Dedicated to Günter Harder, with gratitude for his beautiful mathematics

THREE LECTURES ON MOTIVIC COHOMOLOGY

These are notes from my short course giving an introduction to the parallel theories of motivic cohomology furnished by Bloch's higher Chow groups and Voevodsky's motivic cohomology, as part of the "Conference on Motives and Automorphic Forms in Honour of Günter Harder's 85th Birthday". In addition to describing these constructions of motivic cohomology and some of their categorical homes, we will also discuss relations of motivic cohomology to algebraic K-theory, and to étale cohomology, as well as extensions of the theory over arbitrary base-schemes.

All of this material has been covered in much greater detail in the literature; some references can be found in the bibliography to this article. Our aim here is to give the non-expert an introduction to some of the main ideas that have propelled the development and applications of motivic cohomology over the past thirty years.

Lecture 1: An introduction to higher Chow groups and triangulated categories of motives

In the first lecture, we will introduce Bloch's cycle complex, and Bloch's higher Chow groups. After presenting the construction, we will briefly describe the basic properties of the higher Chow groups as a Bloch-Ogus twisted duality theory: functoriality, homotopy invariance, projective bundle formula, and the localization sequence. We conclude with an introduction to the connection of the higher Chow group with algebraic K-theory via the Chern character and the Bloch-Lichtenbaum spectral sequence.

In the second part of this lecture, we will introduces some categories of motives. We start with Grothendieck's category of Chow motives for smooth projective varieties. Next, we discuss Voevodsky's triangulated category of geometric motives and the resulting theory of motivic cohomology.

Lecture 2: Motivic cohomology and triangulated categories of motives

We discuss Voevodsky's sheaf-theoretic version of motives, the triangulated category of effective motives. We describe Voevodsky's embedding theorem and the categorical description of Suslin homology. We discuss the comparison theorem identifying motivic cohomology with Bloch's higher Chow groups, and the relation of mod n-motivic cohomology with étale cohomology.

We use the comparison theorem to embed the category of Chow motives in the category of geometric motives, and briefly describe realization functors associated to de Rham cohomology and Betti cohomology.

Lecture 3: Applications and perspectives

We introduce the construction by Morel-Voevodsky [70] of $\mathbb{A}^1$-homotopy theory and Voevodsky's construction of the stable version [103], and briefly describe how Voevodsky's triangulated category of motives fits into this picture via the motivic Eilenberg-MacLane spectrum. We discuss two applications of $\mathbb{A}^1$-homotopy theory to motivic cohomology: the slice spectral sequence and the solution of the Bloch-Kato conjectures, concentrating on the case for the prime 2 (the Milnor conjecture).

The second part of this lecture is devoted to extensions of the theory to more general base-schemes. This includes the Déglise-Cisinski category of Beilinson motives, and its use by Spitzweck in constructing a motivic cohomology spectrum over an arbitrary base. We conclude with a description of Hoyois' construction of this motivic cohomology spectrum, which relies on the theory of framed correspondences.

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Contents

0. History and background

We can start the story with Grothendieck's idea of motives of smooth projective varieties as giving a framework for the universal Weil cohomology for smooth projective varieties over an algebraically closed field. This has never been carried out completely, although the construction of motives for an adequate equivalence relation makes perfectly good sense and is very much in use today.

Let me take this opportunity to recall a bit of the early history (and mystery?) of Grothendieck's constructions, without any attempt at completeness. The article of Demazure [20] gives a detailed construction of motives for numerical equivalence, and refers to a 1967 lecture of Grothendieck at the IHES, which never made it into publication, as the original source of this construction, although a 1964 letter from Grothendieck to Serre (reproduced in an appendix in [84]) does use the term "motif" and outlines some of what Grothendieck hoped from such a theory. Around the same time as Grothendieck"s lecture, we have Kleiman's IHES lectures on algebraic cycles and the Weil conjectures [55]. Here Kleiman discusses algebraic cycles, correspondences and various equivalence relations on algebraic cycles, as well as Grothendiecks's standard conjectures. However, beyonds stating that these conjectures "... imply the Weil conjectures and they are basic to Grothendieck's theories of motives ...", the theory of motives itself is not introduced. Shortly thereafter, Grothendieck's two articles [36, 37] appear. The first of these, dealing with his standard conjectures, mentions that these have already been introduced three years previously, by himself and independently by Bombieri. Although the conjectures are all about algebraic cycles and their relation to cohomology, there is no mention of a category of motives until the concluding paragraph:

'**Conclusions**. The proof of the two standard conjectures would yield results going considerably further than Weil's conjectures. They would form the basis of the so-called "theory of motives" which is a systematic theory of "arithmetic properties" of algebraic varieties, as embodied in their groups of classes of cycles for numerical equivalence. We have at present only a very small part of this theory in dimension one, as contained in the theory of abelian varieties.'

The second paper, dealing with the generalized Hodge conjecture, only mentions motives tangentially, giving 'For the reader informed about the yoga of "motives" ... ' a motivic version of the "corrected" generalized Hodge conjecture.

Between Grothendieck's IHES lecture and Demazure's article, Manin [61] used Grothendieck's ideas to give a "motivic" proof of the rationality of the zeta function of a smooth cubic threefold over a finite field. Manin mentions that, assuming the standard conjectures, the category of Grothendieck motives for numerical equivalence would be semi-abelian and the functor sending a smooth projective variety to its corresponding object in Grothendieck motives would form the universal Weil cohomology theory, calling this theory *motivic cohomology.* That Grothendieck motives for numerical equivalence is in fact semi-abelian, without assuming any supplementary conjectures, was proven much later, and to the great surprise of many, by Jannsen [49] (published in 1992).

In both Manin's and Demazure's treatments, realization functors to Weil cohomology theories were discussed, and this structure forms the main thrust in Grothendieck's 1964 letter to Serre. In Manin's treatment it was silently assumed that numerical equivalence implies homological equivalence (for the given Weil cohomology theory), while in Demazure this is stated as an axiom, even though this was already recognized as an open question that had an affirmative answer assuming the standard conjectures. In any case, the emphasis in all of these works was on zeta functions and the Weil conjectures, and very little was said about Chow motives, i.e., motives for rational equivalence. For a more detailed discussion of the history and development of Grothendieck's theory of motives, we refer to Serre's article [84].

The next step was the introduction of what would now be known as "oriented cohomology with additive formal group law" but really goes under the name of Bloch-Ogus twisted duality theory [15]. This consists of assigning a bi-graded ring $X \mapsto \oplus H^a(X, \Lambda(b))$ for X smooth over k, together with Gysin isomorphisms $H^a_Z(X, \Lambda(b)) \cong H^{a-2c}(Z, \Lambda(b-c))$ for smooth closed codimension $Z \subset X$ (plus other stuff, like 1st Chern of line bundles ...). The need for the second grading (twist) arose from concrete examples, like ℓ-adic étale cohomology over a non-algebraically closed field. Beilinson [7] introduced Adams graded algebraic K-theory as a Bloch-Ogus theory with $\mathbb{Q}$-coefficients and used Gillet's theory of Chern classes [35] that this forms the universal theory with $\mathbb{Q}$-coefficients. and the search was soon on for the universal integral theory.

Beilinson [9] (see also Lichtenbaum's discussion [60, §5]) and Lichtenbaum [60, §3] produced axioms for sheaves of complexes $X \mapsto \Gamma_X(q)$ that would produce the universal integral theory by taking hypercohomology; Beilinson used Zariski sheaves, while Lichtenbaum used étale sheaves. The axioms include a close relation with algebraic K-theory (enhanced to an Atiyah-Hirzebruch type spectral sequence giving an integral relation), a connection with the classical Chow ring and with Milnor K-theory of fields, as well as the identification of the mod n theory with a truncated version of étale cohomology.

