Three lectures on quadratic enumerative geometry

Marc Levine

Marc Levine, Universität Duisburg-Essen, Fakultät Mathematik, Campus Essen, 45117 Essen, Germany

Email address: marc.levine@uni-due.de

Abstract. We give an overview of the goals and recent progress in the development of an enumerative geometry with quadratic forms.

Overview

These notes are taken from a three-lecture series I gave at the BIRS Workshop "Moduli, Motives and Bundles – New Trends in Algebraic Geometry", that took place at Casa Matemática Oaxaca, Sept. 18-23, 2022. It was a very enjoyable experience being able to interact face-to-face with my fellow mathematicians in the lovely environment provided by the CMO, especially after the long isolation due to covid. I would like to thank all the participants for making the workshop a success, especially the organizers, Pedro Luis del Angel, Frank Neumann and Alexander Schmitt.

Here is an outline of the talks.

Lecture 1: An introduction to quadratic enumerative geometry

Classical enumerative geometry counts solutions to "geometric problems" in algebraic geometry that are expected to have a finite number of solutions, or more generally compute integer invariants of algebro-geometrical objects. Typical examples include:

- Bézout's theorem: how many points of intersection are there among n hypersurfaces of degrees $d_1, \ldots, d_n$ in $\mathbb{P}^n$, for example two curves C_1, C_2 of degrees d_1, d_2 in $\mathbb{P}^2$?
- Find a formula for the Euler characteristic of a smooth hypersurface of degree d in $\mathbb{P}^n$
- How many lines are there on a (smooth) hypersurface of degree $2n-3$ in $\mathbb{P}^n$, for example, how many lines are there on a smooth cubic surface in $\mathbb{P}^3$?
- How many rational plane curves of degree d pass through $3d-1$ general points in $\mathbb{P}^2$?
- How many conics in $\mathbb{P}^2$ are tangent to 5 general lines?

Usually one looks for an answer to such questions over an algebraically closed field, where essentially discrete, topological invariants will give at least a first approximation to an answer. The goal of "quadratic" enumerative geometry is to refine the typically $\mathbb{Z}$-valued answer to an enumerative problem over an algebraically closed field to an element of the Grothendieck-Witt ring of non-degenerate quadratic forms over a field k over which the problem makes sense, in the hope that

The author gratefully acknowledges support from the DFG through the grant LE 2259/7-2 and from the ERC through the project QUADAG. This paper is part of a project that has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 832833).

erc
European Research Council
Established by the European Commission

this finer invariant will give additional information about the set of solutions over k.

In this first lecture, we will concentrate on the example of the quadratic Euler characteristic, which has an abstract definition, but is also amenable to concrete computations, as we will see on Lecture 3.

Lecture 2: Quadratic intersection theory and characteristic classes

Applying the intersection theory of the Chow ring to fundamental classes of varieties and Chern classes of vector bundles is the main tool used to compute classical enumerative invariants. More recently, this collection of objects has been enlarged by the introduction of virtual fundamental classes in Gromov-Witten theory. In this lecture we will introduce the framework needed to construct quadratic refinements of all of these objects. Here the Milnor-Witt K-sheaves and the sheaf of Witt groups will play an important role. We will illustrate with some examples, for instance, the quadratic Bézout theorem, quadratic counts of lines on hypersurfaces and complete intersections in a projective space, a quadratic Riemann-Hurwitz formula, and the quadratic Gauß-Bonnet theorem.

Lecture 3: Computational methods

As they carry more information than the classical $\mathbb{Z}$-valued invariants, the quadratic invariants are often more difficult to compute. In this lecture, we will go over some of the computational tools that have been developed to enable such computations. The methods include the development of a calculus of characteristic classes of vector bundles with values in Witt sheaf cohomology, algebraic computations of the quadratic Euler characteristics of smooth hypersurfaces in $\mathbb{P}^n$, and localization techniques for computing Euler classes and virtual fundamental classes. As a further example we look at a quadratic count of twisted cubic curves on hypersurfaces and complete intersections in a projective space.

CHAPTER 1

Lecture 1: An introduction to quadratic enumerative geometry

We discuss Euler characteristics from various points of view.

1. Introduction

Intersection theory has a long and interesting history, and is closely tied to questions of *enumerative geometry*, that is, the counting of solutions to geometric problems in algebraic geometry, or more generally, attaching integer invariants to a given variety or finite collection of varieties.

In this lecture, we look at perhaps the most elementary invariant, the Euler characteristic. A topological space T with the homotopy type of a finite CW complex (say dimension d) has its Euler characteristic

$$\chi^{\mathrm{top}}(T) := \sum_{i=0}^{d} \dim_{\mathbb{Q}} H_i(T, \mathbb{Q})$$

In fact, one can use $\dim_F H_i(T, F)$ for any field F. For an algebraic variety X over $\mathbb{C}$, we have the space $X(\mathbb{C})$, so we have its Euler characteristic

$$\chi^{\mathrm{top}}(X) := \chi^{\mathrm{top}}(X(\mathbb{C}))$$

Over an arbitrary algebraically closed field k, we can use instead étale cohomology with $\mathbb{Q}_\ell$ coefficients for a prime ℓ different from the characteristic.

A somewhat more sophisticated definition in the case of a smooth proper scheme X over a field k is to use a version of the *Gauß-Bonnet theorem*.

THEOREM 1.1 (algebraic Gauß-Bonnet). *Let X be a smooth proper scheme of dimension n over a field k. Then*

$$\chi^{\mathrm{top}}(X_{\bar{k}}) = \deg_k c_n(T_{X/k}) = (-1)^n \deg_k c_n(\Omega_{X/k}).$$

Here $T_{X/k}$ is the tangent bundle of X, $\Omega_{X/k}$ is the sheaf of differentials, c_n is the nth Chern class with values in the Chow group $\mathrm{CH}^n(X)$, and $\deg_k$ is the degree map

$$\deg_k : \mathrm{CH}^n(X) \to \mathrm{CH}^0(k) = \mathbb{Z}.$$

One can give a proof using the various versions of the Lefschetz trace formula. We won't be going into all these objects in detail, but let's just list a few useful objects and their properties. For a detailed discussion of the Chow groups, intersection products, and Chern classes, see Fulton's book *Intersection Theory* [**13**].

2. Chow groups and Chern classes

We fix a field k and let $\mathbf{Sch}/k$ denote the category of quasi-projective k-schemes, and $\mathbf{Sm}/k$ the full subcategory of quasi-projective k-schemes, smooth over k. A *variety* is an integral $X \in \mathbf{Sch}/k$. We write dim for $\dim_k$.

$X \in \mathbf{Sch}/k$ has its group of *dimension i algebraic cycles* $Z_i(X)$, the free abelian group on the dimension i subvarieties of X. The subgroup $R_i(X) \subset Z_i(X)$ is generated by cycles of the form $\mathrm{div} f$, with f a non-zero rational function on some dimension $i+1$ subvariety of X. The quotient $\mathrm{CH}_i(X) := Z_i(X)/R_i(X)$ is the dimension i Chow group of X. If X has pure dimension d, we can index by codimension $Z^i(X) := Z_{d-i}(X)$, $\mathrm{CH}^i(X) = \mathrm{CH}_{d-i}(X)$.

For a dimension i subvariety W of X, we denote the class of W in $\mathrm{CH}_i(X)$ by $[W]$.

Each proper map $f : Y \to X$ induces a functorial pushforward map $f_* : Z_i(Y) \to Z_i(X)$ that passes to $f_* : \mathrm{CH}_i(Y) \to \mathrm{CH}_i(X)$. Explicitly, if $W \subset Y$ is a dimension i subvariety, then, as f is proper, $W' := f(W)$ is a closed irreducible subset of X, to which we give the reduced subscheme structure. In case $\dim W' = \dim W$, we have the induced map on the function fields $f^* : k(W') \to k(W)$, making $k(W)$ a finite extension of $k(W')$. Then

$$f_*(W) := \begin{cases} 0 & \text{if } \dim W' < \dim W \\ [k(W) : k(W')] \cdot W' & \text{if } \dim W' = \dim W. \end{cases}$$

and on the Chow groups, one has $f_*([W]) = [f_*(W)]$.

If $f : Y \to X$ is an arbitrary morphism with X and Y smooth, and $W \subset X$ is a codimension i subvariety, we say that the cycle-theoretic pull-back $f^*(W)$ is defined if each irreducible component W' of $f^{-1}(W)$ has codimension i on Y. In this case, one has Serre's intersection multiplicity

$$m(W'; f^*(W)) := \sum_{i \geq 0} (-1)^i \ell_{\mathcal{O}_{Y,W'}}(\mathrm{Tor}_i^{\mathcal{O}_{X,W}}(k(W), \mathcal{O}_{Y,W'}))$$

where $\ell_{\mathcal{O}_{Y,W'}}(-)$ is the length of an $\mathcal{O}_{Y,W'}$-module. This is in fact a finite sum, and $m(W'; f^*(W)) > 0$ (see [**32**] for these facts). One then defines

$$f^*(W) := \sum_{W'} m(W'; f^*(W)) \cdot W'$$

where the sum is over the (finitely many) irreducible components of $f^{-1}(W)$. Letting $Z^i(X)_f \subset Z^i(X)$ be the subgroup generated by those W for which the cycle-theoretic pullback by f is defined, one extends by linearity to give the homomorphism

$$f^* : Z^i(X)_f \to Z^i(Y).$$

In general, $Z^i(X)_f$ is a proper subgroup of $Z^i(X)$. However, the map $Z^i(X)_f \to \mathrm{CH}^i(X)$ is in fact surjective (at least for quasi-projective X), and the partially defined cycle-theoretic pullback descends to

$$f^* : \mathrm{CH}^i(X) \to \mathrm{CH}^i(Y)$$

The surjectivity of $Z^i(X)_f \to \mathrm{CH}^i(X)$, and the fact that f^* does indeed descend is a consequence of *Chow's moving lemma* (see e.g., [**6**, §3, Proposition 1], [**33**]). Fulton *op. cit.* gives a different approach to the construction of f^* and the descent property in the general case.

One has the external product

$$Z^i(X) \otimes_{\mathbb{Z}} Z^j(Y) \to Z^{i+j}(X \times_k Y)$$

which descends to

$$\boxtimes : \mathrm{CH}^i(X) \otimes_{\mathbb{Z}} \mathrm{CH}^j(Y) \to \mathrm{CH}^{i+j}(X \times_k Y).$$

For X smooth, composing $\boxtimes$ with pullback by the diagonal $\Delta_X : X \to X \times_k X$ gives the *intersection product*

$$\cup : \mathrm{CH}^i(X) \otimes \mathrm{CH}^j(X) \to \mathrm{CH}^{i+j}(X)$$

making the graded group $\mathrm{CH}^*(X) := \oplus_{i=0}^{\dim X} \mathrm{CH}^i(X)$ a commutative, $\mathbb{Z}$-graded ring. The unit in $\mathrm{CH}^0(X) = \mathrm{CH}_{\dim X}(X)$ is the fundamental class $[X] = 1 \cdot X$, and for $f : Y \to X$, the map $f^* : \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$ is a ring homomorphism.

For subvarieties W_1, W_2 of X that *intersect properly*, that is, for each integral component W' of $W_1 \cap W_2$, we have

$$\mathrm{codim}_X W' = \mathrm{codim}_X W_1 + \mathrm{codim}_X W_2,$$

the intersection product is given by Serre's intersection formula: let

$$m(W_1, W_2; W') := \sum_{i \geq 0} (-1)^i \ell_{\mathcal{O}_{X,W'}}(\mathrm{Tor}_i^{\mathcal{O}_{X,W'}}(\mathcal{O}_{W_1,W'}, \mathcal{O}_{W_2,W'})). \tag{2.1}$$

Then

$$[W_1] \cdot [W_2] = \sum_{W'} m(W_1 \cap W_2; W') \cdot [W']$$

This follows directly from the definitions of $W_1 \boxtimes W_2$ and of Δ_X^*.

We also have the *criterion of intersection multiplicity one*: with X, W_1, W_2 and W' as above, let $w' \in W'$ be a geometric generic point. Then $m(W_1, W_2; W') = 1$ if and only if, W_1 and W_2 are both smooth over k in a neighborhood of w', and $T_{X,w'}$ is generated (as $k(w')$-vector space) by the subspaces $T_{W_1,w'}$ and $T_{W_2,w'}$.

For f proper, X, Y smooth, we have the projection formula

$$f_*(f^*(x) \cdot y) = x \cdot f_*(y)$$

We have $\mathrm{CH}_0(\mathrm{Spec}\, k) = Z_0(\mathrm{Spec}\, k) = \mathbb{Z}$. For $\pi : X \to \mathrm{Spec}\, k$ proper, we have the *degree map*

$$\deg_k := \pi_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(\mathrm{Spec}\, k) = \mathbb{Z}$$

Explicitly, if $p \in X$ is a closed point, $\deg_k(p)$ is the field extension degree $[k(p) : k]$.

Each vector bundle V (locally free coherent sheaf) on a smooth X has Chern classes

$$c_i(V) \in \mathrm{CH}^i(X), i = 1, 2, \ldots$$

with $f^* c_i(V) = c_i(f^* V)$ for $f : Y \to X$ map of smooth varieties. $c_i(V)$ depends only on the isomorphism class of V and $c_i(V) = 0$ for $i > \mathrm{rank}(V)$; we set $c_0(V) = 1 \in \mathrm{CH}^0(X)$. Sending a line bundle L to $c_1(L) \in \mathrm{CH}^1(X)$ defines an isomorphism

$$c_1 : \mathrm{Pic}(X) \to \mathrm{CH}^1(X).$$

In case $L = \mathcal{O}_X(D)$ for some divisor $D \in Z^1(X)$, we have

$$c_1(\mathcal{O}_X(D)) = [D] \in \mathrm{CH}^1(X).$$

The top Chern class $c_r(V)$ for $r = \mathrm{rank}(V)$ is also called the *Euler class* and is given by

$$c_r(V) = s_2^* s_{1*}([X])$$

with $s_1, s_2 : X \to V$ any two sections. The canonical choice is $s_1 = s_2 = s_0$, the zero-section, but this is not necessary.

The total Chern class $c(V) := \sum_{i=0}^{\mathrm{rank}(V)} c_i(V)$ satisfies the Whitney formula: If

$$0 \to V' \to V \to V'' \to 0$$

is an exact sequence of vector bundles, then $c(V) = c(V')c(V'')$. Also, for the dual bundle $V^\vee$, we have

$$c_i(V^\vee) = (-1)^i c_i(V).$$

Proofs of all these facts can be found in [**13**].

3. Intersections, Chern classes and enumerative problems

We give some examples to show how this machinery is useful in solving enumerative problems.

Bézout's theorem. Start with the simplest case: two curves in the plane, C_1, C_2, with no common components. Let C_i have defining equation $F_i(X_0, X_1, X_2)$, a homogeneous polynomial of degree d_i, so the intersection subscheme $C_1 \cap C_2$ is defined by the ideal (F_1, F_2), and is a finite set of points. A each point $p \in C_1 \cap C_2$, we have the *intersection multiplicity*

$$m(C_1, C_2; p) := \ell_{\mathcal{O}_{\mathbb{P}^2,p}} \mathcal{O}_{C_1 \cap C_2, p}$$

To explain this, we assume k is algebraically closed and take coordinates so that $p = (1, 0, 0) \in \mathbb{P}^2$. We pass to affine coordinates $x_i = X_i/X_0$ for the open subscheme $U_0 = \mathbb{P}^2 \setminus \{X_0 = 0\} = \mathrm{Spec}\, k[x_1, x_2]$, so $\mathcal{O}_{\mathbb{P}^2,p}$ is the local ring $k[x_1, x_2]_{(x_1,x_2)}$. Let $f_i = F_i/X_0^{d_i}$, so f_i is the defining equation of $C_i \cap U_0$, and $(f_1, f_2)\mathcal{O}_{\mathbb{P}^2,p}$ is an (x_1, x_2)-primary ideal. Thus $k[x_1, x_2]_{(x_1,x_2)}/(f_1, f_2)$ is a $k[x_1, x_2]_{(x_1,x_2)}$-module of finite length ℓ, with $\ell = \dim_k k[x_1, x_2]_{(x_1,x_2)}/(f_1, f_2)$, thus

$$m(C_1, C_2; p) = \dim_k k[x_1, x_2]_{(x_1,x_2)}/(f_1, f_2).$$

We note that, in this situation, $\mathrm{Tor}_i^{\mathcal{O}_{\mathbb{P}^2,p}}(\mathcal{O}_{C_1,p}, \mathcal{O}_{C_2,p})$ vanish for $i > 0$ and

$$\ell_{\mathcal{O}_{\mathbb{P}^2,p}} \mathcal{O}_{C_1,p} \otimes_{\mathcal{O}_{\mathbb{P}^2,p}} \mathcal{O}_{C_2,p} = \dim_k k[x_1, x_2]_{(x_1,x_2)}/(f_1, f_2),$$

so our formula for $m(C_1, C_2; p)$ agrees with (2.1). Let

$$C_1 \cdot C_2 = \sum_{p \in C_1 \cap C_2} m(C_1, C_2, p) \cdot p \in Z^2(\mathbb{P}^2).$$

On the other hand, each F_i is a section s_i of $\mathcal{O}_{\mathbb{P}^2}(d_i)$ and another application of Serre's intersection formula gives

$$s_i^* s_{0*}[\mathbb{P}^2] = [C_i],$$

so

$$c_1(\mathcal{O}_{\mathbb{P}^2}(d_i)) = [C_i].$$

Similarly, we have the section (s_1, s_2) of $\mathcal{O}_{\mathbb{P}^2}(d_1) \oplus \mathcal{O}_{\mathbb{P}^2}(d_2)$ and a computation similar to that for $C_1 \cdot C_2$ shows

$$(s_1, s_2)^* s_{0*}[\mathbb{P}^2] = [C_1 \cdot C_2] \in \mathrm{CH}^2(\mathbb{P}^2),$$

so

$$c_2(\mathcal{O}_{\mathbb{P}^2}(d_1) \oplus \mathcal{O}_{\mathbb{P}^2}(d_2)) = [C_1 \cdot C_2].$$

