T}$-module. The work of Serre [35] shows that this is well-defined and is indeed a positive integer.

If now each component T of $Z \cap W$ has the correct codimension, define

$$Z \cdot W := \sum_T m(T; Z \cdot W) \cdot T.$$

This is called the intersection product of cycles.

Theorem 1.6. *Suppose that X is smooth and quasi-projective over k. The partially defined intersection of cycles descends to a well-defined product*

$$\mathrm{CH}^p(X) \otimes \mathrm{CH}^q(X) \to \mathrm{CH}^{p+q}(X),$$

making $\mathrm{CH}^*(X) := \oplus_p \mathrm{CH}^p(X)$ *a graded, commutative ring with unit.*

In fact, this theorem has had a long list of false proofs, before Fulton [11] gave the first completely correct proof using algebraic geometry. Quillen's proof of Bloch's formula (see theorem 9.7) gave an earlier proof using higher K-theory.

1.3. **Functorialities for K_0, G_0 and CH^*.** Let $f : X \to Y$ be a projective morphism of schemes. Define

$$f_* : Z_p(X) \to Z_p(Y)$$

as we did for divisors: $f_*(Z) = [k(Z) : k(f(Z))] \cdot f(Z)$ if $Z \to f(Z)$ is generically finite, and $f_*(Z) = 0$ if $\dim\ Z > \dim\ f(Z)$. This passes to the Chow groups, giving

$$f_* : \mathrm{CH}_*(X) \to \mathrm{CH}_*(Y).$$

Similarly, if $\mathcal{F}$ is a coherent sheaf on X, we have the coherent sheaf $f_*(\mathcal{F})$ on Y. However, $\mathcal{F} \mapsto f_*\mathcal{F}$ does not preserve exact sequences, so does not define a map on G_0. To rectify this, we use the higher direct images $R^i f_*\mathcal{F}$, which are also coherent sheaves. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence, we have the long exact sequence

$$0 \to f_*\mathcal{F}' \to f_*\mathcal{F} \to f_*\mathcal{F}'' \to R^1 f_*\mathcal{F}' \to \ldots \to R^n f_*\mathcal{F} \to R^n f_*\mathcal{F}'' \to 0,$$

where n is for intance the dimension of X. Thus, the assignment

$$\mathcal{F} \mapsto \sum_{i=0}^{\dim\ X} (-1)^i [R^i f_*\mathcal{F}]$$

gives a well-defined map

$$f_* : G_0(X) \to G_0(Y),$$

using the following:

Remark 1.7. If

$$0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \ldots \to \mathcal{F}_m \to 0$$

is an exact sequence of coherent sheaves on X, then $\sum_{i=0}^m (-1)^i [\mathcal{F}_i] = 0$ in $G_0(X)$.

The push-forward f_* on K_0 is more difficult to define and in fact cannot always be defined; we postpone this to later.

Let $f : X \to Y$ be a morphism of schemes. If $\mathcal{F}$ is a coherent sheaf on Y, we have the coherent sheaf $f^*\mathcal{F}$ on X. However, this operation does not preserve exact sequences, unless for example f is flat, or the exact sequence consists of locally free sheaves. This gives functorial pull-back maps

$$f^* : K_0(Y) \to K_0(X)$$

and, if f is flat

$$f^* : G_0(Y) \to G_0(X).$$

If $f : X \to Y$ is of finite Tor-dimension (always the case if Y is smooth), we can define f^* on G_0 the way we did f_*:

$$f^*(\mathcal{F}) = \sum_{i=0}^{\dim\ Y} (-1)^i L_i f^*(\mathcal{F}).$$

Here $L_0 f^* = f^*$, and the $L_i f^*(\mathcal{F}) = \mathcal{T}or_i^{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_X)$ for $i > 0$. It is easy to see that f^* is a ring homomorphism for K_0, and a K_0-module homomorphism for G_0 (when defined). We also have the projection formula:

$$f_*(f^*(a) \cdot b) = a \cdot f_*(b) \tag{1.1}$$

for f projective, $a \in K_0(X)$, $b \in G_0(X)$.

Pullback $f^* : \mathrm{CH}_*(Y) \to \mathrm{CH}_{*+d}(X)$ for a flat morphism f of relative dimension d is defined by sending a subvariety $Z \subset Y$ to the cycle determined by the subscheme $f^{-1}(Z)$. If W is an irreducible component of $f^{-1}(Z)$, we let $m(W; f^{-1}(Z))$ be the length of $\mathcal{O}_{f^{-1}(Z)} \otimes \mathcal{O}_{X,W}$ as an $\mathcal{O}_{X,W}$-module, and define

$$f^*(Z) := \sum_W m(W; f^{-1}(Z)) \cdot W.$$

The $\mathbb{Z}$-linear extension of f^* to $f^* : Z_p(Y) \to Z_{p+d}(X)$ descends to

$$f^* : \mathrm{CH}_p(Y) \to \mathrm{CH}_{p+d}(X).$$

In general, the pull-back $f^* : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$ is defined using the intersection product. Let $\Gamma \subset X \times Y$ be the graph of f. The operation $\cap\Gamma$ defines a map from $\mathrm{CH}^*(X \times Y)$ to $\mathrm{CH}^*(\Gamma) = \mathrm{CH}^*(X)$. We define

$$f^*(Z) := (X \times Z) \cap \Gamma \in \mathrm{CH}^*(X).$$

We have a projection formula for CH as well.

2. *Fundamental properties*

We discuss the important properties of K_0 and G_0. Some of these properties are also shared by CH^* and CH_*, but we will concentrate on K-theory, giving brief indications of the analogues for the Chow groups.

2.1. **Reduction by resolution and filtration.** We work in the category of R-modules for simplicity. Let M be an R-module, and suppose we have a filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M.$$

Putting together all the subquotients gives a bunch of short exact sequences, which shows that

$$[M] = \sum_{i=1}^{n} [M_i/M_{i-1}].$$

Now suppose we have a nilpotent ideal $I \subset R$, and let $\bar{R} = R/I$. Let $i_* : G_0(\bar{R}) \to G_0(R)$ be the evident map. Take a filtration as above such that $IM_i \subset M_{i-1}$. Then each quotient M_i/M_{i-1} is an $\bar{R}$-module (for instance, $M_j = I^j M$), so the sum $\sum_{i=1}^{n}[M_i/M_{i-1}]$ defines a class $[\hat{M}]$ in $G_0(\bar{R})$ with $i_*([\hat{M}]) = [M]$ in $G_0(R)$. As the notation suggests, $[\hat{M}]$ is independent of the choice of filtration (this follows from the butterfly lemma), and sending M to $[\hat{M}]$ descends to a well-defined homomorphism

$$\hat{}: G_0(R) \to G_0(\bar{R})$$

with $i_*([\hat{M}]) = [M]$. If N is already a $\bar{R}$-module, we can take the trivial filtration, so $i_*[\hat{N}] = [N]$. Thus:

Theorem 2.1. *Let I be a nilpotent ideal in R. Then $i_* : G_0(R/I) \to G_0(R)$ is an isomorphism. More generally, let X be a scheme, $i : X_{\mathrm{red}} \to X$ the associated reduced scheme. Then $i_* : G_0(X_{\mathrm{red}}) \to G_0(X)$ is an isomorphism.*

Examples 2.2. (1) Let F be a field. Clearly $\mathcal{M}_F = \mathcal{P}_F$ are both the category of finite dimensional vector spaces over F. As each vector space V is the direct sum of $\dim_F V$ copies of F, sending V to $\dim_F V$ gives an isomorophism of $K_0(F) = G_0(F)$ with $\mathbb{Z}$.
(2) Let $\mathcal{O}$ be a local ring with maximal ideal m. Let $\mathcal{M}_{\mathcal{O}}(m)$ be the subcategory of $\mathcal{M}_{\mathcal{O}}$ consisting of those $\mathcal{O}$-modules which are m^N-torsion for some N. Let $k = \mathcal{O}/m$. Clearly $G_0(\mathcal{M}_{\mathcal{O}}(m)) = \lim_{N\to\infty} G_0(\mathcal{M}_{\mathcal{O}/m^N})$. By theorem 2.1, the inclusion $\mathcal{M}_k \to \mathcal{M}_{\mathcal{O}}(m)$ induces an isomorphism

$$\mathbb{Z} \cong G_0(\mathcal{M}_k) \cong \mathcal{M}_{\mathcal{O}}(m).$$

If M is in $\mathcal{M}_{\mathcal{O}}(m)$, then M admits a filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

with $M_i/M_{i-1} \cong k$; by definition n is the length of M. Thus, sending M to $\ell(M)$ gives the isomorphism $G_0(\mathcal{M}_{\mathcal{O}}(m)) \cong \mathbb{Z}$.

We can also use long exact sequences to relate K_0 and G_0. Suppose X is a regular scheme (e.g., X is smooth over a field k). Then every coherent sheaf $\mathcal{F}$ admits a finite resolution by locally free sheaves:

$$0 \to \mathcal{E}_n \to \ldots \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

(in fact, one can always take $n \leq \dim_k X$ if X is smooth and finite type over k). Send $\mathcal{F}$ to the class $\sum_{i=0}^n (-1)^i [\mathcal{E}_i]$ in $K_0(X)$. If we have a second resolution $\mathcal{E}' \to \mathcal{F} \to 0$, one can always find a third resolution to which the other two map, term-wise injectively, with locally free cokernel. From this, it is easy to see that $\sum_{i=0}^n (-1)^i [\mathcal{E}_i]$ is independent of the choice of resolution, and defines a homomorphism

$$res : G_0(X) \to K_0(X).$$

Letting $can : K_0(X) \to G_0(X)$ be the canonical map, it follows that $can \circ res = \mathrm{id}$, since we can take the identity resolution of a locally free sheaf. By remark 1.7, $res \circ can = \mathrm{id}$. Thus

Theorem 2.3. *Let X be a regular scheme of finite Krull dimension. Then $can : K_0(X) \to G_0(X)$ is an isomorphism, with inverse $res : G_0(X) \to K_0(X)$.*

Using theorem 2.3, we can define push-forward maps $f_* : K_0(X) \to K_0(Y)$ for $f : X \to Y$ a projective morphism with Y smooth over k: take the composition

$$K_0(X) \xrightarrow{can} G_0(X) \xrightarrow{f_*} G_0(Y) \xrightarrow{res} K_0(Y).$$

2.2. **Localization.** Let $i : Z \to X$ be a closed subscheme with open complement $j : U \to X$. The localization sequence gives a way of relating $G_0(X)$, $G_0(U)$ and $G_0(Z)$.

We recall that $\mathcal{M}_X$ is an abelian category. Let $\mathcal{M}_X(Z)$ be the subcategory of $\mathcal{M}_X$ consisting of coherent sheaves which are supported on Z. $\mathcal{M}_X(Z)$ is a *Serre subcategory*, i.e., $\mathcal{M}_Z$ is closed under subquotients and extensions in $\mathcal{M}_X$.

Given an abelian category $\mathcal{A}$ and a Serre subcategory $\mathcal{B} \subset \mathcal{A}$, one can form the *quotient category* $\mathcal{A}/\mathcal{B}$, having the same objects as $\mathcal{A}$, but where we formally invert a morphism $f : M \to N$ if $ker(f)$ and $cok(f)$

are in $\mathcal{B}$. Explicitly, a morphism $g : M \to N$ in $\mathcal{A}/sB$ is given by a diagram in $\mathcal{A}$

$$\begin{tikzcd} & M' \arrow[dl, "i"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

with $ker(i)$ and $cok(i)$ in $\mathcal{B}$, where we identify two such diagrams if there is a commutative diagram

$$\begin{tikzcd} & M' \arrow[dl, "i"'] \arrow[dr, "f"] & \\ M & & N \\ & M'' \arrow[ul, "i'"] \arrow[ur, "f'"'] \arrow[uu, "\phi"] & \end{tikzcd}$$

Composition of $M_1 \xleftarrow{i_1} M_1' \xrightarrow{f_1} M_2$ with $M_2 \xleftarrow{i_2} M_2' \xrightarrow{f_2} M_3$ is given by going around the outside of the diagram

$$\begin{tikzcd} & & M_1' \times_{M_2} M_2' \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\ & M_1' \arrow[dl, "i_1"'] \arrow[dr, "f_1"] & & M_2' \arrow[dl, "i_2"'] \arrow[dr, "f_2"] & \\ M_1 & & M_2 & & M_3. \end{tikzcd}$$

Essentially, this makes all the objects of $\mathcal{B}$ isomorphic to the zero object, in a universal way, so each functor of abelian categories $F : \mathcal{A} \to \mathcal{A}'$ for which $F(B) \cong 0$ for all $B \in \mathcal{B}$ factors uniquely through $\mathcal{A}/\mathcal{B}$. It is not hard to see that $\mathcal{A}/\mathcal{B}$ is also an abelian category, and the canonical functor $p : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is exact.

Theorem 2.4. *Let $\mathcal{A}$ be an abelian category, $i : \mathcal{B} \to \mathcal{A}$ a Serre subcategory, $p : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ the quotient category. Then the sequence*

$$K_0(\mathcal{B}) \xrightarrow{i_*} K_0(\mathcal{A}) \xrightarrow{p_*} K_0(\mathcal{A}/\mathcal{B}) \to 0$$

is exact.

Proof. Since $\mathcal{A}/\mathcal{B}$ and $\mathcal{A}$ have the same objects, p_* is clearly surjective.

Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence in $\mathcal{A}/\mathcal{B}$. Changing the sequence by an isomorphism in $\mathcal{A}/\mathcal{B}$, we can assume that f and g are morphisms in $\mathcal{A}$ and that $g \circ f = 0$ in $\mathcal{A}$. Then the sequence being exact in $\mathcal{A}/\mathcal{B}$ means that $ker(f)$, $ker(g)/im(f)$ and

$cok(g)$ (all taken in $\mathcal{A}$) are in $\mathcal{B}$. The sequence being a complex in $\mathcal{A}$ gives the identity in $K_0(\mathcal{A})$:

$$\begin{aligned}[M]-[M']-[M''] =&[ker(g)]+[im(g)]\\ &-([ker(f)]+[im(f)])-([im(g)]+[cok(g)])\\ =&[ker(g)/im(f)]-[ker(f)]-[cok(g)]\end{aligned}$$

Thus, every relation defining $K_0(\mathcal{A}/\mathcal{B})$ lifts to a relations showing an element of $K_0(\mathcal{A})$ is actually in $K_0(\mathcal{B})$. This together with a simple diagram chase shows that we have a well-defined map $p^*: K_0(\mathcal{A}/\mathcal{B}) \to K_0(\mathcal{A})/K_0(\mathcal{B})$ with $p^*\circ p_*$ the quotient map $K_0(\mathcal{A})\to K_0(\mathcal{A})/K_0(\mathcal{B})$. Thus $ker(p_*) = im(i_*)$. $\square$

If we apply this to the situation $\mathcal{A} = \mathcal{M}_X$, $\mathcal{B} = \mathcal{M}_X(Z)$, we also have the equivalence of categories $\mathcal{M}_X/\mathcal{M}_X(Z) \sim \mathcal{M}_U$. Thus, we have the exact sequence

$$K_0(\mathcal{M}_X(Z)) \to K_0(\mathcal{M}_X) \xrightarrow{j^*} K_0(\mathcal{M}_U) \to 0.$$

Using reduction by filtration, the inclusion $\mathcal{M}_Z \to \mathcal{M}_X(Z)$ induces an isomorphism on K_0, so we have

Theorem 2.5. *Let $i: Z \to X$ be a closed subscheme, $j: U\to X$ the open complement. Then the sequence*

$$G_0(Z) \xrightarrow{i_*} G_0(X) \xrightarrow{j^*} G_0(U) \to 0$$

is exact.

If X is smooth, so is U, and we have the exact sequence

$$G_0(Z) \xrightarrow{i_*} K_0(X) \xrightarrow{j^*} K_0(U) \to 0,$$

using the resolution theorem 2.3.

The analogous result holds for CH_*:

Theorem 2.6. *Let $i: Z \to X$ be a closed subscheme, $j: U\to X$ the open complement. Then the sequence*

$$\mathrm{CH}_*(Z) \xrightarrow{i_*} \mathrm{CH}_*(X) \xrightarrow{j^*} \mathrm{CH}_*(U) \to 0$$

is exact.

Proof. If Z is a subvariety of U, then the closure $\bar{Z}$ is a subvariety of X restricting to Z on U. Thus j^* is surjective. Clearly $j^*i_* = 0$. If $\eta \in \mathrm{CH}_q(X)$ has $j^*\eta = 0$, then $\eta = [\sum_i n_i\bar{Z}_i]$, and there is a dimension $q+1$ subscheme $W \subset U$ and an $f \in k(W)^*$ with

$$i_{W*}(\mathrm{div}(f)) = \sum_i n_i Z_i; \quad Z_i := \bar{Z}_i \cap U.$$

Let $\bar{W}$ be the closure of W in X. Then f is in $k(\bar{W})^* = k(W)^*$, and $j^*(i_{\bar{W}*}(\mathrm{div} f)) = i_{W*}(\mathrm{div}(f))$. Thus there is a cycle $z \in Z_q(Z)$ with

$$i_*(z) + i_{\bar{W}*}(\mathrm{div} f)) = \sum_i n_i \bar{Z}_i,$$

or $\eta = i_*([z])$. □

2.3. **Homotopy.** G_0 enjoys a homotopy invariance property, which K_0 inherits for regular schemes.

Theorem 2.7. *Let X be a scheme, $p : X \times \mathbb{A}^1 \to X$ the projection. Then $p^* : G_0(X) \to G_0(X \times \mathbb{A}^1)$ is an isomorphism.*

Proof. We give the proof for $X = \mathrm{Spec}\, R$ an affine variety over a field k, for simplicity.

For $a \in k$, we have the functor $\phi_a : \mathcal{M}_{R[X]} \to \mathcal{M}_R$, $\phi_a(M) = M \otimes_{R[X]} R[X]/(X-a)$. Take $a = 0$. Since $R[X]/X$ has Tor-dimension 1 over $R[X]$, sending M to $\phi_a^*(M) := [\phi_a(M)] - [Tor_1(M, R[X]/X)]$ gives

$$\phi^* : G_0(R[X]) \to G_0(R).$$

We also have $p^* : \mathcal{M}_R \to \mathcal{M}_{R[X]}$, $p^*(M) = M[X]$, and $\phi^* \circ p^* = id$. Thus p^* is injective.