Another axiom, the so-called Beilinson-Soulé vanishing conjecture[1], may have inspired Beilinson to reframe this conjectural universal theory as arising as Ext-groups in an abelian category of "motivic sheaves" over each scheme X, with a six functor formalism on the derived category [8, §0.3]. Indeed, if one does have such an abelian category MM_X on each X, with "Tate motivic sheaves" $\mathbb{Z}_X(q)$, then the complexes of Zariski sheaves $\Gamma_X(q)$ conjectured to exist in Beilinson's list of axioms would be given as

$$\Gamma_X(q) := R\mathcal{H}om_{MM_X}(\mathbb{Z}_X(0), \mathbb{Z}_X(q)),$$

where $\mathcal{H}om_{MM_X}(-,-)$ is the $\mathrm{Sh}^{\mathbf{Ab}}(X_{\mathrm{Zar}})$-valued mapping sheaf functor. The motivic cohomology would then be the Ext groups

$$H^p(X, \mathbb{Z}(q)) := \mathrm{Ext}^p_{MM_X}(\mathbb{Z}_X(0), \mathbb{Z}_X(q)),$$

which would vanish for $p < 0$. The vanishing for $q > 0$ and $p = 0$ would arise from a suitable theory of weights, analogous to the situation for variations of mixed Hodge structures. Following this framework, the universal Bloch-Ogus cohomology theory became rechristened as "motivic cohomology" in [8, §0.2] and [10, §5.10] (this superseded Manin's use of this term for the theory given by Grothendieck's motives for numerical equivalence, mentioned above).

This vision has not been completely realized, but nearly so. The first construction of complexes that (partly) satisfied the Beilinson-Lichtenbaum axioms was given by

[1]Soulé framed this conjecture in terms of the γ-filtration on rational K-theory [85, §2.9 Conjecture], Beilinson [7, §2.2.2] in terms of Adams-graded rational K-theory. For motivic cohomology, these say that the rational motivic cohomology $H^p(X, \mathbb{Q}(q))$ should vanish for $p < 0$ and $q > 0$. Soulé's version is for an arbitrary affine scheme, while Beilinson's version appearing in *loc. cit.* is for a regular scheme X in the same range. The version encoded in the Beilinson-Lichtenbaum conjectures [9], [60, §3] is the vanishing of integral motivic cohomology $H^p(X, \mathbb{Z}(q))$ in the range $p \leq 0$, $q > 0$.

Bloch, with his cycle complex and higher Chow groups [11]. There were a number of proposals for a categorical framework (see for instance [40, 41, 42]), but Voevodsky (together with Friedlander and Suslin) [104] came up with the most successful version, a triangulated category that has the "feel" of the derived category of the conjectural category of motivic sheaves over a field. Remarkably, the motivic cohomology that arises out of this categorical construction agrees with Bloch's higher Chow groups. Work of others (Cisinski-Déglise [18, 19], Ayoub [3, 4], Röndigs-Østvær [76], Spitzweck [86], Hoyois [45], ...) has extended Voevodsky's triangulated category to a very satisfying theory (actually several different constructions that yield more or less the same theory) over arbitrary base-schemes.

The lack of a Beilinson-Soulé vanishing theorem is at least partly responsible for having all these constructions at the triangulated level, rather than admitting the framework of an abelian category of mixed motivic sheaves following Beilinson's vision. To recover such an abelian category from the triangulated version, one would need a "motivic t-structure", but Voevodsky has shown that, for his triangulated category of effective geometric motives over a given field k, a "reasonable" t-structure does not in general exist. He gives an obstruction involving a conic over k without k-points (see [104, Chap. 5, Proposition 4.3.8]). His obstruction does however vanish if one passes to the category with $\mathbb{Q}$-coefficients, or if k is algebraically closed, so perhaps not all hope is lost.

In these three lectures, I will begin in the first lecture with a description of Bloch's cycle complex and the first of Voevodsky's categorical constructions. This latter has a very straightforward definition, but has the disadvantage that the motivic cohomology resulting from it is nearly impossible to compute or even relate to other theories, such as the classical Chow groups.