The Whitney product formula says $c_2(\mathcal{O}_{\mathbb{P}^2}(d_1) \oplus \mathcal{O}_{\mathbb{P}^2}(d_2)) = c_1(\mathcal{O}_{\mathbb{P}^2}(d_1)) \cup c_1(\mathcal{O}_{\mathbb{P}^2}(d_2))$ and since $c_1 : \mathrm{Pic}(\mathbb{P}^2) \to \mathrm{CH}^1(\mathbb{P}^2)$ is a group homomorphism, we have

$$\begin{aligned}[C_1 \cdot C_2] &= c_2(\mathcal{O}_{\mathbb{P}^2}(d_1) \oplus \mathcal{O}_{\mathbb{P}^2}(d_2)) \\ &= c_1(\mathcal{O}_{\mathbb{P}^2}(d_1)) \cup c_1(\mathcal{O}_{\mathbb{P}^2}(d_2)) \\ &= d_1 d_2 \cdot c_1(\mathcal{O}_{\mathbb{P}^2}(1)) \cdot c_1(\mathcal{O}_{\mathbb{P}^2}(1))\end{aligned}$$

If we now take $d_1 = d_2 = 1$, $F_1 = X_1$, $F_2 = X_2$, we have $C_1 \cdot C_2 = 1 \cdot (1 : 0 : 0)$, so $c_1(\mathcal{O}_{\mathbb{P}^2}(1)) \cup c_1(\mathcal{O}_{\mathbb{P}^2}(1)) = [1 \cdot (1 : 0 : 0)] \in \mathrm{CH}^2(\mathbb{P}^2)$, and thus

$$[C_1 \cdot C_2] = d_1 d_2 \cdot [(1 : 0 : 0)]$$

Applying the pushforward to the point, $\pi : \mathbb{P}^2 \to \mathrm{Spec}\, k$, we have $\pi_*(p) = 1$ for all $p \in \mathbb{P}^2(k)$ and so

$$\begin{aligned}\sum_{p \in C_1 \cap C_2} m(C_1, C_2, p) &= \pi_*(C_1 \cdot C_2) \\ &= \pi_*(d_1 d_2 \cdot [(1 : 0 : 0)]) \\ &= d_1 d_2\end{aligned}$$

which is exactly Bézout's theorem. The case of n hypersurfaces $H_1, \ldots, H_n$ in $\mathbb{P}^n$ that intersect in finitely many points is exactly the same: if these have degrees $d_1, \ldots, d_n$, then

$$\deg_k H_1 \cdots H_n = d_1 \cdots d_n$$

Lines on a cubic surface Consider a smooth cubic surface $S \subset \mathbb{P}^3$, with defining equation $F \in k[X_0, \ldots, X_3]_3$. We want to count the lines on S. For this, consider the Grassmannian of 2-dimensional subspaces of k^4, $\mathrm{Gr}(2, 4)$ (which is the same as lines in $\mathbb{P}^3$), with its tautological subbundle $E_2 \to \mathrm{Gr}(2, 4)$ of $\mathrm{Gr}(2, 4) \times \mathbb{A}^4$: the fiber of E_2 over a point $x \in \mathrm{Gr}(2, 4)$ representing a 2-plane Π in k^4 is $\Pi \subset k^4$. Note that $\mathrm{Gr}(2, 4)$ is a smooth proper variety of dimension 4.

The polynomial F determines a degree 3 polynomial function on each fiber Π of E_2, by restricting F to Π, in other words, F gives a section s_F of $\mathrm{Sym}^3 E_2^\vee$ over $\mathrm{Gr}(2, 4)$. s_F vanishes at $x \in \mathrm{Gr}(2, 4)$ exactly when F vanishes on the corresponding plane Π, in other words, when the line $\ell_x := \mathbb{P}(\Pi) \subset \mathbb{P}^3$ is contained in $V(F) = S$. Noting that $\mathrm{Sym}^3 E_2^\vee$ is a vector bundle of rank 4 on $\mathrm{Gr}(2, 4)$, we thus have

$$\#\{\text{lines in } S\} = \deg_k s_F^* s_{0*}[\mathrm{Gr}(2, 4)] = \deg_k c_4(\mathrm{Sym}^3 E_2^\vee).$$

where we count each line with the appropriate multiplicity (we can try to apply the criterion of multiplicity one to see if we are really just counting the number of lines).

So, we need to find a way to compute Chern classes of symmetric powers.

This is done via the *splitting principle*, which roughly speaking says that for computing Chern classes of a functor (like Sym^3) applied to a vector bundle, we may assume that the vector bundle is a sum of line bundles. So take $E^\vee = M_1 \oplus M_2$. Let $\xi_i = c_1(M_i)$, then $c(E^\vee) = c(M_1) \cdot c(M_2)$, so $c_1(E^\vee) = \xi_1 + \xi_2$, $c_2(E^\vee) = \xi_1 \xi_2$.

$$\mathrm{Sym}^3 E^\vee = M_1^{\otimes 3} \oplus M_1^{\otimes 2} \otimes M_2 \oplus M_1 \otimes M_2^{\otimes 2} \oplus M_2^{\otimes 3},$$

so

$$\begin{aligned}
c_4(\mathrm{Sym}^3 E^\vee) &= c_1(M_1^{\otimes 3}) \cdot c_1(M_1^{\otimes 2} \otimes M_2) \cdot c_1(M_1 \otimes M_2^{\otimes 2}) \cdot c_1(M_2^{\otimes 3}) \\
&= (3\xi_1) \cdot (2\xi_1 + \xi_2) \cdot (\xi_1 + 2\xi_2) \cdot (3\xi_2) \\
&= 9\xi_1\xi_2(2\xi_1^2 + 2\xi_2^2 + 5\xi_1\xi_2) \\
&= 9\xi_1\xi_2(2(\xi_1 + \xi_2)^2 + \xi_1\xi_2) \\
&= 9(\xi_1\xi_2)^2 + 18(\xi_1\xi_2) \cdot (\xi_1 + \xi_2)^2 \\
&= 9c_2(E^\vee)^2 + 18c_2(E^\vee) \cdot c_1(E^\vee)^2.
\end{aligned}$$

The point of the splitting principle is that this identity will hold, even if $E^\vee$ is not a sum of line bundles.

In any case, we now need to compute the degrees of $c_2(E^\vee)^2$ and $c_2(E^\vee) \cdot c_1(E^\vee)^2$. Note that an linear polynomial L in $X_0, \ldots, X_3$ gives a section s_L of $E^\vee$, so $c_2(E^\vee)$ is the class of $V(s_L)$. But $V(s_L)$ is just the variety of lines in $\mathbb{P}^3$ contained in $L = 0$, which is a $\mathbb{P}^2$. Similarly, $c_2(E^\vee)^2$ is the class of $V(s_L) \cdot V(s_{L'})$, in other words, the lines in $V(L) \cap V(L')$, which is just a single line if L and L' are independent. Thus

$$\deg_k c_2(E^\vee)^2 = 1$$

Also $c_2(E^\vee) \cdot c_1(E^\vee)^2$ is just the restriction of $c_1(E^\vee)^2$ to $V(s_L)$, so

$$\deg_k(c_2(E^\vee) \cdot c_1(E^\vee)^2) = \deg_k(c_1(E^\vee_{|\mathbb{P}^2})^2)$$

In general, c_1 of a vector bundle V is the same as c_1 of the line bundle $\det V$ (splitting principle again), so

$$c_1(E^\vee_{|\mathbb{P}^2})^2 = c_1(\det E^\vee_{|\mathbb{P}^2})^2$$

Finally, one shows that $\det E^\vee_{|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$, so using Bézout's theorem we have

$$\deg_k(c_1(\det E^\vee_{|\mathbb{P}^2})^2) = \deg_k(c_1(\mathcal{O}_{\mathbb{P}^2}(1))^2) = 1$$

Putting this altogether gives

$$\#\{\text{ lines in } S\} = \deg_k c_4(E^\vee) = 9 + 18 = 27.$$

If we then want to show that our cubic surface S has exactly 27 lines (over the algebraic closure $\bar{k}$), we need to see that our section s_F is transverse to the zero-section at each point of intersection, and use the criterion of multiplicity one.

This can be done by taking the line ℓ in question to be given by $X_0 = X_1 = 0$, then writing down the general cubic polynomial F that vanishes on ℓ:

$$\begin{aligned}
F = {} & X_0 \cdot Q_0(X_2, X_3) + X_1 \cdot Q_1(X_2, X_3) \\
& + X_0^2 L_{00}(X_2, X_3) + X_0X_1L_{01}(X_2, X_3) + X_1^2 L_{11}(X_2, X_3) + C(X_0, X_1)
\end{aligned}$$

where the Q_i are quadratic, and the L_{ij} are linear, in X_2, X_3, and C is cubic in X_0, X_1. The assumption that $F = 0$ is smooth (along ℓ) implies that $Q_0(X_2, X_3)$ and $Q_1(X_2, X_3)$ have no common factor.

Local coordinates on $\mathrm{Gr}(2, 4)$ near ℓ are $y := (y_1, y_2, y_3, y_4)$ corresponding to the line ℓ_y defined by $X_0 = y_1X_3 + y_2X_4, X_1 = y_3X_3 + y_4X_4$; the fiber $E_{2,y}$ is the 2-plane in $\mathbb{A}^4$ defined by the same equations. Thus, the functions (X_3, X_4) define linear coordinates on $E_{2,y}$ for all y, so a k-basis of cubic polynomials in X_3, X_4 give a framing for $\mathrm{Sym}^3 E_2^\vee$ near ℓ.

The function F restricted to $E_{2,y}$ is given by

$$y_1 \cdot X_3Q_0(X_3,X_4) + y_2X_4Q_0(X_3,X_4) \\ + y_3X_3Q_1(X_3,X_4) + y_4X_4Q_1(X_3,X_4) \\ + \text{ terms of higher order in the } y_i.$$

The fact that Q_0 and Q_1 have no common factors implies that the cubic polynomials X_3Q_0, X_4Q_0, X_3Q_1 and X_4Q_1 are a k-basis of the cubic polynomials in X_3, X_4, ie, these form a framing for E_2 near $y=0$. Thus $ds_F(T_{\mathrm{Gr}(2,4),\ell})$ is transverse to $(T_{\mathrm{Gr}(2,4),\ell}, 0)$ in the tangent space $T_{E_2,\ell,X_3=X_4=0}$ at $y=0, X_3=X_4=0$ in the total space of the bundle E_2, giving the intersection multiplicity one.

The Gauß-Bonnet theorem and the Euler characteristic

For X smooth and proper of dimension n, we have $c_n(T_{X/k}) \in \mathrm{CH}^n(X) = \mathrm{CH}_0(X)$ and thus $\deg_k(c_n(T_{X/k})) = (-1)^n \deg_k(c_n(\Omega_{X/k}))$ is a well-defined integer. The Gauß-Bonnet theorem says that this is exactly the topological Euler characteristic. On the enumerative side, one can compute $\chi^{\mathrm{top}}(X)$ for X a smooth degree d hypersurface in $\mathbb{P}^{n+1}$ explicitly as follows.

We have the Euler sequence for $T_{\mathbb{P}^{n+1}}$

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}} \to \mathcal{O}_{\mathbb{P}^{n+1}}(1)^{\oplus n+2} \to T_{\mathbb{P}^{n+1}} \to 0$$

which, using the Whitney formula, gives

$$c(T_{\mathbb{P}^{n+1}}) = c(\mathcal{O}_{\mathbb{P}^{n+1}}(1)^{\oplus n+2})/c(\mathcal{O}_{\mathbb{P}^{n+1}}) = (1+h)^{n+2}$$

with $h \in \mathrm{CH}^1(\mathbb{P}^{n+1})$ the class of a hyperplane $H \subset \mathbb{P}^{n+1}$. The tangent-normal bundle sequence for $i: X \to \mathbb{P}^{n+1}$ of degree d

$$0 \to T_X \to i^*T_{\mathbb{P}^{n+1}} \to i^*\mathcal{O}_{\mathbb{P}^{n+1}}(d) \to 0$$

gives

$$c(T_X) = i^*\left[c(T_{\mathbb{P}^{n+1}})/c(\mathcal{O}_{\mathbb{P}^{n+1}}(d))\right] = i^*[(1+h)^{n+2}/(1+dh)]$$

Taking the degree n component gives

$$\begin{aligned} \deg_k c_n(T_X) &= \deg_k i_*c_n(T_X) \\ &= \deg_k i_*i^*[h^n \sum_{i+j=n} (-1)^j \binom{n+2}{i} d^j] \\ &= \sum_{i+j=n} (-1)^j \binom{n+2}{i} d^{j+1} \end{aligned}$$

since

$$\deg_k(i_*i^*h^n) = \deg_k(i_*([X]\cdot i^*h^n)) = \deg_k(i_*([X])\cdot h^n) = d$$

Here is a table for the Euler characteristic of a degree d hypersurface X_d^n in $\mathbb{P}^{n+1}$:

n	$\chi^{\mathrm{top}}(X_d^n)$
1	$-d^2+3d$
2	d^3-4d^2+6d
3	$-d^4+5d^3-10d^2+10d$
4	$d^5-6d^4+15d^3-20d^2+15d$

Thinking purely topologically, one has the recursive formula for $\chi^{\mathrm{top}}(X_d^n)$:

$$\chi^{\mathrm{top}}(X_d^n) = (n+1)d - (d-1)\chi^{\mathrm{top}}(X_d^{n-1}).$$

To see this, one can use any family of degree d hypersurfaces in $\mathbb{P}^{n+1}$, so we choose the Fermat hypersurfaces $V(\sum_{i=0}^{n+1} X_i^d) \subset \mathbb{P}^{n+1}$. The projection from $[0,\dots,0,1] \in \mathbb{P}^{n+1}$ represents $V(\sum_{i=0}^{n+1} X_i^d)$ as a d-fold cover of $\mathbb{P}^n$, totally ramified along the hypersurface $V(\sum_{i=0}^{n} X_i^d) \subset \mathbb{P}^n$. Thus $V(\sum_{i=0}^{n+1} X_i^d) \setminus (V(\sum_{i=0}^{n} X_i^d), 0)$ is a d to 1 covering space of $\mathbb{P}^n \setminus V(\sum_{i=0}^{n} X_i^d)$, so

$$\chi^{\mathrm{top}}(X_d^n) - \chi^{\mathrm{top}}(X_d^{n-1}) = d \cdot (\chi^{\mathrm{top}}(\mathbb{P}^n) - \chi^{\mathrm{top}}(X_d^{n-1})),$$

and since $\chi^{\mathrm{top}}(\mathbb{P}^n) = n+1$, we have our formula.

Another consequence of the Gauß-Bonnet theorem is a version of the Riemann-Hurwitz formula

Theorem 3.1. *Let $f : X \to C$ be a morphism of a smooth proper variety X of dimension n to a smooth projective curve C, giving the differential $df : f^*\omega_C \to \Omega_X$. Suppose that the induced section $df : \mathcal{O}_X \to \Omega_X \otimes f^*\omega_C^{-1}$ has isolated zeros $p_1, \dots, p_r$, with multiplicities $m_1, \dots, m_r$. Let X_p be a general (smooth) fiber. Then*

$$\chi^{\mathrm{top}}(X) = \chi^{\mathrm{top}}(X_p) \cdot \chi^{\mathrm{top}}(C) + (-1)^n \cdot \sum_i m_i$$

Proof. Using the splitting principle one shows that for V a rank n bundle and L a line bundle, one has

$$c_n(V \otimes L) = \sum_{i=0}^{n} c_{n-i}(V) \cdot c_1(L)^i$$

Our assumption on df implies that $c_n(\Omega_X \otimes f^*\omega_C^{-1}) = \sum_i m_i$. Also, $c_1(f^*\omega_C^{-1})^i = f^*(c_1(\omega_C^{-1})^i) = 0$ for $i \geq 2$ (since C has dimension 1), so

$$\sum_i m_i = \deg_k(c_n(\Omega_X) + c_{n-1}(\Omega_X) \cdot f^*(c_1(T_C)))$$

Since $\Omega_X = T_X^\vee$, Gauß-Bonnet tells us that

$$\deg_k(c_n(\Omega_X)) = (-1)^n \deg_k(c_n(T_X)) = (-1)^n \chi^{\mathrm{top}}(X).$$

Since the normal bundle to X_p is trivial, we have

$$\Omega_X \otimes \mathcal{O}_{X_p} = \Omega_{X_p} \oplus \mathcal{O}_{X_p}$$

so if $c_1(T_C)) = \sum_i n_i p_i$ with the p_i taken so that X_{p_i} is smooth, we have

$$c_{n-1}(\Omega_X) \cdot f^*(c_1(T_C)) = \sum_i i_{X_{p_i}*}(n_i \cdot c_{n-1}(\Omega_{X_{p_i}}))$$

Each of the smooth fibers X_{p_i} have the same Euler characteristic, so

$$\deg_k c_{n-1}(\Omega_X) \cdot f^*(c_1(T_C)) = (-1)^{n-1} \chi^{\mathrm{top}}(X_p) \cdot \chi^{\mathrm{top}}(C)$$

Putting this altogether gives the result. $\square$

4. Dualizable objects and abstract Euler characteristics

Let $(\mathcal{C}, \otimes, 1, \tau)$ be a symmetric monoidal category with symmetry constraint $\tau_{x,y} : x \otimes y \to y \otimes x$.