Suppose that R is a field F. Let M be a finitely generated $F[X]$-module. Since $F[X]$ is a PID, $M = F[X]^r \oplus T$, where T is a finitely generated torsion module. Each T has a finite filtration

$$0 = T_0 \subset \ldots \subset T_n = T$$

with $T_i/T_{i-1} \cong F[X]/(f)$, where f is an irreducible monic polynomial. Also, we have the exact sequence

$$0 \to F[X] \xrightarrow{\times f} F[X] \to F[X]/(f) \to 0,$$

showing that $[F[X]/(f)] = [F[X]] - [F[X]] = 0$ in $G_0(F[X])$. Thus $[T] = 0$ as well, and $[M] = r \cdot [F[X]] = p^*([F^r])$. Thus $p^* : G_0(F) \to G_0(F[X])$ is surjective.

Now proceed by noetherian induction. Suppose R is an integral domain with quotient field F. Let $\mathcal{M}_R^{(1)}$ be the category of torsion R-modules. Similarly, let $\mathcal{M}_{R[X]}^{(1')}$ be the subcategory of $R[X]$ which are f-torsion for some $f \in R$. Then

$$K_0(\mathcal{M}_R^{(1)}) = \varinjlim_{f \in R - \{0\}} G_0(R/(f))$$

and

$$K_0(\mathcal{M}_{R[X]}^{(1')}) = \varinjlim_{f \in R - \{0\}} G_0(R[X]/(f))$$

Thus $p^* : K_0(\mathcal{M}_R^{(1)}) \to K_0(\mathcal{M}_{R[X]}^{(1')})$ is an isomorphism, by noetherian induction.

By the localization theorem 2.4, we have the commutative diagram with exact rows

$$\begin{array}{ccccccc}
K_0(\mathcal{M}_{R[X]}^{(1')}) & \longrightarrow & G_0(R[X]) & \longrightarrow & G_0(F[X]) & \longrightarrow & 0 \\
\uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle p^*} & & \\
K_0(\mathcal{M}_R^{(1)}) & \longrightarrow & G_0(R) & \longrightarrow & G_0(F) & \longrightarrow & 0
\end{array}$$

The left and right-hand p^* are surjective, thus the middle p^* is surjective as well. We can handle a general R similarly by induction on the number of components. □

Remark 2.8. An argument similar to the above shows that CH_* also enjoys the homotopy property.

Remark 2.9. One can show, more generally, that if $p : E \to X$ is a flat morphism of schemes, with $p^{-1}(x) \cong \mathbb{A}^n_x$ for each point $x \in X$, then $p^* : G_0(X) \to G_0(E)$ is an isomorphism, and similarly for CH_*. The proof uses the localization theorem as above, which shows p^* is surjective. For injectivity one needs to use the projective bundle formula in the next section, or wait for higher K-theory.

2.4. **The first Chern class.** Let $L \to X$ be a line bundle (algebraic) on some smooth k-variety X, and let $s : X \to L$ be a section. If L is trivialized on some open $U \subset X$, $\psi : L_U \cong U \times \mathbb{A}^1$, then we may consider s as a regular function s_ψ on U, and so we have the divisor $\mathrm{div}_U(s_\psi) = \sum_{D \subset U} \mathrm{ord}_D(s_\psi) \cdot D$. If $\phi : L_U \cong U \times \mathbb{A}^1$ is another trivalization, then $s_\psi = v \cdot s_\phi$, where v is a nowhere vanishing regular function on U, so $\mathrm{div}_U(s_\psi) = \mathrm{div}_U(s_\phi)$. Thus, the local divisors of s patch together on X to give $\mathrm{div}(s) \in Z^1(X)$.

Lemma 2.10. *The class of* $\mathrm{div}(s) \in \mathrm{CH}^1(X)$ *is independent of the choice of* s.

Proof. Let s' be another section of L. Then $s' \otimes s^{-1}$ is a rational section of $L \otimes L^{-1} \cong X \times \mathbb{A}^1$, that is $s' \otimes s^{-1}$ is a rational function f on X. By checking in local coordinates, we find

$$\mathrm{div}(f) = \mathrm{div}(s' \otimes s^{-1}) = \mathrm{div}(s') - \mathrm{div}(s),$$

or $\mathrm{div}(s') = \mathrm{div}(s) + \mathrm{div}(f)$, so $\mathrm{div}(s) = \mathrm{div}(s')$ in $\mathrm{CH}^1(X)$. □

Thus, if L has a non-zero section s, we may define $c_1(L) \in \mathrm{CH}^1(X)$ by $c_1(L) := \mathrm{div}(s)$. Since $\mathrm{div}(s \otimes s') = \mathrm{div}(s) + \mathrm{div}(s')$, it follows that $c_1(L \otimes L') = c_1(L) + c_1(L')$ if L and L' have non-zero sections.

Now suppose that $X \subset \mathbb{P}^N$ is quasi-projective and let $O_X(1)$ be the restriction of the hyperplane bundle. If L is an arbitrary line bundle on X, then $L \otimes O_X(n)$ has a non-zero section, for n sufficiently large. Thus, we may define

$$c_1(L) := c_1(L \otimes O_X(n)) - c_1(O_X(n)).$$

Let $\mathrm{Pic}(X)$ be the group of line bundles under tensor product. Then we have defined a homomorphism

$$c_1 : \mathrm{Pic}(X) \to \mathrm{CH}^1(X).$$

We identify line bundles with rank one locally free sheaves by passing from a line bundle to its sheaf of sections.

Proposition 2.11. *For X quasi-projective and smooth over k, $c_1 : \mathrm{Pic}(X) \to \mathrm{CH}^1(X)$ is an isomorphism.*

Proof. Let D be a divisor on X. We have the invertible sheaf $\mathcal{O}_X(D)$ with

$$\mathcal{O}_X(D)(U) := \{f \in k(X) \mid (\mathrm{div}(f) + D) \cap U > 0\}.$$

If $D > 0$, we have the section $s : \mathcal{O}_X \to \mathcal{O}_X(D)$ sending 1 to 1. It is easy to check that $\mathrm{div}(s) = D$. Also $\mathcal{O}_X(D + D') = \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')$, and if $D = \mathrm{div}(f) + D'$, then multiplication by f gives an isomorphism $\mathcal{O}_X(D') \to \mathcal{O}_X(D)$. Finally, if an invertible sheaf $\mathcal{L}$ has a section s with $\mathrm{div}(s) = D$, then $\mathcal{L} \cong \mathcal{O}_X(D)$. Thus, sending D to $\mathcal{O}_X(D)$ defines an inverse to c_1. □

Remark 2.12. We also have the functoriality:

$$c_1(f^*L) = f^*c_1(L)$$

for $L \to X$ a line bundle $f : Y \to X$ a morphism.

2.5. **The projective bundle formula for** CH^***.** Let $E \to X$ be a vector bundle on X, $q : \mathbb{P}(E) \to X$ the projective space bundle of lines in E, $O(-1) \to \mathbb{P}(E)$ the tautological subbundle of q^*E, and $q^*E^\vee \to O(1)$ the dual quotient bundle.

Let $\xi \in \mathrm{CH}^1(\mathbb{P}(E))$ be $c_1(O(1))$.

Lemma 2.13. *If E has rank $r + 1$, the $q_*(\xi^r) = 1 \cdot X \in \mathrm{CH}^0(X)$.*

Proof. We may suppose X irreducible. Then $CH^0(X) = \mathbb{Z} \cdot [X]$, so for each open $j : U \to X$, the restriction $j^* : \mathrm{CH}^0(X) \to \mathrm{CH}^0(U)$ is an isomorphism. Thus, it suffices to show that $j^*(q_*(\xi^r)) = 1 \cdot [U]$ for some open U.

Take U so that $E_U \to U$ is trivial, $E_U \cong U \times \mathbb{A}^{r+1}$. Then $O(1)$ is the hyperplane bundle on $\mathbb{P}^r \times U$, with sections the free $\Gamma(U, \mathcal{O}_U)$-module on the standard coordinates $X_0, \ldots, X_r$. Thus, ξ is represented by the hyperplane $X_i = 0$ for any i, and ξ^r is thus represented by the transverse intersection $X_1 = \ldots = X_r = 0$. Thus ξ^r is represented by the codimension r subvariety $(1:0:\ldots:0) \times U$; clearly $q_*((1:0:\ldots:0) \times U) = 1 \cdot [U]$, proving the result. $\square$

Theorem 2.14. *If E has rank $r+1$, then $\mathrm{CH}^*(\mathbb{P}(E))$ is a free $\mathrm{CH}^*(X)$-module with basis $1, \xi, \ldots, \xi^r$.*

Proof. We first consider the case of a trivial bundle $E = O_X^{r+1}$, so $\mathbb{P}(E) = \mathbb{P}^r \times X$. Let $Z = \mathbb{P}^{r-1} \times X$ be the closed subscheme defined by $X_r = 0$, with inclusion $i : Z \to \mathbb{P}(E)$. Let $\bar{\xi} = c_1(O(1)_{|Z})$. By our computation of ξ^i above, we see that

$$i_*(\bar{\xi}^i) = \xi^{i+1}$$

for $i = 0, 1, \ldots$. Let $j : \mathbb{P}^r \times X - Z = \mathbb{A}^r \times X \to \mathbb{P}^r \times X$ be the inclusion. Since Z is defined by $X_r = 0$, $j^*(O(1))$ is the trivial bundle, so $j^*(\xi) = 0$. We have the exact localization sequence

$$\mathrm{CH}_*(\mathbb{P}^{r-1} \times X) \xrightarrow{i_*} \mathrm{CH}_*(\mathbb{P}^r \times X) \xrightarrow{j^*} \mathrm{CH}_*(\mathbb{A}^r \times X) \to 0.$$

By induction $\mathrm{CH}^*(\mathbb{P}^{r-1} \times X)$ is generated over $\mathrm{CH}^*(X)$ by $1, \ldots, \xi^{r-1}$, and $\mathrm{CH}^*(\mathbb{A}^r \times X)$ is generated by 1, by the homotopy property. Thus $\mathrm{CH}^*(\mathbb{P}^r \times X)$ is generated by $1, \ldots, \xi^r$.

In general, let $i : Z \to X$ be the complement of $j : U \to X$ such that j^*E is trivial. Then we have the commutative diagram, where the rows are the exact localization sequences, and the vertical arrows

$$\begin{tikzcd}
\mathrm{CH}_*(\mathbb{P}(i^*E))) \arrow[r, "\tilde{i}_*"] & \mathrm{CH}_*(\mathbb{P}(E)) \arrow[r, "\tilde{j}^*"] & \mathrm{CH}^*(\mathbb{P}(j^*E)) \arrow[r] & 0 \\
\mathrm{CH}_{*-r+i}(Z) \arrow[r, "i_*"'] \arrow[u, "\xi_Z^i \cup q^*"] & \mathrm{CH}_{*-r+i}(X) \arrow[r, "j^*"'] \arrow[u, "\xi_X^i \cup q^*"] & \mathrm{CH}_{*-r+i}(U) \arrow[r] \arrow[u, "\xi_U^i \cup q^*"] & 0
\end{tikzcd}$$

the case of the trivial bundle and noetherian induction shows that $\mathrm{CH}_*(\mathbb{P}(E))$ is generated over $\mathrm{CH}^*(X)$ by $1, \ldots, \xi^r$.

Now suppose that $\sum_{i=0}^s q^*(a_i)\xi^i = 0$ for $a_0, \ldots, a_s \in \mathrm{CH}^*(X)$, with $a_s \neq 0$ and $s \leq r$. Multiply by ξ^{r-s} and take q_*. Then

$$\begin{aligned} 0 &= q_*(\sum_{i=0}^s q^*(a_i)\xi^{i+r-s}) \\ &= \sum_{i=0}^s a_i \cdot q_*(\xi^{i+r-s}). \end{aligned}$$

By dimension reasons, $q_*(\xi^j) = 0$ if $0 \leq j < r$, so we have

$$0 = a_s \cdot q_*(\xi^r) = a_s,$$

contradicting our choice of s. $\square$

2.6. **The projective bundle formula for G_0.** For a vector bundle $E \to X$, we have the projective bundle $q : \mathbb{P}(E) \to X$, with fiber $q^{-1}(x)$ the space of lines in E_x through 0. This gives the *tautological line bundle* $O(-1) \to \mathbb{P}(E)$, with inclusion $O(-1) \to E$, giving the exact sequence

$$0 \to O(-1) \to q^*E \to Q \to 0,$$

with Q a vector bundle.

Suppose E has rank $r+1$. Let $\xi_j : G_0(X) \to G_0(\mathbb{P}(E))$ be the map

$$\xi_j(x) = O(-j) \otimes q^*(x)$$

where $O(-j) = O(-1)^{\otimes j}$. Let $\xi : \oplus_{j=0}^r G_0(X) \to G_0(\mathbb{P}(E))$ be the product $\prod_{j=0}^r \xi_j$.

Theorem 2.15. *$\xi : G_0(X)^{r+1} \to G_0(\mathbb{P}(E))$ is an isomorphism.*

Proof. We give the "motivic" proof, due to Beilinson. Write $\mathbb{P}$ for $\mathbb{P}(E)$. Let $\Delta \subset \mathbb{P} \times_X \mathbb{P}$ be the diagonal. Dualize the tautological sequence, giving

$$0 \to Q^\vee \to q^*E^\vee \to O(1) \to 0.$$

For $x, y \in \mathbb{P}$, we have the bilinear map

$$Q^\vee_x \times O(-1)_y \to E^\vee \times E \xrightarrow{<-,->} F,$$

where $<-,->$ is the canonical pairing. It is easy to see that the map on the tensor product $Q^\vee_x \otimes O(-1)_y \to F$ is surjective if $x \neq y$ and 0 if $x = y$. In fact, taking the associated locally free sheaves $\mathcal{Q}$ and $\mathcal{O}(-1)$, the map $p_1^*\mathcal{Q} \otimes p_2^*\mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}\times_X\mathbb{P}}$ has image exactly the ideal sheaf of Δ. Since $\mathcal{E} := p_1^*\mathcal{Q} \otimes p_2^*\mathcal{O}(-1)$ is a locally free sheaf of rank r, the Koszul complex

$$0 \to \Lambda^r\mathcal{E} \to \ldots \to \Lambda^1\mathcal{E} \to \mathcal{O}_{\mathbb{P}\times\mathbb{P}} \to \mathcal{O}_\Delta \to 0$$

is exact. Thus, in $K_0(\mathbb{P} \times \mathbb{P})$, $[\mathcal{O}_\Delta] = \sum_{i=0}^r (-1)^i [\Lambda^i \mathcal{E}]$.

Now, define the map $\phi : G_0(\mathbb{P}) \to G_0(\mathbb{P})$ by

$$\phi(x) = p_{2*}(p_1^*(x) \otimes [\mathcal{O}_\Delta]).$$

Since $p_i : \Delta \to \mathbb{P}$ is an isomorphism, $\phi(x) = x$. By our formula, we can write ϕ as a sum $\phi = \sum_{i=0}^r \phi_i$, where

$$\phi_i(x) := (-1)^i p_{2*}(p_1^*(x) \otimes [\Lambda^i\mathcal{E}]).$$

Now $\Lambda^i(\mathcal{E}) = p_1^*(\Lambda^i(\mathcal{Q}^\vee)) \otimes p_2^*(\mathcal{O}(-i))$, so

$$\phi_i(x) = (-1)^i q_*(x \otimes \Lambda^i(\mathcal{Q}^\vee)) \otimes \mathcal{O}(-i).$$

Thus the classes $[\mathcal{O}], [\mathcal{O}(-1)], \ldots, [\mathcal{O}(-r)]$ generate $G_0(\mathbb{P})$ over $G_0(X)$. Also, $q_*(\mathcal{O}(-j)) = 0$ if $0 < j \leq r$, and $= [\mathcal{O}_X]$ if $j = 0$. If we have $x_i \in G_0(X)$ with $\sum_{i=0}^r x_i[\mathcal{O}(-i)] = 0$, then multiplying by $\mathcal{O}(i)$, $i = 0, \ldots, r$, and applying q_*, we see that $x_i = 0$, $i = 0, \ldots, r$. This proves the theorem. $\square$

The same proof shows

Theorem 2.16. *Let $E \to X$ be a rank $r+1$ vector bundle on X. Then $\xi : K_0(X)^{r+1} \to K_0(\mathbb{P}(E))$ is an isomorphism, so $K_0(\mathbb{P}(E))$ is a free $K_0(X)$-module with basis $1, [\mathcal{O}(-1)], \ldots, [\mathcal{O}(-r)]$.*

3. *Relating* K_0 *and* CH

3.1. **Chern classes.** We recall Grothendieck's method of defining Chern classes of vector bundles (see [13]).

Let $E \to X$ be a vector bundle on X of rank r. We have the projective bundle $q : \mathbb{P}(E^\vee) \to X$ and the canonical quotient bundle $q^*E \to O(1)$. Let $\xi = c_1(O(1))$. By theorem 2.14, $\mathrm{CH}^*(\mathbb{P}(E^\vee))$ is a free $\mathrm{CH}^*(X)$-module with basis $1, \xi, \ldots, \xi^{r-1}$. Thus, there are unique elements $a_i \in \mathrm{CH}^i(X)$, $i = 1, \ldots, r$, with

$$\xi^r + \sum_{i=1}^r (-1)^i q^*(a_i)\xi^{r-i} = 0. \tag{3.1}$$

The element a_i is denoted $c_i(E)$, and is called the ith *Chern class* of E. We let $c(E) = 1 + c_1(E) + \ldots + c_r(E)$, the *total* Chern class of E.

Suppose that $E = L$ is a line bundle on X. Then $\mathbb{P}(L^\vee) = X$, and the canonical quotient $L \to O(1)$ is an isomorphism. Then $\xi = c_1(L)$ and the relation (3.1) gives $\xi = a_1$, so this method recoves the original definition of $c_1(L)$.

Proposition 3.1 (Properties of the Chern classes)**.** *Let X be a smooth k-variety. Then*

1. *Let E be a vector bundle on X, and $f : Y \to X$ a morphism of smooth varieties. Then $f^*(c_i(E)) = c_i(f^*E)$.*
2. *Let $0 \to E' \to E \to E'' \to 0$ be an exact sequence of vector bundles on X. Then $c(E) = c(E') \cdot c(E'')$.*

Proof. (1) follows from the naturality of the quotient bundle $O(1)$ and the operation of taking c_1 of a line bundle.

