Motivic Postnikov towers

Motives and Algebraic Cycles
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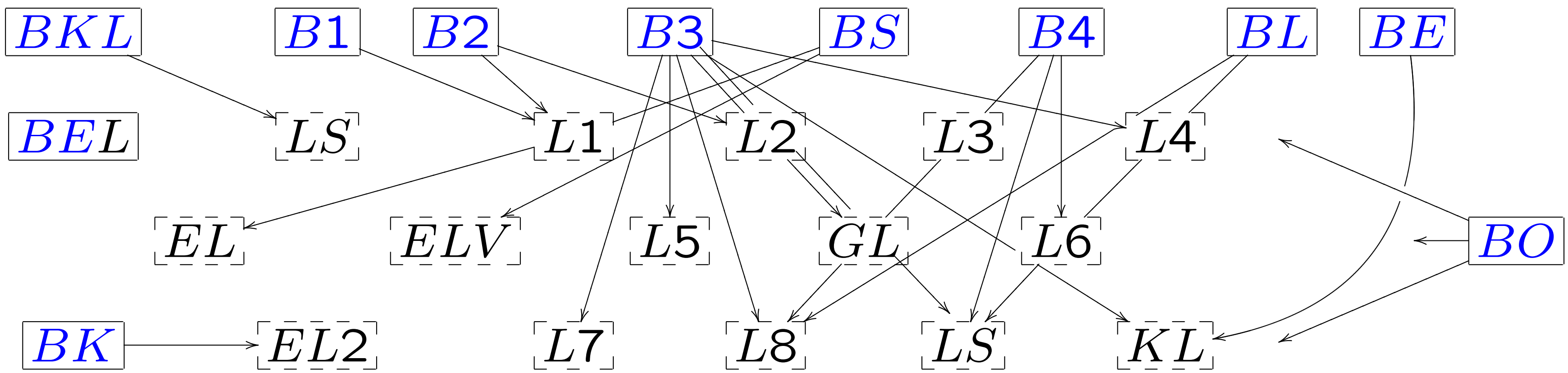
A tribute to Spencer



BO (with Ogus): Gersten's conjecture and the homology of schemes
BKL (with Kas and Lieberman): Zero cycles on surfaces with $p_g = 0$
$B1$: K_2 and algebraic cycles
$B2$: Irvine Lecture Notes
$B3$: Algebraic cycles and higher K-theory
BS (with Srinivas): Remarks on correspondences and algebraic cycles
$B4$: The moving lemma for higher Chow groups
BL (with Lichtenbaum): A spectral sequence for motivic cohomology
BE (with Esnault): An additive version of higher Chow groups
BK (with Kriz): Mixed Tate motives
BEL (with Esnault and Levine): Decomposition of the diagonal ...

LS (with Srinivas): 0-cycles on certain singular elliptic surfaces
$L1$: Bloch's formula for singular surfaces
$L2$: The indecomposable K_3 of fields
$L3$: Relative Milnor K-theory
$L4$: Bloch's higher Chow groups revisited
EL (with Esnault): Surjectivity of cycle maps
ELV (with Esnault and Viehweg): small degree
$L5$: Mixed motives
GL (with Geisser): The K-theory of fields in char p
$L6$: Techniques of localization
$L7$: Chow's moving lemma
$L8$: Homotopy coniveau
LS (with Serpé): spectral sequence for G-equivarient K-theory
$EL2$ (with Esnault): Motivic π_1 and Tate motives
KL (with Krishna): Additive higher Chow groups of schemes

Outline

- Homotopy theory and motivic homotopy theory
- Postnikov towers
- The homotopy coniveau tower
- Computations and examples
- The Postnikov tower for motives

Homotopy theory

and
motivic homotopy theory

Homotopy theory in 60 seconds

SH is the *stable homotopy category*: The localization of the category $\mathbf{Spt}$ of *spectra* with respect to stable weak equivalence.

A spectrum is a sequence of pointed spaces $E = (E_0, E_1, \ldots)$ plus bonding maps $\Sigma E_n \to E_{n+1}$. A map $f : E \to F$ is a stable weak equivalence if f induces an isomorphism on the stable homotopy groups:

$$\pi_n^s(E) := \lim_N \pi_{n+N}(E_N).$$

A spectrum E gives a *generalized cohomology theory* by

$$E^n(X) := \mathsf{Hom}_{\mathsf{SH}}(\Sigma^\infty X_+, \Sigma^n E)$$

with X a space (simplicial set),

$$\Sigma^\infty X_+ := (X_+, \Sigma X_+, \ldots, \Sigma^n X_+, \ldots)$$

and

$$\Sigma^n(E_0, E_1, \ldots) := (E_n, E_{n+1}, \ldots)$$

We go from spaces to spectra by taking Σ^∞. Conversely, sending a spectrum $E = (E_0, E_1, \ldots)$ to its *0th-space*

$$\Omega^\infty E := \lim_n \Omega^n E_n$$

gives a right adjoint to the infinite suspension functor Σ^∞.

SH **and** $D(\mathbf{Ab})$

For us, a *space* is a simplicial set, $\mathbf{Spc}$ is the category of spaces. Replacing simplicial sets with simplicial abelian groups and repeating the above construction, we get the unbounded derived category $D(\mathbf{Ab})$ together with a (non-full!) embedding

$$D(\mathbf{Ab}) \to \mathsf{SH}.$$

This allows one to think of stable homotopy theory as an extension of homological algebra.