In the second lecture, I will discuss Voevodsky's great innovation, which was to put this rather naive theory into a sheaf-theoretic context and, combining fairly classical constructions with the cohomological methods made available by the use of sheaves and the derived category, was able to realize the categorically-defined motivic cohomology as the hypercohomology of a sheaf of complexes, just as envisaged by Beilinson and Lichtenbaum. Moreover, a similar combination of geometry and sheaf theory allowed him (with Friedlander and Suslin) to connect the motivic cohomology with Bloch's higher Chow groups, and to embed Grothendieck's category of Chow motives (motives for rational equivalence) as a full subcategory of the triangulated category of motives.

In the third lecture, I will look at the interplay of the categories of motives with a finer theory: motivic homotopy theory. This will clarify the relation of motivic cohomology with algebraic K-theory, as well as providing many of the tools used by Voevodsky and others to verify the Beilinson-Lichtenbaum conjectures describing the relation of mod n motivic cohomology with mod n étale cohomology. In addition, this gives a framework for expanding the theory over a field to one over an arbitrary base-scheme, using methods that are interesting in their own right.

Here is a partial table of the historical development of motivic cohomology and its categorical framework:

Contributor	Cohomology/ representing object	Category (sometimes conjectural)
Weil	Weil cohomology	
Grothendieck	universal Weil cohomology	motives for smooth projective varieties
Bloch, Ogus	twisted duality theories	
Beilinson, Lichtenbaum	universal cohomology, motivic complexes	
Beilinson	motivic cohomology	abelian categories of motivic sheaves
Bloch	Cycle complexes, higher Chow groups	
Suslin	algebraic singular complex, algebraic homology	
Voevodsky	motivic cohomology on $\mathbf{Sm}_k$	triangulated categories of motives over a field
Morel-Voevodsky, Voevodsky	generalized motivic cohomology	motivic homotopy categories
Röndigs-Østvær	motivic cohomology on $\mathbf{Sm}_k$, $H_{\mathrm{mot}}\mathbb{Z} \in \mathrm{SH}(k)$	modules over $H_{\mathrm{mot}}\mathbb{Z}$
Cisinski-Déglise	rational motivic cohomology over B, $H_B^{\mathbb{B}} \in \mathrm{SH}(B)$	Beilinson motives: modules over $H_B^{\mathbb{B}}$
Spitzweck	motivic cohomology over B, $M\mathbb{Z}_B \in \mathrm{SH}(B)$	modules over $M\mathbb{Z}_B$
Hoyois	motivic cohomology over B, $M\mathbb{Z}_B^{fr} \in \mathrm{SH}^{fr}(B)$	modules over $M\mathbb{Z}_B^{fr}$

1. Lecture 1: An introduction to higher Chow groups, motivic cohomology and the triangulated category of motives

1.1. **Higher Chow groups.**

1.1.1. *Motivation.* The idea behind Bloch's construction of his higher Chow groups [11] is to give a "resolution" of the classical Chow group of dimension q cycles on a variety (reduced, separated, finite-type k-scheme) X modulo rational equivalence, that encodes what one might call "higher rational equivalences".

We first recall some of the elementary structures available on algebraic cycles. We work in the category $\mathbf{Sch}_k$ of separated, finite-type k-schemes, and in the full subcategory $\mathbf{Sm}_k$ of smooth k-schemes. We call a reduced scheme $X \in \mathbf{Sch}_k$ a *k-variety*.

For $X \in \mathbf{Sch}_k$, we have $S_q(X)$, the set of integral, dimension q closed subschemes of X and $Z_q(X)$, the group of dimension q algebraic cycles, i.e., the free abelian group on $S_q(X)$:

$$Z_q(X) := \oplus_{W \in S_q(X)} \mathbb{Z} \cdot W$$

Here the *dimension* of an integral finite type k-scheme W means equivalently the Krull dimension, or the transcendence dimension over k of the function field $k(W)$. By passing to the generic point, we often identify $S_q(X)$ with the set $X_{(q)}$ of points $x \in X$ with closure $\bar{x}$ of dimension q.