For (2), first suppose that $E' = \sum_{i=1}^s L_i$, $E'' = \sum_{i=s+1}^r L_i$ and $E = \sum_{i=1}^r L_i$. Let $\xi_i = c_1(L_i)$. It suffices to show that

$$c(E) = \prod_{i=1}^r (1 + c_1(L_i)),$$

that is, that $c_i(E)$ is the ith elementary function σ_i in the Chern classes $c_1(L_1), \ldots, c_1(L_r)$.

For each i, the projection $\sum_{j=1}^r L_j \to \sum_{j\neq i} L_j$ gives the inclusion $\mathbb{P}(\sum_{j\neq i} L_j^\vee) \to \mathbb{P}(E^\vee)$, call this divisor D_i. D_i is defined by the vanishing of the composition $q^*L_i \to q^*E \to O(1)$, i.e., $\mathcal{O}(D_i) \cong q^*L_i^\vee \otimes O(1)$. Thus $D_i = c_1(q^*L_i^\vee \otimes O(1)) = \xi - q^*\xi_i$.

Since $\cap_{i=1}^r D_i = \emptyset$, we have $\prod_{i=1}^r(\xi - q^*\xi_i) = 0$. Thus

$$\sum_{i=0}^{r}(-1)^i q^*(\sigma_i(\xi_1, \ldots, \xi_r))\xi^{r-i} = 0,$$

so $c_i(E) = \sigma_i(\xi_1, \ldots, \xi_r)$, as desired.

In general, let $p : \mathbb{F}\ell(E) \to X$ be the *full flag variety* of E, i.e. the variety of filtrations

$$0 = E_0 \subset E_1 \subset \ldots \subset E_r = E,$$

where E_i is a vector bundle of rank i. Clearly p^*E admits a filtration by subbundles with quotients $E_i/E_{i-1} = L_i$ line bundles. Also, we can construct $\mathbb{F}\ell(E)$ by first passing to $\mathbb{P}(E)$, taking the quotient $E^1 := E/O(-1)$, passing to $\mathbb{P}(E^1)$, etc. Thus $\mathrm{CH}^*(\mathbb{F}\ell(E))$ is free $\mathrm{CH}^*(X)$-module; in particular, $p^* : \mathrm{CH}^*(X) \to \mathrm{CH}^*(\mathbb{F}\ell(E))$ is injective. Thus, if we want to check identities in $\mathrm{CH}^*(X)$, we can pass to $\mathrm{CH}^*(\mathbb{F}\ell(E))$.

Thus, we may assume that E has a filtration as above and that E' and E'' are given by a sub and quotient in the filtration. However, for each i, the Ext-group $Ext^1(E_{i-1}, L_i)$ is a k-vector space, and the sequence $0 \to E_{i-1} \to E_i \to L_i \to 0$ is an element $\eta_i \in Ext^1(E_{i-1}, L_i)$. We may thus take a family of vector bundles over $X \times \mathbb{A}^1$, with value at $t = 1$ the vector bundle E, and at $t = 0$ the vector bundle $\sum_{i=1}^r L_i$; similarly for E' and E''. By the homotopy property, we have

$$c(E) = c(\sum_{i=1}^{r} L_i),$$

and also for E' and E'', which reduces us to the case we have already handled. □

Corollary 3.2. *Let X be a smooth k-variety. The assignment $E \mapsto c_p(E)$ descends uniquely to a map of pointed sets $c_p : K_0(X) \to \mathrm{CH}^p(X)$.*

Proof. Make $\prod_{i=1}^{\dim_k X} \mathrm{CH}^i(X)$ into a group by sending $(z_1, \ldots, z_r)$ to the sum $1 + z_1 + \ldots + z_r \in \mathrm{CH}^*(X)$, and defined the addition $\star$ by using the product in $\mathrm{CH}^*(X)$. Call this group $\widehat{\mathrm{CH}}(X)$. Setting $\tilde{c}(E) = (c_1(E), \ldots, c_d(E))$, $d = \dim_k X$, we have

$$\tilde{c}(E) = \tilde{c}(E') \star \tilde{c}(E'')$$

if there is an exact sequence $0 \to E' \to E \to E'' \to 0$. Thus $\tilde{c}$ respects the relations defining $K_0(X)$, and thus descends to a group homomorphism

$$\tilde{c} : K_0(X) \to \widehat{\mathrm{CH}}(X).$$

Taking the component c_p proves the result. □

3.2. **The topological filtration.** Let X be a smooth variety. Sending a subvariety $Z \subset X$ to the class of the structure sheaf $\mathcal{O}_Z$ in $G_0(X) = K_0(X)$ defines a homomorphism

$$\mathrm{cl} : Z^p(X) \to K_0(X).$$

However, this map does not pass to rational equivalence. To understand the situation, we first note that the subgroup $R^p(X) \subset Z^p(X)$ can be described by cycles on $X \times \mathbb{A}^1$: Let f be a rational function on a codimension $p-1$ subvariety $W \subset X$, and let $\Gamma_f \subset X \times \mathbb{P}^1$ be the closure of the graph of X. The open subset $\mathbb{P}^1 - \{1\}$ of $\mathbb{P}^1$ is isomorphic to $\mathbb{A}^1$ via $t \in \mathbb{A}^1 \mapsto (t-1 : t) \in \mathbb{P}^1 - \{1\}$. Letting $\Gamma_f^0 \subset X \times \mathbb{A}^1$ be the restriction of Γ_f, we have

$$\mathrm{div}(f) = pr_X[(X \times 0 - X \times 1) \cdot \Gamma_f^0].$$

Conversely, let Γ be an arbitrary codimension p subvariety of $X \times \mathbb{A}^1$. Then by the homotopy property, there is a codimension p cycle z on X with $\Gamma \sim p_1^* z$. Thus, in $\mathrm{CH}^p(X)$, we have

$$pr_X[(X \times 0 - X \times 1) \cdot \Gamma] = pr_X[i_0^* p_1^*(z) - i_1^* p_1^* z] = z - z = 0,$$

where $i_0, i_1 : X \to X \times \mathbb{A}^1$ are the 0 and 1 sections. Thus, the cycle $pr_X[(X \times 0 - X \times 1) \cdot \Gamma]$ is in $R^p(X)$. In short, $R^p(X)$ is the subgroup of $Z^p(X)$ consisting of cycles of the $pr_X[(X \times 0 - X \times 1) \cdot \Gamma]$, where Γ is a codimension p cycle on $X \times \mathbb{A}^1$.

By the homotopy property of $G_0(X)$, we similarly have

$$i_0^*(\mathcal{O}_\Gamma) = i_1^*(\mathcal{O}_\Gamma)$$

for each subvariety Γ of $X \times \mathbb{A}^1$. However, $i_0^*(\mathcal{O}_\Gamma)$ is *not* in general the same as $\mathrm{cl}(i_0^*(\Gamma))$. It does follow from the computation in example 2.2 and the localization theorem for G_0 that $i_0^*(\mathcal{O}_\Gamma) \equiv \mathrm{cl}(i_0^*(\Gamma))$ modulo the image of $G_0(Z)$ for some closed subset Z of X of codimension $\geq p+1$. This motivates the following

Definition 3.3. Let X be a scheme. Define $F^q_{\mathrm{top}} G_0(X) \subset G_0(X)$ to be the subgroup of $G_0(X)$ generated by the images $G_0(Z) \to G_0(X)$, as Z runs over closed subschemes of X of codimension $\geq q$.

If X is regular, we define $F^q_{\mathrm{top}} K_0(X)$ as the image of $F^q_{\mathrm{top}} G_0(X)$ via the isomorphism $K_0(X) \cong G_0(X)$.

Clearly $\mathrm{cl}(Z^p(X)) \subset F^p_{\mathrm{top}}G_0(X)$; our computation above shows that cl descends to

$$\mathrm{cl}^p : \mathrm{CH}^p(X) \to gr^p_{\mathrm{top}}G_0(X).$$

Lemma 3.4. $\mathrm{cl}^p : \mathrm{CH}^p(X) \to gr^p_{\mathrm{top}}G_0(X)$ *is surjective.*

Proof. Let η be in $F^p_{\mathrm{top}}G_0(X)$. Then there is a pure codimension p subscheme $i : Z \to X$ and an element $\eta' \in G_0(Z)$ with $\eta = i_*(\eta')$. If Z has irreducible components $Z_1, \ldots, Z_s$, then $G_0(k(Z_i)) = \mathbb{Z}$, so there are integers $n_1, \ldots, n_s$ such that $\eta' - \sum_i n_i[\mathcal{O}_{Z_i}]$ goes to zero in $\oplus_i G_0(k(Z_i))$. But then there is an open subscheme U of Z, containing the generic point of each Z_i such that $\eta' - \sum_i n_i[\mathcal{O}_{Z_i}]$ goes to zero in $G_0(U)$. Let $\bar{Z} = Z \setminus U$. Then $\bar{Z}$ has codimension $\geq p+1$ on X, and by the localization sequence

$$G_0(\bar{Z}) \xrightarrow{\bar{i}} G_0(Z) \to G_0(U) \to 0,$$

there is an element $\alpha \in G_0(\bar{Z})$ with

$$\eta' = \sum_i n_i[\mathcal{O}_{Z_i}] + \bar{i}_*(\alpha)$$

in $G_0(Z)$. Pushing forward to $G_0(X)$ gives

$$\eta = \sum_i n_i[\mathcal{O}_{Z_i}] + \alpha',$$

where α' is the image of α under $G_0(\bar{Z}) \to G_0(X)$. But $\sum_i n_i[\mathcal{O}_{Z_i}] = \mathrm{cl}(\sum_i n_i \cdot Z_i)$, and α' is in $F^{p+1}_{\mathrm{top}}G_0(X)$, proving the result. □

3.3. **GRR.** In fact, cl^p is almost an isomorphism. The almost inverse is given by the pth Chern class. This follows from a special case of what is known as the *Grothendieck-Riemann-Roch theorem*. Here is the special case we need (for a proof, see the original article of Borel-Serre [9], or the more modern treatment in [11]):

Theorem 3.5 (Grothendieck-Riemann-Roch)**.** *Let $i : Z \to X$ be the inclusion of an integral closed codimension p subscheme of a smooth k variety X, giving the class $[\mathcal{O}_Z]$ in $G_0(X) = K_0(X)$. Then $c_q([\mathcal{O}_Z]) = 0$ for $q < p$, and $c_p([\mathcal{O}_Z]) = (-1)^{p-1}(p-1)! \cdot Z$ in $\mathrm{CH}^p(X)$.*

Corollary 3.6. *Let X be a smooth k-variety.*

(1) *The map $c_p : K_0(X) \to \mathrm{CH}^p(X)$ sends $F^{p+1}_{\mathrm{top}}K_0(X)$ to zero, and defines a homomorphism*

$$c_p : gr^p_{\mathrm{top}}K_0(X) \to \mathrm{CH}^p(X).$$

(2) $c_p \circ \mathrm{cl}^p = (-1)^{p-1}(p-1)! \cdot \mathrm{id}$ *and* $\mathrm{cl}^p \circ c_p = (-1)^{p-1}(p-1)! \cdot \mathrm{id}$.

Proof. That c_p descends to a set map on gr^p follows from GRR. Also, if a and b are in $F^p_{\text{top}}K_0(X)$, then $c(a+b) = c(a)c(b)$ and GRR implies that $c_p(a+b) = c_p(a) + c_p(b)$ (since the lower Chern classes are zero). This proves (1). The first formula in (2) follows also from GRR. For the second, $\mathrm{cl}^p : Z^p(X) \to gr^p_{\text{top}}K_0(X)$ is surjective by lemma 3.4. Thus, the first formula implies the second. □

3.4. **Curves and surfaces.** For X a smooth curve over k, GRR gives us the short exact sequence

$$0 \to \mathrm{CH}^1(X) \to K_0(X) \to \mathrm{CH}^0(X) \to 0.$$

$\mathrm{CH}^0(X)$ is just the free abelian group on the components of X, and $\mathrm{CH}^1(X)$ is the classical group of divisors modulo linear equivalence, which we have already seen is isomorphic to the Picard group $\mathrm{Pic}(X)$.

Suppose X is projective. Then we have the degree homomorphism $\mathrm{CH}^1(X) \to \mathbb{Z}$, which is surjective if for example X has a k-rational point. The kernel of the degree homomorphism is the group of k-points of the Jacobian variety of X.

Now suppose X is a surface. We have the two-step filtration of $K_0(X)$:

$$F^2_{\text{top}}K_0(X) \subset F^1_{\text{top}}K_0(X) \subset K_0(X).$$

By GRR, we have $gr^0_{\text{top}}K_0(X) = \mathrm{CH}^0(X)$, $gr^1_{\text{top}}K_0(X) = \mathrm{CH}^1(X)$ and $gr^2_{\text{top}}K_0(X) = \mathrm{CH}^2(X)$. The story for $\mathrm{CH}^0(X)$ is just as for curves, and that of $gr^1_{\text{top}}K_0(X) = \mathrm{CH}^1(X)$ is similar: if X is smooth and projective, we have the intersection product $\deg(D \cdot D') \in \mathbb{Z}$ for $D, D' \in \mathrm{CH}^1(X)$. Call D *numerically equivalent to zero* if $\deg(D \cdot D') = 0$ for all D'. Then $\mathrm{CH}^1(X)/num$ (num = the group of divisors numerically equivalent to zero). is a finitely generated group. num contains a subgroup of finite index alg, and $alg \subset \mathrm{CH}^1(X)$ is the group of k-points on the *Picard variety* of X, which is a projective group variety over k. In fact, there is a very similar description of $\mathrm{CH}^1(X)$ for all X smooth and projective over a field k.

The situation for $\mathrm{CH}^2(X)$ is radically different, in general. We will take this up in some detail in the next paragraph.

3.5. **Topological and analytic invariants.** Now suppose the base field $k = \mathbb{C}$. By the localization sequence, it suffices to understand $K_0(X)$ or $\mathrm{CH}^*(X)$ for X smooth and projective over $\mathbb{C}$; using GRR, we can restrict our attention to CH^*.

First of all, taking the topological class of an algebraic cycle defines the map

$$\mathrm{cl}^p_{\text{top}} : \mathrm{CH}^p(X) \to H^{2p}(X, \mathbb{Z}).$$

It is known that $\mathrm{cl}^p_{\mathrm{top}}$ is not in general surjective. In fact, the $\mathbb{C}$-cohomology of X has the *Hodge decomposition*

$$H^n(X,\mathbb{C}) = \oplus_{p+q=n;\ p,q\geq 0} H^{p,q}(X),$$

with $H^{p,q}(X) = \overline{H^{q,p}(X)}$ and with

$$H^{p,q} \cong H^q(X, \Omega^p_X).$$

It is easy to see that the image of $\mathrm{cl}^p_{\mathrm{top}}$ lands in $H^{2p}(X,\mathbb{Z})\cap(2\pi i)^{-p}H^{p,p}$, and it is easy to construct examples where this is not all of $H^{2p}(X,\mathbb{Z})$. We write $\mathrm{CH}^p(X)_{\mathrm{hom}}$ for the kernel of $\mathrm{cl}^p_{\mathrm{top}}$.

The *Hodge conjecture* asserts that the image of $(2\pi i)^p\mathrm{cl}^p$ is of finite index in $(2\pi i)^p H^{2p}(X,\mathbb{Z})\cap H^{p,p}$. This is known to be the case for $p=1$ (in this case one gets all of $(2\pi i)H^2(X,\mathbb{Z})\cap H^{1,1}$ as the image), but is not known in general (there is a million dollar prize for a proof or counter-example!)

In the case of $p=1$, the kernel of $\mathrm{cl}^1_{\mathrm{top}}$ is the same as the subgroup *alg* defined above, so $\mathrm{CH}^1(X)_{\mathrm{hom}}$ is the Picard variety of X. There is an analytic discription of this variety: let

$$J^1(X) = H^{0,1}(X)/(2\pi i)H^1(X,\mathbb{Z}).$$

Then $J^1(X)$ is isomorphic to the Picard variety of X. Griffiths generalized this construction, defining complex torii $J^p(X)$ by

$$J^p(X) := \frac{H^{0,2p-1}(X)\oplus\ldots\oplus H^{p-1,p}(X)}{(2\pi i)^p H^{2p-1}(X,\mathbb{Z})}.$$

There is a cycle class map

$$\mathrm{cl}^p_{\mathrm{hom}} : \mathrm{CH}^p(X)_{\mathrm{hom}} \to J^p(X),$$

generalizing the isomorphism $\mathrm{CH}^1(X)_{\mathrm{hom}} \cong J^1(X)$. However, except in some special cases, the map $\mathrm{cl}^p_{\mathrm{hom}}$ is neither injective nor surjective. If $H^{a,b}(X)\neq 0$ for some a,b with $a+b=2p-1$ and $|a-b|>1$, then $\mathrm{cl}^p_{\mathrm{hom}}$ is not surjective. Also, the image of $\mathrm{cl}^p_{\mathrm{hom}}$ can be quite complicated; examples of Clemens and others show that the image can be an uncountably generated group, with trivial connected component of 0.

For codimension$=\dim X = d$ (i.e., dimension zero), $\mathrm{CH}^d(X)_{\mathrm{hom}}$ is just the group of dimension zero cycles of degree zero, $J^d(X)$ is the *Albanese* variety, and $\mathrm{cl}^d_{\mathrm{hom}} : \mathrm{CH}^d(X)_{\mathrm{hom}} \to J^d(X)$ is induced by the Albanese morphism of X, in particular $\mathrm{cl}^d_{\mathrm{hom}}$ is surjective.

3.6. **Infinite dimensionality.** For $d = 1$, $\mathrm{cl}^1_{\mathrm{hom}} : \mathrm{CH}^1(X)_{\mathrm{hom}} \to J^1(X)$ is an isomorphism, and $J^1(X)$ is an algebraic variety. For CH^2, the situation is very different, as first pointed out in a fundamental result of Mumford [28].

Definition 3.7. For a given variety X, we say that $\mathrm{CH}^p(X)_{\mathrm{hom}}$ is *finite dimensional* if there is a variety T over $\mathbb{C}$, together with a codimension p cycle $\mathcal{W}$ on $X \times T$, such that the set of codimension p cycles on X, $\{pr_X(\mathcal{W} \cdot (X \times t)) \mid t \in T(\mathbb{C})\}$ is all of $\mathrm{CH}^p(X)_{\mathrm{hom}}$. We say that $\mathrm{CH}^p(X)_{\mathrm{hom}}$ is *infinite dimensional* if it is not finite dimensional.