For example, the object of SH corresponding to the complex $A[n]$ is the *Eilenberg-Maclane spectrum* $EM(A[n])$, characterized by

$$\pi^s_m(EM(A[n])) = \begin{cases} 0 & \text{for } m \neq n \\ A & \text{for } m = n \end{cases}$$

The cohomology theory represented by $EM(A)$ is singular cohomology:

$$H^n(X, A) \cong \mathrm{Hom}_{\mathrm{SH}}(\Sigma^\infty X_+, EM(A[n])).$$

Summary

$$\overset{\Sigma}{\underset{\Omega}{\circlearrowright}}\qquad\qquad\qquad\overset{\Sigma}{\underset{\Omega}{\circlearrowright}}$$

$$\mathbf{Spc} \xrightarrow{+} \mathbf{Spc}_* \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathbf{Spt}$$

$$\Big\downarrow \text{invert weak equivalences}$$

$$\overset{\Sigma}{\underset{\Omega}{\circlearrowright}}\qquad\qquad\qquad\overset{\Sigma}{\underset{\Omega}{\circlearrowright}}$$

$$\mathcal{H} \xrightarrow{+} \mathcal{H}_* \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathsf{SH} \qquad \supset \qquad D(\mathbf{Ab})$$

$\Omega = \Sigma^{-1}$ on SH. SH is a triangulated category with distinguished triangles the homotopy (co)fiber sequences.

Motivic stable homotopy theory

The motivic version of stable homotopy theory follows the same pattern, with changes:

$\mathbf{Spc} \rightsquigarrow \mathbf{Spc}(k)$: *presheaves of spaces* on $\mathbf{Sm}/k$.

There are two basic functors: the constant presheaf functor $c : \mathbf{Spc} \to \mathbf{Spc}(k)$ and the representable presheaf functor $\mathbf{Sm}/k \to \mathbf{Spc}(k)$.

$\mathbf{Spc}(k)$ inherits the operations in $\mathbf{Spc}$ by performing them point-wise: e.g. pushouts. The pointed category $\mathbf{Spc}_*(k)$ has e.g. wedge products.

$\mathbf{Spt} \rightsquigarrow \mathbf{Spt}_T(k)$: *$T$-spectra*. Let $T = \mathbb{P}^1 \cong S^1 \wedge \mathbb{G}_m$. A T-spectrum $\mathcal{E}$ is

$$\mathcal{E} = (\mathcal{E}_0, \mathcal{E}_1, \ldots) \;+\; \text{ bonding maps } \Sigma_T \mathcal{E}_n \to \mathcal{E}_{n+1}$$

$\Sigma_T E := E \wedge \mathbb{P}^1$.

We have adjoint functors $\Sigma_T^\infty : \mathbf{Spc}_*(k) \leftrightarrow \mathbf{Spt}_T(k) : \Omega_T^\infty$.

One localizes with respect to

1. the Nisnevich topology
2. $\mathbb{A}^1$-homotopy equivalence

For $\mathbf{Spc}_*(k)$, this localization is the *unstable motivic homotopy category* $\mathcal{H}(k)$. For $\mathbf{Spt}_T(k)$, this localization is the *stable motivic homotopy category* $\mathrm{SH}(k)$.

T-spectra and cohomology theories

Cohomology represented by a T-spectrum is

1. Bigraded. Since $\Sigma_T \cong \Sigma_{\mathbb{G}_m} \circ \Sigma_{S^1}$, we have two independent, invertible suspension operators on $\mathsf{SH}(k)$. So, generalized motivic cohomology is bi-graded ($X \in \mathbf{Sm}/k$):

$$\mathcal{E}^{n,m}(X) := \mathrm{Hom}_{\mathsf{SH}(k)}(\Sigma^\infty X_+, \Sigma^m_{\mathbb{G}_m} \Sigma^{n-m}_{S^1} \mathcal{E}).$$

2. Satisfies Nisnevich Mayer-Vietoris

3. Is $\mathbb{A}^1$-homotopy invariant

The localization performed imposes Nisnevich Mayer-Vietoris and $\mathbb{A}^1$-homotopy on $\mathcal{E}^{*,*}$.

Motives

In motivic stable homotopy theory, the triangulated category of motives $DM(k)$, plays the role that $D(\mathbf{Ab})$ does in the classical theory.

There is a motivic Eilenberg-Maclane functor

$$EM : DM(k) \to \mathsf{SH}(k)$$

The T-spectrum $\mathcal{HZ} := EM(\mathbb{Z})$ represents motivic cohomology:

$$\mathcal{HZ}^{p,q}(X) = H^p(X, \mathbb{Z}(q)).$$

S^1-spectra Rather than inverting Σ_T on $\mathbf{Spc}_*(k)$, one can just invert Σ_{S^1}.

Definition $\mathbf{Spt}_{S^1}(k)$ is the category of *presheaves of spectra on* $\mathbf{Sm}/k$: objects are sequences $X = (X_0, X_1, \ldots)$ in $\mathbf{Spc}_*(k)$ plus bonding maps $\epsilon_n : \Sigma X_n \to X_{n+1}$.

Localizing $\mathbf{Spt}_{S^1}(k)$ to impose Nisnevich Mayer-Vietoris and $\mathbb{A}^1$-homotopy invariance gives the homotopy category of S^1 spectra over k, $\mathsf{SH}_s(k)$. This is a triangulated category with shift induced by the usual suspension of spectra.

By forming T-spectra in $\mathbf{Spt}_{S^1}(k)$, one constructs the category of S^1-T bi-spectra, $\mathbf{Spt}_{s,t}(k)$, with homotopy category equivalent to $\mathsf{SH}(k)$. So, we can freely pass between T spectra and S^1-T bi-spectra.