DEFINITION 4.1. (1) The *dual* of an object x in $\mathcal{C}$ is a triple $(x^\vee, \delta, ev)$ with $x^\vee$ in $\mathcal{C}$, and $\delta : 1 \to x \otimes x^\vee$, $ev : x^\vee \otimes x \to 1$ morphisms such that both compositions

$$x \cong 1 \otimes x \xrightarrow{\delta\otimes\mathrm{Id}} x \otimes x^\vee \otimes x \xrightarrow{\mathrm{Id}\otimes ev} x \otimes 1 \cong x$$

$$x^\vee \cong x^\vee \otimes 1 \xrightarrow{\mathrm{Id}\otimes\delta} x^\vee \otimes x \otimes x^\vee \xrightarrow{ev\otimes\mathrm{Id}} 1 \otimes x^\vee \cong x^\vee$$

are identity morphisms.

(2) Suppose x has dual $(x^\vee, \delta, ev)$ and let $f : x \to x$ be an endomorphism. Define the *trace* $\mathrm{Tr}_x(f) \in \mathrm{End}_{\mathcal{C}}(1)$ as the composition

$$1 \xrightarrow{\delta} x \otimes x^\vee \xrightarrow{f\otimes\mathrm{Id}} x \otimes x^\vee \xrightarrow{\tau_{x,x^\vee}} x^\vee \otimes x \xrightarrow{ev} 1$$

The Euler characteristic $\chi_{\mathcal{C}}(x)$ is by definition $\mathrm{Tr}_{\mathcal{C}}(\mathrm{Id}_x)$.

REMARK 4.2. If $x \in \mathcal{C}$ admits a dual, we say that *x is dualizable.*

The dual $(x^\vee, \delta, ev)$ of an object x, if it exists, is unique up to unique isomorphism. This implies that the trace $\mathrm{Tr}_x(f)$ of an endomorphism $f : x \to x$, and Euler characteristic $\chi_{\mathcal{C}}(x)$, for dualizable x, are well-defined elements of $\mathrm{End}_{\mathcal{C}}(1)$, independent of the choice of dual.

EXAMPLES 4.3. 1. Let $\mathcal{C} = k - \mathbf{Vec}$, the category of k-vector spaces, with $\otimes = \otimes_k$, unit k and $\tau(a \otimes b) = b \otimes a$. Then $V \in k - \mathbf{Vec}$ is dualizable if and only if $\dim_k V < \infty$, the dual is the usual dual vector space, $ev : V^\vee \otimes_k V \to k$ is the evaluation map $f \otimes v \mapsto f(v)$, and $\delta : k \to V \otimes_k V^\vee$ sends $1 \in k$ to $\sum_i e_i \otimes e^i$, where $e_1, \ldots, e_n$ is a basis of V with dual basis $e^1, \ldots, e^n$. The trace is the usual trace and $\chi(V) = \dim_k V$ as an element of $\mathrm{End}_k(k) \cong k$.

2. For $\mathcal{C} =$ graded k-vector spaces, we have a similar story, except that $\tau(a \otimes b) = (-1)^{|a||b|} b \otimes a$, for a, b homogeneous of degrees $|a|$, $|b|$. If $V = \oplus_n V_n$, then $\chi(V) = \sum_n (-1)^n \dim_k V_n$.

3. For $\mathcal{C} = D(k - \mathbf{Vec})$, the derived category, the dualizable objects are the complexes K_* such that the homology $H_*(K_*) = \oplus_n H_n(K_*)$ is finite dimensional over k and $\chi(K_*) = \sum_n (-1)^n \dim_k H_n(K_*)$, again as an element of $\mathrm{End}(k) \cong k$. Sending a finite CW complex T to its singular chain complex $C_*(T, k)$ we see that

$$\chi(C_*(T, k)) = \chi^{\mathrm{top}}(T)$$

in k. We have a similar computation for $\mathcal{C} = D(\mathbf{Ab})$ and for the integral singular chain complex $C_*(T, \mathbb{Z})$, giving $\chi(C_*(T, \mathbb{Z})) = \chi^{\mathrm{top}}(T) \in \mathbb{Z} = \mathrm{End}_{D(\mathbf{Ab})}(\mathbb{Z})$.

4. We may take $\mathcal{C}$ to be the homotopy category SH of the category Sp of spectra. SH is symmetric monoidal with unit the sphere spectrum $\mathbb{S}$. Note that $\mathrm{End}_{\mathrm{SH}}(\mathbb{S})$ is the 0th stable homotopy group of spheres, which is $\mathbb{Z}$, and that the dualizable objects form the thick subcategory generated by the suspension spectra of finite CW complexes. One recovers the usual topological Euler characteristic as the categorical Euler characteristic

$$\chi_{\mathrm{SH}}(\Sigma^\infty T_+) = \chi^{\mathrm{top}}(T).$$

One can see this by applying the symmetric monoidal functor $C_*^{\mathrm{Sing}} : \mathrm{SH} \to D(\mathbb{Z})$ sending the suspension spectrum of a topological space T to the singular chain complex $C_*^{\mathrm{sing}}(T, \mathbb{Z})$, which induces the identity map $\mathbb{Z} = \mathrm{End}_{\mathrm{SH}}(\mathbb{S}) \to \mathrm{End}_{D(\mathbb{Z})}(\mathbb{Z}) = \mathbb{Z}$.

5. Morel's theorem and the quadratic Euler characteristic

Morel and Voevodsky have defined a homotopy theory where finite sets in the classical theory get replaced by smooth algebraic varieties over a given field k. The replacement of the stable homotopy category is the *motivic stable homotopy category over* k, $\mathrm{SH}(k)$. This is a symmetric monoidal category with unit the *motivic sphere spectrum* $\mathbb{S}_k$. The operation of $\mathbb{P}^1$ suspension, $\Sigma_{\mathbb{P}^1}$, is formally inverted in $\mathrm{SH}(k)$.

For each pair of integers a, b one has the associated suspension functor $\Sigma^{a,b}$; for $a \geq b \geq 0$, this is smash product with $S^{a-b} \wedge \mathbb{G}_m^{\wedge b}$ and for arbitrary (a, b), this is defined as

$$\Sigma^{a,b} = \Sigma^{a+2N,b+N} \Sigma_{\mathbb{P}^1}^{-N}; \quad N >> 0.$$

The fact that $S^1 \wedge \mathbb{G}_m \cong \mathbb{P}^1$ implies that this is well-defined, independent of N.

To construct the *Grothendieck-Witt ring over* k, $\mathrm{GW}(k)$ one starts with the set of isomorphism classes of non-degenerate symmetric bilinear forms over k (this is the same as non-degenerate quadratic forms over k if $1/2 \in k$). This is a commutative monoid under orthogonal direct sum, and $\mathrm{GW}(k)$ is the group completion, that is elements are formal differences of non-degenerate symmetric bilinear forms (up to isomorphism), with the relation $a - b = (a \perp c) - (b \perp c)$.

$\mathrm{GW}(k)$ is a commutative ring, with product induced by tensor product of symmetric bilinear forms: for $b : V \times V \to k$, $b' : W \times W \to k$, we have $b \otimes b' : (V \otimes W) \times (V \otimes W) \to k$ with $b \otimes b'(v \otimes w, v' \otimes w') = b(v, v')b'(w, w')$.

We will usually work away from characteristic 2, and so will speak mainly of quadratic forms.

A non-degenerate form q has its *rank*, namely, the dimension of the vector space on which it is defined. Sending q to $\mathrm{rank}\, q$ defines a ring homomorphism $\mathrm{rank} : \mathrm{GW}(k) \to \mathbb{Z}$.

For $u \in k^\times$, we have the rank 1 form $\langle u \rangle$ with $\langle u \rangle(x) = ux^2$, more generally, we have the rank n form $\sum_{i=1}^n \langle u_i \rangle$ with $\sum_{i=1}^n \langle u_i \rangle(x_1, \ldots, x_n) = \sum_{i=1}^n u_i x_i^2$. Away from characteristic 2, every quadratic form is isomorphic to such a "diagonal" form. The hyperbolic form is $H(x, y) = x^2 - y^2 = \langle 1 \rangle + \langle -1 \rangle$. For a form q, we have $q \cdot H = \mathrm{rank}(q) \cdot H$. The Witt ring $W(k)$ is defined by

$$W(k) := \mathrm{GW}(k)/(H).$$

For k algebraically closed, the rank homomorphism is an isomorphism $\mathrm{GW}(k) \cong \mathbb{Z}$. For $k = \mathbb{R}$, Sylvester's theorem of inertia says that each $q \in \mathrm{GW}(\mathbb{R})$ is uniquely of the form $q = a \cdot \langle 1 \rangle + b \cdot \langle -1 \rangle$, $a, b \in \mathbb{Z}$, and the signature homomorphism

$$\mathrm{sig} : \mathrm{GW}(\mathbb{R}) \to \mathbb{Z}$$

is given by $\mathrm{sig}(a \cdot \langle 1 \rangle + b \cdot \langle -1 \rangle) = a - b$.

Crucial to our discussion is Morel's theorem [**27**, Theorem 6.4.1, Remark 6.4.2].

THEOREM 5.1 (Morel). *There is a natural isomorphism*

$$\mathrm{GW}(k) \cong \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k)$$

Each smooth proper variety over k, X, defines a dualizable object $\Sigma^\infty_{\mathbb{P}^1}X_+$ in $\mathrm{SH}(k)$ [**16**, Corollary 6.13], so one has the associated Euler characteristic

$$\chi(X/k) := \chi_{\mathrm{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1}X_+) \in \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k) = \mathrm{GW}(k)$$

If we assume that k has characteristic zero, or if we invert p if k has characteristic $p > 0$, $\Sigma^\infty_{\mathbb{P}^1}U_+$ is dualizable for all smooth U, so the definition of $\chi(X/k)$ extends to arbitrary smooth U over k. Under the same assumptions, $\chi(X/k)$ extends to the Euler characteristic with compact support, $\chi_c(Z/k)$ for arbitrary finite type k-schemes, with $\chi(X/k) = \chi_c(X/k)$ for X smooth and proper.

The formal properties of categorical Euler characteristics and additional structural properties of $\mathrm{SH}(k)$ yield a number of properties of these Euler characteristics: For $u \in k^\times$, let $\langle u\rangle$ denote the rank one form $\langle u\rangle(x, y) = uxy$.

- $\chi(\Sigma^{a,b}X/k) = (-1)^a(\langle -1\rangle)^b \cdot \chi(X/k)$
- If Z contains a closed subscheme W with open complement U, then
$$\chi_c(Z/k) = \chi_c(U/k) + \chi_c(W/k)$$
If Z and W are smooth, and W has codimension c in Z, then
$$\chi(Z/k) = \chi(U/k) + \langle -1\rangle^c\chi(W/k)$$
- If $E \to B$ is a fiber bundle with fiber F, locally trivial in the Nisnevich topology, and E, B and F are smooth, then
$$\chi(E/k) = \chi(B/k) \cdot \chi(F/k)$$
- For X a smooth k-scheme, we have $\mathrm{rank}\chi(X/k) = \chi^{\mathrm{top}}(X)$. If $k = \mathbb{C}$, this says $\mathrm{rank}\chi(X/\mathbb{C}) = \chi^{\mathrm{top}}(X(\mathbb{C}))$. If $k = \mathbb{R}$, we have $\mathrm{sig}\chi(X/\mathbb{R}) = \chi^{\mathrm{top}}(X(\mathbb{R}))$.
- Suppose X is *cellular*: there is a stratification $\emptyset = X_{-1} \subset X_0 \subset \ldots \subset X_n = X$ with $X_i \subset X$ closed of dimension i, such that $X_i \setminus X_{i-1}$ is a disjoint union of affine spaces $\mathbb{A}^i_k$. Then $\mathrm{CH}^j(X)$ is a free abelian group of finite rank for each j, and letting $r_+ = \sum_{j \text{ even}} \mathrm{rank}\mathrm{CH}^j(X)$, $r_- = \sum_{j \text{ odd}} \mathrm{rank}\mathrm{CH}^j(X)$, we have
$$\chi(X/k) = r_+ \cdot \langle 1\rangle + r_- \cdot \langle -1\rangle.$$
For example
$$\chi(\mathbb{P}^n/k) = \sum_{i=0}^{n}\langle -1\rangle^i$$
- Let $Z \subset X$ be a smooth closed subscheme of a smooth k-scheme X, of codimension c and let $\tilde{X}$ be the blow-up of X along Z. Then
$$\chi(\tilde{X}/k) = \chi(X/k) + (\sum_{i=1}^{c-1}\langle -1\rangle^i) \cdot \chi(Z/k).$$

Since the rank n form $\sum_{i=0}^{n-1}\langle -1\rangle^i$ comes up a lot, we denote this by n_ϵ.

6. A version for K_0

The Gauß-Bonnet theorem is quite flexible, one can replace the Chow groups of algebraic cycles with another theory with similar formal properties. One such theory is the Grothendieck group of vector bundles $K_0(X)$.

DEFINITION 6.1. Let X be a (finite-type, separated) scheme over a field k. $K_0(X)$ is defined as the free abelian group on isomorphism classed of vector bundles (locally free coherent sheaves) on X, modulo relations of the form $[V] = [V'] + [V'']$ for each exact sequence

$$0 \to V' \to V \to V'' \to 0$$

of vector bundles on X. Tensor product of locally free sheaves defines a commutative ring structure on $K_0(X)$ with unit $[\mathcal{O}_X]$. For $f : Y \to X$ a morphism of k-schemes, sending a locally free sheaf V on X to the pullback f^*V on Y descends to a ring homomorphism $f^* : K_0(X) \to K_0(Y)$.

For example, sending k-vector space V to its dimension defines an isomorphism $K_0(\operatorname{Spec} k) \cong \mathbb{Z}$.

We now restrict to smooth k-schemes. The pushforward for a proper map $f : Y \to X$ is defined by taking finite resolutions by vector bundles of the higher direct images $P_{i,*} \to R^i f_* V$ and then taking the alternating sum $f_*[V] := \sum_{i,j}(-1)^{i+j}[P_{i,j}]$. This gives us the degree map

$$\deg_k^K : K_0(X) \to \mathbb{Z}$$

for $\pi : X \to \operatorname{Spec} k$ smooth and proper over k by $\deg_k^K(x) := \pi_*(x) \in K_0(\operatorname{Spec} k) \cong \mathbb{Z}$.

The Euler class of a rank r vector bundle $p : V \to X$ has the explicit form

$$c_r^K(V) = \sum_{i=0}^{r}(-1)^i[\Lambda^i V^\vee]$$

since $s_*\mathcal{O}_X$ has the resolution as the Koszul complex

$$0 \to \Lambda^r p^* V^\vee \to \ldots \to \Lambda^2 p^* V^\vee \to p^* V^\vee \xrightarrow{can} s_*\mathcal{O}_X \to 0$$

and $R^i s_* \mathcal{O}_X = 0$ for $i > 0$. Thus

$$c_r^K(V) := \sum_{i=0}^{r}(-1)^i s^*[\Lambda^i p^* V^\vee] = \sum_{i=0}^{r}(-1)^i[\Lambda^i V^\vee].$$

For the case of the tangent bundle on X of dimension n, we get

$$c_n^K(T_X) = \sum_{i=0}^{n}(-1)^i[\Omega_X^i].$$

Since $R^j \pi_{X*}(\Omega^i)$ is the k-vector space $H^j(X, \Omega_X^i)$, the Gauß-Bonnet theorem gives

$$\chi^{\text{top}}(X) = \deg_k^K(c_n^K(T_X)) = \sum_{i,j=0}^{n}(-1)^{i+j}\dim_k H^j(X, \Omega_X^i) \in \mathbb{Z} = K_0(\operatorname{Spec} k),$$

7. Computing the quadratic Euler characteristic

The definition of $\chi(X/k)$ is very abstract, so except perhaps for cellular varieties, it is not clear how to compute it. The motivic version of Gauß-Bonnet theorem is valid in a wide range of contexts; the most general version is due to Déglise-Jin-Khan [**7**, Theorem 4.6.1]. A consequence of their motivic Gauß-Bonnet theorem is the following.

THEOREM 7.1 (Levine-Raksit [**25**]). *Let X be a smooth proper k-scheme of dimension n. Let q_{hdg} be the quadratic form on $\oplus_{p,q} H^q(X, \Omega^p_{X/k})[p-q]$ induced by the cup product map*

$$H^q(X, \Omega^p_{X/k}) \otimes H^{n-q}(X, \Omega^{n-p}_{X/k}) \to H^n(X, \Omega^n_{X/k})$$

followed by the canonical trace map given by Serre duality

$$Tr_X : H^n(X, \Omega^n_{X/k}) \to k.$$

Then $\chi(X/k) \in \mathrm{GW}(k)$ is the class of q_{hdg}.

We will discuss the main ideas in the proof of this result in the third lecture, and we will also see how to make the computation of $\chi(X/k)$ quite explicit for X a smooth hypersurface in a projective space, using the *Jacobian ring*.

CHAPTER 2

Lecture 2: Quadratic intersection theory

We introduce some basic notions about a quadratic refinement of intersection theory and characteristic classes.

1. Introduction

We have seen that the Chow groups, with their intersection product and the Chern classes of vector bundles, gives a path to computing enumerative invariants for geometric problems over an algebraically closed field. Here we refine this to a setting where the invariants live in the Grothendieck-Witt ring. This gives information on enumerative problems over the reals by taking the signature, and other invariants of quadratic forms, such as the discriminant, gives information over other fields.