For example if X is smooth and projective, $\mathrm{CH}^1(X)_{\mathrm{hom}}$ is finite dimensional, as taking $\mathcal{W}$ to be the Poincaré divisor on $J^1(X) \times X$ realizes the isomorphism $\mathrm{cl}^1_{\mathrm{hom}} : \mathrm{CH}^1(X)_{\mathrm{hom}} \to J^1(X)$.

Theorem 3.8 (Mumford)**.** *Let X be a smooth projective surface over $\mathbb{C}$. Suppose that $H^0(X, \Omega^2_{X/\mathbb{C}}) \neq 0$, that is, that X has a non-zero global two-form. Then $\mathrm{CH}^2(X)$ is infinite dimensional.*

Roitman [32] has generalized Mumford's result to show

Theorem 3.9. *Let X be a smooth projective variety over $\mathbb{C}$. Suppose that $H^0(X, \Omega^p_X) \neq 0$ for some $p > 1$. Then the kernel of $\mathrm{cl}^q_{\mathrm{hom}}$ is infinite dimensional.*

So, bad news. To date no one has been able to give a coherent description of $\mathrm{CH}^d(X)_{\mathrm{hom}}$ in case $H^0(X, \Omega^p_X) \neq 0$ for some $p > 1$. There is the famous conjecture of Bloch:

Conjecture 3.10. Let X be a smooth projective surface over $\mathbb{C}$ with $H^0(X, \Omega^2_X) = 0$. Then $\mathrm{cl}^2_{\mathrm{hom}} : \mathrm{CH}^2(X)_{\mathrm{hom}} \to J^2(X)$ is an isomorphism.

This has been settled for surfaces not of general type in [6], and for many surfaces of general type by a number of authors, but without any "structural" proof, the full conjecture still remains quite open. The converse of Bloch's conjecture, that the injectivity of the cycle class map implies the that cohomology/Hodge theory of X is particularly simple, has been generalized by Jannsen [17] and Esnault-Levine [10] as follows:

Theorem 3.11 (Jannsen)**.** *Let X be a smooth projective variety over $\mathbb{C}$. Suppose that $\mathrm{cl}^p_{\mathrm{top}} : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}(X, \mathbb{Q})$ is injective for all p. Then $H^{2p+1}(X, \mathbb{Q}) = 0$ for all p, and $\mathrm{cl}^p_{\mathrm{top}} : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}(X, \mathbb{Q})$ is an isomorphism for all p. In particular, $H^{a,b}(X) = 0$ for $a \neq b$.*

Theorem 3.12 (Esnault, Levine)**.** *Let X be a smooth projective variety over $\mathbb{C}$. Suppose that $\mathrm{cl}^p_{\mathrm{hom}} : \mathrm{CH}^p(X)_{\mathrm{hom}\mathbb{Q}} \to J^p(X)_{\mathbb{Q}}$ is injective for* *all p. Then $\mathrm{cl}^p_{\mathrm{hom}} : \mathrm{CH}^p(X)_{\mathrm{hom}\mathbb{Q}} \to J^p(X)_{\mathbb{Q}}$ is an isomorphism for all p, and $H^{a,b}(X) = 0$ for $|a-b| > 1$.*

Both of these results have finer versions stating that, if cl^p is injective for $p \geq s$, then cl^p is surjective for $p \leq s+1$.

On the positive side, the torsion seems to be quite well behaved. Roitman [33] has shown (he actually proves a more general result valid for an arbitrary algebraically closed field k instead of $\mathbb{C}$):

Theorem 3.13. *Let X be a smooth projective variety of dimension d over $\mathbb{C}$. Then the map*

$$\mathrm{cl}^d_{\mathrm{hom}} : \mathrm{CH}^d(X)_{\mathrm{hom}} \to J^d(X)$$

induces an isomorphism on the torsion subgroups.

3.7. **Complete intersections.** For an affine variety over a field, $X = \operatorname{Spec} R$, there is a close connection between K_0 and problems in commutative algebra. One example is the question: Is a given ideal I a complete intersection in R, i.e., is $I = (f_1, \ldots, f_n)$, where $f_1, \ldots, f_n$ form a regular sequence in R?

One condition is clearly that I must be a local complete intersection: for each prime ideal p, the image I_p in the local ring R_p must be a complete intersection, but in general this is not enough. If I is a local complete intersection, then R/I admits a finite projective resolution, so has a class $[R/I] \in K_0(R)$. If I is a complete intersection, one has the Koszul resolution

$$0 \to \Lambda^n R^n \to \ldots \to \Lambda^2 R^n \to R^n \to R \to R/I \to 0,$$

from which it follows $[R/I] = 0$. What about the converse? One can even ask the more difficult question: Let $X = \operatorname{Spec} R$, and let $Z \subset X$ be a pure codimension p subscheme. Suppose that Z is a local complete intersection in X and that the associated cycle $|Z| \in Z^p(X)$ goes to zero in $\mathrm{CH}^p(X)$. Is Z a complete intersection subscheme in X?

Since one would expect that going to zero in $\mathrm{CH}^p(X)$ is a weaker condition than going to zero in $K_0(X)$, this may seem to be too much to require. However, one has the following result:

Theorem 3.14 (Murthy, Levine, Srinivas)**.** *Let X be a reduced affine variety of dimension d over an algebraically closed field k. Then $\widetilde{\mathrm{CH}}^d(X)$ is torsion free, hence the map $\widetilde{\mathrm{CH}}^d(X) \to K_0(X)$ is injective.*

Here, I should explain $\widetilde{\mathrm{CH}}^d$ is a modified version of $\mathrm{CH}^d(X)$ (constructed in [18]) which maps to $K_0(X)$ even if X is not smooth over k; $\widetilde{\mathrm{CH}}^d$ is CH^d if X is smooth. The case of a smooth X was proved by

Mohan Kumar and Murthy [27], relying on Roitman's theorem 3.13; Levine [19] proved the case of X smooth in codimension one, and Srinivas [37] proved the general case.

As for the complete intersection question, if Z has pure codimension one and is a local complete intersection, the ideal sheaf $\mathcal{I}_Z$ is a rank one locally free sheaf, and $[\mathcal{O}_Z] = [\mathcal{O}_X] - [\mathcal{I}_X]$, so $[\mathcal{O}_Z] = 0$ if and only if $[\mathcal{I}_Z] = [\mathcal{O}_X]$. However, sending $\mathcal{E}$ to $\Lambda^{\mathrm{rank}\mathcal{E}}\mathcal{E}$ defines a map of sets $\det : K_0(X) \to \mathrm{Pic}(X)$, splitting the map $\mathrm{Pic}(X) \to K_0(X)$. Thus, two rank one sheaves have the same $K_0(X)$-class if and only if they are isomorphic. So, $[\mathcal{O}_Z] = 0$ implies $\mathcal{I}_Z \cong \mathcal{O}_X$, so Z is defined by a single equation (the image of $1 \in \mathcal{O}_X$ in $\mathcal{I}_Z$).

For codimension two, Serre proved

Theorem 3.15. *Let X be a smooth affine surface over an algbraically closed field, and let Z be a codimension two closed subscheme. Suppose that Z is a local complete intersection and that the associated cycle $|Z|$ vanishes in $\mathrm{CH}^2(X)$. Then Z is a complete intersection.*

Relying on theorem 3.14, Murthy and Mohan Kumar [26] extended this to codimension two subschemes of smooth threefolds:

Theorem 3.16. *Let X be a smooth affine scheme of dimension 3 over an algbraically closed field, and let Z be a codimension 2 closed subscheme. Suppose that Z is a local complete intersection and that the associated cycle $|Z|$ vanishes in $\mathrm{CH}^2(X)$. Then Z is a complete intersection.*

A related problem is: Let $\mathcal{E}$ be a locally free sheaf of rank r on an affine variety X of dimension r. If $\mathcal{E}$ admits a nowhere vanishing section (that is, $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{E}'$ for some rank $r-1$ sheaf $\mathcal{E}'$), it follows from the Whitney product formula for Chern classes that the top Chern class $c_r(\mathcal{E})$ is zero in $\mathrm{CH}^r(X)$. The converse for X a variety over an algebraically closed field is the following theorem of Murthy [29] (the case of dimension three was settled in [26])

Theorem 3.17. *Let $X = \mathrm{Spec}\, R$ be a reduced affine variety of dimension r over an algebraically closed field k, and let P be a projective R-module of rank r. Let $\mathcal{P}$ denote the associated locally free sheaf on X. If $c_r(\mathcal{P}) = 0$ in $\widetilde{\mathrm{CH}}^r(X) \subset K_0(X)$, then $P = Q \oplus R$ for some rank $r-1$ projective R-module Q.*

Part 2. Higher K-theory of rings

We now turn to higher K-theory, with some historical background on K_1 and K_2 of rings, followed by a sketch of Quillen's plus construction.

4. K_1 *of a ring*

4.1. **Matrices and elementary matrices.** Let R be a ring. We have the ring of $n \times n$ matrices $M_n(R)$, and the group of units $\mathrm{GL}_n(R)$. The stabilization map $\mathrm{GL}_n(R) \xrightarrow{\rho_n} \mathrm{GL}_{n+1}(R)$ is defined by

$$(a_{ij}) \mapsto \begin{pmatrix} a_{ij} & 0 \\ 0 & 1 \end{pmatrix};$$

define $\mathrm{GL}(R)$ to be the limit

$$\mathrm{GL}(R) := \varinjlim_n \mathrm{GL}_n(R).$$

For indices $1 \leq i \neq j \leq n$ and element $\lambda \in R$, let e_{ij}^{λ} be the $n \times n$ matrix with 1's down the diagonal, λ in the ith row and jth column, and all other entries zero. We let $E_n(R)$ be the subgroup of $\mathrm{GL}_n(R)$ generated by the e_{ij}^{λ}. The stabilization maps send $E_n(R)$ into $E_{n+1}(R)$, so we can define $E(R)$ as the limit of the $E_n(R)$. These are all called the group of *elementary* matrices.

Remark 4.1. Let $Uni_n(R)$ be the group of upper triangular matrices with 1's down the diagonal. Then $Uni_n(R) \subset E_n(R)$, in fact $Uni_n(R)$ is the subgroup of $\mathrm{GL}_n(R)$ generated by the e_{ij}^{λ} with $i < j$. Indeed, left multiplication by e_{ij}^{λ} gives the elementary row operation of adding λ times the jth row to the ith row, from which our assertion easily follows. Similarly, the lower triangular matrices with 1's on the diagonal are in $E_n(R)$, being the subgroup generated by the e_{ij}^{λ} with $i > j$.

4.2. **The Whitehead lemma.** The basic elementary matrices satisfy the following identities (we ignore the size n): Take λ, μ in R.

(1) if i, j, k, l are all distinct, then e_{ij}^{λ} and e_{kl}^{μ} commute.
(2) $e_{ij}^{\lambda} e_{ij}^{\mu} = e_{ij}^{\lambda+\mu}$.
(3) if i, j, k are distinct, then the commutator $[e_{ij}^{\lambda}, e_{jk}^{\mu}]$ is $e_{ik}^{\lambda\mu}$ (here $[a, b] = a^1 b^{-1} ab$).

In consequence, we have

Lemma 4.2. *For $n \geq 3$, $E_n(R) = [E_n(R), E_n(R)]$.*

Proof. Take $i \neq j$ between 1 and n, and take $\lambda \in R$. Since $n \geq 3$, there is a k in $\{1, \ldots, n\}$ distinct from i and j. By the relation (3), we have

$$e_{ij}^{\lambda} = [e_{ik}^{1}, e_{kj}^{\lambda}],$$

so $E_n(R) \subset [E_n(R), E_n(R)]$. The other inclusion is evident. $\square$

Lemma 4.3. *Let A be in $\mathrm{GL}_n(R)$. Then $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ is in $E_{2n}(R)$.*

Proof. We have seen in remark 4.1 that, for $M \in M_n(R)$, the matrices

$$\begin{pmatrix} I_n & M \\ 0 & I_n \end{pmatrix}, \quad \begin{pmatrix} I_n & 0 \\ M & I_n \end{pmatrix}$$

are in $E_{2n}(R)$. We have

$$\begin{pmatrix} 0 & A \\ -A^{-1} & 0 \end{pmatrix} = \begin{pmatrix} I_n & A \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -A^{-1} & I_n \end{pmatrix} \begin{pmatrix} I_n & A \\ 0 & I_n \end{pmatrix} \in E_{2n}(R)$$

Taking $A = -I_n$, we see that $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is also in $E_{2n}(R)$. Thus

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 0 & A \\ -A^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

is in $E_{2n}(R)$. $\square$

Theorem 4.4 (Whitehead)**.** $E(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$. *In particular, $E(R)$ is a normal subgroup of $GL(R)$.*

Proof. Let A and B be in $\mathrm{GL}_n(R)$. Then the image of $[A, B]$ in $\mathrm{GL}_{2n}(R)$ is the product

$$\begin{pmatrix} (BA)^{-1} & 0 \\ 0 & BA \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix},$$

which is in $E_{2n}(R)$ by the previous lemma. $\square$

Definition 4.5. Let R be a ring. The group $K_1(R)$ is the abelianization of $\mathrm{GL}(R)$:

$$K_1(R) := \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)].$$

By the Whitehead theorem, we have the alternate description of $K_1(R)$ as

$$K_1(R) = \mathrm{GL}(R)/E(R).$$

Examples 4.6. (1) Let R be a commutative ring. The determinant homomorphisms $\det : \mathrm{GL}_n(R) \to R^\times$ define a homomorphism $\det : \mathrm{GL}(R) \to R^\times$. Since clearly $\det(E(R)) = 1$, we have the surjective homomorphism

$$\det : K_1(R) \to R^\times.$$

The kernel of det is denoted $SK_1(R)$.
(2) Let R be a field F. Noting from the proof of lemma 4.3 that

the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is in $E_2(R)$, is easy to see that every invertible matrix can be made into a diagonal matrix by the elementary row operations $e_{ij}^\lambda \times -$. Noting that $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ is also in $E_2(R)$ for $u \in F^\times$ (by lemma 4.3), we see that $G \in GL_n(F)$ is equivalent to a matrix of the form

$$\begin{pmatrix} u & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

for some $u \in F^\times$. Clearly $u = \det G$, i.e. $\det : K_1(F) \to F^\times$ is an isomorphism. An analogous argument shows that $K_1(R) = R^\times$ for R a local ring, or R a Euclidean ring (but not in general for R a PID!). In a famous paper of Bass-Milnor-Serre [2], it is shown that $SK_1(\mathcal{O}_{S,F}) = 0$ where $\mathcal{O}_{S,F}$ is the ring of S-integers in a number field F.

5. *K_2 of a ring*

We have the exact sequence $1 \to E(R) \to \mathrm{GL}(R) \to K_1(R) \to 0$; $K_2(R)$ continues this "unwinding" of $\mathrm{GL}(R)$.

5.1. **The Steinberg group.** Fix an integer $n \geq 3$ and a ring R. The *Steinberg group* $\mathrm{St}_n(R)$ is the free group on symbols x_{ij}^λ, with $1 \leq i \neq j \leq n$, $\lambda \in R$, modulo the following relations: Take λ, μ in R.
(5.1)

1. If i, jk, l are all distinct, then $[x_{ij}^\lambda, x_{kl}^\mu] = 1$.
2. $x_{ij}^\lambda x_{ij}^\mu = x_{ij}^{\lambda+\mu}$.
3. If i, j, k are distinct, then $[x_{ij}^\lambda, x_{jk}^\mu] = x_{ik}^{\lambda\mu}$.

We have the evident homomorphisms $\mathrm{St}_n(R) \to \mathrm{St}_{n+1}(R)$; we let $\mathrm{St}(R)$ be the limit of the $\mathrm{St}_n(R)$, i.e., $\mathrm{St}(R)$ is the free group on generators x_{ij}^λ with $1 \leq i \neq j$, $\lambda \in R$, modulo the relations (1)-(3).

Since the elementary matrices e_{ij}^λ satisfy the relations (1)-(3), sending x_{ij}^λ to e_{ij}^λ defines the homomorphisms $\mathrm{St}_n(R) \to E_n(R)$, $\mathrm{St}(R) \to E(R)$.

Definition 5.1. Let R be a ring. $K_2(R)$ is defined to be the kernel of $\mathrm{St}(R) \to E(R)$.

The following result is crucial:

Theorem 5.2. *$K_2(R)$ is an abelian group. In fact, $K_2(R)$ is the center of* $\mathrm{St}(R)$ *and the sequence*

$$0 \to K_2(R) \to \mathrm{St}(R) \to E(R) \to 1$$

is the universal central extension of $E(R)$.

For a proof of this result, see [25, §5].

Since the central extensions of a group G are classified by $H_2(G, \mathbb{Z})$, we have

Corollary 5.3. *$K_2(R)$ is canonically isomorphic to $H_2(E(R), \mathbb{Z})$.*

Because $K_2(R)$ is an abelian group, we usually write the group law additively.

Example 5.4. We have already seen that

$$e_{12}^1 e_{21}^{-1} e_{12}^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Calling this matrix A, it is clear that $A^4 = I_2$. Thus $\eta := (x_{12}^1 x_{21}^{-1} x_{12}^1)^4 \in \mathrm{St}(\mathbb{Z})$ is in fact in $K_2(\mathbb{Z})$. It turns out that $\eta \neq 0$, $2\eta = 0$, and $K_2(\mathbb{Z}) = <\eta>$.

5.2. **Symbols and Matsumoto's theorem.** Let U, V be commuting elements of $E(R)$. If we lift U and V to elements $\tilde{U}, \tilde{V}$ of $\mathrm{St}(R)$, then clearly the commutator $[\tilde{U}, \tilde{V}]$ is in $K_2(R)$. Since $K_2(R)$ is the center of $\mathrm{St}(R)$, this commutator depends only on U and V; we denote it $<U, V>$.

For example, take units u and v in R, and assume that R is a commutative ring. Define the *symbol* $\{u, v\} \in K_2(R)$ by

$$\{u, v\} := < \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^{-1} \end{pmatrix} >.$$

This symbol has the following properties:
(5.2)

(1) The assignment $(u, v) \mapsto \{u, v\}$ is bilinear, with respect to group law of multiplication on $R^\times$.
(2) If u and $1-u$ are in $R^\times$, then $\{u, 1-u\} = 0$.