Effective motives

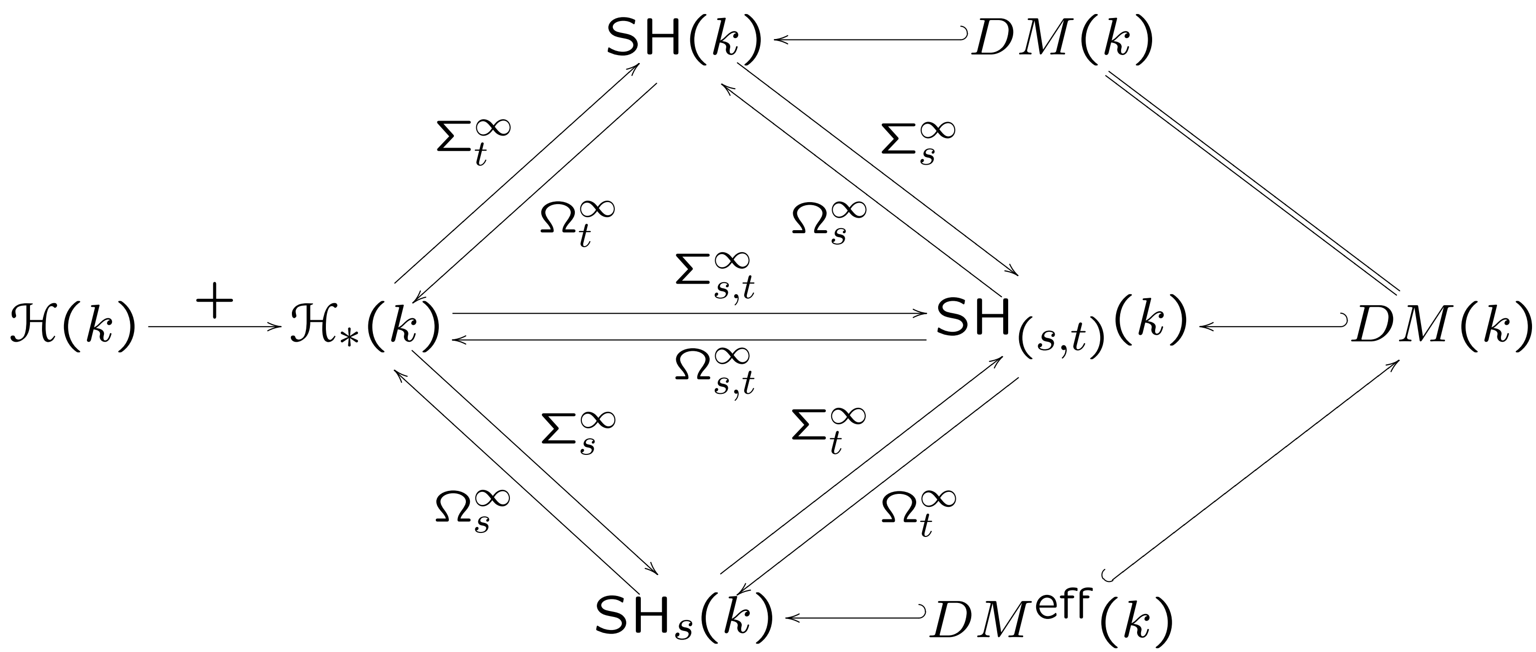
Just as $\mathsf{SH}(k)$ contains the category of motives $DM(k)$, $\mathsf{SH}_s(k)$ contains the category of *effective motives* $DM^{\mathsf{eff}}(k)$.

Note: Voevodsky's cancellation theorem says that the canonical functor

$$DM^{\mathsf{eff}}(k) \to DM^{\mathsf{eff}}(k)$$

is faithful. It is not known if $\mathsf{SH}_s(k) \to \mathsf{SH}(k)$ is faithful.

Summary


$\mathsf{SH}(k)$
$DM(k)$
Σ_t^∞
Σ_s^∞
Ω_t^∞
Ω_s^∞
$\Sigma_{s,t}^\infty$
$\mathcal{H}(k)$
$+$
$\mathcal{H}_*(k)$
$\mathsf{SH}_{(s,t)}(k)$
$DM(k)$
$\Omega_{s,t}^\infty$
Σ_s^∞
Σ_t^∞
Ω_s^∞
Ω_t^∞
$\mathsf{SH}_s(k)$
$DM^{\mathsf{eff}}(k)$

Postnikov towers

The classical Postnikov tower

Let E be a spectrum, n an integer. The *n-1-connected cover* $E{<}n{>} \to E$ of E is a map of spectra such that

1. $\pi^s_m(E{<}n{>}) = 0$ for $m \leq n-1$ and
2. $\pi^s_m(E{<}n{>}) \to \pi^s_m(E)$ is an isomorphism for $m \geq n$.

One can construct $E{<}n{>} \to E$ by killing all the homotopy groups of E in degrees $\geq n$, $E \to E(n)$ (successively coning off each element) and then take the homotopy fiber.

There is a structural approach as well: Let $\mathsf{SH}^{\mathsf{eff}} \subset \mathsf{SH}$ be the full subcategory of -1 connected spectra; this is the same as the smallest subcategory containing all suspension spectra $\Sigma^\infty X$ and closed under colimits.

Let $i_n : \Sigma^n \mathsf{SH}^{\mathsf{eff}} \to \mathsf{SH}$ be the inclusion of the nth suspension of $\mathsf{SH}^{\mathsf{eff}}$. The Brown representability theorem shows that the functor on $\Sigma^n \mathsf{SH}^{\mathsf{eff}}$

$$A \mapsto \mathrm{Hom}_{\mathsf{SH}}(i_n(A), E)$$

is representable in $\Sigma^n \mathsf{SH}^{\mathsf{eff}}$; the representing object is the n-1-connected cover $E{<}n{>} \to E$.

Forming the tower of subcategories

$$\ldots \subset \Sigma^{n+1}\mathsf{SH}^{\mathsf{eff}} \subset \Sigma^{n}\mathsf{SH}^{\mathsf{eff}} \subset \ldots \subset \mathsf{SH}$$

we have for each E the corresponding *Postnikov tower* of n-1 connected covers

$$\ldots E{<}n+1{>} \to E{<}n{>} \to \ldots \to E$$

natural in E.

The layers

Form the cofiber $E{<}n+1{>} \to E{<}n{>} \to E{<}n/n+1{>}$. Clearly

$$\pi_m^s(E{<}n/n+1{>}) = \begin{cases} 0 & \text{for } m \neq n \\ \pi_n^s(E) & \text{for } m = n. \end{cases}$$

Obstruction theory gives an isomorphism of $E{<}n/n{+}1{>}$ with the Eilenberg-Maclane spectrum $\Sigma^n(EM(\pi_n^s(E))) = EM(\pi_n^s(E)[n])$.

Roughly speaking, the Postnikov tower shows how a spectrum is built out of Eilenberg-Maclane spectra.