2. Milnor-Witt K-sheaves

There is a rather sophisticated description of the Chow ring of a smooth variety X as sheaf cohomology:

$$\mathrm{CH}^n(X) = H^n(X, \mathcal{K}^M_n) \tag{2.1}$$

where $\mathcal{K}^M_*$ is the sheaf of *Milnor K-groups*. For a local ring R (with infinite residue field), $K^M_*(R)$ is the tensor algebra on the group of units $R^\times$ modulo the Steinberg relation

$$K^M_*(R) := (R^\times)^{\otimes_{\mathbb{Z}}}/\langle u \otimes 1-u \mid u, 1-u \in R^\times\rangle$$

$K^M_*(R) = \oplus_{n\geq 0} K^M_n(R)$ is a graded ring with multiplication induced from the multiplication in the tensor algebra. This construction extends to a sheaf of graded rings $\mathcal{K}^M_*$ on a scheme X with stalk at $x \in X$ $K^M_*(\mathcal{O}_{X,x})$; note that $\mathcal{K}^M_1 = \mathcal{O}_X^\times$ and $\mathcal{K}^M_0$ is the constant sheaf $\mathbb{Z}$. The identity (2.1) is known as *Bloch's formula*; this is the classical identity

$$H^1(X, \mathcal{O}_X^\times) = \mathrm{Pic}(X) = \mathrm{CH}^1(X)$$

for $n = 1$; for $n = \dim_k X$, this was proven by Kato [**18**, [§0, Theorem], and in general by Kerz [**20**, Theorem 7.5] (assuming the base-field has more than a certain finite number M_n of elements). The main point is to show that $\mathcal{K}^M_n$ admits a flasque resolution of the form

$$\begin{aligned}0 \to \mathcal{K}^M_n \to \oplus_{x\in X^{(0)}} i_{x*}K^M_n(k(x)) \xrightarrow{\partial} \oplus_{x\in X^{(1)}} i_{x*}K^M_{n-1}(k(x)) \xrightarrow{\partial} \dots\\ \xrightarrow{\partial} \oplus_{x\in X^{(n-1)}} i_{x*}K^M_1(k(x)) \xrightarrow{\partial} \oplus_{x\in X^{(n)}} i_{x*}K^M_0(k(x)) \to 0\end{aligned}$$

with $X^{(q)}$ the set of codimension q points of X, so

$$\begin{aligned} H^n(X, \mathcal{K}^M_n) &= \operatorname{coker}[\oplus_{x \in X^{(n-1)}} K^M_1(k(x)) \xrightarrow{\partial} \oplus_{x \in X^{(n)}} K^M_0(k(x))] \\ &= \operatorname{coker}[\oplus_{x \in X^{(n-1)}} k(x)^\times \xrightarrow{\mathrm{div}} \oplus_{x \in X^{(n)}} \mathbb{Z}] \\ &= \mathrm{CH}^n(X). \end{aligned}$$

See [**18**, §3, Theorem 1], [**34**, Theorem 6.1], [**8**, Proposition 4.3], [**20**, Theorem 1.3] for the successive stages in the proof of this result.

The quadratic refinement, the *Chow-Witt groups*, were first defined by Barge and Morel [**3**]. Later one, Hopkins and Morel (see [**27**, §6.3] defined the *Milnor-Witt K-groups*, which lead to a definition of the Chow-Witt groups completely parallel to Bloch's formula.

Following Hopkins-Morel, for a field F, $K^{MW}_*(F)$ is the graded, associative $\mathbb{Z}$-algebra defined by generators and relations

- Generators:
  - $[u]$ in degree 1 for $u \in F^\times$
  - η in degree -1.
- Relations:
  - $[u]\eta = \eta[u]$ for all $u \in F^\times$
  - $[u][1-u] = 0$ for $u, 1-u \in F^\times$
  - $[uv] = [u] + [v] + \eta[u][v]$
  - let $h := 2 + \eta[-1]$. Then $\eta \cdot h = 0$

Morel [**26**] shows that the $K^{MW}_*(F)$ extend to define a Nisnevich sheaf of graded rings $\mathcal{K}^{MW}_*$ on a smooth k-scheme X, or even on the category of smooth, separated, finite-type k schemes, $\mathbf{Sm}/k$. Here is a resumé of some of the first properties of this construction.

PROPOSITION 2.1. *Let X be a smooth k-scheme.*

1. Let $\mathcal{GW}$, $\mathcal{W}$ denote sheaves of Grothendieck-Witt rings, resp. Witt groups, on X. There is natural isomorphism $\mathcal{K}^{MW}_0 \cong \mathcal{GW}$ and for $n < 0$ a natural isomorphism $\mathcal{K}^{MW}_n \cong \mathcal{W}$.

2. The element η defines a global section of $\mathcal{K}^{MW}_{-1}$ and $\mathcal{K}^{MW}_*/(\eta) \cong \mathcal{K}^M_*$.

3. Let $\mathcal{I} \subset \mathcal{GW}$ be the kernel of the rank homomorphism. Then for all $n \geq 0$, the surjection $\mathcal{K}^{MW}_n \to \mathcal{K}^M_n$ has kernel $\mathcal{I}^{n+1}$.

4. The assignment $X \mapsto \mathcal{K}^{MW}_{n,X}$ extends to a sheaf on $\mathbf{Sm}/k$: Let $f : Y \to X$ be a morphism of smooth k-schemes. There is a natural pullback map of sheaves $f^* : f^{-1}\mathcal{K}^{MW}_{n,X} \to \mathcal{K}^{MW}_{n,Y}$, with $(fg)^* = g^* f^*$. The items (1)-(3) are natural with respect to f^*.

3. Chow-Witt groups and Witt sheaf cohomology

DEFINITION 3.1. Let X be a smooth k-scheme. For $n \geq 0$, the nth Chow-Witt group $\widetilde{\mathrm{CH}}^n(X)$ is defined as

$$\widetilde{\mathrm{CH}}^n(X) := H^n(X, \mathcal{K}^{MW}_n)$$

Via the surjection $\mathcal{K}^{MW}_n \to \mathcal{K}^M_n$, we have the map $\widetilde{\mathrm{CH}}^n(X) \to \mathrm{CH}^n(X)$, with kernel and cokernel arising from $H^*(X, \mathcal{I}^{n+1})$, which gives the new "quadratic" information. The pullback maps $f^* : f^{-1}\mathcal{K}^{MW}_{n,X} \to \mathcal{K}^{MW}_{n,Y}$ for $f : Y \to X$ induces pullbacks $f^*: \widetilde{\mathrm{CH}}^n(X) \to \widetilde{\mathrm{CH}}^n(Y)$ compatible with the pullbacks $f^*: \mathrm{CH}^n(X) \to \mathrm{CH}^n(Y)$. There are also pushforward maps for proper maps, but here we need to introduce a new ingredient: *orientations* and *twisting*.

Given an invertible sheaf $\mathcal{L}$ on X, we can form the twisted version $\mathcal{GW}(\mathcal{L})$ of $\mathcal{GW}$, this being the sheaf of quadratic forms with values in $\mathcal{L}$ (instead of in $\mathcal{O}_X$). $\mathcal{GW}(L)$ is a $\mathcal{GW} = \mathcal{K}_0^{MW}$ module by multiplication, and we can define the twisted Milnor-Witt sheaf by

$$\mathcal{K}_n^{MW}(\mathcal{L}) = \mathcal{K}_n^{MW} \otimes_{\mathcal{K}_0^{MW}} \mathcal{GW}(\mathcal{L})$$

We can think of a section of $\mathcal{K}_n^{MW}(\mathcal{L})$ as locally in the form $s \cdot \lambda$, with s a section of $\mathcal{K}_n^{MW}$ and λ a nowhere zero section of $\mathcal{L}$, with the relation

$$s \cdot (u\lambda) = (\langle u \rangle \cdot s) \cdot \lambda$$

for u a unit.

Definition 3.2. The $\mathcal{L}$-twisted Chow-Witt groups are defined by

$$\widetilde{\mathrm{CH}}^n(X; \mathcal{L}) := H^n(X, \mathcal{K}_n^{MW}(\mathcal{L}))$$

There is a Gersten-type resolution of the Milnor-Witt sheaves, which gives an interpretation of the Chow-Witt groups as "cycles with coefficients in the Grothendieck-Witt group". This is called the *Rost-Schmid resolution* and looks like this ($d = \dim_k X$)

$$\begin{aligned}
0 \to \mathcal{K}_n^{MW} \to \oplus_{x \in X^{(0)}} K_n^{MW}(k(x)) &\xrightarrow{\partial} \oplus_{x \in X^{(1)}} K_{n-1}^{MW}(k(x); \det{}^{-1} \mathfrak{m}_x/\mathfrak{m}_x^2) \xrightarrow{\partial} \ldots \\
&\xrightarrow{\partial} \oplus_{x \in X^{(q)}} K_{n-q}^{MW}(k(x); \det{}^{-1} \mathfrak{m}_x/\mathfrak{m}_x^2) \xrightarrow{\partial} \ldots \\
\xrightarrow{\partial} \oplus_{x \in X^{(d-1)}} K_{n-d+1}^{MW}(k(x); \det{}^{-1} \mathfrak{m}_x/\mathfrak{m}_x^2) &\xrightarrow{\partial} \oplus_{x \in X^{(d)}} K_{n-d}^{MW}(k(x); \det{}^{-1} \mathfrak{m}_x/\mathfrak{m}_x^2) \to 0
\end{aligned}$$

See [**26**] and [**11, 12**] for details. Looking at the terms in degree $n-1, n, n+1$, ones sees that an element x of $\widetilde{\mathrm{CH}}^n(X)$ is represented by a finite formal sum

$$\sum_j q_j \cdot Z_j$$

where the Z_j are codimension n subvarieties of X, q_j is in $\mathrm{GW}(k(Z_j), \det \mathcal{N}_j)$, and $\mathcal{N}_j$ is the restriction to $\operatorname{Spec} k(Z_j)$ of the normal sheaf $(\mathcal{I}_{Z_j}/\mathcal{I}_{Z_j}^2)^\vee$. There is the coboundary condition $\partial(\sum_j q_j \cdot Z_j) = 0$, living in the twisted Witt groups of codimension one points of the Z_js, and all this is modulo the boundary of elements of the twisted K_1^{MW} of generic points of codimension $n-1$ subvarieties. One should think of these relations as a quadratic version of the divisor of rational functions, but, as $\mathcal{K}_{-1}^M = 0$, there is no analog in the Chow groups of the additional "coboundary condition" that one has for the Chow-Witt groups.

Since $\langle u^2 v \rangle = \langle v \rangle$, we have canonical isomorphisms

$$\mathrm{CH}^n(X; \mathcal{L} \otimes \mathcal{M}^{\otimes 2}) \cong \mathrm{CH}^n(X; \mathcal{L})$$

For $f: Y \to X$ a proper map of smooth varieties of relative dimension d, and $\mathcal{L}$ an invertible sheaf on X we have the pushforward map

$$f_*: H^p(Y, \mathcal{K}_q^{MW}(\omega_f \otimes f^*\mathcal{L})) \to H^{p-d}(X, \mathcal{K}_{q-d}^{MW}(\mathcal{L}))$$

Here ω_f is the *relative dualizing sheaf* $\omega_f := \omega_{Y/k} \otimes f^*\omega_{X/k}^{-1}$, and $\omega_{Y/k} = \Omega_{Y/k}^{\dim Y}$ is the sheaf of top degree differential forms (similarly for $\omega_{X/k}$). This gives

$$f_* : \widetilde{\mathrm{CH}}^n(Y, \omega_f \otimes f^*\mathcal{L}) \to \widetilde{\mathrm{CH}}^{n-d}(X, \mathcal{L})$$

One can view this extra twisting by ω_f as an analog of the introduction of the relative orientation sheaf needed to define proper pushforward for cohomology of unoriented manifolds.

For a rank r vector bundle $p : V \to X$ with zero section $s_0 : X \to V$, we have

$$\omega_{s_0} = \det V$$

giving the pushforward

$$s_{0*} : \widetilde{\mathrm{CH}}^m(X) \to \widetilde{\mathrm{CH}}^{m+r}(V, p^* \det^{-1} V)$$

and the *Euler class*

$$e(V) := s^* s_{0*}(1_X) \in \widetilde{\mathrm{CH}}^r(X, \det^{-1} V).$$

For $p_X : X \to \mathrm{Spec}\, k$ smooth and proper of dimension n we have the *quadratic degree*

$$\widetilde{\deg}_k := p_{X*} : \widetilde{\mathrm{CH}}^n(X, \omega_{X/k}) \to \widetilde{\mathrm{CH}}^0(\mathrm{Spec}\, k) = \mathrm{GW}(k)$$

An *orientation* for a vector bundle $V \to X$ is an isomorphism $\rho : \det^{-1} V \xrightarrow{\sim} \omega_X \otimes \mathcal{L}^{\otimes 2}$ for some invertible sheaf $\mathcal{L}$. Given an orientation on a vector bundle V of rank $n = \dim_k X$, we have $\widetilde{\deg}_k(e(V)) \in \mathrm{GW}(k)$ defined by applying the composition

$$\widetilde{\mathrm{CH}}^n(X, \det^{-1} V) \xrightarrow{\rho_*} \widetilde{\mathrm{CH}}^n(X, \omega_X \otimes \mathcal{L}^{\otimes 2}) \cong \widetilde{\mathrm{CH}}^n(X, \omega_X) \xrightarrow{p_{X*}} \widetilde{\mathrm{CH}}^0(\mathrm{Spec}\, k) = \mathrm{GW}(k).$$

to $e(V)$.

The surjection $\mathcal{K}_*^{MW} \to \mathcal{K}_*^M$ extends to a surjection $\mathcal{K}_*^{MW}(\mathcal{L}) \to \mathcal{K}_*^M$, giving the map

$$\widetilde{\mathrm{CH}}^n(X, \mathcal{L}) \to \mathrm{CH}^n(X)$$

In another direction, the isomorphisms $\mathcal{K}_n^{MW}(\mathcal{L}) \to \mathcal{W}(\mathcal{L})$ for $n < 0$ are compatible with multiplication by η, $\times\eta : \mathcal{K}_n^{MW}(\mathcal{L}) \to \mathcal{K}_{n-1}^{MW}(\mathcal{L})$, so extends to a map

$$\times\eta^N : \mathcal{K}_n^{MW}(\mathcal{L}) \to \mathcal{W}(\mathcal{L}), \quad N >> 0$$

giving the map

$$\widetilde{\mathrm{CH}}^n(X, \mathcal{L}) \to H^n(X, \mathcal{W}(\mathcal{L}))$$

One has the functorialities for $H^n(X, \mathcal{W}(\mathcal{L}))$ similar to those for the twisted Chow-Witt groups, and the two comparison maps

$$\mathrm{CH}^n(X) \leftarrow \widetilde{\mathrm{CH}}^n(X, \mathcal{L}) \to H^n(X, \mathcal{W}(\mathcal{L}))$$

are compatible with f^* and f_*. For the case of the degree maps, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^n(X) \arrow[d, "\deg_k"] & \widetilde{\mathrm{CH}}^n(X, \omega_{X/k}) \arrow[l] \arrow[r] \arrow[d, "\widetilde{\deg}_k"] & H^n(X, \mathcal{W}(\omega_{X/k})) \arrow[d, "\overline{\deg}_k"] \\
\mathbb{Z} & \mathrm{GW}(k) \arrow[l, "\mathrm{rank}"'] \arrow[r, "\pi"] & W(k)
\end{tikzcd}$$

for X smooth and proper of dimension n over k, with $\overline{\deg}_k$ the pushforward to the point,

$$\overline{\deg}_k := p_{X*} : H^n(X, \mathcal{W}(\omega_{X/k})) \to H^0(\operatorname{Spec} k, \mathcal{W}) = W(k),$$

and with $\pi : \mathrm{GW}(k) \to W(k)$ the quotient map.

Noting that an element of $x \in \mathrm{GW}(k)$ is determined by $\operatorname{rank}(x) \in \mathbb{Z}$ and $\pi(x) \in W(k)$, it is often easier to work with the somewhat simpler Witt sheaf cohomology if one is mainly interested in "quadratic part" of enumerative invariants. Here are some examples.

Quadratic Bézout theorem This was first discussed by McKean [**19**]; we give here a slightly different treatment.

The global part is very simple

PROPOSITION 3.3. *Let $V \to X$ be a vector bundle of odd rank r. Then $e^{\mathcal{W}}(V) \in H^r(X, \mathcal{W}(\det^{-1} V))$ is zero.*

The Euler class is multiplicative with respect to direct sums (or exact sequences), so

$$e^{\mathcal{W}}(\oplus_j L_j) = 0$$

for line bundles L_j. However, for the quadratic Bézout theorem, one also needs the quadratic analog of the intersection multiplicities. This can be supplied by the Euler class with support and the purity theorem.

Let $V \to X$ be a rank r vector bundle, $s : X \to V$ a section and $Z \subset X$ a closed subset containing the locus $s = 0$. Then $e(V) := s^* s_{0*}(1_X) \in H^r(X, \mathcal{K}_r^{MW}(\det^{-1} V))$ lifts canonically to the *Euler class with support* $e_Z(V, s) \in H^r_Z(X, \mathcal{K}_r^{MW}(\det^{-1} V))$.

The purity theorem is the following

THEOREM 3.4. *Suppose $i : Z \to X$ is the inclusion of a smooth subvariety Z of a smooth variety X of codimension c, and let $\mathcal{L}$ be an invertible sheaf on X. Then the pushforward $i_* : H^{p-c}(Z, \mathcal{K}_{q-c}^{MW}(i^*\mathcal{L} \otimes \omega_i) \to H^p(X, \mathcal{K}_q^{MW}(\mathcal{L}))$ factors through an isomorphism*

$$i_* : H^{p-c}(Z, \mathcal{K}_{q-c}^{MW}(i^*\mathcal{L} \otimes \omega_i) \xrightarrow{\sim} H^p_Z(X, \mathcal{K}_q^{MW}(\mathcal{L}))$$

via the forget the support map $H^p_Z(X, \mathcal{K}_q^{MW}(\mathcal{L})) \to H^p(X, \mathcal{K}_q^{MW}(\mathcal{L}))$.