No discussion of K_2 is complete without Matsumoto's theorem:

Theorem 5.5. *Let F be a field. Then sending u, v to $\{u, v\}$ gives a surjective homomorphism $F^\times \otimes_{\mathbb{Z}} F^\times \to K_2(F)$, with kernel the subgroup generated by elements of the form $u \otimes (1-u)$, with $u \in F$, $u \neq 0, 1$.*

Remark 5.6. Let $u \neq 0$ be in F. We have

$$0 = \{\frac{1}{u}, 1 - \frac{1}{u}\} = -\{u, \frac{1-u}{-u}\} = \{u, -u\}.$$

In particular, the bilinear map $(u, v) \mapsto \{u, v\}$ is alternating: $\{u, v\} = -\{v, u\}$.

Remark 5.7. The relation $\{a, 1-a\} = 0$ is called the *Steinberg relation*. It crops up in many situations; Matsumoto's theorem then allows one to insert K-theoretic machinery. For example, let k be a field, let $W(k)$ denote the Witt ring of quadratic forms over k, modulo hyperbolic forms, and let $I \subset W(k)$ be the ideal of forms of even dimension. Sending $a \in k^\times$ to the class $<a>$ of the form $x^2 - ay^2$ defines a homomorphism $k^\times \to I/I^2$; using the ring structure in $W(k)$, we have $k^\times \otimes k^\times \to I^2/I^3$ by $a \otimes b \mapsto <a, b> := <a> \cdot <b>$. One can show that $<a, 1-a> = 0$, giving the map $K_2(k) \to I^2/I^3$. It is known that I^2/I^3 is a 2-torsion group, so we have $K_2(k)/2 \to I^2/I^3$; a fundamental theorem of Merkurjev [22] says that this map is an isomorphism.

As another example, let F/k be an extension of fields. We have the group homomorphism $d\ln : F^\times \to \Omega^1_{F/k}$ by $a \mapsto (1/a)da$. This induces $d\ln \wedge d\ln : F^\times \otimes F^\times \to \Omega^2_{F/k}$, $a \otimes b \mapsto d\ln(a) \wedge d\ln(b)$; clearly $d\ln(a) \wedge d\ln(1-a) = 0$, giving $d\ln \wedge d\ln : K_2(F) \to \Omega^2_{F/k}$. This map has important applications in relating K-theory to de Rham cohomology.

Finally, the vanishing of $a \cup (1-a)$ in $H^2(F, \mu_n^{\otimes 2})$ (discussed below) allows for the formulation of a central result in K-theory, the *Bloch-Kato conjecture.*

6. BGL^+

6.1. **Categories and simplicial sets.** We let **Ord** denote the category with objects the non-empty finite totally ordered sets, with morphisms being order-preserving maps of sets. As each finite ordered set of cardinality $n+1 \geq 1$ is uniquely isomorphic to the ordered set $[n] := \{0, \ldots, n\}$ with the standard order, **Ord** is equivalent to the category with objects $[n]$ $n = 0, 1, \ldots$, with order-preserving maps.

A functor $S : \mathbf{Ord}^{\mathrm{op}} \to \mathbf{Sets}$ is called a *simplicial set*, $S([n])$ is the *set of n-simplices of S*. We can just as well replace **Sets** with an arbitrary category $\mathcal{C}$, giving the notion of a simplicial object of $\mathcal{C}$. A *cosimplicial object of* $\mathcal{C}$ is a functor $T : \mathbf{Ord} \to \mathcal{C}$; $T([n])$ is the n-cosimplices of T.

Remark 6.1. The morphisms in **Ord** are generated by the *coboundary* maps $\delta_i^n : [n] \to [n+1]$, $i = 0, \dots, n+1$,

$$\delta_i^n(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i, \end{cases}$$

and the *codegeneracy* maps $\sigma_i^n : [n] \to [n-1]$, $i = 1, \dots, n$,

$$\sigma_i^n(j) = \begin{cases} j & \text{if } j < i \\ j-1 & \text{if } j \geq i. \end{cases}$$

These satisfy certain relations, which we don't specify here. A simplicial object S is thus often given by defining the n-simplices S_n, the boundary maps $\partial_i^n = S(\delta_i^{n-1}) : S_n \to S_{n-1}$, and the degeneracy maps $s_i^n = S(\sigma_i^{n+1}) : S_n \to S_{n+1}$.

The fundamental example of a cosimplicial space ($\mathcal{C} = \mathbf{Top}$) is $\Delta :$ $\mathbf{Ord} \to \mathbf{Top}$. $\Delta([n])$ is the standard topological n-simplex:

$$\Delta([n]) = \Delta_n := \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\}.$$

Δ_n has vertices $v_0^n, \dots, v_n^n$, where v_i^n has $t_i = 1$, $t_j = 0$ for $j \neq i$; clearly Δ_n is the convex hull of its vertices. Let $g : [n] \to [m]$ be a map of sets (order-preserving). We send Δ_n to Δ_m by sending the vertex v_i^n to $v_{g(i)}^m$ and then taking the convex-linear extension:

$$\sum_i t_i v_i^n \mapsto \sum_i t_i v_{g(i)}^m.$$

This defines the functor $\Delta : \mathbf{Ord} \to \mathbf{Top}$.

Now let $S : \mathbf{Ord}^{\mathrm{op}} \to \mathbf{Sets}$ be a simplicial set. Define the *geometric realization* of S, $|S|$, as the topological space

$$|S| = \coprod_{n=0}^{\infty} S_n \times \Delta_n / \sim$$

where the gluing data $\sim$ is defined by

$$S(g)(s) \times x \sim s \times \Delta(g)(x),$$

for all $s \in S_n$, $x \in \Delta_m$ and $g : [m] \to [n]$ in **Ord**.

Now let $\mathcal{C}$ be a small category ($\mathcal{C}$ has a set of objects). Define a simplicial set $\mathcal{NC}$, the *nerve* of $\mathcal{C}$ with 0-simplices the objects of $\mathcal{C}$, and with n-simplices (for $n > 0$) the set of composable morphisms $(f_1, \dots, f_n)$:

$$a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} a_{n-1} \xrightarrow{f_n} a_n.$$

If $h : [m] \to [n]$ is order-preserving define $h((f_1, \dots, f_n)) = (g_1, \dots, g_m)$, where $g_i : a_{h(i-1)} \to a_{h(i)}$ is the composition of $f_{h(i-1)+1}, \dots, f_{h(i)}$ if $h(i-1) < h(i)$, and the identity on $a_{h(i-1)}$ if $h(i-1) = h(i)$. The *classifying space* of the category $\mathcal{C}$ is defined as

$$B\mathcal{C} := |\mathcal{N}\mathcal{C}|.$$

Examples 6.2. (1) Let X be a set. Let $\mathcal{E}(X)$ be the category with objects X, and with a unique morphism $x \to y$ for each $x, y \in X$. Clearly $\mathcal{N}(\mathcal{E}(X))_n = X^{n+1}$, with

$$\partial_i^n(x_0, x_1, \dots, x_n) = (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

We write EX for $|\mathcal{E}(X)|$; it is not hard to see that EX is contractible.

(2) Let G be a group. We may consider G as a category $\mathcal{B}(G)$ with a single object $*$, where $Hom_{\mathcal{B}(G)}(*, *) = G$, and the composition is $f \circ g = gf$. Thus, the nerve of $\mathcal{B}(G)$ has n-simplices G^n, and the ith boundary is given by

$$\partial_i^n(g_1, \dots, g_n) = (g_1, \dots, g_{i-2}, g_{i-1}g_i, g_{i+1}, \dots, g_n)$$

for $0 < i \leq n$, and

$$\partial_0^n(g_1, \dots, g_n) = (g_2, \dots, g_n).$$

We write BG for $|\mathcal{B}(G)|$. We have the isomorphism of simplicial sets

$$\mathcal{N}(\mathcal{B}(G)) \cong G \backslash \mathcal{N}(\mathcal{E}(G)),$$

where $g(g_0, \dots, g_n) = (gg_0, \dots, gg_n)$, (and we send $(g_1, \dots, g_n) \in \mathcal{N}(\mathcal{B}(G))_n$ to $(1, g_1, g_1g_2, \dots, g_1 \cdot \ldots \cdot g_n) \in G \backslash \mathcal{N}(\mathcal{E}(G)))$ which extends to the spaces

$$BG = G \backslash EG.$$

Also G acts freely on EG, so we have the covering space $EG \to BG$ with group G. Since EG is contractible, BG has $\pi_1 = G$, $\pi_0 = *$, and $\pi_i = 0$ for $i > 1$. BG is the *classifying space* of the group G.

Either by definition, or by identifying the chain complex of BG with the standard complex computing the homology $H_*(G, \mathbb{Z})$, we see that $H_*(G, \mathbb{Z})$ is canonically isomorphic to $H_*(BG, \mathbb{Z})$. More generally if M is a G-module, since $\pi_1(BG) = G$, we have the associated local system $\mathcal{M}$ on BG, and $H_*(G, M) = H_*(BG, \mathcal{M})$.

6.2. **The plus construction.** Quillen's plus construction is in some sense the topological version of taking the quotient of a group π by a normal subgroup N. In order for the more subtle topological operation to work, one needs to assume that N is *perfect*, that is, $N = [N, N]$. We will apply this with $\pi = \mathrm{GL}(R)$, $N = E(R)$ to define the space $B\mathrm{GL}(R)^+$, whose homotopy groups are the higher K-groups of R.

Now for the general construction (we follow the description given in [1]). Start with a connected pointed space $(X, *)$ and let $N \subset \pi_1(X, *)$ be a perfect normal subgroup. We wish to construct a pointed map $i : (X, *) \to (X^+, *)$ with the following properties:
(6.1)

1. X^+ is connected, and the map $i : \pi_1(X, *) \to \pi_1(X^+, *)$ is the quotient map $\pi_1(X, *) \to \pi_1(X, *)/N$.
2. Let $\mathcal{L}$ be a local system on X^+. Then $i_* : H_*(X, i^*\mathcal{L}) \to H_*(X^+, \mathcal{L})$ is an isomorphism.

In other words, the plus construction kills exactly N in $\pi_1(X, *)$, but leaves the homology of X alone. Using obstruction theory, one can easily show that (6.1) characterizes the map $i : X \to X^+$ uniquely up to homotopy equivalence (if $i : X \to X^+$ is a relative CW complexe; up to weak equivalence in general).

To construct X^+, let $p : \tilde{X} \to X$ be the covering space corresponding to the subgroup $N \subset \pi_1(X, *)$, and let $G = \pi_1(X, *)/N$. Thus G acts freely on $\tilde{X}$ over X, $X = G\backslash\tilde{X}$ and $\pi_1(\tilde{X}) = N$.

Take $x \in N$. Since $N = [N, N]$, we can write $x = \prod_i [y_i, z_i]$, $y_i, z_i \in N$. Let $X^1, \tilde{X}^1$ be the 1-skeleton of $X, \tilde{X}$. Since $\pi_1(\tilde{X}^1, *) \to \pi_1(\tilde{X}, *)$ is surjective, we can lift y_i, z_i to $\bar{y}_i, \bar{z}_i$ in $\pi_1(\tilde{X}^1, *)$. Attach a two-cell D^2_x to X by the attaching map $p_*(\prod_i [\bar{y}_i, \bar{z}_i])$, forming the space Y. Similarly, for each $g \in G$, attach a two cell $D^2_{x,g}$ to $g(\prod_i [\bar{y}_i, \bar{z}_i])$, forming the space $\tilde{Y}$. Extend the G action to $\tilde{Y}$ by sending $D^2_{x,g}$ to $D^2_{x,g'g}$ via the identity. This makes $\tilde{Y} \to Y$ a covering space with group G.

Continue doing this for enough $x \in N$ to generate N as a normal subgroup, and denote again by $\tilde{Y} \to Y$ the resulting spaces. Then $\tilde{Y}$ is connected and simply connected, hence $\pi_1(Y, *) = G$.

By the Hurewicz theorem,

$$\pi_2(\tilde{Y}, *) = H_2(\tilde{Y}, \mathbb{Z}).$$

Also, for each cell $D^2_{x,g}$ we attached, we have $\partial(D^2_{x,g}) = 0$ in homology, so

$$H_2(\tilde{Y}, \mathbb{Z}) = H_2(\tilde{X}, \mathbb{Z}) \oplus F,$$

where F is the free $\mathbb{Z}$-module on the $D^2_{x,g}$. Thus F is a free $\mathbb{Z}[G]$-module. Let $f_\alpha : S^2 \to \tilde{Y}$, $\alpha \in A$, be maps which form a $\mathbb{Z}[G]$-basis for F.

Form X^+ by attaching 3-cells D^3_α to Y by the attaching maps $p_* f_\alpha$. Form $\tilde{X}^+$ similarly by attaching $D^3_{\alpha,g}$ with attaching maps $g \cdot f_\alpha$, $g \in G$. Then $\tilde{X}^+ \to X^+$ is again covering space with group G.

Since $\pi_1(X^+) = \pi_1(Y)$, we have $\pi_1(X^+) = G$. Now let $\mathcal{L}$ be a local system on X^+, and let L be the corresponding G-module. As the relative chain complex $C_*(\tilde{X}^+, \tilde{X})$ is clearly

$$0 \to \ldots \to F \xrightarrow{\cong} F \to \ldots \to 0,$$

the relative chain complex $C_*(X^+, X; \mathcal{L})$ is thus

$$0 \to \ldots \to F \otimes_{\mathbb{Z}[G]} L \xrightarrow{\cong} F \otimes_{\mathbb{Z}[G/N]} L \to \ldots \to 0,$$

hence $i^* : H_*(X, i^*\mathcal{L}) \to H_*(X^+, \mathcal{L})$ is an isomorphism. Thus $i : X \to X^+$ verifies (6.1).

We now take the case $X = B\mathrm{GL}(R)$, $N = E(R)$, forming $i : B\mathrm{GL}(R) \to B\mathrm{GL}(R)^+$. By our identification of $H_*(\mathrm{GL}(R), \mathbb{Z})$ with $H_*(B\mathrm{GL}(R), \mathbb{Z})$, we thus have the canonical identification

$$H_*(B\mathrm{GL}(R)^+, \mathbb{Z}) \cong H_*(\mathrm{GL}(R), \mathbb{Z}).$$

Definition 6.3. Let R be a ring. The higher K-groups $K_i(R)$, $i = 1, 2, \ldots$ are defined by

$$K_i(R) := \pi_i(B\mathrm{GL}(R)^+, *).$$

6.3. K_1 **and** K_2**.** We need to reconcile this definition with the "classical" definition of K_1 and K_2.

Proposition 6.4. *For $i = 1, 2$, the new definition of $K_i(R)$ agrees with the old one.*

Proof. For $i = 1$, this follows from the property (6.1)(1). For $i = 2$, consider the covering

$$p : \widetilde{B\mathrm{GL}(R)}^+ \to B\mathrm{GL}(R)^+$$

in the construction above. Clearly $\widetilde{B\mathrm{GL}(R)}^+ = BE(R)^+$, where we use the perfect subgroup $E(R)$ of $E(R)$ for the second $+$-construction. Since p is a covering space, p_* is an isomorphism on π_2, so it suffices to show that $\pi_2(BE(R)^+) = K_2(R)$.

For this, $BE(R)^+$ is simply connected, so by the Hurewicz theorem,

$$\pi_2(BE(R)^+) = H_2(BE(R)^+, \mathbb{Z}).$$

But by the property (6.1)(2),

$$H_2(BE(R)^+, \mathbb{Z}) = H_2(BE(R), \mathbb{Z}) = H_2(E(R), \mathbb{Z}) = K_2(R).$$

□

6.4. **Sums and products.** Given two matrices A and B, one can form the direct sum

$$A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

However, this is clearly not compatible with stabilization. For $Gl(R)$, we do the following: Reorder the basis $e_1, \dots, e_{2n}$ of R^{2n} by taking all the odd vectors first, followed by the even ones, both sets in increasing order. After this basis change, $A \oplus B$ becomes the shuffled sum $A \oplus_\sigma B$. The sum $\oplus_\sigma$ is stable and thus defines an operation $\oplus_\sigma : B\mathrm{GL}(R) \times B\mathrm{GL}(R) \to B\mathrm{GL}(R)$. However, $\oplus_\sigma$ is evidently not associative, and the unit matrix doesn't act as the identity for this operation; these statements are only true after a further reordering of the basis. As a change of basis matrix τ acts on $\pi_1(B\mathrm{GL}(R)) = \mathrm{GL}(R)$ by conjugation, this reordering is not even homotopically trivial, so one can't hope to define an H-group structure on $B\mathrm{GL}(R)$ this way.

If one passes to $B\mathrm{GL}(R)^+$, then, at least on π_1, conjugation acts trivially, so there is some hope. In fact

Lemma 6.5. *Take $g \in \mathrm{GL}(R)$. Then the conjugation action by g on $B\mathrm{GL}(R)$ extends to an action on $B\mathrm{GL}(R)^+$ which is homotopic to the identity.*

Additionally, one can show that the stable sum $\oplus_\sigma$ extends to an operation $\oplus : B\mathrm{GL}(R)^+ \times B\mathrm{GL}(R)^+ \to B\mathrm{GL}(R)^+$ which makes $B\mathrm{GL}(R)^+$ into an H-group, with the identity matrix as unit.

Loday [21] has shown that the tensor product operation on matrices $(A, B) \mapsto A \otimes B$ can be modified to define a product

$$B\mathrm{GL}(R)^+ \wedge B\mathrm{GL}(R)^+ \to B\mathrm{GL}(R)^+$$

which makes the graded group $\oplus_{i=0}^{\infty} K_i(R)$ into a graded ring.

6.5. **Milnor K-theory of fields.** Let F be a field. We have $K_1(F) = F^\times$ and $K_2(F) = F^\times \otimes F^\times / \{a \otimes (1-a)\}$ (by Matsumoto's theorem). Milnor [24] defined an extension of this in a universal way to a ring-valued functor on fields, now called *Milnor K-theory*.

Definition 6.6. Let F be a field. The graded ring

$$K_*^M(F) := \oplus_{p=0}^{\infty} K_p^M(F)$$

is defined as the quotient of the tensor algebra over $\mathbb{Z}$ on the abelian group $F^\times$, $\oplus_{p=0}(F^\times)^{\otimes p}$, modulo the two-sided ideal generated by elements of the form $a \otimes (1-a)$, $a \in F$, $a \neq 0, 1$.

The image of an element $a_1 \otimes \ldots \otimes a_n$ in $K_n^M(F)$ is denoted $\{a_1, \dots, a_n\}$.