From the point of view of the cohomology theory represented by E, the Postnikov tower yields the *Atiyah-Hirzebruch spectral sequence*

$$E_2^{p,q} := H^p(X, \pi^s_{-q}(E)) \Longrightarrow E^{p+q}(X)$$

(there are convergence problems in general).

This is constructed just like the the spectral sequence for a filtered complex, by linking all the long exact sequences coming from applying $\mathrm{Hom}_{\mathsf{SH}}(\Sigma^\infty X_+, -)$ to the cofiber sequence $E{<}n+1{>} \to E{<}n{>} \to E{<}n/n+1{>}$.

The sketal filtration

For a CW complex X, one can recover the A-H spectral sequence by applying E to the skeletal filtration of X:

$$\emptyset = X_{-1} \subset X_0 \subset \ldots \subset X_n \subset \ldots \subset X$$

Applying E to the cofiber sequences $X_{p-1} \to X_p \to X_p/X_{p-1} \to \Sigma X_{p-1}$ gives the long exact sequence

$$\ldots \to E^{p+q-1}(X_{p-1}) \to E^{p+q}(X_p/X_{p-1}) \\ \to E^{p+q}(X_p) \to E^{p+q}(X_{p-1}) \to \ldots$$

which link together to give a spectral sequence.

The universal property of the $E{<}n{>}$ identifies the skeletal spectral sequence with the A-H spectral sequence.

The motivic Postnikov tower

Voevodsky has defined the Tate analog of the Postnikov tower:

Let $\mathsf{SH}^{\mathsf{eff}}(k)$ be the smallest full triangulated subcategory of $\mathsf{SH}(k)$ containing all the T-suspension spectra $\Sigma_t^\infty A$, $A \in \mathbf{Spc}_*(k)$, and closed under colim.

Taking T-suspensions gives the tower of full triangulated localizing subcategories

$$\ldots \subset \Sigma_t^{n+1}\mathsf{SH}^{\mathsf{eff}}(k) \subset \Sigma_t^{n}\mathsf{SH}^{\mathsf{eff}}(k) \subset \ldots \subset \mathsf{SH}(k)$$

$n \in \mathbb{Z}$.

Lemma *The inclusion functor* $i_n : \Sigma_t^n \mathsf{SH}^{\mathsf{eff}}(k) \to \mathsf{SH}(k)$ *admits an exact right adjoint* $r_n : \mathsf{SH}(k) \to \Sigma_t^n \mathsf{SH}^{\mathsf{eff}}(k)$*.*

This follows by Neeman's "Brown representability" theorem applied to the functor on $\Sigma_t^n \mathsf{SH}^{\mathsf{eff}}(k)$

$$F \mapsto \mathsf{Hom}_{\mathsf{SH}(k)}(i_n(F), E)$$

for each $E \in \mathsf{SH}(k)$.

Define $f_n := i_n r_n : \mathsf{SH}(k) \to \mathsf{SH}(k)$, giving the *motivic Postnikov tower*

$$\ldots \to f_{n+1}E \to f_nE \to \ldots \to E.$$

The layers

$$s_nE := \mathrm{cofib}(f_{n+1}E \to f_nE)$$

are Voevodsky's *slices*.

The Postnikov tower in $\mathsf{SH}_s(k)$

Just as one can form an unstable Postnikov tower in $\mathcal{H}_*$, we have the "semi-stable" motivic Postnikov tower in $\mathsf{SH}_s(k)$.

Take the tower of full triangulated subcategories

$$\ldots \subset \Sigma_t^{n+1}\mathsf{SH}_s(k) \subset \Sigma_t^n\mathsf{SH}_s(k) \subset \ldots \subset \Sigma_t\mathsf{SH}_s(k) \subset \mathsf{SH}_s(k)$$

The inclusions $i_{n,s} : \Sigma_t^n\mathsf{SH}_s(k) \to \mathsf{SH}_s(k)$ have a right adjoint $r_{n,s} : \mathsf{SH}_s(k) \to \Sigma_t^n\mathsf{SH}_s(k)$, giving us the truncation functors

$$f_{n,s} : \mathsf{SH}_s(k) \to \mathsf{SH}_s(k),$$

and for $E \in \mathsf{SH}_s(k)$, the S^1*-motivic Postnikov tower*

$$\ldots \to f_{n+1,s}E \to f_{n,s}E \to \ldots \to f_{1,s}E \to E.$$

Let $s_{n,s}E$ be the cofiber of $f_{n+1,s}E \to f_{n,s}E$.

The homotopy coniveau tower

This construction, based on the Bloch-Lichtenbaum construction of the spectral sequence for K-theory, gives an algebraic version of the (co)skeletal filtration of a CW complex.

Notation:

$\Delta^n := \mathrm{Spec}\, k[t_0, \ldots, t_n]/\sum_i t_i - 1$
A *face* F of Δ^n is a closed subscheme defined by
$t_{i_1} = \ldots = t_{i_r} = 0$.

$n \mapsto \Delta^n$ extends to the cosimplicial scheme

$$\Delta^* : \mathbf{Ord} \to \mathbf{Sm}/k.$$

For $E \in \mathbf{Spt}(k)$, $X \in \mathbf{Sm}/k$, $W \subset X$ closed, set

$$E^W(X) := \mathrm{fib}(E(X) \to E(X \setminus W)).$$

• For $X \in \mathbf{Sm}/k$:

$$\mathcal{S}_X^{(p)}(n) := \{W \subset X \times \Delta^n, \text{closed, } \mathrm{codim}_{X\times F} W \cap (X \times F) \geq p\}.$$

• For $E \in \mathbf{Spt}(k)$:

$$E^{(p)}(X,n) := \operatorname*{hocolim}_{W \in \mathcal{S}_X^{(p)}(n)} E^W(X \times \Delta^n).$$

• This gives the simplicial spectrum $E^{(p)}(X)$: $n \mapsto E^{(p)}(X,n)$, and the *homotopy coniveau tower*

$$\ldots \to E^{(p+1)}(X) \to E^{(p)}(X) \to \ldots \to E^{(0)}(X) = E^{(-1)}(X) = \ldots$$

Remark: $X \mapsto E^{(p)}(X)$ is functorial in X for *flat* maps.