To apply this to Bézout's theorem, take our two curves C_1, C_2 defined by sections $F_i : \mathbb{P}^2 \to O_{\mathbb{P}^2}(d_i)$ and with $C_1 \cap C_2 = \{p_1, \ldots, p_r\}$. Let $Z = \{p_1, \ldots, p_r\}$. The section $s := (F_1, F_2)$ of $V := O_{\mathbb{P}^2}(d_1) \oplus O_{\mathbb{P}^2}(d_2)$ gives the Euler class with support

$$e_Z(V, s) \in H^2_Z(\mathbb{P}^2, \mathcal{K}_2^{MW}(\mathcal{O}_{\mathbb{P}^2}(-d_1 - d_2)) \cong \oplus_j H^0(p_j, \mathcal{GW}(\mathcal{O}_{\mathbb{P}^2}(-d_1 - d_2) \otimes \omega_{\mathbb{P}^2}^{-1}) \otimes k(p_j))$$

Now suppose that $-d_1 - d_2$ is odd, and recall that $\omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$. Then $\mathcal{GW}(\mathcal{O}_{\mathbb{P}^2}(-d_1 - d_2) \otimes \omega_{\mathbb{P}^2}^{-1}) \cong \mathcal{GW}$, and we have

$$e_Z(V, s) = \prod_j \tilde{m}(F_1, F_2, p_j) \in \oplus_j \mathrm{GW}(p_j)$$

defining the quadratic intersection multiplicity $\tilde{m}(s_1, s_2, p_j) \in \mathrm{GW}(p_j)$. Using the functoriality of pushforward, and the fact that the pushforward for $p_j \to \operatorname{Spec} k$ is the trace map $\mathrm{Tr}_{k(p_j)/k} : \mathrm{GW}(k(p_j)) \to \mathrm{GW}(k)$, we find

$$\widetilde{\deg}_k(e(V)) = \sum_j \mathrm{Tr}_{k(p_j)/k}(\tilde{m}(F_1, F_2, p_j))$$

But since $e^{\mathcal{W}}(V) = 0$, this says that $\pi(\widetilde{\deg}_k(e(V))) = 0$ in $W(k)$, that is,

$$\widetilde{\deg}_k(e(V)) = m \cdot H \in \mathrm{GW}(k).$$

Comparing with the classical Bézout theorem via the rank map, we know that $m = d_1d_2/2$, which is an integer, since exactly one of d_1, d_2 is even. This gives us the quadratic Bézout theorem.

THEOREM 3.5 (McKean [**19**]). *Suppose we have plane curves $C_1, C_2 \subset \mathbb{P}^2_k$ of degree d_1, d_2, with no common components. Suppose in addition that $d_1 + d_2$ is odd Then*

$$\sum_j Tr_{k(p_j)/k}(\tilde{m}(F_1, F_2, p_j)) = \frac{d_1d_2}{2} \cdot H$$

To round things out, it would be nice if we had a more explicit description of the quadratic intersection multiplicity. This is given by a quadratic refinement of the formula

$$m(C_1, C_2, p) = \dim_k \mathcal{O}_{\mathbb{P}^2,p}/(f_1, f_2)$$

where (f_1, f_2) are local defining equations for C_1, C_2 near an intersection point p.

For this, we need to make clear how our (canonical) isomorphism $\omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$ gives rise to the isomorphism $\mathcal{GW}(\mathcal{O}_{\mathbb{P}^2}(-d_1 - d_2) \otimes \omega_{\mathbb{P}^2}^{-1}) \cong \mathcal{GW}$.

The isomorphism $\omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$ is given by choosing the global generator for $\omega_{\mathbb{P}^2}(3)$ to be the differential form

$$\Omega := X_0dX_1dX_2 - X_1dX_0dX_2 + X_2dX_0dX_1$$

so we have $\mathcal{O}_{\mathbb{P}^2}(-3) \cong \omega_{\mathbb{P}^2}$ by sending a local section λ of $\mathcal{O}_{\mathbb{P}^2}(-3)$ to the local section $\lambda \cdot \Omega$ of $\omega_{\mathbb{P}^2}$. This gives the isomorphism $\mathcal{O}_{\mathbb{P}^2}(-d_1 - d_2 + 3) \cong \omega_{\mathbb{P}^2}^{-1}(-d_1 - d_2)$ similarly. Writing $-d_1 - d_2 + 3 = 2m$, we have the isomorphism

$$\phi : \mathcal{O}_{\mathbb{P}^2}(m)^{\otimes 2} \xrightarrow{\sim} \omega_{\mathbb{P}^2}^{-1}(-d_1 - d_2),$$

and a *distinguished local section* of $\omega_{\mathbb{P}^2}^{-1}(-d_1 - d_2)$ is a section of form $\phi(\lambda^2)$ for λ a local section of $\mathcal{O}_{\mathbb{P}^2}(m)$.

Take $p = p_j$ for some j and let $L = L(X_0, X_1, X_2)$ be a linear form with $L(p) \neq 0$. Choose local parameters t_1, t_2 generating $\mathfrak{m}_p \subset \mathcal{O}_{\mathbb{P}^2,p}$ such that

$$(L^{d_1+d_2} \cdot dt_1 \wedge dt_2)^{-1}$$

is a distinguished local section of $\omega_{\mathbb{P}^2}^{-1}(-d_1 - d_2)$ and let $f_i = F_i/L^{d_i} \in \mathfrak{m}_p$. Choose $a_{ij} \in \mathcal{O}_{\mathbb{P}^2,p}$ so that

$$f_i = a_{i1}t_1 + a_{i2}t_2$$

and let e be the image of $\det(a_{ij})$ in $\mathcal{O}_p = \mathcal{O}_{\mathbb{P}^2,p}/(f_1, f_2)$. $\mathcal{O}_p$ is a Artin local ring with residue field $k(p)$, so the surjection $\mathcal{O}_p \to k(p)$ admits a (non-unique) splitting, making $\mathcal{O}_p$ a finite dimensional $k(p)$-algebra.

The following result comes from work of Scheja-Storch [**35**], Kass-Wickelgren [**17**], and Bachman-Wickelgren [**2**].

PROPOSITION 3.6. *1. e is independent of the choice of the a_{ij} and generates the socle of J as $k(p)$-vector space.*

2. Let $\ell : \mathcal{O}_p \to k(p)$ be a $k(p)$-linear form with $\ell(e) = 1$. Then $\tilde{m}(F_1, F_2, p) \in \mathrm{GW}(k(p))$ is represented by the quadratic form q_{SS} associated to the bilinear form

$$b_{SS}(x, y) := \ell(xy),$$

that is $q_{SS}(x) = \ell(x^2)$.

EXAMPLE 3.7. The simplest case is when C_1 and C_2 intersect transversely at p and p is a k-point, so $\mathcal{O}_p = k$. In this case, the image of a_{ij} in J is just $(\partial f_i/\partial t_j)(p)$, so e is the determinant of the Jacobian matrix $(\partial f_i/\partial t_j)(p)$, and q_{SS} is the rank one form $\langle 1/e\rangle \sim \langle e\rangle$.

Exercise Assume that at p, using coordinates (x, y) and a certain L gives a distinguished local section of $\omega_{\mathbb{P}^2}^{-1}(-d_1 - d_2)$ at p, and that $f_i = F_i/L^{d_i}$. Compute the quadratic intersection multiplicity at $p = (0, 0) \in \operatorname{Spec} k[x, y]$ for the given (f_1, f_2)

a. $(f_1, f_2) = (x, 3y)$
b. $(f_1, f_2) = (x, y^2)$
c. $(f_1, f_2) = (y - x^2, y^2 - x^3)$
d. $(f_1, f_2) = (yx^2, y^2 - x^3)$.

Lines on a hypersurface

As for the Chow group, one can compute the quadratic count of the number of lines on a hypersurface $X \subset \mathbb{P}^n$ of appropriate degree d by computing the degree of the Euler class of $\mathrm{Sym}^d(E_2^\vee)$, where $E_2 \to \mathrm{Gr}(2, n + 1)$ is the tautological rank 2 subbundle of the trivial rank $n + 1$ bundle. Since $\dim_k \mathrm{Gr}(2, n + 1) = 2n - 2$ and $\mathrm{Sym}^d(E_2^\vee)$ has rank $d + 1$, the condition on d is $d = 2n - 3$. In this case $\mathrm{Sym}^d(E_2^\vee)$ has even rank $2n$, so one has the possibility of a non-zero Euler class. We need to check the orientation condition.

One has the Euler sequence for $\mathrm{Gr}(2, n + 1)$:

$$0 \to E_2 \otimes E_2^\vee \to \mathcal{O}_{\mathrm{Gr}(2,n+1)}^{n+1} \otimes E_2^\vee \to T_{\mathrm{Gr}(2,n+1)} \to 0$$

$\det E_2^\vee = \mathcal{O}_{\mathrm{Gr}(2,n+1)}(1)$ with respect to the Plücker embedding, and $\det E_2 \otimes E_2^\vee$ is trivial, so we have

$$\det T_{\mathrm{Gr}(2,n+1)} = \mathcal{O}_{\mathrm{Gr}(2,n+1)}(n + 1),\ \omega_{\mathrm{Gr}(2,n+1)} = \mathcal{O}_{\mathrm{Gr}(2,n+1)}(-n - 1).$$

We can compute $\det \mathrm{Sym}^d(E_2^\vee)$ by using the splitting principle again: If $E_2^\vee = M_1 \oplus M_2$, then

$$\mathrm{Sym}^d(E_2^\vee) = \oplus_{i=0}^d M_1^{\otimes d-i} \otimes M_2^{\otimes i}$$

so

$$\det \mathrm{Sym}^d(E_2^\vee) = (M_1 \otimes M_2)^{\otimes \sum_{i=1}^d i} = \mathcal{O}_{\mathrm{Gr}(2,n+1)}\Big(\frac{d(d+1)}{2}\Big)$$

Since $d = 2n - 1$, this is $\mathcal{O}_{\mathrm{Gr}(2,n+1)}((2n - 3)(n - 1))$ and so

$$\det{}^{-1} \mathrm{Sym}^d(E_2^\vee) \cong \omega_{\mathrm{Gr}(2,n+1)} \otimes \mathcal{O}_{\mathrm{Gr}(2,n+1)}((n - 1)^2 + 1)^{\otimes 2}$$

which gives the orientation condition. We thus have

$$\begin{aligned} e^{\mathcal{W}}(\mathrm{Sym}^d(E_2^\vee)) \in H^{2n-2}(&\mathrm{Gr}(2, n + 1), \mathcal{W}(\det{}^{-1} \mathrm{Sym}^d(E_2^\vee))) \\ &\cong H^{2n-2}(\mathrm{Gr}(2, n + 1), \mathcal{W}(\omega_{\mathrm{Gr}(2,n+1)})) \end{aligned}$$

so we have

$$\widetilde{\deg}_k(e^{\mathcal{W}}(\mathrm{Sym}^d(E_2^\vee))) \in W(k).$$

To compute this, we use the following general result

THEOREM 3.8 ([**22**, Theorem 8.1]). *Let $V \to X$ be a rank 2 vector bundle. Then for d odd*

$$e^{\mathcal{W}}(\mathrm{Sym}^d V) = d!!e(V)^{d+1/2} \in H^{d+1}(X, \mathcal{W}(\det^{-1}\mathrm{Sym}^d V))$$

Here $d!! = d \cdot (d-2) \cdots 3 \cdot 1$.

In our case, we have

$$e^{\mathcal{W}}(\mathrm{Sym}^d(E_2^\vee)) = d!!e^{\mathcal{W}}(E_2^\vee)^{n-1} \in H^{2n-2}(\mathrm{Gr}(2, n+1), \mathcal{W}(\mathcal{O}_{\mathrm{Gr}(2,n+1)}(n-1)))$$

Wendt [**38**] has computed the intersection ring of $H^*(\mathrm{Gr}(2, n+1), \mathcal{W}(*))$ and shows that

$$\widetilde{\deg}_k(e^{\mathcal{W}}(E_2^\vee)^{n-1}) = \langle 1\rangle \in W(k)$$

so

$$\widetilde{\deg}_k(e^{\mathcal{W}}(\mathrm{Sym}^d(E_2^\vee))) = d!! \cdot \langle 1\rangle \in W(k).$$

If we let $N_1(n) = \deg_k(c_{2n-2}(\mathrm{Sym}^{2n-3}(E_2^\vee))) \in \mathbb{Z}$, then we have the full quadratic degree

$$\widetilde{\deg}_k(e^{CW}(\mathrm{Sym}^d(E_2^\vee))) = d!! \cdot \langle 1\rangle + \frac{N_1(n) - d!!}{2} \cdot H \in \mathrm{GW}(k)$$

For the case of the cubic surface in $\mathbb{P}^3$, we have

$$\widetilde{\deg}_k(e^{CW}(\mathrm{Sym}^3(E_2^\vee))) = 3 \cdot \langle 1\rangle + 12 \cdot H \in \mathrm{GW}(k)$$

This recovers the first such computation, by Kass-Wickelgren, who used a more explicit computation of the Euler class via the quadratic local multiplicities.

REMARK 3.9. An amusing but as yet unexplained fact is that this "quadratic" count $n_d := d!!$ is comparable with the classical count N_d of the the number of lines on a degree $d = 2n-3$ hypersurface in $\mathbb{P}^n$ via the following

$$\lim_{d\to\infty} \frac{\log N_d}{\log n_d} = 2.$$

To see this, one has the following formula for $N_d := \deg(c_{d+1}(\mathrm{Sym}^d E_2^\vee))$:

$$N_d = ((d!!)^2 \cdot \sum_{r=0}^{\frac{d-1}{2}} \frac{(2r)!}{(r+1)!r!} \cdot \left(\sum_{1 \le i_1 < \ldots < i_r \le \frac{d-1}{2}} \prod_{j=1}^{r} \frac{i_j(d-i_j)}{(d-2i_j)^2} \right) \tag{3.1}$$

This follows by first using the splitting principle to give the expression

$$c_{d+1}(\mathrm{Sym}^d E_2^\vee) = \prod_{i=0}^{\frac{d-1}{2}} ((d-2i)c_2 + i(d-i)c_1^2)$$

where $c_i := c_i(E_2^\vee)$, or

$$c_{d+1}(\mathrm{Sym}^d E_2^\vee) = (d!!)^2 \cdot \sum_{r=0}^{\frac{d-1}{2}} c_2^{(d+1)/2-r} c_1^{2r} \left(\sum_{1 \le i_1 < \ldots < i_r \le \frac{d-1}{2}} \prod_{j=1}^{r} \frac{i_j(d-i_j)}{(d-2i_j)^2} \right).$$

The degree of $c_2^{(d+1)/2-r}c_1^{2r}$ is the degree of $\mathrm{Gr}(2, r+2)$ with respect to the Plücker embedding, and the Schubert calculus tells us that this is the number of ways of filling in a $2 \times r$ matrix with the integers $1, \ldots, 2r$, increasing in both rows and in all columns. By the "hook-length formula" (see e.g. [**14**, Formula 4.12]) this gives

$$\deg c_2^{(d+1)/2-r} c_1^{2r} = \frac{(2r)!}{(r+1)!r!}.$$

In an appendix to the article by Grünberg-Moree [**15**] on the asymptotic properties of the numbers N_d, Zagier rewrites N_d as an integral

$$N_d = \frac{2}{\pi} d^{d+1} \int_{-\infty}^{\infty} \phi_d(t) \frac{t^2}{(1+t^2)^2} dt,$$

where

$$\phi_d(t) = \prod_{i=0}^{\frac{d-1}{2}} \frac{1 + \frac{(d-2i)^2}{d^2} t^2}{1+t^2} < 1.$$

This enables us to compare the asymptotics for the classical count N_d and the quadratic count $d!!$. The formula (3.1) gives the inequality

$$2 < \frac{\log(N_d)}{\log(d!!)}.$$

In the other direction, we have

$$\int_{-\infty}^{\infty} \phi_d(t) \frac{t^2}{(1+t^2)^2} dt < 2 + 2 \cdot \int_1^{\infty} t^{-2} dt = 4$$

giving the upper bound

$$\log(N_d) < \log(\frac{8}{\pi}) + (d+1)\log(d).$$

Similarly, we have

$$2\log(d!!) > \log(d!) > d(\log(d) - 1)$$

so (with $C := \log(\frac{8}{\pi})$)

$$\log(N_d) < C + 2\log(d!!) + d + \log(d).$$

Thus

$$2 < \frac{\log(N_d)}{\log(d!!)} < 2 + \frac{C}{\log(d!!)} + \frac{2}{\log(d) - 1} + \frac{2}{d(1 - (\log(d))^{-1})},$$

and hence

$$\lim_{d \to \infty} \frac{\log(N_d)}{\log(n_d)} = 2.$$

Thanks to Kirsten Wickelgren for pointing out the paper [**15**] and to Sabrina Pauli for discussions on this topic.

4. Quadratic Gauß-Bonnet and the quadratic Riemann-Hurwitz formula

The motivic Gauß-Bonnet theorem gives as special cases

THEOREM 4.1. *Let X be smooth and proper over a field k. Then*

$$\chi(X/k) = \widetilde{\deg}_k(e^{CW}(T_{X/k})) \in \mathrm{GW}(k)$$

and the image $\pi(\chi(X/k))$ of $\chi(X/k)$ in $W(k)$ is given by

$$\pi(\chi(X/k)) = \overline{\deg}_k(e^{\mathcal{W}}(T_{X/k})) \in W(k)$$

Note: This says in particular that $\chi(X/k) = m\cdot H$ for some integer m if $\dim_k X$ is odd.