Let $Z \subset X$ be a closed subscheme such that each irreducible component of Z has dimension q, and let $z_1, \ldots, z_r$ be the generic points of Z. We have the associated cycle $\mathrm{cyc}_X(Z) \in Z_q(X)$ defined by $\mathrm{cyc}_X(Z) = \sum_{i=1}^r m_i \bar{z}_i$, where $m_i > 0$ is the length of $\mathcal{O}_{Z,z_i}$ over $\mathcal{O}_{X,z_i}$.

For $f : Y \to X$ a proper map, there is an induced map $f_* : Z_q(Y) \to Z_q(Y)$, with

$$f_*(1 \cdot W) = \begin{cases} 0 & \text{if } \dim f(W) < \dim W \\ [k(W) : k(f(W))] \cdot f(W) & \text{if } \dim f(W) = \dim W. \end{cases}$$

We have $(fg)_* = f_* g_*$. There is an external product

$$\times : Z_q(X) \times Z_r(Y) \to Z_{q+r}(X \times_k Y)$$

defined on $W_1 \in S_q(X)$, $W_2 \in S_r(Y)$ by $W_1 \times W_2 := \mathrm{cyc}_{X \times_k Y}(W_1 \times_k W_2)$.

For $f : Y \to X$ a flat morphism, of relative dimension d, there is a well-defined functorial pullback $f^* : Z_q(X) \to Z_{q+d}(Y)$.

If $D \subset X$ is an effective Cartier divisor, let $Z_q(X)_D \subset Z_q(X)$ be the subgroup generated by integral W with W not contained in D. There is a well-defined intersection map

$$D \cdot (-) : Z_q(X)_D \to Z_{q-1}(D)$$

defined for integral $W \in Z_q(X)_D$ by

$$D \cdot W := \mathrm{cyc}_D(D \cap W),$$

where $D \cap W$ is the scheme-theoretic intersection, and then extending to $Z_q(X)_D$ by linearity.

More generally, if $W \in S^q(X), W' \in S^{q'}(X)$ are integral closed subschemes on some $X \in \mathbf{Sm}_k$, we say that W and W' *intersect properly* if each integral component W_i of $W \cap W'$ has codimension $q + q'$ on X. If this is the case, we have the well-defined cycle $W \cdot W' \in Z^{q+q'}(X)$ defined by

$$W \cdot W' = \sum_{i=1}^r m_i W_i$$

where $W_1, \ldots, W_r$ are the integral components of $W \cap W'$, and the positive integer m_i is given by Serre's intersection multiplicity formula:

$$m_i = \sum_{j=0}^{\dim_k X} (-1)^j \mathrm{lng}_{\mathcal{O}_{X,W_i}} \mathrm{Tor}_j^{\mathcal{O}_{X,W_i}}(\mathcal{O}_{W,W_i}, \mathcal{O}_{W',W_i})$$

The Chow group $\mathrm{CH}_q(X)$ is the quotient $Z_q(X)/R_q(X)$, where $R_q(X) \subset Z_q(X)$ is the subgroup of cycles rationally equivalent to zero. $R_q(X)$ has many equivalent definitions, but for us, let's use

$$R_q(X) := \{i_0^*(W) - i_1^*(W) \mid W \in Z_{q+1}(\mathbb{A}^1 \times X)_{0 \times X + 1 \times X}\}$$

Here $i_j^*(W) := p_{j*}(j \times X) \cdot W$, $j = 0, 1$, with $p_j : j \times X \to X$ the projection. In other words, we have the coequalizer sequence

$$Z_{q+1}(\mathbb{A}^1 \times X)_{0,1} \underset{i_0^*}{\overset{i_1^*}{\rightrightarrows}} Z_q(X) \to \mathrm{CH}_q(X) \tag{1.1}$$

The proper pushforward map descends to $f_* : \mathrm{CH}_q(Y) \to \mathrm{CH}_q(X)$ and the flat pullback map descends to $f^* : \mathrm{CH}_q(X) \to \mathrm{CH}_{q+d}(Y)$. If X is equi-dimensional over k of dimension d, we can index by codimension, with $S^q(X) := S_{d-q}(X)$, $X^{(q)} = X_{(d-q)}$, $Z^q(X) := Z_{d-q}(X)$, and similarly for $R^q(X)$ and $\mathrm{CH}^q(X)$.