Properties of the HC tower

Fix an $E \in \mathbf{Spt}(k)$. We will assume 2 basic properties hold for E:

1. *homotopy invariance:* For all $X \in \mathbf{Sm}/k$, $E(X) \to E(X \times \mathbb{A}^1)$ is a stable weak equivalence.

2. *Nisnevic excision:* Let $f : Y \to X$ be an étale map in $\mathbf{Sm}/k$. Suppose $W \subset X$ is a closed subset such that $f : f^{-1}(W) \to W$ is an isomorphism. Then $f^* : E^W(X) \to E^{f^{-1}(W)}(Y)$ is a stable weak equivalence.

We also assume that k is an infinite perfect field.

Theorem *Let* E *be in* $\mathbf{Spt}(k)$ *satisfying properties 1 and 2. Then*

(1) $X \mapsto E^{(p)}(X)$ *extends (up to weak equivalence) to a functor* $E^{(p)} : \mathbf{Sm}/k^{\mathrm{op}} \to \mathbf{Spt}$.

(2) Localization. *Let* $i : W \to X$ *be a closed codimension* d *closed embedding in* $\mathbf{Sm}/k$, *with trivialized normal bundle, and open complement* $j : U \to X$. *There is a natural homotopy fiber sequence in* SH

$$(\Omega_t^d E)^{(p-d)}(W) \to E^{(p)}(X) \xrightarrow{j^*} E^{(p)}(U)$$

(3) Delooping. *There is a natural weak equivalence*

$$(\Omega_t^m E)^{(n)} \xrightarrow{\sim} \Omega_t^m(E^{(n+m)})$$

(1) Functoriality: this is proven using Chow's moving lemma, just as for Bloch's cycle complexes.

(2) Localization: this is proven using Bloch's moving lemma (blowing up) just as for Bloch's cycle complexes.

(3) Delooping follows from the localization sequence:

$$(\Omega_t E)^{(n)}(X \times 0) \to E^{(n+1)}(X \times \mathbb{P}^1) \to E^{(n+1)}(X \times \mathbb{A}^1)$$

and the natural weak equivalence

$$\mathrm{fib}(F(X \times \mathbb{P}^1) \to F(X \times \mathbb{A}^1)) \cong (\Omega_t F)(X).$$

For $q \geq p$, set $E^{(p/q)}(X) :=$ cofib$(E^{(q)}(X) \to E^{(p)}(X))$.

Corollary (Birationality) *Take* $E \in \mathbf{Spt}(k)$, $X \in \mathbf{Sm}/k$, *Then*

$$E^{(0/1)}(X) \cong E^{(0/1)}(k(X)).$$

Proof: Take $W \subset X$ smooth with trivial normal bundle, codim $d > 0$. Let $F = \Omega_t^d E$, $U = X \setminus W$. Localization $\Longrightarrow$ we have a fiber sequence

$$F^{(0-d/1-d)}(W) \to E^{(0/1)}(X) \to E^{(0/1)}(U)$$

But $1 - d \leq 0$, so $F^{(-d)}(W) = F^{(1-d)}(W) = F(W)$ and thus $F^{(0-d/1-d)}(W) \sim *$.

For general W, the same follows by stratifying.

The comparison theorem

Theorem *(1) For* E *satisfying (1) and (2),* $E^{(n)}$ *is in* $\Sigma_T^n \mathsf{SH}_s(k)$.

(2) The map $E^{(n)} \to f_{n,s}E$ *induced by* $E^{(n)} \to E$ *is an isomorphism*

The motivic Postnikov tower is just a homotopy invariant version of the coniveau filtration.

The delooping identity $\Omega_t(E^{(n+1)}) \cong (\Omega_t E)^{(n)}$ gives

Corollary $\Omega_t \circ f_{n+1,s} \cong f_{n,s} \circ \Omega_t$.

This yields the motivic Freudenthal suspension theorem:

Theorem $E \in \Sigma_t^n \mathsf{SH}_s(k) \Longrightarrow \Omega_t \Sigma_t E \in \Sigma_t^n \mathsf{SH}_s(k)$

This allows one to use the semi-stable Postnikov tower to compute the stable one via

Corollary $E \in \Sigma_t^n \mathsf{SH}_s(k) \Longrightarrow \Omega_t^\infty \Sigma_t^\infty E \in \Sigma_t^n \mathsf{SH}_s(k)$

The stable homotopy coniveau tower

Let

$$\mathcal{E} := (E_0, E_1, \ldots, E_n, \ldots)$$
$$\epsilon_n : E_n \to \Omega_T E_{n+1}$$

be an (s,t)-spectrum over k. We assume that the ϵ_n are weak equivalences.

For each n, m we have the weak equivalence $\epsilon_n^{<m>}$:

$$E_n^{(n+m)} \xrightarrow{(\epsilon_n)^{(n+m)}} (\Omega_T E_{n+1})^{(n+m)} \xrightarrow{\text{deloop}} \Omega_T(E_{n+1}^{(n+m+1)})$$

Set:

$$\mathcal{E}^{<m>} := (E_0^{(m)}, E_1^{(m+1)}, \ldots, E_n^{(m+n)}, \ldots)$$

with bonding maps $\epsilon_n^{<m>}$.

The homotopy coniveau towers

$$\ldots \to E_n^{(m+n+1)} \to E_n^{(m+n)} \to \ldots$$

fit together to form the T-*stable homotopy coniveau tower*

$$\ldots \to \mathcal{E}{<}m+1{>} \to \mathcal{E}{<}m{>} \to \ldots \to \mathcal{E}{<}0{>} \to \mathcal{E}{<}-1{>} \to \ldots \to \mathcal{E}.$$

in $\mathsf{SH}(k)$.

The stable comparison theorem

Theorem *(1) For* $\mathcal{E} \in \mathsf{SH}(k)$, $\mathcal{E}{<}n{>}$ *is in* $\Sigma_t^n \mathsf{SH}^{\mathsf{eff}}(k)$.