We will say a bit about the proof in Lecture 3. A consequence is a quadratic version of the Riemann-Hurwitz formula

THEOREM 4.2. *Let $f : X \to C$ be a morphism of a smooth proper k-scheme X of dimension n to a smooth projective curve C. Suppose that the induced section $df : \mathcal{O}_X \to \Omega_X \otimes f^*\omega_C^{-1}$ has isolated zeros $p_1, \ldots, p_r$ with quadratic multiplicities $\tilde{m}_i \in W(k(p_i))$. If n is odd, we suppose in addition that $\omega_C \cong \mathcal{L}^{\otimes 2}$ for some invertible sheaf on C. Then*

$$\pi(\chi(X/k)) = \sum_i Tr_{k(p_i)/k}\tilde{m}_i \in W(k).$$

Since $\det(\Omega_X \otimes f^*\omega_C^{-1}) = \omega_X \otimes f^*\omega_C^{-n}$, our assumption that $\omega_C \cong \mathcal{L}^{\otimes 2}$ if n is odd says that we have the orientation condition needed to define the local quadratic multiplicities

$$\tilde{m}_i := e^{\mathcal{W}}_{p_i}(\Omega_X\otimes f^*\omega_C^{-1}, df) \in H^n_{p_i}(X, \mathcal{W}(\omega_X\otimes f^*\omega_C^{-n})) \cong H^n_{p_i}(X, \mathcal{W}(\omega_X)) \cong W(k(p_i))$$

The proof follows the same idea as for the classical case: one computes the quadratic degree $\overline{\deg}_k e^{\mathcal{W}}(\Omega_{X/k} \otimes f^*\omega_{C/k}^{-1})$ as $\sum_i \mathrm{Tr}_{k(p_i)/k}\tilde{m}_i$ and then uses

PROPOSITION 4.3. *Let V be a rank r vector bundle on a smooth k-scheme X and let L be a line bundle on X. If r is odd, we suppose that $L \cong M^{\otimes 2}$ for some line bundle M. Then*

$$e^{\mathcal{W}}(V \otimes L) = e^{\mathcal{W}}(V) \in H^{2r}(X, \mathcal{W}(\det^{-1} V)) \cong H^{2r}(X, \mathcal{W}(\det^{-1}(V \otimes L)))$$

One also has an explicit formula for the $\tilde{m}_i$ using the quadratic form on the *local Jacobian rings* $J(df)_{p_i}$:

$$J(df)_{p_i} = \mathcal{O}_{X,p_i}/(\ldots, \partial f/\partial t_i, \ldots),$$

with respect to suitably chosen coordinates $t_1, \ldots, t_n$ at p_i. In fact, take $p = p_i$ a point with $df = 0$. Let $q = f(p)$ and let $t \in \mathfrak{m}_q \subset \mathcal{O}_{C,q}$ be a local parameter. Let $x_1, \ldots, x_n \in \mathfrak{m}_p \subset \mathcal{O}_{X,p}$ be local parameters. If n is odd, we let $\rho : \mathcal{L}^{\otimes 2} \to \omega_C$ be the chosen orientation, and we assume that the local generator dt of $\omega_{C,q}$ is of the form $\rho(\lambda^2)$ for λ a local generator of $\mathcal{L}$ near q. Let $g = f^*(t) \in \mathfrak{m}_p$, giving the partial derivatives $\partial g/\partial x_i$, $i = 1, \ldots, n$. Let $J(f,p) = \mathcal{O}_{X,p}/(\partial g/\partial x_1, \ldots, \partial g/\partial x_n)$ and choose elements $a_{ij} \in \mathcal{O}_{X,p}$ with

$$\partial g/\partial x_i = \sum_{j=1}^n a_{ij}x_j$$

Let $e_{SS} \in J(f,p)$ be the image of $\det(a_{ij})$. The fact that df has an isolated zero at p implies that $J(f,p)$ is an Artin k-algebra, so contains the residue field $k(p)$. Let $\ell : J(f,p) \to k(p)$ be a $k(p)$ linear map with $\ell(e_{SS}) = 1$ and define the quadratic form $q^{SS}_{f,p}$ on $J(f,p)$ with values in $k(p)$ by

$$q^{SS}_{f,p}(x) = \ell(x^2).$$

Then the local Euler class $\tilde{m}_i^{CW} := e^{CW}_{p_i}(\Omega_X \otimes f^*\omega_C^{-1}, df) \in \mathrm{GW}(k(p))$ is represented by $q^{SS}_{f,p}$. This type of formula for the local indices appears in [**17**] and is systematically developed in [**2**].

Exercise

Suppose X and C are both smooth curves and $f : X \to C$ a finite cover. Take $p \in X$ and suppose we have local parameters x at p and t at $q := f(p)$ such that $f^*(t) = ux^n$ for $u \in \mathcal{O}_{X,p}^{\times}$ a unit. Suppose that n is prime to the characteristic and that dt satisfies the appropriate orientation condition. Compute the quadratic multiplicity $e_{p_i}^{CW}(\Omega_X \otimes f^*\omega_C^{-1}, df) \in \mathrm{GW}(k(p))$.

CHAPTER 3

Lecture 3: Computational methods

We discuss computing the quadratic Euler characteristic via Hodge cohomology and the Jacobian ring, as well as using normalizer localization to compute degrees of quadratic Euler classes.

1. Introduction

As they carry more information than the classical $\mathbb{Z}$-valued invariants, the quadratic invariants are often more difficult to compute. In this lecture, we will go over some of the computational tools that have been developed to enable such computations. The methods include the development of a calculus of characteristic classes of vector bundles with values in Witt sheaf cohomology, algebraic computations of the quadratic Euler characteristics of smooth hypersurfaces in $\mathbb{P}^n$, and localization techniques for computing Euler classes and virtual fundamental classes. As a further example we look at a quadratic count of twisted cubic curves on hypersurfaces and complete intersections in a projective space.

2. The motivic Gauß-Bonnet theorem and computations of the quadratic Euler characteristic

In this section, we will explain a bit about the motivic Gauß-Bonnet theorem and its proof, and then discuss some computational methods. For the first topic, we need a bit a background about the motivic stable homotopy category $\mathrm{SH}(k)$ a field k.

$\mathrm{SH}(k)$ is a triangulated, symmetric monoidal category, with product $\wedge$ and with translation functor $\Sigma_{S^1} := - \wedge S^1$. $\mathbb{G}_m$-suspension $\Sigma_{\mathbb{G}_m}$ is also invertible and $\mathbb{P}^1$-suspension $\Sigma_{\mathbb{P}^1}$ is the same as $\Sigma_{S^1}\Sigma_{\mathbb{G}_m} = \Sigma_{\mathbb{G}_m}\Sigma_{S^1}$. One defines the family of suspension operations

$$\Sigma^{a,b} := \Sigma_{S^1}^{a-b}\Sigma_{\mathbb{G}_m}^{b}.$$

We have the category of *pointed spaces over* k, $\mathbf{Spc}_\bullet(k)$, this being the category of pointed simplicial presheaves on $\mathbf{Sm}_k$, with the Yoneda embedding $\mathbf{Sm}_k \to \mathbf{Spc}_\bullet(k)$ sending X to the representable presheaf X_+ of sets, with an added basepoint.

A $\mathbb{P}^1$-*spectrum* $\mathcal{E}$ is a sequence of pointed spaces over k, $(\mathcal{E}_0, \mathcal{E}_1, \ldots)$ together with "bonding maps"

$$\epsilon_n : \Sigma_{\mathbb{P}^1}\mathcal{E}_n \to \mathcal{E}_{n+1}$$

As an example, if we start with some $\mathcal{X} \in \mathbf{Spc}_\bullet(k)$, we have the $\mathbb{P}^1$-suspension spectrum

$$\Sigma^\infty_{\mathbb{P}^1}(\mathcal{X}) := (\mathcal{X}, \Sigma_{\mathbb{P}^1}\mathcal{X}, \ldots, \Sigma^n_{\mathbb{P}^1}(\mathcal{X}), \ldots)$$

with ϵ_n the identity map $\Sigma_{\mathbb{P}^1}\Sigma^n_{\mathbb{P}^1}\mathcal{X} = \Sigma^{n+1}_{\mathbb{P}^1}\mathcal{X}$. This gives rise to the $\mathbb{P}^1$-suspension spectrum functor

$$\Sigma^\infty_{\mathbb{P}^1}(-) : \mathbf{Spc}_\bullet(k) \to \mathrm{SH}(k); \quad \mathcal{X} \mapsto \Sigma^\infty_{\mathbb{P}^1}\mathcal{X}.$$

in particular, we have $\Sigma^\infty_{\mathbb{P}^1}X_+ \in \mathrm{SH}(k)$ for each $X \in \mathbf{Sm}_k$, but also objects such as $\Sigma^\infty_{\mathbb{P}^1}X/X \setminus Z$ for $Z \subset X$ an arbitrary closed subset. The unit for the smash product $\wedge$ is the motivic sphere spectrum $\mathbb{S}_k := \Sigma^\infty \mathrm{Spec}\, k_+$.

Each $\mathcal{E} \in \mathrm{SH}(B)$ defines a bi-graded cohomology theory on $\mathbf{Spc}_\bullet(k)$ by setting

$$\mathcal{E}^{a,b}(\mathcal{X}) := \mathrm{Hom}_{\mathrm{SH}(B)}(\Sigma^\infty_{\mathbb{P}^1}\mathcal{X}, \Sigma^{a,b}\mathcal{E}),$$

giving the functor

$$\mathcal{E}^{a,b} : \mathbf{Spc}_\bullet(k)^{\mathrm{op}} \to \mathbf{Ab}.$$

For $\mathcal{X} = X_+$ this is usual $\mathcal{E}$-cohomology, $\mathcal{E}^{a,b}(X)$, and for $\mathcal{X} = X/X \setminus Z$, this gives the $\mathcal{E}$-cohomology with supports $\mathcal{E}^{a,b}_Z(X)$, with the long exact sequence

$$\ldots \to \mathcal{E}^{a,b}_Z(X) \to \mathcal{E}^{a,b}(X) \to \mathcal{E}^{a,b}(X \setminus Z) \xrightarrow{\delta} \mathcal{E}^{a+1,b}_Z(X) \to \ldots$$

We usually work with commutative rings $\mathcal{E}$ in $\mathrm{SH}(k)$, with unit $u : \mathbb{S}_k \to \mathcal{E}$ and product $\mathcal{E} \wedge \mathcal{E} \to \mathcal{E}$. This makes $\mathcal{E}^{*,*}(X) := \oplus_{a,b}\mathcal{E}^{a,b}(X)$ into a bi-graded ring with unit $1^{\mathcal{E}}_X \in \mathcal{E}^{0,0}(X)$, $1_X := p_X^*(u)$, $p_X : X \to \mathrm{Spec}\, k$ the structure map.

We will work with two special types of $\mathcal{E}$ in $\mathrm{SH}(k)$: the *oriented* spectra and the SL-*oriented spectra*; these "simplify" the $\mathcal{E}$-cohomology in the following way. There is a canonical isomorphism

$$\Sigma^\infty_{\mathbb{P}^1}(\mathbb{A}^r \times X/(\mathbb{A}^r \setminus \{0\}) \times X) \cong \Sigma^{2r,r}X_+$$

giving the canonical isomorphism, for $V \to X$ the trivial rank r vector bundle on X,

$$\mathcal{E}^{a+2r,b+r}_{0_V}(V) \cong \mathcal{E}^{a,b}(X)$$

If $\mathcal{E}$ is oriented, one has canonical and natural isomorphisms

$$\mathcal{E}^{a+2r,b+r}_{0_V}(V) \xrightarrow[\sim]{\phi_V} \mathcal{E}^{a,b}(X)$$

for *arbitrary* $V \to X$ ($r = \mathrm{rank}V$). If $\mathcal{E}$ is SL-oriented, one has canonical and natural isomorphisms

$$\mathcal{E}^{a+2r,b+r}_{0_V}(V) \xrightarrow[\sim]{\phi_{V,\rho}} \mathcal{E}^{a,b}(X)$$

for each isomorphism $\rho : \det V \xrightarrow{\sim} O_X$ (if such exists). An oriented theory is also SL-oriented, and the isomorphism $\phi_{V,\rho}$ is independent of ρ.

Definition 2.1. Let $\mathcal{E}$ be an SL-oriented spectrum. $L \to X$ a line bundle on $X \in \mathbf{Sm}_k$. Let $\mathcal{L}$ be the invertible sheaf of section of L. Define the $\mathcal{L}$-twisted $\mathcal{E}$-cohomology by

$$\mathcal{E}^{a,b}(X; \mathcal{L}) := \mathcal{E}^{a+2,b+1}_{0_L}(L)$$

Note that $\mathcal{E}^{a,b}(X; \mathcal{L}) = \mathcal{E}^{a,b}(X)$ if $\mathcal{E}$ is oriented.

An SL-oriented theory $\mathcal{E}$ admits proper pushforward maps similar to those we have seen for $\widetilde{\mathrm{CH}}$: given a proper morphism $f : Y \to X$ in $\mathbf{Sm}_k$, of relative dimension d, and $\mathcal{L}$ an invertible sheaf on X, we have

$$f_* : \mathcal{E}^{a,b}(Y, f^*\mathcal{L} \otimes \omega_{Y/k}) \to \mathcal{E}^{a-2d,b-d}(X, \mathcal{L}).$$

with $(gf)_* = g_* f_*$, and a projection formula if $\mathcal{E}$ is a commutative ring spectrum: $f_*(f^*(x)\cdot y) = x\cdot f_*(y)$. Thus, we also have Euler classes $e^{\mathcal{E}}(V) \in \mathcal{E}^{2r,r}(X, \det^{-1} V)$ for $V \to X$ a rank r vector bundle

$$e^{\mathcal{E}}(V) = s^* s_{0*}(1_X)$$

for $s : X \to V$ any section. For $\mathcal{E}$ oriented, we have f_* as above, without needing any twists, and in addition to the Euler class, we have all the Chern classes $c_i^{\mathcal{E}}(V) \in \mathcal{E}^{2i,i}(X)$, with $c_r^{\mathcal{E}}(V) = e^{\mathcal{E}}(V)$ for $r = \mathrm{rank}(V)$.

For details on oriented and SL-oriented theories, we refer the reader to [**1**], [**2**, §4.3], [**25**, §3], [**29, 30**].

We can now state a version of the motivic Gauß-Bonnet theorem. Recall that $\chi(X/k) \in \mathrm{GW}(k)$ is defined by taking the categorical Euler characteristic

$$\chi_{\mathrm{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1} X_+) \in \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k)$$

for the dualizable object $\Sigma^\infty_{\mathbb{P}^1} X_+$ of the symmetric monoidal category $\mathrm{SH}(k)$, and then using Morel's theorem, giving the isomorphism $\mathrm{GW}(k) \cong \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k)$.

Here is the version of the motivic Gauß-Bonnet theorem appearing in [**25**].

THEOREM 2.2. *Let $\mathcal{E}$ be an* SL*-oriented ring spectrum with unit $u : \mathbb{S}_k \to \mathcal{E}$, and let $p_X : X \to \mathrm{Spec}\, k$ be a smooth proper k-scheme. Applying u to $\chi(X/k) \in \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k)$ gives $u_*(\chi(X/k)) \in \mathcal{E}^{0,0}(k) = \mathrm{Hom}_{\mathrm{SH}(k)}(\mathbb{S}_k, \mathcal{E})$. Then*

$$u_*(\chi(X/k)) = p_{X*}(e^{\mathcal{E}}(T_{X/k}))$$

This is a special case of a more general result, applicable to arbitrary commutative ring spectra $\mathcal{E}$, due to Déglise-Jin-Khan [**7**].

SKETCH OF PROOF OF MOTIVIC GAUSS-BONNET. Let $s_0 : X \to T_{X/k}$ be the zero-section. We have the Thom space $\mathrm{Th}_X(T_{X/k}) := T_{X/k}/T_{X/k} \setminus s_0(X)$ and the map $s_X : X \to \mathrm{Th}_X(T_{X/k})$ given by $s_0 : X \to T_{X/k}$ followed by the quotient map $T_{X/k} \to \mathrm{Th}_X(T_{X/k})$.

The proof relies on the following facts:

- For $\pi_X : X \to \mathrm{Spec}\, k$ smooth and proper of dimension d over k, the dual of $\Sigma^\infty_{\mathbb{P}^1} X_+$ is $\pi_{X\#}\mathrm{Th}_X(-T_{X/k}) := \pi_{X\#}\Sigma^{-T_{X/k}}(\mathbb{S}_X)$ [**16**, Corollary 6.13].
- Let $\Delta_X : X \to X \times_k X$ be the diagonal morphism, with normal bundle N_{Δ_X}.The Morel-Voevodsky purity theorem [**28**, Theorem 3.2.23], and the canonical isomorphism $\Delta_{X*}(T_{X/k}) \cong N_{\Delta_X}$, gives the isomorphism
$$\mathrm{Th}_X(T_{X/k}) \cong X \times_k X/(X \times_k X \setminus \Delta_X(X)).$$
in $\mathcal{H}_\bullet(X)$.

These allow one to rewrite the endomorphism $\chi(X/k) \in \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k)$, defined as the composition (with $x = \Sigma^\infty_{\mathbb{P}^1} X_+$)

$$\mathbb{S}_k \xrightarrow{\delta_x} x \otimes x^\vee \xrightarrow{\tau_{x,x^\vee}} x^\vee \otimes x \xrightarrow{ev_x} \mathbb{S}_k,$$

as the following composition

$$\mathbb{S}_k \xrightarrow{\pi_X^\vee} \pi_{X\#}\mathrm{Th}_X(-T_{X/k}) \xrightarrow{\beta_X} \Sigma^\infty_{\mathbb{P}^1} X_+ \xrightarrow{\pi_X} \mathbb{S}_k.$$

Here $\pi_X^\vee : \mathbb{S}_k \to \pi_{X\#}\mathrm{Th}_X(-T_{X/k}) = (\Sigma^\infty_{\mathbb{P}^1} X_+)^\vee$ is the dual of the natural map $\pi_X : \Sigma^\infty_{\mathbb{P}^1} X_+ \to \mathbb{S}_k$ induced by π_X. The map β_X is $\pi_{X\#}$ applied the composition

$$\Sigma^{-T_{X/k}}(\mathbb{S}_X)) \xrightarrow{\Sigma^{-T_{X/k}}(s_X)} \Sigma^{-T_{X/k}}(\mathrm{Th}_X(T_{X/k})) \cong \Sigma^{-T_{X/k}}\Sigma^{T_{X/k}}(\mathbb{S}_X) = \mathbb{S}_k$$

See the proof of [**25**, Lemma 2.15] for details.