Thus, $K_0^M(F) = \mathbb{Z} = K_0(F)$, $K_1^M(F) = F^\times = K_1(F)$ and $K_2^M(F) = K_2(F)$. One can show that the Matsumoto isomorphism $K_2^M(F) = K_2(F)$ is induced by the product in K-theory

$$\cup : F^\times \otimes F^\times = K_1(F) \otimes K_1(F) \to K_2(F).$$

Moreover, as the elements $a \cup (1-a)$ are thus zero in $K_2(F)$, the product in K-theory gives rise to a unique ring homomorphism $K_*^M(F) \to K_*(F)$ with $\{a_1, \ldots, a_n\}$ going to $a_1 \cup \ldots \cup a_n$.

It is thus reasonable to ask if $K_n^M(F) \to K_n(F)$ is an isomorphism for all n, and the answer is no, in general (in fact, $K_3^M(F) \to K_3(F)$ is never surjective). One way to see this is to use an additional structure on K_*, namely the Adams operations.

Just as tensor product induces the product in K-theory, the wedge product operations on matrices, $g \mapsto \Lambda^i g$, induce operations in K-theory, $\lambda^i : K_n(R) \to K_n(R)$, which satisfy the "same" relations as the universal relations among the representations Λ^i (see [16] for a construction of these operations). If $P_n(\sigma_1, \ldots, \sigma_n)$ is the polynomial with $\mathbb{Z}$-coefficients that expresses the symmetric function $\sum_i t_i^n$ in terms of the elementary symmetric functions $\sigma_1(t_1, t_2, \ldots), \ldots, \sigma_n(t_1, t_2, \ldots)$, we have the *Adams operation*

$$\psi_n := P_n(\lambda^1, \ldots, \lambda^n).$$

It turns out that the $\mathbb{Q}$-vector space $K_p(R)_\mathbb{Q}$ breaks up into simultaneous eigenspaces for the Adams operations:

$$K_p(R)_\mathbb{Q} = \oplus_{q=0}^p K_p(R)^{(q)},$$

where

$$K_p(R)^{(q)} = \{x \in K_p(R)_\mathbb{Q} \mid \psi_k(x) = k^q \cdot x \text{ for all/some } k \geq 2\}.$$

If $p > 0$, the term $K_p(R)^{(0)}$ is zero, and if $p > 1$, the term $K_p(R)^{(1)} = 0$ as well; if $R = F$ is a field, then $K_1(F) = K_1(F)^{(1)}$ (even integrally). In addition, the ψ_k are ring homomorphisms, so the image of $K_n^M(F)_\mathbb{Q}$ is contained in $K_n(F)^{(n)}$. Thus, if $K_n(F)^{(m)} \neq 0$ for some $m < n$, then $K_n^M(F) \to K_n(F)$ cannot be surjective.

For a number field F, the weight-spaces $K_n(F)^{(q)}$ have been calculated by Borel [8], and the answer is (for $n \geq 2$): If $n \geq 2$ is even, then $K_n(F)_\mathbb{Q} = 0$. If $n = 2p-1 \geq 3$ is odd, then $K_n(F)^{(q)} = 0$ for $q \neq p$, and

$$\dim_\mathbb{Q}(K_{2p-1}(F)^{(p)}) = \begin{cases} r_2 & \text{for } p \text{ even,} \\ r_1 + r_2 & \text{for } p \text{ odd.} \end{cases}$$

Here r_1 is the number of real embeddings $F \to \mathbb{R}$, and r_2 is the number of pairs of complex conjugate embeddings $F \to \mathbb{C}$, $r_1 + 2r_2 = [F : \mathbb{Q}]$.

In particular every number field F has $K_5(F) \neq K_5^M(F)$, even after tensoring with $\mathbb{Q}$.

For a finite field, Quillen has computed

$$K_n(\mathbb{F}_q) = \begin{cases} 0 & \text{for } n \geq 2 \text{ even,} \\ \mathbb{Z}/q^p - 1 & \text{for } n = 2p - 1 \geq 1. \end{cases}$$

In particular, since $K_2^M(F) = K_2(F) = 0$, $K_n^M(F) = 0$ for all $n \geq 2$; so $K_n^M(F) \neq K_n(F)$ for $n > 2$. In fact, even for a number field $K_n^M(F) \to K_n(F)$ is not surjective for all odd $n > 2$, since one can construct torsion elements in $K_n(F)$ which obviously don't come from $K_n^M(F)$. Also, for a number field, Bass and Tate [3] have shown that $K_n^M(F)$ is a finite two-torsion group for all $n > 2$.

Suslin [39] has investigated the kernel of $K_n^M(F) \to K_n(F)$ for an arbitrary field F, proving

Theorem 6.7. *The kernel ker of the natural map $K_n^M(F) \to K_n(F)$ satisfies $(n-1)!(ker) = 0$.*

We conclude with a theorem of Quillen:

Theorem 6.8 ([31])**.** *Let F be a number field, $\mathcal{O} = \mathcal{O}_{F,S}$ the ring of S-integers in F, for some finite set of primes S. Then $K_n(\mathcal{O})$ is finitely generated for all n.*

Together with the localization sequence discussed below, and the computation of the K-groups of finite fields, this gives some idea about the structure of the K-groups of number fields.

6.6. **Some conjectures.** The results mentioned in the previous section have inspired a number of conjectures, some of which remain open to this day.

Conjecture 6.9 (Bass). Let R be a commutative ring which is finitely generated over $\mathbb{Z}$. Then $K_0(R)$ is a finitely generated group.

The Bass conjecture for R the ring of S-integers in a number field follows from the finitenes of the class group. For R the ring of a curve over a finite field, the conjecture follows from the representability of the Picard group of a smooth projective curve by the Jacobian times $\mathbb{Z}$. A deep result of Bloch [5], relying an the Mordell-Weil theorem, extends this to curves over a ring of S-integers, but after this, there are only scattered results. In short, the general Bass conjecture remains wide open.

What about the higher K-groups? In fact, the Bass conjecture implies the finite generation of all the K-groups of a regular commutative ring which is finitely generated over $\mathbb{Z}$, and in fact a similar finite generation for the K-groups of a regular scheme of finite type over $\mathbb{Z}$.

The next conjecture involves the weight spaces $K_n^{(q)}$.

Conjecture 6.10 (Beilinson, Soulé). Let F be a field. Then $K_p(F)^{(q)} = 0$ if $p > 0$, $2q \leq p$.

As mentioned above, Soulé showed that $K_p(F)^{(1)} = 0$ for $p > 1$, which verifies the conjecture for $p \leq 3$. Except for number fields, finite fields and function fields of curves over finite field, and some trivial extensions of these examples, the conjecture is unknown in the first interesting cases $n = 4, 5$, that is, the weight two part is not known to vanish. The Beilinson-Soulé vanishing conjecture is in turn related to other conjectures of Bloch and Beilinson concerning the existence of a category of *mixed motives* with certain properties.

In the study of the torsion of the K-groups, the most central conjectures are the Quillen-Lichtenbaum conjectures, which give a relation of the torsion orders in various K-groups, and certain "regulators" constructed out of the free part of the K-groups, to the values of the zeta function of the given number field. This conjecture can be broken into two parts: the first concerning the relationship of the values of zeta functions to the étale cohomology of $\mathcal{O}_F$, and the second relating the étale cohomology to the K-theory. On the zeta function side, the conjecture is verified, at least a certain class of fields, the totally real fields [43], with some results for imaginary quadratic fields [34] and CM-fields [14] as well. For the part of the conjecture relating the K-groups to étale cohomology groups, the chapter is almost closed; we'll give a quick resumé of the story below.

For a field F of characteristic prime to n, we have the Kummer sequence

$$1 \to \mu_n \to \bar{F}^\times \xrightarrow{x^n} \bar{F}^\times \to 1$$

where $\bar{F}$ is the separable closure. This gives the identity

$$H^1(F, \mu_n) \cong F^\times/(F^\times)^n,$$

where H^1 is the Galois (or étale) cohomology. The right-hand side is $K_1^M(F)/n$, so we have the isomorphism

$$\vartheta^1_{F,n} : K_1^M(F)/n \to H^1(F, \mu_n).$$

One can show that $\vartheta(a) \cup \vartheta(1-a) = 0$ in $H^2(F, \mu_n^{\otimes 2})$, so we have the *Galis symbol*

$$\vartheta^q_{F,n} : K_q^M(F)/n \to H^q(F, \mu_n^{\otimes q}).$$

We have

Conjecture 6.11 (Bloch-Kato). $\vartheta^q_{F,n}$ is an isomorphism for all q and all n prime to the characteristic of F.

In fact, Milnor made this conjecture for $n = 2^\nu$ in [24]. Merkurjev proved the Milnor version for $q = 2$ in [22] and Merkurjev and Suslin gave a proof of the Bloch-Kato conjecture for $q = 2$ and all F and n in [23]. Voevodsky proved the full Milnor conjecture in [41], and together with Rost, there is now a proof of the full Bloch-Kato conjecture.

Work of Suslin-Voevodsky [40] and Geisser-Levine [12], plus the "motivic spectral sequence" of Bloch-Lichtenbaum [7], show how the Bloch-Kato conjecture implies the part of the Quillen-Lichtenbaum conjecture relating K-theory and étale cohomology.

Finally, let me just mention the conjectures of Beilinson [4] which used higher algebraic K-theory and Deligne cohomology to simultaneously generalize the Hodge conjecture and the Birch/Sinnerton-Dyer conjecture.

Part 3. Higher K-theory of schemes

Quillen's Q-construction in [30] laid the general basis for a wide-ranging application of K-theory to commutative algebra, algebraic geometry and number theory. In this part, we describe the Q-construction and outline its basic properties, mostly without proof, and give a glimpse into the consequences for the algebraic K-theory of schemes. Those interested in the details are encouraged to look at Quillen's beautiful paper [30].

7. *The Q-construction*

7.1. **Exact categories.** We follow the discussion given in [30]. Let $\mathcal{E}$ be a full additive subcategory of an abelian category $\mathcal{A}$. We suppose that $\mathcal{E}$ is closed under extensions in $\mathcal{A}$, that is, if M' and M'' are in $\mathcal{E}$, and $0 \to M' \to M \to M'' \to 0$ is an exact sequence in $\mathcal{A}$, then M is in $\mathcal{E}$. In particular, $\mathcal{E}$ is closed under isomorphisms and finite direct sums in $\mathcal{A}$.

Let $\underline{\mathcal{E}}$ be the collection of sequences

$$0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$$

in $\mathcal{E}$ which are exact in $\mathcal{A}$. A map in $\mathcal{E}$ which occurs as a map i in such a sequence is called an *admissible monomorphism*; a map which occurs as a map j is called an *admissible epimorphism*. This data satisfies the following properties

(7.1)

(1) Any sequence in $\mathcal{E}$ which is isomorphic to a sequence in $\underline{\mathcal{E}}$ is in $\underline{\mathcal{E}}$. For all objects M', M'' in $\mathcal{E}$, the sequence

$$0 \to M' \xrightarrow{(\mathrm{id},0)} M' \oplus M'' \xrightarrow{p_2} M'' \to 0$$

is in $\underline{\mathcal{E}}$. For each sequence in $\underline{\mathcal{E}}$, i is a kernel for j, and j is a cokernel for i in the additive category $\mathcal{E}$.

(2) The classes of admissible monomorphisms and admissible epimorphisms are both closed under composition. Admissible epimorphisms are closed under base-change by an arbitrary morphism in $\mathcal{E}$; admissible monomorphisms are closed under cobase-change by an arbitrary morphism in $\mathcal{E}$.

(3) Let $M \to M''$ be a morphism possessing a kernel in $\mathcal{E}$. If there exists a map $N \to M$ in $\mathcal{E}$ such that $N \to M \to M''$ is an admissible epimorphism, then $M \to M''$ is an admissible epimorphism. Dually for admissible monomorphisms.

We denote an admissible monomorphism by $M' \rightarrowtail M$ and an admissible epimorphism by $M \twoheadrightarrow M''$.

Definition 7.1. An additive category $\mathcal{E}$ with a class of sequences $\underline{\mathcal{E}}$ satisfying (7.1) is called an *exact* category. An exact functor $F : \mathcal{E} \to \mathcal{E}'$ of exact categories is an additive functor which sends $\underline{\mathcal{E}}$ to $\underline{\mathcal{E}'}$.

Remark 7.2. In fact, if $\mathcal{E}$ is an exact category, there is an abelian category $\mathcal{A}$ for which $\mathcal{E}$ is a full additive subcategory of $\mathcal{A}$, closed under extensions in $\mathcal{A}$, and where $\underline{\mathcal{E}}$ is the class of sequences in $\mathcal{E}$ which are exact in $\mathcal{A}$. One can thus take this as a definition instead of using the properties (7.1).

Examples 7.3. (1) An abelian category $\mathcal{A}$ with the collection of all exact sequences in $\mathcal{A}$ is an exact category; we will always use this structure on an abelian category. For example, $\mathcal{M}_X$ is an exact category.
(2) Taking the full subcategory $\mathcal{P}_X$ of $\mathcal{M}_X$ gives the exact category $\mathcal{P}_X$.
(3) Let $\mathcal{H}_X$ be the full subcategory of $\mathcal{M}_X$ consisting of coherent sheaves $\mathcal{F}$ which admit a finite resolution by locally free sheaves. Then $\mathcal{H}_X$ is closed under extensions in $\mathcal{M}_X$, hence defines an exact category.

7.2. **The definition of Q.** Let $\mathcal{E}$ be an exact category. Form a new category $Q\mathcal{E}$ with the same objects as $\mathcal{E}$. A morphism $M \to N$ in $Q\mathcal{E}$ is an equivalence class of diagrams

$$\begin{tikzcd} N' \arrow[r, tail, "i"] \arrow[d, two heads, "j"'] & N \\ M & \end{tikzcd} \tag{7.2}$$

As the notation suggests, i is an admissible monomorphism and j is an admissible epimorphism; two diagrams are equivalent if there is a diagram of the form

$$\begin{tikzcd} N' \arrow[r, tail, "i"] \arrow[d, two heads, "j"'] & N \\ M & N'' \arrow[l, two heads, "j'"] \arrow[u, tail, "i'"'] \arrow[ul, "\phi"'] \end{tikzcd}$$

with ϕ an isomorphism. We write the morphism given by (7.2) as $i_! j^!$. Composition is given via the diagram

$$\begin{tikzcd}
N' \times_N P' \arrow[r, tail, "p_2"] \arrow[d, two heads, "p_1"'] & P' \arrow[r, tail, "i'"] \arrow[d, two heads, "j'"'] & P \\
N' \arrow[r, tail, "i"] \arrow[d, two heads, "j"'] & N & \\
M & &
\end{tikzcd}$$

by setting $(i'_! j'^!) \circ (i_! j^!) = (i' p_2)_! (j p_1)^!$.

If $i : N' \to N$ is an admissible monomorphism or if $j : N' \to M$ is an admissible epimorphism, we write $i_! : N' \to N$ for $i_! \mathrm{id}^!$ and $j^! : M \to N'$ for $Id_! j^!$. Then $i_! \circ j^! = i_! j^!$.

7.3. **The K-groups of an exact category.**

Definition 7.4. Let $\mathcal{E}$ be an exact category. The K-groups of $\mathcal{E}$, $K_i(\mathcal{E})$, are defined as

$$K_i(\mathcal{E}) := \pi_{i+1}(BQ(\mathcal{E}), 0).$$

Clearly, the K-groups of $\mathcal{E}$ are functorial with respect to exact functors: if $F : \mathcal{E} \to \mathcal{E}'$ is an exact functor, we write $F_* : K_p(\mathcal{E}) \to K_p(\mathcal{E}')$ for the induced map on the K-groups.

Since $\mathcal{E}$ is an additive category, we have the direct sum operation $\oplus : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$. This defines a functor $Q\oplus : Q(\mathcal{E}) \times Q(\mathcal{E}) \to Q(\mathcal{E})$. Taking the classifying space, we have the operation

$$\oplus : BQ(\mathcal{E}) \times BQ(\mathcal{E}) \to BQ(\mathcal{E});$$

one can easily show that this makes $BQ(\mathcal{E})$ into an H-group.

We now have two definitions of K_0 of an exact category, one from the Grothendieck construction, and one from the Q-construction. We temporarily write $K_0^Q(\mathcal{E})$ for $\pi_1(BQ(\mathcal{E}))$.

Let M be an object in an exact category $\mathcal{E}$. We have the canonical admissible monomorphism $i_M : 0 \to M$ and admissible epimorphism $j_M : M \to 0$. This gives us the path $(j_M^!)^{-1} \circ i_{M!}$ from 0 to 0 in $BQ(\mathcal{E})$. Thus, we have a map

$$\phi : \mathrm{Obj}(\mathcal{E}) \to \pi_1(BQ(\mathcal{E})) = K_0^Q(\mathcal{E}).$$

Proposition 7.5. *The map ϕ descends to an isomorphism $K_0(\mathcal{E}) \to \pi_1(BQ(\mathcal{E}))$.*

Proof. For a small category $\mathcal{C}$, let $X \to B\mathcal{C}$ be a covering space. For each $x \in \mathcal{C}$ and morphism $f : x \to y$. we have the path γ_f from x to y in $B\mathcal{C}$ corresponding to f, which gives the isomorphism

$$f_* : X_x \to X_y.$$

Given $g : y \to z$, we have the 2-simplex in $B\mathcal{C}$ with faces f, g and gf, so $(gf)_* = g_* \circ f_*$. Thus, X determines a functor

$$\hat{X} : \mathcal{C} \to \mathbf{Sets}; \quad x \mapsto X_x,\ f \mapsto f_*.$$

$\hat{X}$ is *morphism-inverting*, that is, $\hat{X}(f)$ is an isomorphism for all f.

Conversely, let $F : \mathcal{C} \to \mathbf{Sets}$ be a morphism inverting functor. Form the category $\mathcal{C}|F$, with objects the pair (x, y) with $x \in \mathcal{C}$, $y \in F(x)$, where a morphism $f : (x, y) \to (x', y')$ is just a morphism $f : x \to x'$ in $\mathcal{C}$, and where we have $y' = f(y)$. It is clear that the projection $(x, y) \mapsto x$ determines a functor $p : \mathcal{C}|F \to \mathcal{C}$, that $Bp : B\mathcal{C}|F \to B\mathcal{C}$ is a covering space, and that $\widehat{Bp} = F$. Thus, the category of covering spaces of $B\mathcal{C}$ is equivalent to the category of morphism-inverting functors $F : \mathcal{C} \to \mathbf{Sets}$. If $B\mathcal{C}$ is connected, and $0 \in \mathcal{C}$ is an object, we have a canonical equivalence of the category of covering spaces of $B\mathcal{C}$ with the category of $\pi_1(B\mathcal{C}, 0)$-sets.