(2) For each $\mathcal{E} \in \mathsf{SH}(k)$, *the canonical map* $h : \mathcal{E}{<}n{>} \to f_n\mathcal{E}$ *is an isomorphism.*

These results follow easily from the S^1 results.

Some results

1. $s_0(S_k) = \mathcal{H}\mathbb{Z}$ (a theorem of Voevodsky), $S_k := \Sigma_t^\infty \mathrm{Spec}\, k_+$.

2. $s_n(\mathcal{K}) = \Sigma_t^n(\mathcal{H}\mathbb{Z})$. This yields the Atiyah-Hirzebruch spectral sequence for K-theory:

$$E_2^{p,q} := H^{p-q}(X, \mathbb{Z}(-q)) \Longrightarrow K_{-p-q}(X)$$

This is the same one as constructed by Bloch-Lichtenbaum (for fields) and extended to arbitrary X by Friedlander-Suslin.

3. The layers s_nE are all *motives*: There is an equivalence of categories (Østvær-Röndigs)

$$EM : DM(k) \to \mathcal{H}\mathbb{Z}\text{-Mod}$$

Since each $E \in \mathsf{SH}(k)$ is an S_k-modules, s_nE is thus an $s_0(S_k) = \mathcal{H}\mathbb{Z}$-module.

In fact, there is a canonical *birational motive* $\pi_n^\mu(E)$ in $DM(k)$ with

$$\Sigma_t^n EM(\pi_n^\mu(E)) = EM(\pi_n^\mu(E)(n)[2n]) = s_nE.$$

A birational motive M (following Kahn-Sujatha) is one that is locally constant in the Zariski topology on $\mathbf{Sm}/k$: the restriction map from X to an open subscheme U induces an isomorphism

$$\mathrm{Hom}_{DM(k)}(M_{\mathrm{gm}}(X), M[i]) \to \mathrm{Hom}_{DM(k)}(M_{\mathrm{gm}}(U), M[i])$$

We can think of $\pi_n^\mu(E)$ as the *nth homotopy motive* of E.

4. The slice tower yields the *motivic Atiyah-Hirzebruch spectral sequence*

$$E_2^{p,q} := H^{p-q}(X, \pi^{\mu}_{-q}(E)(-q)) \Longrightarrow E^{p+q}(X)$$

Here

$$H^p(X, \pi^{\mu}_{-q}(E)(-q)) := \mathrm{Hom}_{DM(k)}(M(X), \pi^{\mu}_{-q}(E)(-q)[p-q]).$$

The change in cohomological index comes from the shift $[-2q]$ rather than $[-q]$ in the topological version.

Computations and examples

The birational homotopy motives

For presheaf of spectra E, we have the birational motive $\pi_n^\mu(E)$ and the identity

$$s_n(E) = EM(\pi_n^\mu(E)(n)[2n]).$$

This allows us to decribed $s_n(E)$ as a "generalized cycle complex".

Let $X^{(n)}(m)$ be the set of points $w \in X \times \Delta^m$ with closure $\bar{w}$ in good position.

Theorem *Take $E \in \mathbf{Spt}_{S^1}(k)$ satisfying properties 1 and 2 and take $X \in \mathbf{Sm}/k$. Then*

1. $\pi_n^\mu(E)(X) = s_0(\Omega_t^n E)(X) \cong (\Omega_t^n E)^{(0/1)}(k(X))$

2. There is a simplicial spectrum $E_{s.l.}^{(n)}(X)$, with

$$E_{s.l.}^{(n)}(X)(m) \cong \bigoplus_{w \in X^{(n)}(m)} s_0(\Omega_t^n E)(w)$$

and with $s_n E(X)$ is isomorphic in SH *to $E_{s.l.}^{(n)}(X)$.*

The homotopy groups $\pi_m(s_n E(X))$ of $s_n E(X)$ are the higher Chow groups of X with coefficients $\pi_n^\mu(E)$.

The semi-local Δ

For a field F, let $\Delta^n_{F,0} = \mathrm{Spec}\,(\mathcal{O}_{\Delta^n_F,v})$, the "semi-local" n simplex.

It follows directly from the comparison theorem that the coefficient motive $\pi^\mu_n(E)$ is given by

$$\pi^\mu_n(E)(X) \cong (\Omega^n_t E)^{(0/1)}(k(X)) = (\Omega^n_t E)(\Delta^*_{k(X),0})$$

The nth homotopy motive of E is $\Omega^n_t E$ made $k(t)$-homotopy invariant.

Some examples

(1) One can calculate $s_n K(X)$ directly using these results. It is not hard to see that

$$(\Omega_t^n K)^{(0/1)}(w) = K^{(0/1)}(w) = EM(K_0(k(w))) = EM(\mathbb{Z}),$$

so we get $K_{s.l.}^{(n)}(X) = z^n(X,*)$. In terms of the homotopy motives, this gives

$$\pi_n^{\mu}(K) = \mathbb{Z}$$

just like for topological K-theory.

(2) The coefficient spectrum $s_0(\Omega_t^n E)$ has been computed explicitly for some other E, for example $E = K_{\mathcal{A}}$, $K_{\mathcal{A}}(X) := K(X; \mathcal{A})$, for $\mathcal{A}$ a c.s.a. over k (w. Bruno Kahn). We get

$$(\Omega_t^n K_{\mathcal{A}})^{(0/1)}(w) = K_{\mathcal{A}}^{(0/1)}(w) = EM(K_0(k(w) \otimes_k \mathcal{A})).$$

In terms of motives, this gives

$$\pi_n^{\mu}(K_{\mathcal{A}}) = \mathbb{Z}_{\mathcal{A}}$$

where $\mathbb{Z}_{\mathcal{A}}$ is the birational homotopy invariant presheaf with value $K_0(k(X) \otimes_k \mathcal{A})$ on X.