For $f : Y \to X$ a morphism in $\mathbf{Sch}_k$ with $X \in \mathbf{Sm}_k$ and Y locally equi-dimensional over k, there is also a partially defined pullback map $f^* : Z^q_f(X) \to Z^q(Y)$, with $Z^q_f(X) \subset Z^q(X)$ the subgroup generated by integral closed $W \subset X$ such that each component of $f^{-1}(W)$ has codimension q on Y. Explicitly, we take the graph $\Gamma_f \subset Y \times X$, giving the cycle

$$\Gamma_f \cdot (Y \times W) \in Z^q(\Gamma_f)$$

and then use the projection $p_1 : \Gamma_f \xrightarrow{\sim} Y$ to transform this to the cycle $f^*(W) \in Z^q(Y)$. This descends to a well-defined pullback $f^* : \mathrm{CH}^q(X) \to \mathrm{CH}^q(Y)$, functorial on $\mathbf{Sm}_k$.

For $X \in \mathbf{Sm}_k$, external product and pullback by the diagonal $\Delta_X : X \times X \to X$ gives the graded group $\mathrm{CH}^*(X) := \oplus_q \mathrm{CH}^q(X)$ the structure of a commutative ring, with intersection product

$$\alpha \cdot \beta := \Delta_X^*(\alpha \times \beta),$$

and unit $1_X := 1 \cdot X$. If α and β are represented by cycles Z, Z' on X that intersect properly, then $\alpha \cdot \beta$ is the class in $\mathrm{CH}^*(X)$ represented by the well-defined cycle intersection $Z \cdot Z' \in Z^*(X)$. Moreover, for $f : Y \to X$ a morphism in $\mathbf{Sm}_k$, the pullback map f^* is a ring homomorphisms. Finally, one has the projection formula

$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta)$$

for $f : Y \to X$ a proper map in $\mathbf{Sm}_k$. See e.g. [28, Chap. 1] for more details on the basic properties of the cycle groups and the Chow groups and more of [28] for details on the pullback maps for arbitrary $f \in \mathbf{Sm}_k$.

If $p : E \to X$ is an affine space bundle (torsor for a vector bundle) with fiber dimension d, then $p^* : \mathrm{CH}_q(X) \to \mathrm{CH}_{q+d}(E)$ is an isomorphism for all $q \in \mathbb{Z}$. Ff E is in fact a vector bundle with zero section $s : X \to E$, one can define $s^* := (p^*)^{-1} : \mathrm{CH}_{q+d}(E) \to \mathrm{CH}_q(X)$, even if X is not smooth; this does agree with the pullback map s^* defined above if X is smooth.

If $p : L \to X$ is a line bundle with zero-section $s : X \to L$, we have the 1st Chern class operator

$$\tilde{c}_1(L) : \mathrm{CH}_q(X) \to \mathrm{CH}_{q-1}(X)$$

defined by $\tilde{c}_1(L)(\alpha) = s^*(s_*(\alpha))$. For $X \in \mathbf{Sm}_k$ we have $c_1(L) \in \mathrm{CH}^1(X)$ defined by $c_1(L) := \tilde{c}_1(L)(1_X)$ and in this case, $\tilde{c}_1(L)(\alpha) = c_1(L) \cdot \alpha$.