Let $\mathcal{E}$ be our SL-oriented spectrum. The Thom isomorphisms $\phi_{V,\rho}$ extend to virtual bundles, giving the Thom isomorphism

$$\phi_{-T_{X/k}}:\mathcal{E}^{2d,d}(X;\omega_{X/k})\xrightarrow{\sim}\mathcal{E}^{0,0}(Th_X(-T_{X/k}))$$

One then shows that

$$\phi_{-T_{X/k}}(e^{\mathcal{E}}(T_{X/k})=\beta_X^*(1_X)$$

Since the pushfoward map $\pi_{X*}:\mathcal{E}^{2d,d}(X;\omega_{X/k})\to\mathcal{E}^{0,0}(\mathbb{S}_k)$ is the composition $(\pi_X^\vee)^*\circ\phi_{-T_{X/k}}$, we thus have

$$\pi_{X*}(e^{\mathcal{E}}(T_{X/k}))=(\pi_X^\vee)^*(\beta_X^*(1_X^{\mathcal{E}}))=(\pi_X\circ\beta_X\circ\pi_X^\vee)^*(u),$$

which, by our factorization of $\chi(X/k)$ is exactly $u_*(\chi(X/k))$. See the proof of [**25**, Theorem 5.3] for details. □

Examples 2.3. Take $p_X:X\to k$ smooth and proper of dimension n.

1. $\mathcal{E}=H\mathbb{Z}$ representing motivic cohomology. $H\mathbb{Z}$ is an oriented ring spectrum and $H\mathbb{Z}^{2n,n}(X)=\mathrm{CH}^n(X)$. The unit map $u_{H\mathbb{Z}}:\mathrm{End}(\mathbb{S}_k)\to H\mathbb{Z}^{0,0}(k)$ is the rank map $\mathrm{rank}:\mathrm{GW}(k)\to\mathbb{Z}$, and we thus have

$$\mathrm{rank}(\chi(X/k))=u_{H\mathbb{Z}*}(\chi(X/k))=p_{X*}(e^{\mathrm{CH}}(T_{X/k}))=\deg_k(c_n^{\mathrm{CH}}(T_{X/k}))$$

in other words, $\mathrm{rank}(\chi(X/k))=\chi^{\mathrm{top}}(X)$.

2. $\mathcal{E}=\widetilde{H\mathbb{Z}}$ representing "Milnor-Witt motivic cohomology", $\widetilde{H\mathbb{Z}}$ is an SL-oriented ring spectrum and $\widetilde{H\mathbb{Z}}^{2n,n}(X,\mathcal{L})=\widetilde{\mathrm{CH}}^n(X;\mathcal{L})$. $u_{\widetilde{H\mathbb{Z}}}$ induces the identity map $\mathrm{GW}(k)=\mathrm{End}(\mathbb{S}_k)\to\widetilde{H\mathbb{Z}}^{0,0}(k)=\widetilde{\mathrm{CH}}^n(k)=\mathrm{GW}(k)$, so

$$\chi(X/k)=u_{\widetilde{H\mathbb{Z}}*}(\chi(X/k))=\widetilde{\deg}_k(e^{CW}(T_{X/k}))$$

3. $H^*(-,\mathcal{W})$ is represented by the SL-oriented ring spectrum $\mathrm{EM}(\mathcal{W}_*)$ via

$$\mathrm{EM}(\mathcal{W}_*)^{a,b}(X;\mathcal{L})=H^{a-b}(X,\mathcal{W}(\mathcal{L}))$$

and we thus have

$$\pi(\chi(X/k))=u_{\mathrm{EM}(\mathcal{W}_*)*}(\chi(X/k))=\overline{\deg}_k(e^{\mathcal{W}}(T_{X/k}))$$

where $\pi:\mathrm{GW}(k)\to W(k)$ is the canonical surjection.

4. $\mathcal{E}=\mathrm{KGL}$, representing algebraic K-theory $\mathrm{KGL}^{a,b}(X)=K_{2b-a}(X)$. KGL is oriented and $u_{\mathrm{KGL}*}$ induces the rank map $\mathrm{GW}(k)\to\mathbb{Z}$, so

$$\chi^{\mathrm{top}}(X)=\mathrm{rank}(\chi(X/k))=u_{\mathrm{KGL}*}(\chi(X/k))=p_{X*}(e^K(T_{X/k}))$$

The pushforward in K_0 is defined by taking the derived pushforward of coherent sheaves, then taking a resolution by locally free sheaves. For $p:V\to X$ a rank r vector bundle, with 0-section $s_0:X\to V$, we have $s_{0*}(1_X)=s_{0*}(\mathcal{O}_X)$, which has the Koszul resolution

$$0\to\Lambda^rp^*\mathcal{V}^\vee\to\ldots\to\Lambda^jp^*\mathcal{V}^\vee\to\ldots\to p^*\mathcal{V}^\vee\to s_{0*}(\mathcal{O}_X)\to 0$$

where $\mathcal{V}$ is the sheaf of sections of V, so

$$e^K(V)=\sum_{j=0}^r(-1)^j[\Lambda^j\mathcal{V}^\vee]$$

and

$$p_{X*}(e^K(T_{X/k})) = \sum_{i,j=0}^{\dim X} (-1)^{i+j} \dim_k H^i(X, \Omega^j_{X/k}).$$

Thus

$$\chi^{\mathrm{top}}(X) = \sum_{i,j=0}^{\dim X} (-1)^{i+j} \dim_k H^i(X, \Omega^j_{X/k}).$$

For X a $\mathbb{C}$-scheme, this identity also follows from classical Hodge theory:

$$H^n(X(\mathbb{C}), \mathbb{C}) \cong \oplus_{p+q=n} H^q(X, \Omega^p_{X/\mathbb{C}}).$$

Let $n = \dim_k X$. We have the k-bilinear form b^{hdg}

$$b^{hdg} : \oplus_{i,j} H^i(X, \Omega^j_{X/k})[j-i] \times \oplus_{i,j} H^i(X, \Omega^j_{X/k})[j-i] \to k$$

defined by composing the product

$$H^i(X, \Omega^j_{X/k})[j-i] \times H^{n-i}(X, \Omega^{n-j}_{X/k})[i-j] \to H^n(X, \Omega^n_{X/k})$$

with the canonical trace map

$$\mathrm{Tr}_{X/k} : H^n(X, \Omega^n_{X/k}) \to k,$$

and is zero on other factors. Here $H^i(X, \Omega^j_{X/k})[j-i]$ is the object of the category of graded k-vector spaces consisting of $H^i(X, \Omega^j_{X/k})$ supported in degree $i-j$. Since the commutativity constraint on graded k-vector spaces is defined by $\tau_{a,b}(a \otimes b) = (-1)^{\deg(a)\deg(b)} b \otimes a$, the bilinear form b^{hdg} is symmetric, giving us the associated quadratic form q^{hdg} on $\oplus_{i,j} H^i(X, \Omega^j_{X/k})[j-i]$, and its class $[q_{hdg}] \in \mathrm{GW}(k)$.

Theorem 2.4 (L.-Raksit). $\chi(X/k) = [q^{hdg}] \in \mathrm{GW}(k)$

Proof. We apply the motivic Gauß-Bonnet formula to $\mathcal{E} = KQ$, the ring spectrum representing hermitian K-theory (K-theory of quadratic forms). The unit map induces the identity

$$\mathrm{GW}(k) = \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k) \to KQ^{0,0}(k) = \mathrm{GW}(k)$$

Let $n = \dim_k X$. From work of Calmés-Hornbostel [**4**], the (derived) pushforward $s_{0*}(T_{X/k})$ is represented by the Koszul resolution of the sheaf $s_{0*}(T_{X/k})$ (as for K-theory), with quadratic form induced by the exterior product

$$- \wedge - : p^*\Omega^i_{X/k}[i] \otimes_k p^*\Omega^{n-i}_{X/k}[n-i] \to p^*\omega_{X/k}[n],$$

so $e^{KQ}(T_{X/k}) = s_0^* s_{0*}(T_{X/k})$ is given by $\oplus_{i=0}^n * \Omega^i_{X/k}[i]$ with quadratic from induced by

$$- \wedge - : \Omega^i_{X/k}[i] \otimes_k \Omega^{n-i}_{X/k}[n-i] \to \omega_{X/k}[n].$$

The pushforward $p_{X*}(e^{KQ}(T_{X/k}))$ then given by $\oplus_{i,j} H^j(X, \Omega^i_{X/k})[i-j]$, with quadratic form induced by that of $e^{KQ}(T_{X/k})$, using Serre duality:

$$\begin{tikzcd} H^j(X, \Omega^i_{X/k})[i-j] \otimes_k H^{n-j}(X, \Omega^{n-i}_{X/k})[j-i] \arrow[r, "- \cup -"] \arrow[dr, "q^{hdg}"'] & H^n(X, \Omega^n_{X/k}) \arrow[d, "\mathrm{Tr}_{X/k}"] \\ & k. \end{tikzcd}$$

$\square$

REMARK 2.5. Serre duality says that $\mathrm{Tr}_{X/k} \circ (- \cup -)$ identifies $H^i(X, \Omega^j_{X/k})$ with the dual of $H^{n-i}(X, \Omega^{n-j}_{X/k})$, so q^{hdg} is a sum of hyperbolic forms for $i+j<n$ and $i+j=n$, $i<j$,

$$q^{hdg}_{i,j} : H^i(X, \Omega^j_{X/k})[j-i] \oplus H^{n-i}(X, \Omega^{n-j}_{X/k})[i-j] \to k$$

and in addition, in case $n = 2m$, the form

$$q^{hdg}_{m,m} : H^m(X, \Omega^m_{X/k}) \to k.$$

Thus, letting

$$b_{hyp} := \sum_{i+j<n} \dim_k H^i(X, \Omega^j_{X/k}) + \sum_{i<j, i+j=n} \dim_k H^i(X, \Omega^j_{X/k}),$$

we have

$$\chi(X/k) = \begin{cases} b_{hyp} \cdot H \in \mathrm{GW}(k) & \text{if } n \text{ is odd,} \\ [q_{m,m}] + b_{hyp} \cdot H \in \mathrm{GW}(k) & \text{if } n = 2m \text{ is even.} \end{cases}$$

Applying $\pi : \mathrm{GW}(k) \to W(k)$, we have

$$\pi(\chi(X/k)) = \begin{cases} 0 \in W(k) & \text{if } n \text{ is odd,} \\ [q_{m,m}] \in W(k) & \text{if } n = 2m \text{ is even.} \end{cases}$$

3. Explicit computations for a hypersurface

Except for cellular varieties, this cup product on Hodge cohomology is not easy to compute explicitly. For hypersurfaces however, the primitive Hodge cohomology is computable algebraically via the *Jacobian ring*.

DEFINITION 3.1. Let $F \in k[X_0, \ldots, X_{n+1}]$ be a degree d homogeneous polynomial. The Jacobian ring $J(F)$ is

$$J(F) := k[X_0, \ldots, X_{n+1}]/(\partial F/\partial X_0, \ldots, \partial F/\partial X_{n+1})$$

If the zero-subscheme $X_F := V(F)$ is smooth over k and $d \geq 2$ is prime to the characteristic of k, then $J(F)$ is a graded Artin ring, that is, a finite dimensional k-algebra. $J(F)$ has highest non-zero degree $(d-2)(n+2)$, and the component $J(F)_{(d-2)(n+2)}$ (the socle) has dimension one over k. There is a canonical choice of generator for $J(F)_{(d-2)(n+2)}$, the *Scheja-Storch element* e_{SS}, defined as follows: write (non-uniquely!)

$$\partial F/\partial X_i = \sum_{i=0}^{n+1} a_{ij} X_j$$

for homogeneous a_{ij} of degree $d-2$. This gives us the $n+1 \times n+1$ matrix (a_{ij}) and e_{SS} is the image in $J(F)$ of $\det(a_{ij})$. It turns out that e_{SS} is uniquely defined, independent of the choice of the a_{ij}. Let $\mathrm{Tr}_{SS} : J(F) \to k$ be the k-linear map sending $J(F)_e$ to zero for $e \neq (d-2)(n+2)$ and mapping e_{SS} to one.

We have the (non-degenerate) quadratic form

$$q_{SS} : J(F) \to k, q_{SS}(x) = \mathrm{Tr}_{SS}(x^2).$$

Going back to work of Carlson-Griffiths, the Jacobian ring of F is closely related to the so-called *primitive* Hodge cohomology of the hypersurface $X := V(F)$.

For L a line bundle on some smooth Y over k, we have the 1st Chern class $c_1^{hdg}(\mathbb{L}) \in H^1(Y, \Omega^1_{Y/k})$ defined as $d\log([L])$, with $[L] \in H^1(Y, \mathcal{O}_Y^*)$ the cohomology class defining L and $d\log : \mathcal{O}_Y^* \to \Omega^1_{Y/k}$ the map $d\log u = du/u$.

We recall that the Hodge cohomology of $\mathbb{P}^{n+1}_k$ is computed as

$$H^a(\mathbb{P}^{n+1}_k, \Omega^b_{\mathbb{P}^{n+1}_k}) = \begin{cases} 0 & \text{for } a \neq b \\ k \cdot h^a & \text{for } a = b, \end{cases}$$

where $h := c_1^{hdg}(\mathcal{O}_{\mathbb{P}^{n+1}}(1))$ is the hyperplane class in $H^1(\mathbb{P}^{n+1}_k, \Omega^1_{\mathbb{P}^{n+1}_k})$.

Definition 3.2. Let $i : X \hookrightarrow \mathbb{P}^{n+1}_k$ be a smooth hypersurface; we assume that the characteristic of k does not divide the degree of X. We have the pushforward map

$$i_* : H^q(X, \Omega^p_{X/k}) \to H^{q+1}(\mathbb{P}^{n+1}, \Omega^{p+1}_{\mathbb{P}^{n+1}/k})$$

which is an isomorphism for $p+q \neq n$. The primitive Hodge cohomology $H^*(X, \Omega^*_{X/k})_{prim}$ is defined to be the kernel of i_*.

Explicitly, $H^q(X, \Omega^p_{X/k})_{prim} = 0$ for $p + q \neq n$, and for $p + q = n$ and $p \neq q$, $H^q(X, \Omega^p_{X/k})_{prim} = H^q(X, \Omega^p_{X/k})$. If $n = 2r$ is even and $p = r = q$, then

$$H^r(X, \Omega^r_{X/k}) = H^r(X, \Omega^r_{X/k})_{prim} \oplus k \cdot i^* h^r.$$

If n is odd, then

$$H^*(X, \Omega^*_{X/k})_{prim} = \oplus_{p+q=n} H^q(X, \Omega^p_{X/k}).$$

Together with Simon Pepin Lehalleur and Vasudevan Srinivas, and building on work of Carlson-Griffiths [**5**] and other, we relate the natural cup product/trace pairing on Hodge cohomology to the quadratic form q_{SS} on the Jacobian ring.

Theorem 3.3 (Levine-Pepin Lehalleur-Srinivas [**24**]). *Let $X \subset \mathbb{P}^{n+1}_k$ be a smooth hypersurface of degree $d \geq 2$.*

There are canonical isomorphisms

$$\psi_q : J(F)_{d(q+1)-n-2} \xrightarrow{\sim} H^q(X, \Omega^{n-q}_{X/k})_{prim}$$

such that, for $A \in J(F)_{d(q+1)-n-2}$, $B \in J(F)_{d(n-q+1)-n-2}$,

$$Tr_X(\psi_q(A) \cup \psi_{n-q}(B)) = \langle -d \rangle \cdot q_{SS}(AB)$$

Corollary 3.4. *Let $X \subset \mathbb{P}^{n+1}_k$ be a smooth hypersurface of degree $d \geq 2$ with inclusion $i : X \to \mathbb{P}^{n+1}_k$; we assume that the characteristic of k does not divide d. Let q_{SS}^{hdg} be the restriction of q_{SS} to $\oplus_{q=0}^n J(F)_{d(q+1)-n-2} \subset J(F)$. Then*

$$\chi(X/k) = \begin{cases} \langle d \rangle + \langle -d \rangle \cdot q_{SS}^{hdg} + \frac{n}{2} \cdot H & \textit{if } n \textit{ is even} \\ \langle -d \rangle \cdot q_{SS}^{hdg} + \frac{n+1}{2} \cdot H & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. We have the orthogonal decomposition of $\oplus_{p,q} H^q(X, \Omega^p_{X/k})$ with respect to the trace form as

$$\oplus_{p,q} H^q(X, \Omega^p_{X/k}) = \oplus_{p \neq q} H^q(X, \Omega^p_{X/k}) | \oplus_{p=0}^n H^p(X, \Omega^p_{X/k})$$

If n is odd, the summand $\oplus_{p=0}^n H^p(X, \Omega^p_{X/k})$ is $(n+1)/2 \cdot H$ and the first summand is $\langle -d \rangle \cdot q_{SS}^{hdg}$ by Theorem 3.3.