(3) (with C.Serpé) Let a finite group G act on a (smooth) k-scheme X. Consider the presheaf $K_{G,X}$

$$K_{G,X}(Y) := K(G, X \times Y)$$

the K-theory of the category of G-bundles over the $G \times \mathrm{id}$ action on $X \times Y$. Then (for $W \subset X \times Y$)

$$(\Omega_t^n K_{G;X})^{(0/1)}(w) = K_{G;X}^{(0/1)}(w) = EM(K_0(k(w)^{tw}[G])),$$

with $k(w)^{tw}[G]$ the twisted group ring. We denote this motive by $R_{G;X}$. This gives

$$\pi_n^{\mu} K_{G,X} = R_{G,X}$$

All three examples give strongly convergent A-H spectral sequences.

We concentrate on the example $K_{\mathcal{A}}$:

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}_{\mathcal{A}}(-q)) \Longrightarrow K_{-p-q}(X, \mathcal{A}).$$

So: $K_0(\mathcal{A}) = H^0(k, \mathbb{Z}_{\mathcal{A}})$, $K_1(\mathcal{A}) = H^1(k, \mathbb{Z}_{\mathcal{A}}(1))$.

For $X = \operatorname{Spec} k$, and $\deg \mathcal{A} = p$ prime, $H^n(k, \mathbb{Z}_{\mathcal{A}}(1)) = 0$ for $n \neq 1$, so

$$K_2(\mathcal{A}) = H^2(k, \mathbb{Z}_{\mathcal{A}}(2)),$$

and we have an exact sequence

$$0 \to H^1(k, \mathbb{Z}_s A(2) \to K_3(\mathcal{A}) \to H^3(k, \mathbb{Z}_{\mathcal{A}}(3)) \to 0.$$

The inclusion $\mathbb{Z}_{\mathcal{A}} \to \mathbb{Z}$ induces the *reduced norm*

$$H^p(k, \mathbb{Z}_{\mathcal{A}}(q)) \to H^p(k, \mathbb{Z}(q))$$

which is the usual reduced norm on K-theory for $(p, q) = (0, 0), (1, 1), (2, 2)$.

Oriented higher Chow groups?

One can apply this machinery to hermitian K-theory/Grothendieck-Witt theory. It's not clear what one gets.

Questions: What is the "coefficient spectrum" $(\Omega_p^t GW)(\Delta^*_{k(X),0})$? Is it an Eilenberg-Maclane spectrum? Is

$$\widetilde{CH}^p(X) = H^{2p}(X, \pi_p^\mu(GW)(p))?$$

For a field F, is

$$J^p(F) = H^p(F, \pi_p^\mu(GW)(p))?$$

The Postnikov tower for motives

One defines the motivic Postnikov tower inside $DM(k)$ or $DM^{\text{eff}}(k)$ directly by using

$$\ldots \subset DM^{\text{eff}}(k)(n{+}1) \subset DM^{\text{eff}}(k)(n) \subset \ldots \subset DM^{\text{eff}}(k) \subset \ldots \subset DM(k)$$

The cancellation theorem gives a simple formula for $f_n = f_{n,s}$ (for $E \in DM^{\text{eff}}(k)$):

$$f_n M = \mathcal{H}om_{DM^{\text{eff}}}(\mathbb{Z}(n), M)(n)$$

(Kahn).

The homotopy coniveau approach also works.

The slices for $M(X)$

Since $DM^{\mathrm{eff}}(k)$ is a category of complexes of sheaves on $\mathbf{Sm}/k$, we have the cohomology sheaves $\mathcal{H}^m$ of a motive. Recall:

$$\begin{aligned}\pi_n^{\mu}(M) &:= s_n(M)(-n)[-2n] \\ &= \mathrm{cofib}[\mathcal{H}om_{DM^{\mathrm{eff}}}(\mathbb{Z}(n+1)[2n], M)(1) \\ &\qquad \xrightarrow{ev} \mathcal{H}om_{DM^{\mathrm{eff}}}(\mathbb{Z}(n)[2n], M)]\end{aligned}$$

For X projective over k, we have the birational sheaf $\underline{CH}_r(X)$

$$\underline{CH}_r(X)(Y) := \mathrm{CH}_r(X_{k(Y)}).$$

Proposition (Huber-Kahn-Sujatha) *Let X be smooth projective over k.*

1. *For* $0 \leq n \leq \dim X$, $\mathcal{H}^m(\pi_n^\mu(M(X))) = 0$ *for* $m > 0$ *and*

$$\mathcal{H}^0(\pi_n^\mu(M(X))) = \underline{CH}_n(X).$$

2. *For* $n > \dim X$, $f_n(M(X)) = 0$.

Note: In general, $\mathcal{H}^m(\pi_n^\mu(M(X))) \neq 0$ for $m < 0$. But

$$\pi_n^\mu(M(\mathbb{P}^N)) = \mathbb{Z}$$

for $0 \leq n \leq N$.

Theorem (Kahn-L.) *Let* $X = SB(\mathcal{A})$*, deg*$(\mathcal{A}) = p$*. Then*

$$\pi_n^\mu(M(X)) = \mathbb{Z}_{\mathcal{A}^{\otimes d-n}} = \underline{CH}_n(X)$$

$0 \leq n \leq d = p - 1$*.*

Sketch of proof: For E a (fibrant) presheaf of spectra, we have the presheaf $R\mathcal{H}om(X, E)$:

$$R\mathcal{H}om(X, E)(Y) := E(X \times Y)$$

One shows: $s_0 R\mathcal{H}om(X, f_m E) \sim *$ for $m > \dim X$. Applying $s_0 R\mathcal{H}om(X, -)$ to the Postnikov tower for E

$$\ldots \to f_{m+1}E \to f_m E \to \ldots \to E$$

gives the finite tower

$$s_0 R\mathcal{H}om(X, f_d E) \to \ldots \to s_0 R\mathcal{H}om(X, E)$$

with layers $s_0 R\mathcal{H}om(X, s_n E)$, $n = 0, \ldots d$.