Since $BQ(\mathcal{E})$ is evidently connected (use $i_{M!} : 0 \to M$, for example), we are thus reduced to showing that the category of morphism-inverting functors $F : Q(\mathcal{E}) \to \mathbf{Sets}$ is equivalent to the category of $K_0(\mathcal{E})$-sets.

For this, let $F : B\mathcal{E} \to \mathbf{Sets}$ be a morphism-inverting functor. F is clearly canonically equivalent to a functor with $F(0) = F(M)$ and with $F(i_{M!}) = \mathrm{id}_M$ for all M, so we need only consider such functors. Now let S be a $K_0(\mathcal{E})$-set. Let $F_S : Q\mathcal{E} \to \mathbf{Sets}$ be the functor with $F_S(M) = S$, and with $F_S(i_! j^!) : S \to S$ multiplication by $[\ker j] \in K_0(\mathcal{E})$ (note that the isomorphism class of $\ker j$ depends only on the morphism $i_! j^!$). Conversely, let $F : Q\mathcal{E} \to \mathbf{Sets}$ be a morphism-inverting functor with $F(0) = F(M)$ and $F(i_{M!}) = \mathrm{id}_M$ for all M. Given $i : M' \rightarrowtail M$, we have $i \circ i_{M'} = i_M$, so $F(i) = \mathrm{id}$.

Suppose we have an exact sequence

$$0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0.$$

Then $j^! \circ i_{M''!} = i_! j^!_{M'}$, so $F(j^!) = F(j^!_{M'})$. Also $j^!_M = j^! \circ j^!_{M''}$, so

$$F(j^!_M) = F(j^!)F(j^!_{M''}) = F(j^!_{M'})F(j^!_{M''}).$$

By the universal property of K_0, there is a unique group homomorphism of $K_0(\mathcal{E})$ to $Aut(F(0))$ with $[M] \mapsto F(j^!_M)$. This gives the inverse to the transformation constructed above, proving the result. $\square$

In fact, this can be generalized to the following important property of the Q-construction:

Proposition 7.6 ([30, Corollary 1, §3]). *Let* $0 \to F' \to F \to F'' \to 0$ *be an exact sequence of exact functors* $\mathcal{E} \to \mathcal{E}'$. *Then* $F_* = F'_* + F''_*$ *as maps* $K_p(\mathcal{E}) \to K_p(\mathcal{E}')$.

7.4. **Fundamental properties of the K-groups.** Three fundamental properties of the functor K_0 extend in a much stronger fashion to the higher K-groups. To explain this, we first recall a basic notion from algebraic topology.

Let $f : (X, *) \to (Y, *)$ be a continuous map of pointed topological spaces. The *homotopy fiber* of f is the space $Fib(f)$ consisting of pairs (x, γ), where x is in X, and γ is a path from $f(x)$ to $* \in Y$; the topology on $Fib(f)$ is induced from that of X and Y. The base-point on $Fib(f)$ is $(*_X, \mathrm{id}_{*_Y})$. We have the map $q : (Fib(f), *) \to (X, *)$ by sending (x, γ) to x; clearly the paths γ give a canonical homotopy of $f \circ q$ to the map $Fib(f) \to *$. Also, taking $x = *$, we have an inclusion $i : \Omega Y \to Fib(f)$. The sequence

$$\Omega Y \xrightarrow{i} Fib(f) \xrightarrow{q} X \xrightarrow{f} Y$$

thus gives a sequence of maps on homotopy groups

$$\ldots \to \pi_n(Y) \to \pi_{n-1}(Fib(f)) \to \pi_{n-1}(X) \to \pi_{n-1}(Y) \to \ldots,$$

which is in fact *exact* (at least down to π_1).

We call a sequence $F \to X \to Y$ a *weak homotopy fiber sequence* if $F \to Y$ is contractible, and the map $F \to Fib(X \to Y)$ induced by the choice of a contraction is a weak equivalence. Thus, constructing a weak homotopy fiber sequence is a method for giving a long exact sequence of homotopy groups.

We can now state the main theorems for the K-theory of exact categories.

Theorem 7.7 ([30, Theorem 4, §5]). *Let* $\mathcal{A}$ *be an abelian category,* $i : \mathcal{B} \to \mathcal{A}$ *a full abelian subcategory, closed under taking subquotients in* $\mathcal{A}$. *Suppose that each object* M *of* $\mathcal{A}$ *admits a finite filtration*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

with quotients M_i/M_{i-1} *all in* $\mathcal{B}$. *Then* $BQi : BQ(\mathcal{B}) \to BQ(\mathcal{A})$ *induces an isomorphism* $i_* : K_p(\mathcal{B}) \to K_p(\mathcal{A})$ *for all* p.

Theorem 7.8 ([30, Theorem 3, §4]). *Let* $i : \mathcal{E}_0 \to \mathcal{E}_1$ *be a full exact subcategory of an exact subcategory* $\mathcal{E}_1$, *with* $\mathcal{E}_0$ *closed under extensions* *in $\mathcal{E}_1$. Suppose that each object M of $\mathcal{E}_1$ admits a finite resolution*

$$0 \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$$

with the P_i in $\mathcal{E}_0$. Then $BQi : BQ(\mathcal{E}_0) \to BQ(\mathcal{E}_1)$ induces an isomorphism $i_* : K_p(\mathcal{E}_0) \to K_p(\mathcal{E}_1)$ for all p.

Theorem 7.9 ([30, Theorem 5, §5])**.** *Let $i : \mathcal{B} \to \mathcal{A}$ be the inclusion of a Serre subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$, and let $j : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ be the canonical quotient map. Then*

$$BQ(\mathcal{B}) \xrightarrow{BQi} BQ(\mathcal{A}) \xrightarrow{BQj} BQ(\mathcal{A}/\mathcal{B})$$

is a weak homotopy fiber sequence, so we have a long exact sequence of K-groups

$$\ldots \to K_{p+1}(\mathcal{A}/\mathcal{B}) \xrightarrow{\partial} K_p(\mathcal{B}) \xrightarrow{i_*} K_p(\mathcal{A}) \xrightarrow{j_*} K_p(\mathcal{A}/\mathcal{B}) \to \ldots$$

In addition $K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{B})$ is surjective.

8. *K-theory and G-theory of schemes*

Definition 8.1. Let X be a scheme. Recall the abelian cateogory $\mathcal{M}_X$ of coherent sheaves on X, and the exact subcategory $\mathcal{P}_X$ of locally free coherent sheaves. Define

$$K_p(X) := K_p(\mathcal{P}_X); \quad G_p(X) := K_p(\mathcal{M}_X).$$

Let $f : Y \to X$ be a morphism of schemes. Then $f^* : \mathcal{P}_X \to \mathcal{P}_Y$ is exact, so we have $f^* : K_p(X) \to K_p(Y)$; if f is flat, we similarly have $f^* : G_p(X) \to G_p(Y)$. With some technical fiddling, one can define $f_* : G_p(Y) \to G_p(X)$ if f is projective.

8.1. **Devissage, resolution and localization.**

The main theorems of the previous section have the following applications for $K_p(X)$ and $G_p(X)$.

Theorem 8.2. *Let $i : Z \to X$ be the inclusion of a closed subscheme, giving the full embedding of abelian categories $i_* : \mathcal{M}_Z \to \mathcal{M}_X(Z)$. Then $i_* : G_p(Z) \to K_p(\mathcal{M}_X(Z))$ is an isomorphism for all p.*

Indeed, each coherent sheaf $\mathcal{F}$ supported on Z has a finite filtration with quotient sheaves $\mathcal{O}_Z$-modules. We then apply theoreom 7.7. As one consequence, let X be a scheme, and take $Z = X_{\mathrm{red}}$. Then $\mathcal{M}_X(Z) = \mathcal{M}_X$, and thus $G_p(X_{\mathrm{red}}) \to G_p(X)$ is an isomorphism.

Theorem 8.3. *Let X be a regular noetherian scheme (e.g., X is a smooth scheme over a field). Then the inclusion $\mathcal{P}_X \to \mathcal{M}_X$ induces an isomorphism $K_p(X) \to G_p(X)$.*

Indeed, if X is regular and noetherian, each coherent sheaf $\mathcal{F}$ on X has a finite resolution by locally free sheaves. We then apply theorem 7.8.

Theorem 8.4. *Let $i : Z \to X$ be the inclusion of a closed subscheme, $j : U \to X$ the open complement. Then the sequence*

$$BQ(\mathcal{M}_Z) \xrightarrow{BQi_*} BQ(\mathcal{M}_X) \xrightarrow{BQj^*} BQ(\mathcal{M}_U)$$

is a weak homotopy fiber sequence, so we have a long exact sequence

$$\ldots \to G_{p+1}(U) \xrightarrow{\partial} G_p(Z) \xrightarrow{i_*} G_p(X) \xrightarrow{j^*} G_p(U) \to \ldots$$

for $p \geq 0$. Also, $j^* : G_0(X) \to G_0(U)$ is surjective.

Proof. We have the equivalence of categories $\mathcal{M}_U \sim \mathcal{M}_X/\mathcal{M}_X(Z)$. This gives the homotopy fiber sequence

$$BQ(\mathcal{M}_X(Z)) \xrightarrow{BQi_*} BQ(\mathcal{M}_X) \xrightarrow{BQj^*} BQ(\mathcal{M}_U)$$

by theorem 7.9. By theorem 8.2, $BQ(\mathcal{M}_Z) \to BQ(\mathcal{M}_X(Z))$ is a weak equivalence, which proves the theorem. $\square$

Remark 8.5. We now have two possible definitions of the K-theory of a commutative ring R, namely $K_p(R)$ and $K_p(\mathrm{Spec}\, R)$. In [15], it is shown that there is a natural isomorphism of these two, even as H-spaces.

8.2. **Mayer-Vietoris.** A nice consequence of the localization property for G-theory is the *Mayer-Vietoris* property:

Theorem 8.6. *Let X be a scheme, $j_U : U \to X$, $j_V : V \to X$ open subschemes with $X = U \cup V$. Let $j_0 : U \cap V \to U$, $j_1 : U \cap V \to V$ be the inclusions. Then*

$$BQ(\mathcal{M}_X) \xrightarrow{(BQj_U^*, BQj_V^*)} BQ(\mathcal{M}_U) \times BQ(\mathcal{M}_V) \xrightarrow{BQj_1^* - BQj_2^*} BQ(\mathcal{M}_{U\cap V})$$

is a weak homotopy fiber sequence, giving the Mayer-Vietoris sequence

$$\ldots \to G_{p+1}(U \cap V) \xrightarrow{\partial} G_p(X)$$
$$\xrightarrow{(j_U^*, j_V^*)} G_p(U) \oplus G_p(V) \xrightarrow{j_1^* - j_2^*} G_p(U \cap V) \to \ldots$$

Proof. Let $i : Z \to X$ be the complement of U. Since $U \cup V = X$, Z is also the complement of $U \cap V$ in V, $i^V : Z \to V$. Thus, the homotopy fibers of

$$BQj_U^* : BQ(\mathcal{M}_X) \to BQ(\mathcal{M}_U); \quad BQj_2^* : BQ(\mathcal{M}_V) \to BQ(\mathcal{M}_{U\cap V})$$

are both weakly equivalent to $BQ(\mathcal{M}_Z)$. By standard homotopy theory, this shows that we have the weak homotopy fiber sequence we wanted (or just patch together the two localization sequences along $G_p(Z)$ to get the Mayer-Vietoris sequence). $\square$

8.3. **Homotopy.** In addition to these structural properties, the G-theory of a scheme satisfies a homotopy invariance property, generalizing that of G_0.

Theorem 8.7. *Let X be a scheme. Then $p_1 : X \times \mathbb{A}^1 \to X$ induces an isomorphism $p_1^* : G_p(X) \to G_p(X \times \mathbb{A}^1)$.*

If we combine this with the localization property and use noetherian induction, we have the extended homotopy property for G-theory:

Theorem 8.8. *Let $p : E \to X$ be a flat morphism such that $p^{-1}(x)$ is an affine space $\mathbb{A}^N_x$ for each $x \in X$. Then $p^* : G_p(X) \to G_p(E)$ is an isomorphism.*

8.4. **Ring structure.** One cannot expect a nice product (like tensor product on $\mathcal{P}_X$) give a product structure on the Q-construction. Indeed, since $K_i(\mathcal{E}) = \pi_{i+1} BQ(\mathcal{E})$, a product of spaces

$$BQ(\mathcal{E}) \wedge BQ(\mathcal{E}) \to BQ(\mathcal{E})$$

would induce $K_i(\mathcal{E}) \otimes K_j(\mathcal{E}) \to K_{i+j+1}(\mathcal{E})$. What in fact occurs comes from Waldhausen's *multiple* Q-construction. Without going into details, one can iterate the Q-construction, forming for each n an n-category $Q^n(\mathcal{E})$. The nerve of an n-category is not a simplicial set, but an n-simplicial set, where the models are not simplices, but products of n-simplices (possibly of different dimensions). Using these models, one has the geometric realization $|\mathcal{N}_n(Q^n(\mathcal{E}))|$, which we write as $BQ^n(\mathcal{E})$.

Waldhausen [42] shows there is a natural weak equivalence

$$\Omega^m BQ^{n+m}(\mathcal{E}) \sim BQ^n(\mathcal{E})$$

for all $n \geq 1$, $m \geq 0$. (This shows that $BQ(\mathcal{E})$ is an infinite loop space, and defines the K-theory spectrum:

$$K(\mathcal{E})_n := BQ^{n-1}(\mathcal{E}).)$$

Also, a bilinear exact pairing $\cup : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ induces a map of bisimplicial sets

$$\mathcal{N}Q(\mathcal{E}_1) \wedge \mathcal{N}Q(\mathcal{E}_2) \to \mathcal{N}_2 Q^2(\mathcal{E}_2).$$

Taking the geometric realizations and using Waldhausen's theorem gives the map

$$\cup : BQ(\mathcal{E}_1) \wedge BQ(\mathcal{E}_2) \to \Omega BQ(\mathcal{E}_3),$$

after possibly inverting some weak equivalences. Thus, we have a map on the homotopy groups

$$\cup_{ij} : K_i(\mathcal{E}_1) \otimes K_j(\mathcal{E}_2) \to K_{i+j}(\mathcal{E}_3).$$

Using the tensor product $\mathcal{P}_X \times \mathcal{P}_X \to \mathcal{P}_X$ or $\mathcal{P}_X \times \mathcal{M}_X \to \mathcal{M}_X$, this gives $K_*(X)$ the structure of a graded-commutative ring, and $G_*(X)$ a graded $K_*(X)$-module. These structures generalize the ones we already have for K_0 and G_0. For $X = \operatorname{Spec} R$, this product on $K_*(X) = K_*(R)$ agrees with the product on $K_*(R)$ defined by Loday. The $K_*(X)$-ring structure is natural with respect to the pull-back maps f^*, and, for $f : Y \to X$ projective, we have the projection formula:

$$f_*(f^*(a) \cdot b) = a \cdot f_*(b);\ \ a \in K_i(X), b \in G_j(Y).$$

8.5. **Projective bundles.** Both K-theory and G-theory satisfy a projective bundle formula: for $E \to X$ a vector bundle of rank $r + 1$, $K_*(\mathbb{P}(E))$ is a free $K_*(X)$-module with basis $1, [\mathcal{O}(-1)], \ldots, [\mathcal{O}(-r)]$, and similarly for G-theory; the proof is essentially the same as for G_0 and K_0. There is an interesting extension of this to "twisted forms" of projective bundles, the so-called Severi-Brauer varieties over K. This formula of Quillen's was important in the argument of Merkurjev and Suslin [23] proving that the Galois symbol

$$\vartheta^2_{F,n} : K_2(F)/n \to H^2_{\text{ét}}(F, \mu_n^{\otimes 2})$$

is an isomorphism for all fields F of characteristic prime to n.

9. *Gersten's conjecture and Bloch's formula*

9.1. **The topological filtration.** Let X be a noetherian scheme, $Z \subset X$ a closed subscheme. The codimension of Z in X is the minimum of the Krull dimension of the local rings $\mathcal{O}_{X,z}$, as z runs over the generic points of Z. Define $\mathcal{M}^p_X$ to be the full subcategory of $\mathcal{M}_X$ with objects the coherent sheaves $\mathcal{F}$ such that $\operatorname{supp}(\mathcal{F})$ has codimension $\geq p$. $\mathcal{M}^p_X$ is a Serre subcategory of $\mathcal{M}_X$, giving the sequence of Serre subcategories

$$0 = \mathcal{M}_X^{\dim X+1} \to \mathcal{M}_X^{\dim X} \to \ldots \to \mathcal{M}^p_X \to \ldots \to \mathcal{M}^0_X = \mathcal{M}_X.$$

We let $\mathcal{M}_X^{p/q}$ denote the quotient category $\mathcal{M}^p_X/\mathcal{M}^q_X$ for $q \geq p$.

We can now state Gersten's conjecture:

Conjecture 9.1 (Gersten). Suppose $X = \operatorname{Spec}(\mathcal{O})$, where $\mathcal{O}$ is a regular local ring. Then for each $p \geq 0$, the inclusion $\mathcal{M}^{p+1}_{\mathcal{O}} \to \mathcal{M}^p_{\mathcal{O}}$ induces the zero map $K_q(\mathcal{M}^{p+1}_{\mathcal{O}}) \to K_q(\mathcal{M}^p_{\mathcal{O}})$ for all q.