If $n = 2r$ is even, $\oplus_{p=0,p\neq m}^{n} H^p(X, \Omega^p_{X/k})$ is $(n/2)H$, and, with respect to the pairing $\langle a, b\rangle = \mathrm{Tr}_X(a \cup b)$ we have the orthogonal decomposition

$$H^r(X, \Omega^r_{X/k}) = k \cdot i^* h^r \cdot k | H^m(X, \Omega^m_{X/k})_{prim}.$$

Since $\mathrm{Tr}_X((i^*h^r)^2) = \deg h^n \cdot X = d$, the first term contributes the factor $\langle d\rangle$ and the sum $\oplus_{p+q=n} H^q(X, \Omega^p_{X/k}$ contributes the factor $\langle -d\rangle \cdot q^{hdg}_{SS}$ by Theorem 3.3. $\square$

Remark 3.5. If n is odd, then $q^{hdg}_{SS} = (b_n/2) \cdot H$, where

$$b_n = \sum_{p+q=n} \dim_k H^q(X, \Omega^p_{X/k}),$$

and thus $\langle -d\rangle q^{hdg}_{SS} = (b_n/2) \cdot H$ as well. Indeed, the perfect pairing on $J(F)$, $\langle x, y\rangle := \mathrm{Tr}_{SS}(xy)$, identifies $J(F)_{d(q+1)-n-2}$ with the dual of $J(F)_{d(n-q+1)-n-2}$, and since n is odd, there is no q with $q = n - q$.

Remark 3.6. There is also version of Theorem 3.3 for hypersurfaces in a weighted projective space, see [**24**, Theorem 4.5]. Anneloes Viergever [**37**] has extended Theorem 3.3 to the case of a smooth complete intersection $X \subset \mathbb{P}^{n+r}_k$ defined by r homogeneous polynomials, all of the same degree.

4. An example and some exercises

Take $n = 2m$ even and $X \subset \mathbb{P}^{n+1}$ a smooth degree d hypersurface defined by $F = \sum_i a_i X_i^d$ (a generalized Fermat hypersurface). We assume that d is prime to the characteristic. Then

$$J(F) = k[X_0, \ldots, X_{n+1}]/((X_0^{d-1}, \ldots, X_{n+1}^{d-1})$$

and

$$e_{SS} = \prod_i a_i d^{n+2} X_0^{d-2} \cdots X_{n+1}^{d-2}$$

The interesting part of q^{hdg}_{SS} is in degree $(d-2)(m+1)$ (the other degrees contribute a hyperbolic term). Two monomials $\prod_j X_i^{a_i}$, $\prod_j X_i^{b_i}$ of degree $(d-2)(m+1)$ have product a non-zero multiple of e_{SS} if and only if $a_i + b_i = d - 2$ for all i. If d is odd, then one of a_i, b_i is $\geq d - 1$, so one of the monomials is already zero in $J(F)$. If $d = 2e$ is even, the only non-zero contribution comes from the monomial $A := \prod_{i=0}^{n+1} X_i^{e-1}$. Since

$$q_{SS}(A) = 1/d^{n+2}$$

we see that $q^{hdg}_{SS} = \langle \prod_i a_i/d^{n+2}\rangle + b \cdot H \sim \langle \prod_i a_i\rangle + b \cdot H$ for some non-negative integer b, and thus

$$\chi(X/k) = \langle d\rangle + \langle -d \cdot \prod_i a_i\rangle + a \cdot H$$

for some positive integer a.

Exercises Let k be a field of characteristic $\neq 2$.

1. Compute $\chi(X/k)$ for $X = V(F)$,

i. $F = a_0 X_0^3 X_1 + a_1 X_1^3 X_2 + a_2 X_2^3 X_3 + a_3 X_3^3 X_0 \in k[X_0, \ldots, X_3]$
ii. $F = \sum_{i=0}^{3} a_i X_i^4 - \prod_{i=0}^{4} X_i \in k[X_0, \ldots, X_3]$
iii. $F = \lambda \cdot X_0^3 + X_1^3 + X_2^3 + X_3^3 - (\sum_{i=0}^{3} X_i)^3 \in k[X_0, \ldots, X_3]$

with the constants chosen (in k) so $V(F)$ is smooth over k.

2. Let $A = (a_{ij}) \in M_{n+2,n+2}(k)$ be a symmetric matrix with non-zero determinant δ and with n even. Let

$$F(X_0, \ldots, X_{n+1}) = \sum_{i,j=0}^{n+1} a_{ij} X_i X_j$$

and let $V(F) = X$. Show that $\chi(X/k) = \langle 2 \rangle + \langle -2\delta \rangle + (n/2)H$. Hint: use that fact that a quadratic form over k can be diagonalized.

5. Localization in Witt-sheaf cohomology

Torus localization is a powerful technique for computing degrees of characteristic classes. The basic idea is to endow a (smooth) k-scheme X with an action by a torus $T = \mathbb{G}_m^n$ and apply the Atiyah-Bott localization theorem (in this setting proven by Edidin-Graham [**10**]). First one needs to define the T-equivariant Chow groups. Following Totaro [**36**] and Edidin-Graham [**9**], this is done using an algebraic approximation of a contractible space ET on which T acts freely, and then defining $\mathrm{CH}^*_T(X) := \mathrm{CH}^*(X \times ET/T)$ (roughly speaking). Each T-equivariant vector bundle $V \to X$ defines a vector bundle $V \times ET/T \to X \times ET/T$ and thus has Chern classes

$$c_i^T(V) \in \mathrm{CH}^*_T(X)$$

Taking $X = \operatorname{Spec} k$, a T-equivariant vector bundle is just a representation $\rho : T \to \mathrm{Aut}_k(V)$ on some k-vector space V. Letting $x_i = c_1^T(\pi_i)$, where $\pi_i : T \to \mathbb{G}_m = \mathrm{Aut}_k(k)$ is the character given by the ith projection, we have

$$\mathrm{CH}^*(BT) := \mathrm{CH}^*_T(\operatorname{Spec} k) = \mathbb{Z}[x_1, \ldots, x_n]$$

One can also define $\mathrm{CH}_n^T(X) = \mathrm{CH}_T^{\dim X - n}(X)$.

THEOREM 5.1. *Let $i : X^T \to X$ be the inclusion of the fixed points. Then there is a non-zero homogeneous polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]_d$ for some $d > 0$ such that*

$$i_* : \mathrm{CH}_*^T(X^T) \to CH_*^T(X)$$

is an isomorphism after inverting P.

Allied with this is the Bott residue theorem, which says, for an equivariant vector bundle $V \to X$, we have

$$i_*(c_i^T(i^*V)/c_m(N_i)) = c_i^T(V)$$

after inverting perhaps a larger P. Here m is the codimension of X^T in X and N_i is the normal bundle.

We would like to apply this to computations in equivariant Witt sheaf cohomology, but there is a problem: $H^*(BT, \mathcal{W}) = W(k)$, concentrated in degree 0, so the equivariant Euler classes $e^T(\pi_i)$ are all zero. Inverting a polynomial P as above would just be inverting 0, leading to the valid but uninteresting identity $0 = 0$.

Instead, we use a slight enlargement of $\mathbb{G}_m$, namely, let $N \subset \mathrm{SL}_2$ be the normalizer of the torus

$$\mathbb{G}_m = \{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \neq 0 \} \subset \mathrm{SL}_2 \,.$$

N is generated by this $\mathbb{G}_m$, together with an additional element

$$\sigma := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $e \in H^2(BN, \mathcal{W})$ be the Euler class of the rank two vector bundle associated to the representation $N \subset \mathrm{SL}_2 \subset \mathrm{GL}_2$. Then

$$H^*(BN, \mathcal{W})[1/e] = W(k)[e, 1/e].$$

In fact $H^*(BN, \mathcal{W})$ is almost $W(k)[e]$, except there is one extra element $q \in H^0(BN, \mathcal{W})$, which we won't care about.

Replacing T with N^n, we have a nearly direct analog of the Atiyah-Bott localization theorem and the Bott residue formula. Unfortunately, the localization will in general kill the (very interesting) two-primary torsion in $W(k)$, but will at least let us get at the signature information coming from total orderings on k. For details, we refer the reader to our paper [**21**].

With Sabrina Pauli [**23**], we have applied this to compute the quadratic counts for twisted cubics on hypersurfaces and complete intersections in a $\mathbb{P}^n$. One has the closure H_n of the locus of smooth twisted cubics in a suitable Hilbert scheme. H_n is a smooth projective variety of dimension $4n$, with universal bundle $p : \mathcal{C}_n \to H_n$, and universal map $q : \mathcal{C}_n \to \mathbb{P}^n$. As in the case of lines, we have the locally free sheaf $\mathcal{E}_{m,n} = p_* q^* \mathcal{O}_{\mathbb{P}^n}(m)$, whose Euler class counts the twisted cubics on a hypersurface of degree m. Since $\mathcal{E}_{m,n}$ has rank $3m+1$, the condition for finiteness is

$$3m + 1 = 4n,$$

for example, a quintic in $\mathbb{P}^4$. There are additional orientation conditions:

$$n \equiv 0 \mod 2, \ m \equiv 1 \mod 4;$$

there are similar numerical and orientation conditions for complete intersections of multi-degree $(m_1, \ldots, m_r)$. Using the equivariant machinery, we developed an algorithm for computing the signature of $\overline{\deg}_k(e^{\mathcal{W}}(\mathcal{E}_{m,n}))$, which yields the following table of examples.

n	degree(s)	signature	rank
4	(5)	765	317206375
5	(3,3)	90	6424326
10	(13)	768328170191602020	794950563369917462703511361114326425387076
11	(3,11)	4407109540744680	31190844968321382445502880736987040916
11	(5,9)	313563865853700	163485878349332902738690353538800900
11	(7,7)	136498002303600	31226586782010349970656128100205356
12	(3,3,9)	43033957366680	3550223653760462519107147253925204
12	(3,5,7)	5860412510400	67944157218032107464152121768900
12	(5,5,5)	1833366298500	6807595425960514917741859812500

Here is another table (both tables kindly generated by Sabrina Pauli) that looks at the asymptotics of the count over $\mathbb{C}$ (rank) vs. the count over $\mathbb{R}$ (signature).

n	degree(s)	signature	rank	log(rank)/log(signature)
4	5	765	317206375	2.948106807
5	(3,3)	90	6424326	3.483614515
10	13	7,68328E+17	7.94951E+41	2.342692717
11	(3,11)	4.40711E+15	3.11908E+37	2.396679776
11	(5,9)	3.13564E+14	1.63486E+35	2.429131369
11	(7,7)	1.36498E+14	3,12266E+34	2.440340737
12	(3,3,9)	4.3034E+13	3.55022E+33	2.460812682
12	(3,5,7)	5.86041E+12	6.79442E+31	2.493133706
12	(5,5,5)	1.83337E+12	6.8076E+30	2.51425973
13	(3,3,5,5)	2.51455E+11	1.47998E+29	2,558690964
13	(3,3,3,7)	8.03807E+11	1.47694E+30	2.534143421
14	(3,3,3,3,5)	34474614120	3.2204E+27	2.610478
15	(3,3,3,3,3,3)	4725144720	7.01415E+25	2.671580138
17	(11,11)	5.6486E+28	8.16894E+67	2.362002804
17	(9,13)	9.62195E+28	2.36638E+68	2.359088565
17	(7,15)	4.92716E+29	6.16951E+69	2.350426005
17	(5,17)	8.57205E+30	1.84302E+72	2.336188936
17	(3,19)	6.7189E+32	1.09541E+76	2.31635201
16	21	5.07635E+35	5.40713E+81	2,28908285

Questions Does $\lim_{n\to\infty} \log(\mathrm{rank})/\log(\mathrm{signature})$ exist? Is it equal to 2?

Bibliography

[1] A. Ananyevskiy, *SL-oriented cohomology theories*. Contemp. Math., 745 American Mathematical Society, [Providence], RI, 2020, 1–19.

[2] T. Bachmann, K. Wickelgren, *Euler classes: six-functors formalism, dualities, integrality and linear subspaces of complete intersections*. J. Inst. Math. Jussieu 22 (2023), no. 2, 681–746.

[3] J. Barge, F. Morel, *Groupe de Chow des cycles orientés et classe d'Euler des fibrés vectoriels*. C. R. Acad. Sci. Paris Sér. I Math. 330 (2000), no. 4, 287–290.

[4] Calmès, B., Hornbostel, J., *Pushforwards for Witt groups of schemes*. Comment. Math. Helv. 86 (2011), no. 2, 437–468.

[5] J.A. Carlson, P.A. Griffiths, *Infinitesimal variations of Hodge structure and the global Torelli problem*. Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, pp. 51-76, Sijthoff & Noordhoff, Alphen aan den Rijn–Germantown, Md., 1980.

[6] C. Chevalley, *Les classes d'équivalence rationnelle, II*. Séminaire Claude Chevalley, tome 3 (1958), exp. no 3, p. 1–18.

[7] F. Déglise, F. Jin, A.A. Khan, *Fundamental classes in motivic homotopy theory*. J. Eur. Math. Soc. (JEMS) 23 (2021), no. 12, 3935–3993.

[8] P. Elbaz-Vincent, S. Müller-Stach, *Milnor K-theory of rings, higher Chow groups and applications*. Invent. Math.148(2002), no.1, 177–206.

[9] D. Edidin, W. Graham, *Equivariant intersection theory*. Invent. Math. 131 (1998), no. 3, 595-634.

[10] D. Edidin, W. Graham, *Localization in equivariant intersection theory and the Bott residue formula*. Amer. J. Math. 120 (1998), no. 3, 619-636.

[11] N. Feld. *Morel homotopy modules and Milnor-Witt cycle modules*. Doc. Math. 26 (2021), 617–659.

[12] N. Feld. *Milnor-Witt cycle modules*. J. Pure Appl. Algebra 224 (2020), no. 7, 106298, 44 pp.

[13] W. Fulton, **Intersection Theory**. Ergeb. Math. Grenzgeb. (3), 2. Springer-Verlag, Berlin, 1998.

[14] W. Fulton, J. Harris, **Representation theory. A first course**. Grad. Texts in Math., 129. Springer-Verlag, New York, 1991.

[15] D.B. Grünberg, P. Moree, *Sequences of enumerative geometry: congruences and asymptotics*. With an appendix by Don Zagier Experiment. Math.17(2008), no.4, 409–426.

[16] M. Hoyois, *The six operations in equivariant motivic homotopy theory*. Adv. Math. 305 (2017), 197–279.

[17] J. Kass, K. Wickelgren, *An arithmetic count of the lines on a smooth cubic surface*. Compos. Math. 157 (2021), no. 4, 677–709.

[18] K. Kato, *Milnor K-theory and the Chow group of zero cycles*. Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), 241–253. Contemp. Math., 55 American Mathematical Society, Providence, RI, 1986.

[19] S. McKean, *An arithmetic enrichment of Bézout's Theorem*. Math. Ann. 379 (2021), no. 1-2, 633–660.

[20] M. Kerz, *The Gersten conjecture for Milnor K-theory*. Invent. Math.175(2009), no.1, 1–33.

[21] M. Levine, *Atiyah-Bott localization in equivariant Witt cohomology*. Preprint 2022. arXiv:2203.13882.

[22] M. Levine, *Motivic Euler characteristics and Witt-valued characteristic classes*. Nagoya Math. J. 236 (2019), 251–310.

[23] M. Levine, S. Pauli, *Quadratic Counts of Twisted Cubics*. Preprint 2022. arXiv:2206.05729

[24] M. Levine, S. Pepin Lehalleur, V. Srinivas, *Euler characteristics of homogeneous and weighted-homogeneous hypersurfaces.* Preprint 2021. arXiv:2101.00482.

[25] M. Levine, A. Raksit, *Motivic Gauss-Bonnet formulas.* Algebra Number Theory14(2020), no.7, 1801–1851.

[26] F. Morel, **$\mathbb{A}^1$-algebraic topology over a field**. Lecture Notes in Math., 2052 Springer, Heidelberg, 2012.

[27] F. Morel, *An introduction to $\mathbb{A}^1$-homotopy theory.* ICTP Lect. Notes, XV Abdus Salam International Centre for Theoretical Physics, Trieste, 2004, 357–441.

[28] F. Morel, V. Voevodsky, *$\mathbb{A}^1$-homotopy theory of schemes.* Publ. Math. IHES 90 (1999), 45–143.

[29] I. Panin, *Oriented cohomology theories of algebraic varieties. II (After I. Panin and A. Smirnov).* Homology Homotopy Appl. 11 (2009), no. 1, 349–405.

[30] I. Panin, *Oriented cohomology theories of algebraic varieties.* K-Theory 30 (2003), no. 3, 265–314.

[31] J. Riou, *Dualité de Spanier-Whitehead en géométrie algébrique.* C. R. Math. Acad. Sci. Paris 340 (2005), no. 6, 431–436.

[32] J.P. Serre, **Algèbre locale. Multiplicités**. Lecture Notes in Math., 11 Springer-Verlag, Berlin-New York, 1965.

[33] J. Roberts, *Chow's moving lemma.* Appendix 2 to: "Motives" by Steven L. Kleiman. Algebraic geometry, Oslo 1970 (Proc. Fifth Nordic Summer School in Math.). Groningen, Wolters-Noordhoff. pp. 89–96.

[34] M. Rost, *Chow Groups with Coefficients*, Doc. Math. 1 (1996), No. 16, 319–393.

[35] G. Scheja, U. Storch, *Über Spurfunktionen bei vollständigen Durchschnitten.* J. Reine Angew. Math. 278(279) (1975), 174-190.

[36] B. Totaro, *The Chow ring of a classifying space.* Algebraic K-theory (Seattle, WA, 1997), 249-281, Proc. Sympos. Pure Math., 67, Amer. Math. Soc., Providence, RI, 1999.

[37] A.M. Viergever, *The quadratic Euler characteristic of a smooth projective same-degree complete intersection.* Preprint 2023. arXiv:2306.16155.

[38] M. Wendt, *Oriented Schubert calculus in Chow-Witt rings of Grassmannians* Contemp. Math., 745 American Mathematical Society, [Providence], RI, 2020, 217–267.