Let $p : V \to X$ be a rank $n + 1$ vector bundle, with associated projective space bundle $f : \mathbb{P}(V) \to X$ and tautological (quotient) line bundle $\mathcal{O}(1)$. Let $\alpha_i : \mathrm{CH}_q(X) \to \mathrm{CH}_{q+n-i}(\mathbb{P}(V))$ be the map

$$\alpha_i(\beta) := \tilde{c}_1(\mathcal{O}(1))^i(f^*(\beta))$$

Then

$$\sum_{i=0}^n \alpha_i : \oplus_{i=0}^n \mathrm{CH}_{q-n+i}(X) \to \mathrm{CH}_q(\mathbb{P}(V))$$

is an isomorphism. If X is smooth, this says that $\mathrm{CH}^*(\mathbb{P}(V))$ is a free $\mathrm{CH}^*(X)$-module (via f^*) with basis $\{c_1(\mathcal{O}(1))^i\}_{i=0,\dots,n}$. This is known as the *projective bundle formula.*

1.1.2. *Bloch's cycle complex and higher Chow groups.* Bloch's idea is to extend the co-equalizer sequence (1.1) defining $\mathrm{CH}_q(X)$ to the left, using cycles on algebraic versions of the topological n-simplices to give a complex whose higher homology describes the "higher relations" arising from the relation of rational equivalence of cycles on X.

Definition 1.1. 1. For $n \geq 0$ an integer let

$$\Delta^n_k \subset \mathbb{A}^{n+1}_k := \operatorname{Spec} k[t_0, \ldots, t_n]$$

be the affine hyperplane defined by $\sum_i t_i = 1$. A *codimension c face* of Δ^n_k is a subscheme $F \subset \Delta^n_k$ defined by equations of the form $t_{i_1} = \ldots = t_{i_c} = 0$, with $0 \leq i_1 < \ldots < i_c \leq n$, $c \leq n$.
2. Let X be in $\mathbf{Sch}_k$. For $n, q \geq 0$, let $z_q(X, n) \subset Z_{q+n}(\Delta^n_k \times X)$ be the subgroup freely generated by integral closed subschemes W such that, for each c and each codimension c face F of Δ^n_k, each irreducible component of $W \cap X \times F$ has dimension $q + n - c$.

Remarks 1.2. 1. Since for each codimension c face F of Δ^n, the closed subscheme $F \times X$ of $\Delta^n \times X$ is a complete intersection, it follows that for each integral closed subscheme W of $\Delta^n \times X$ of dimension $q+n$, each irreducible component of $W \cap X \times F$ has dimension $\geq q + n - c$. The property that each irreducible component of $W \cap X \times F$ has the minimal dimension $q + n - c$ is often stated as saying that W intersects $X \times F$ *properly.*
2. Letting $i_F : F \hookrightarrow \Delta^n_k$ be the inclusion, the fact that F is a complete intersection in Δ^n_k means we can still use Serre's intersection formula to define

$$(\mathrm{Id}_X \times i_F)^*(W) := (X \times F) \cdot W \in Z_{q+n-c}(X \times_k F)$$

for each $W \in z_q(X, n)$.

The collection of k-varieties $\{\Delta^n_k\}_{n \geq 0}$ forms a smooth, cosimplicial scheme over k as follows. Letting **Ord** be the category with objects the finite ordered sets $[n] := \{0, \ldots, n\}$ (with the standard order) and maps the order-preserving maps of sets, we have

$$\Delta_k : \mathbf{Ord} \to \mathbf{Sm}_k$$

by $\Delta_k([n]) = \Delta^n_k$ and for $g : [n] \to [m]$ an order-preserving map, the map

$$\Delta_k(g) : \Delta^n_k \to \Delta^m_k$$

is the affine-linear map

$$\Delta_k(g)(t_0, \ldots, t_n) = (\Delta_k(g)_0, \ldots, \Delta_k(g)_m); \quad \Delta_k(g)_j = \sum_{i \in g^{-1}(j)} t_i$$

(recall that the empty sum is defined to be 0).