Evaluating at some $Y \in \mathbf{Sm}/k$, we have the strongly convergent spectral sequence

$$E^1_{a,b} = \pi_{a+b} s_0 R\mathcal{H}om(X, s_a E)(Y) \Longrightarrow \pi_{a+b} s_0 R\mathcal{H}om(X, E)(Y). \tag{*}$$

By Quillen's computation of the K-theory of SB varieties, we have (for $X = SB(\mathcal{A})$)

$$R\mathcal{H}om(X, K) = \oplus_{i=0}^{d} K_{\mathcal{A}^{\otimes i}}.$$

For $E = K$, Adams operations act on (*): it degenerates at E_1 giving

$$\begin{aligned} \oplus_{i=0}^{d} \mathbb{Z}_{\mathcal{A}^{\otimes i}} &= s_0(\oplus_{i=0}^{d} K_{\mathcal{A}^{\otimes i}}) \\ &= s_0 R\mathcal{H}om(X, K) \\ &= \oplus_{i=0}^{d} s_0 R\mathcal{H}om(X, s_a K) \end{aligned}$$

By our computations of the slices of K-theory, we have ($a \le d$)

$$\begin{aligned} R\mathcal{H}om(X, s_a K) &= R\mathcal{H}om(X, EM(\mathbb{Z}(a)[2a])) \\ &= \mathrm{Hom}_{DM^{\mathrm{eff}}}(M(X), \mathbb{Z}(a)[2a]) \\ &= \mathrm{Hom}_{DM^{\mathrm{eff}}}(M(X)(d-a)[2d-2a], \mathbb{Z}(d)[2d]) \\ &= \mathrm{Hom}_{DM^{\mathrm{eff}}}(\mathbb{Z}(d-a)[2d-2a], M(X)) \\ &= f_{d-a}(M(X))(a-d)[2a-2d] \end{aligned}$$

Taking s_0 gives

$$s_0 R\mathcal{H}om(X, s_a K) \cong \pi^{\mu}_{d-a}(M(X))$$

so

$$\oplus_{a=0}^{d} \pi^{\mu}_{d-a}(M(X)) \cong \oplus_{i=0}^{d} \mathbb{Z}_{\mathcal{A}^{\otimes i}}$$

hence

$$\mathcal{H}^m(\pi^{\mu}_{d-a}(M(X))) = 0 \text{ for } m \neq 0$$

The rest is bookkeeping.

Corollary *Let $\mathcal{A}$ be a c.s.a over k of prime rank. Then*

$$Nrd : K_2(\mathcal{A}) \to K_2(k)$$

is injective. (Assume BK in weight 3)

Sketch of proof:

$$K_2(\mathcal{A}) = H^2(k, \mathbb{Z}_{\mathcal{A}}(2)) = \mathrm{Hom}_{DM}(\mathbb{Z}, \mathbb{Z}_{\mathcal{A}}(2)[2]).$$

Let $X = SB(\mathcal{A})$. The Postnikov tower

$$f_d M(X) \to \ldots \to f_1 M(X) \to M(X)$$

has layers $s_{d-a} M(X) = \mathbb{Z}_{\mathcal{A}^{\otimes a}}(a)[2a]$. Applying $\mathrm{Hom}_{DM}(\mathbb{Z}, -)$ to $M(X)(3-d)[4-2d]$ gives

$$\mathrm{Hom}_{DM}(\mathbb{Z}, \mathbb{Z}_{\mathcal{A}}(2)[2]) = \mathrm{Hom}_{DM}(\mathbb{Z}, M(X)(3-d)[4-2d])$$

Using duality, this gives

$$K_2(\mathcal{A}) = \mathrm{Hom}_{DM}(M(X), \mathbb{Z}(3)[4]) = H^4(X, \mathbb{Z}(3)).$$

By Beilinson-Lichtenbaum, we have

$$H^4(X, \mathbb{Z}(3)) = H^4_{\text{ét}}(X, \mathbb{Z}(3)) = \mathrm{Hom}_{DM}(M(X)_{\text{ét}}, \mathbb{Z}(3)_{\text{ét}}[4])$$

But $M(X)_{\text{ét}}$ has slices $(\mathbb{Z}_{\mathcal{A}^{\otimes i}}(i)[2i])_{\text{ét}} = \mathbb{Z}(i)_{\text{ét}}[2i]$ and the spectral sequence for the Postnikov tower of $M(X)_{et}$ gives

$$0 \to H^4_{\text{ét}}(X, \mathbb{Z}(3)) \to H^2_{\text{ét}}(k, \mathbb{Z}(2)) \to H^5_{\text{ét}}(k, \mathbb{Z}(3))$$

By Beilinson-Lichtenbaum again,

$$H^2_{\text{ét}}(k, \mathbb{Z}(2)) = H^2(k, \mathbb{Z}(2)) = K_2(k)$$

giving

$$0 \to K_2(\mathcal{A}) \to K_2(k) \to H^5_{\text{ét}}(k, \mathbb{Z}(3))$$

Singular cohomology

Ayoub pointed out that $H^*_{\text{ét}}(-,\mathbb{Z}/n)$ has *all* slices 0 (for $k \supset \mu_n$):

$$H^*_{\text{ét}}(-,\mathbb{Z}/n) = \lim_{n\to\infty} H^*(-,\mathbb{Z}/n(q))$$

so is effective and equal to its own Tate twist.

The same is not true for $H^*_{sing}(-,\mathbb{Z})$ (for $k = \mathbb{C}$): using Hodge theory one can show that the 0th slice is non-zero when evaluated at e.g. an elliptic curve.

Probably this is also true for $H^*_{\text{ét}}(-,\mathbb{Z}_\ell)$ or $H^*_{\text{ét}}(-,\mathbb{Q}_\ell)$?

Questions:

1. How can one describe the cohomology theories $f_n H_{\text{sing}}(-, \mathbb{Z})$, $s_n H_{\text{sing}}(-, \mathbb{Z})$? Do these have something to do with cycles mod algebraic equivalence (via Bloch's formula)?

2. What is the relation with the coniveau filtration on $H_{sing}(-, \mathbb{Z})$?

3. What about the generalized Hodge conjecture (cf. work of Huber)?

Thank you,

and

Happy Birthday, Spencer!