9.2. **The Quillen spectral sequence.** Gersten's conjecture can be viewed as a kind of local triviality of the K-theory functor for regular schemes, anaologous to the assertion that the singular cohomology of an open disk is trivial. In fact, we can patch together the long exact localization sequences arising from the sequence $\mathcal{M}_X^{p+1} \to \mathcal{M}_X^p \to \mathcal{M}_X^{p/p+1}$:

(9.1)
$$\ldots \to K_q(\mathcal{M}_X^{p+1}) \to K_q(\mathcal{M}_X^p) \xrightarrow{j_p^*} K_q(\mathcal{M}_X^{p/p+1}) \xrightarrow{\partial_p} K_{q-1}(\mathcal{M}_X^{p+1}) \to \ldots$$

with the similar one coming from the sequence $\mathcal{M}_X^{p+2} \to \mathcal{M}_X^{p+1} \to \mathcal{M}_X^{p+1/p+2}$ to define the map $d_1^{p,-p-q} : K_q(\mathcal{M}_X^{p/p+1}) \to K_{q-1}(\mathcal{M}_X^{p+1/p+2})$ as the composition

$$K_q(\mathcal{M}_X^{p/p+1}) \xrightarrow{\partial_p} K_{q-1}(\mathcal{M}_X^{p+1}) \xrightarrow{j_{p+1}^*} K_{q-1}(\mathcal{M}_X^{p+1/p+2}).$$

Linking all these long exact sequences together gives an exact couple

$$\begin{tikzcd}
\oplus_{p,q} K_q(\mathcal{M}_X^p) \arrow[rr, "p \mapsto p-1"] & & \oplus_{p,q} K_q(\mathcal{M}_X^p) \arrow[dl, "j^*"] \\
& \oplus_{p,q} K_q(\mathcal{M}^{p/p+1}) \arrow[ul, "\partial_p", "+1"'] &
\end{tikzcd}$$

which by standard machinery defines the *Quillen spectral sequence*

$$E_1^{p,q} = K_{-p-q}(\mathcal{M}_X^{p/p+1}) \Longrightarrow K_{-p-q}(\mathcal{M}_X) = G_{-p-q}(X).$$

The E_1-differentials $d_1^{p,q} : K_{-p-q}(\mathcal{M}_X^{p/p+1}) \to K_{-p-q}(\mathcal{M}_X^{p+1/p+2})$ are the ones defined above.

This spectral sequence is useful, since the E_1-terms can be expressed in terms of the K-theory of the residue fields of X. Indeed, since, for closed subsets $Z \subset W \subset X$, the quotient category $\mathcal{M}_X(W)/\mathcal{M}_X(Z)$ is equivalent to the category $\mathcal{M}_{X \setminus Z}(W \setminus Z)$, we see that $\mathcal{M}_X^{p/p+1}$ is equivalent to the direct sum of the categories $\mathcal{M}_{\mathcal{O}_{X,x}}(x)$, where x runs over the codimension p points of X, and $\mathcal{O}_{X,x}$ is the local ring of functions on X regular at x. As the inclusion $\mathcal{M}_{k(x)} \to \mathcal{M}_{\mathcal{O}_{X,x}}(x)$ induces an isomorphism on the K-groups, by the filtration theorem 8.2, we have a canonical isomorphism

$$K_q(\mathcal{M}_X^{p/p+1}) \cong \oplus_{x \in X^{(p)}} K_q(k(x)),$$

where $X^{(p)}$ is the set of points $x \in X$ with closure $\bar{x} \subset X$ having codimension p. Thus, we have the spectral sequence

$$E_1^{p,q}(X) = \oplus_{x \in X^{(p)}} K_{-p-q}(k(x)) \Longrightarrow K_{-p-q}(\mathcal{M}_X) = G_{-p-q}(X). \tag{9.2}$$

Now the relation with Gersten's conjecture:

Lemma 9.2. *The following are equivalent:*

(1) *For all p and q, the map $K_q(\mathcal{M}_X^{p+1}) \to K_q(\mathcal{M}_X^p)$ is zero.*
(2) *For all q, $E_2^{p,-q}(X) = 0$ if $p \neq 0$, and the edge homomorphism $G_q(X) \to E_2^{0,-q}(X)$ is an isomorphism.*
(3) *For all q, the complex*

$$0 \to G_q(X) \to \oplus_{x\in X^{(0)}} K_q(k(x)) \xrightarrow{d_1} \oplus_{x\in X^{(1)}} K_{q-1}(k(x)) \xrightarrow{d_1} \dots$$

is exact, where d_1 is the E_1-differential in (9.2).

Indeed, all three conditions are equivalent with the long exact sequence (9.1) breaking up into short exact sequences

$$0 \to K_q(\mathcal{M}_X^p) \xrightarrow{j_p^*} K_q(\mathcal{M}_X^{p/p+1}) \xrightarrow{\partial_p} K_{q-1}(\mathcal{M}_X^{p+1}) \to 0$$

for all p and q.

9.3. **Cohomology of K-sheaves.** One can go even further with this if one considers the *K-sheaves* $\mathcal{K}_p$ on X, defined as the sheaf associated to the presheaf $U \mapsto K_p(U)$. The stalk $\mathcal{K}_{p,x}$ at $x \in X$ is just $K_p(\mathcal{O}_{X,x})$.

We may similarly sheafify the E_1-complex of lemma 9.2, giving the complex of sheaves on X

$$(9.3)\quad 0 \to \mathcal{K}_q \to \oplus_{x\in X^{(0)}} i_{x*}(K_q(k(x))) \xrightarrow{d_1} \oplus_{x\in X^{(1)}} i_{x*}(K_{q-1}(k(x))) \xrightarrow{d_1} \dots \xrightarrow{d_1} \oplus_{x\in X^{(q)}} i_{x*}(K_0(k(x))).$$

Here $i_x : x \to X$ is the inclusion, and we consider $K_n(k(x))$ as the constant sheaf on the one-point space x.

Now, if Gersten's conjecture is true for all the local rings $\mathcal{O}_{X,x}$, then the complex of sheaves (9.3) is *exact*. Since $i_{x*}S$ is a flasque sheaf (here we are relying on the Zariski topology!), (9.3) gives a flasque resolution of the sheaf $\mathcal{K}_q$. Thus

Proposition 9.3. *Suppose Gersten's conjecture is true for all the local rings $\mathcal{O}_{X,x}$. Then $H^p(X, \mathcal{K}_q)$ is isomorphic to the E_2-term $E_2^{p,-q}(X)$ in the Quillen spectral sequence* (9.2). *In particular, $H^p(X, \mathcal{K}_q) = 0$ if $p > q$.*

Indeed, the complex $(E_1^{*,-q}(X), d_1^{*,-q})$ is the complex of global sections of the sheaf complex (9.3) (after deleting $\mathcal{K}_q$).

Example 9.4. Take the case $q = 1$. Then $\mathcal{K}_1$ is just the sheaf of units $\mathcal{O}_X^\times$, and the sheafified Gersten complex is (assume X is irreducible with generic point η)

$$1 \to \mathcal{O}_X^\times \to i_{\eta*}k(X)^\times \to \oplus_{x\in X^{(1)}} i_{x*}\mathbb{Z} \to 0.$$

At a particular point $y \in X$, the stalks of this complex at y are

$$1 \to \mathcal{O}_{X,y}^{\times} \to k(X)^{\times} \xrightarrow{\partial} \oplus_{x \in X^{(1)}, y \in \bar{x}} \mathbb{Z} \to 0.$$

Assuming that ∂ is just the map $f \mapsto \operatorname{div}(f)_{|\operatorname{Spec}(\mathcal{O}_{X,y})}$, which we verify in lemma 9.6 below, the exactness of this sequence is just saying that $\mathcal{O}_{X,y}^{\times}$ is the subgroup of $k(X)^{\times}$ consisting of those rational functions f with $\operatorname{div}(f)$ having no component containing y, and that every divisor containing y is principal in a neighborhood of y. This is all true if and only if $\mathcal{O}_{X,y}$ is a UFD. If X is regular, this is true for all y, since a regular local ring is a UFD by the well-known theorem of Auslander-Buchsbaum.

Taking global sections, we have the complex

$$k(X)^{\times} \xrightarrow{\operatorname{div}} Z^1(X),$$

which has kernel $\Gamma(X, \mathcal{O}_X^{\times}) = H^0(X, \mathcal{K}_1)$, and cokernel $\mathrm{CH}^1(X) \cong \operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^{\times}) = H^1(X, \mathcal{K}_1)$.

9.4. **Quillen's theorem.** In [30], Quillen gave a proof of Gersten's conjecture in the "geometric" case.

Theorem 9.5 ([30, Theorem 5.11, §7])**.** *Let $\mathcal{O}$ be the local ring of a point x on a regular scheme of finite type over a field k. Then Gersten's conjecture holds.*

Proof. We assume k is perfect and infinite for simplicity; it is not hard to reduce to this case.

If η is an element of $K_q(\mathcal{M}_{\mathcal{O}}^{p+1})$, there is a smooth affine k-scheme $X \subset \mathbb{A}_k^N$, a point $x \in X$, a closed codimension $p+1$ reduced closed subscheme $Z \subset X$ containing x and an element $\eta_Z \in G_q(Z)$ such that $\mathcal{O} = \mathcal{O}_{X,x}$ and η is the image of η_Z in $K_q(\mathcal{M}_{\mathcal{O}}^{p+1})$. It suffices to show that η_Z dies in $K_q(\mathcal{M}_U^p)$ for some open subset U of X containing x. Say X has dimension $d+1$.

Let D be a codimension one subvariety of X containing Z. By Noether normalization, a generic linear projection $\pi : \mathbb{A}^N \to \mathbb{A}^d$ induces a finite morphism $\pi_{|D} : D \to \mathbb{A}^d$. We can also assume that π is smooth in a neighborhood U of x in X, as a general linear map $T_x(X) \to T_{\pi(x)}(\mathbb{A}^d)$ is surjective. This gives us the diagram

$$\begin{tikzcd} D \times_{\mathbb{A}^d} X \arrow[r, "p_2"] \arrow[d, shift right, "p_1"'] & X \arrow[d, "\pi_{|X}"] \\ D \arrow[u, shift right, "s"'] \arrow[r, "\pi_{|D}"'] & \mathbb{A}^d, \end{tikzcd}$$

where s is the section induced by the inclusion $D \to X$.

Let $V = p_2^{-1}(U)$. Then $p_1 : V \to D$ is smooth, with fiber dimension one, thus $s(D) \cap V$ is a codimension one subscheme of V, and the ideal sheaf $\mathcal{I}_{s(D)}$ is locally principal on V. Also, the map $p_2 : V \to U$ is finite. Shrinking U, we may assume that U and V are affine, and that the ideal I defining $s(D) \cap V$ is principal, $I = (t) \subset R = \Gamma(V, \mathcal{O}_V)$, with t a non-zero divisor.

Let $i : s(D) \cap V \to V$ be the inclusion. Since t is a non-zero divisor, the map

$$\times t : p_1^* M \to p_1^* M$$

is injective for all $M \in \mathcal{M}_{D\cap U}$. We thus have the functorial exact sequence

$$0 \to p_1^* M \xrightarrow{\times t} p_1^* M \to i_* M \to 0,$$

giving the exact sequence of exact functors from $\mathcal{M}^p_{D\cap U}$ to $\mathcal{M}^p_V$

$$0 \to p_1^*(-) \xrightarrow{\times t} p_1^*(-) \to s_*(-) \to 0.$$

Applying p_{2*} gives the exact sequence of functors from $\mathcal{M}^p_{D\cap U}$ to $\mathcal{M}^p_U$

$$0 \to p_{2*}p_1^*(-) \xrightarrow{\times t} p_{2*}p_1^*(-) \to i_*(-) \to 0.$$

Thus, $i_* : K_q(\mathcal{M}^p_{D\cap U}) \to K_q(\mathcal{M}^p_U)$ is zero, by the additivity property proposition 7.6. As $Z \cap U$ has codimension p on $D \cap U$, the composition $K_q(\mathcal{M}_Z) \xrightarrow{i_{Z*}} K_q(\mathcal{M}^{p+1}_X) \xrightarrow{j^*} K_q(\mathcal{M}^{p+1}_U) \to K_q(\mathcal{M}^p_U)$ factors through

$$K_q(\mathcal{M}_Z) \to K_q(\mathcal{M}^p_D) \to K_q(\mathcal{M}^p_{D\cap U}) \xrightarrow{i_*} K_q(\mathcal{M}^p_U).$$

Thus, η_Z goes to zero in $K_q(\mathcal{M}^p_U)$, completing the proof. □

9.5. **Bloch's formula.** The results of the previous section show how to relate the cohomology of the K-sheaves to the Chow ring.

Let F be a field. We have already seen that $K_0(F) = \mathbb{Z}$ and $K_1(F) = F^\times$. This shows that the end of the Gersten complex $E_1^{*,-q}(X)$, for $q \le \dim X$, looks like

$$\ldots \to \oplus_{x\in X^{(q-1)}} k(x)^\times \xrightarrow{d_1^{q-1,-1}} \oplus_{x\in X^{(q)}} \mathbb{Z}.$$

The term in degree q is just $Z^q(X)$.

Lemma 9.6. *Let X be a scheme of finite type over a field k, and let $W \subset X$ be an integral closed subscheme of codimension $q-1$ with generic point w. Let $i_w : k(W)^\times \to \oplus_{x\in X^{(q-1)}} k(x)^\times$ be the inclusion as the summand indexed by w. Then the composition*

$$k(W)^\times \xrightarrow{i_w} \oplus_{x\in X^{(q-1)}} k(x)^\times \xrightarrow{d_1^{q-1,-1}} \oplus_{x\in X^{(q)}} \mathbb{Z} \cong Z^q(X)$$

is the map sending $f \in k(W)^\times$ to $\pm i_{W*}(\mathrm{div}(f)) \in Z^q(X)$, for a universal choice of sign.

Proof. Let Z be a codimension p integral closed subscheme of X. First suppose that Z is not contained in D. If we remove a closed subset of X of codimension $> q$, it does not affect the Gersten complex $E_1^{*,-q}$, so we may assume that $Z \cap D = \emptyset$. Since the Quillen spectral sequence is functorial for flat morphisms, we may restrict to the open neighborhood $X \setminus D$ of Z to compute the composition

$$k(W)^\times \xrightarrow{i_w} \oplus_{x \in X^{(q-1)}} k(x)^\times \xrightarrow{d_1^{q-1,-1}} \oplus_{x \in X^{(q)}} \mathbb{Z} \xrightarrow{p_z} \mathbb{Z},$$

where p_z is the projection on the summand indexed by the generic point z of Z. Since the term $k(w)^*$ clearly goes to zero upon this restriction, it follows that $p_z \circ d_1 \circ i_w(f) = 0$.

Now suppose that $Z \subset D$. We may again use functoriality of the localization sequence with respect to finite morphisms to reduce to the case $X = D$. We may then localize to assume that $X = \operatorname{Spec} \mathcal{O}$ is local, dimension one, and $Z = \operatorname{Spec} F$, where F is the residue field. Again using functoriality with respect to finite morphisms, we may assume that $\mathcal{O}$ is normal, so $\mathcal{O}$ is a discrete valuation ring with residue field F. Let L be the quotient field of $\mathcal{O}$. We have the localization sequence

$$K_1(\mathcal{O}) \to K_1(L) \xrightarrow{\partial} K_0(F) \to K_0(\mathcal{O}) \to K_0(L).$$

Since $K_0(\mathcal{O}) = K_0(L) = \mathbb{Z}$ by rank, ∂ is surjective to $K_0(F) \cong \mathbb{Z}$.

Let t be a generator for the maximal ideal of $\mathcal{O}$. Then $L^\times \cong t^{\mathbb{Z}} \times \mathcal{O}^\times$. Since the image of $\mathcal{O}^\times$ in $L^\times$ comes from $K_1(\mathcal{O})$, we have $\partial(\mathcal{O}^\times) = 0$. Thus $\partial(t^{\mathbb{Z}})$ must be all of $K_0(F) = \mathbb{Z}$, so $\partial(t)$ is a generator. Since $\operatorname{ord}_Z(t) = 1$, we have $\partial(t) = \epsilon(\operatorname{ord}_Z(t))$, with $\epsilon = \pm 1$. For an arbitrary $f \in L^\times$, write $f = u \cdot t^n$ with $u \in \mathcal{O}^\times$. Then $\operatorname{ord}_Z(f) = n = \epsilon\partial(u \cdot t^n)$, as desired.

To see that the sign ϵ is universal, note that we have the flat k-algebra homomorphism $k[X]_{(X)} \to \mathcal{O}$, $X \mapsto t$. This reduces the computation further to the case of $\mathcal{O} = k[X]_{(X)}$, $t = X$, so there is a universal choice of sign. $\square$

In fact, this shows that $E_2^{q,-q}(X) \cong \mathrm{CH}^q(X)$ for all X. As $E_2^{q,-q}(X) \cong H^q(X, \mathcal{K}_q)$ for X smooth over a field by proposition 9.3 and theorem 9.5, we have shown

Theorem 9.7 (Bloch's formula, [30, Theorem 5.19, §7])**.** *Let X be a smooth variety over a field. Then $\mathrm{CH}^q(X) \cong H^q(X, \mathcal{K}_q)$ for all $q \geq 0$.*

The Gersten complex gives "cycle-theoretic" descriptions of other K-cohomology groups. For example, $H^p(X, \mathcal{K}_{p+1})$ is generated by elements $\sum_i (Z_i, f_i)$, where Z_i is a codimension p subscheme of X, f_i is in $k(Z_i)^\times$, and $\sum_i \operatorname{div}(f_i) = 0$ as a codimension $p+1$ cycle on X. The relations are generated by elements of the form $T(D, \{f, g\})$, where D is a codimension $p-1$ subvariety of X, f, g are in $k(D)^\times$, $\{f, g\}$ is the symbol in $K_2(k(D))$. T is the *tame symbol*:

$$T(\{f, g\}) = \sum_Z i_{Z*}\big((-1)^{\mathrm{ord}_Z(f)\mathrm{ord}_Z(g)}(\frac{f^{\mathrm{ord}_Z(g)}}{g^{\mathrm{ord}_Z(f)}})_{|Z}\big),$$

where the sum is over all codimension one subvarieties of the normalization D^N of D, and $i_Z : Z \to X$ is the composition $Z \subset D^N \to D \subset X$. A number of authors, starting with Bloch and Beilinson, have attached analytic invariants to elements of $H^p(X, \mathcal{K}_{p+1})$, by associating to $\sum_i (Z_i, f_i)$ the current

$$\omega \mapsto \sum_i \int_{Z_i} \ln(f_i)\omega + (2\pi i)\int_\Delta \omega,$$

where Δ is a $2p$-chain (with $\mathbb{Q}$-coefficients) with boundary the $(2p-1)$-cycle $\sum_i f_i^{-1}([0, \infty])$.

Similarly, $H^0(X, \mathcal{K}_2)$ is given by elements $\eta = \sum_i \{f_i, g_i\} \in K_2(k(X))$ such that $T(\eta) = \sum_i T(\{f_i, g_i\}) = 0$ in $\oplus_{x \in X^{(1)}} k(x)^*$. Using this description, Bloch has constructed interesting elements in $H^0(E, \mathcal{K}_2)$, and Beilinson [4] has constructed analogous elements in $H^0(C, \mathcal{K}_2)$, where C is a modular curve, and related these elements to values of the L-function of C.

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