Note that $\Delta_k(g)$ factors as a smooth map $p(g) : \Delta^n_k \to F$ followed by the inclusion $i_F : F \hookrightarrow \Delta^m_k$ for some face F, so we have a well-defined pullback

$$g^* := (\mathrm{Id}_X \times \Delta_k(g))^* := (\mathrm{Id}_X \times p(g))^* \circ (\mathrm{Id}_X \times i_F)^* : z_q(X, m) \to z_q(X, n)$$

This gives us the simplicial abelian group $[n] \mapsto z_q(X, n)$, with corresponding homological complex $(z_q(X, *), d)$; as usual, $d_n : z_q(X, n) \to z_q(X, n-1)$ is the map $\sum_{i=0}^n (-1)^i \delta_i^{n*}$, with $\delta^n_i : [n-1] \to [n]$ the unique injective order-preserving map with i not in the image.

Definition 1.3. Take X in $\mathbf{Sch}_k$. The complex $(z_q(X,*),d)$ is Bloch's *dimension q cycle complex* and the homology is Bloch's *higher Chow group*

$$\mathrm{CH}_q(X,n) := H_n((z_q(X,*),d)).$$

If X is equi-dimensional over k of dimension d, we may index by codimension, defining $(z^q(X,*),d) := (z_{d-q}(X,*),d)$, $\mathrm{CH}^q(X,n) = \mathrm{CH}_{d-q}(X,n)$; we extend the notation to X locally equi-dimensional over k (e.g., X smooth over k) by taking the direct sum over the connected components of X.

1.1.3. *Basic properties.* Here is a list of the fundamental properties of the cycle complexes and higher Chow groups.

Theorem 1.4. *1. Let $z_q(X,0) \to \mathrm{CH}_q(X,0)$, $Z_q(X) \to \mathrm{CH}_q(X)$ be the canonical surjections. There is a unique isomorphism $\mathrm{CH}_q(X,0) \cong \mathrm{CH}_q(X)$ making*

$$\begin{tikzcd} z_q(X,0) \arrow[r, equal] \arrow[d] & Z_q(X) \arrow[d] \\ CH_q(X,0) \arrow[r, "\sim"] & \mathrm{CH}_q(X) \end{tikzcd}$$

commute.
2. Let $f : Y \to X$ be a proper map in $\mathbf{Sch}_k$. The push-forward maps $(\mathrm{Id} \times f)_* : Z_{n+q}(\Delta^n_k \times Y) \to Z_{n+q}(\Delta^n_k \times X)$ define a functorial pushforward map of complexes

$$f_* : (z_q(Y,*),d) \to (z_q(X,*),d)$$

and on homology, $f_* : \mathrm{CH}_q(Y,n) \to \mathrm{CH}_q(X,n)$.

The inclusion $i : X_{\mathrm{red}} \to X$ induces an isomorphism $i_* : (z_q(X_{\mathrm{red}},*),d) \to (z_q(X,*),d)$.
3. Let $f : Y \to X$ be a flat map in $\mathbf{Sch}_k$, of relative dimension d. The flat pullback maps

$$(\mathrm{Id} \times f)^* : Z_{q+n}(\Delta^n_k \times X) \to Z_{q+n+d}(\Delta^n_k \times Y)$$

give rise to well-defined functorial maps of complexes

$$f^* : (z_q(X,*),d) \to (z_{q+d}(Y,*),d)$$

and on homology $f^* : \mathrm{CH}_q(X,n) \to \mathrm{CH}_{q+d}(Y,n)$.
3'. Let $f : Y \to X$ be a morphism with X smooth and Y locally equi-dimensional over k. The "partially defined" pullback maps

$$(\mathrm{Id} \times f)^* : Z^q(\Delta^n_k \times X)_{\mathrm{Id}\times f} \to Z^q(\Delta^n_k \times Y)$$

give rise to a well-defined pullback map

$$f^* : z^q(X,*) \to z^q(Y,*)$$

in the derived category $D^-(\mathbf{Ab})$ and on the homology $f^* : \mathrm{CH}^q(X,n) \to \mathrm{CH}^q(Y,n)$, which are functorial on $\mathbf{Sm}_k$.
4. Let $p : E \to X$ be an affine-space bundle (torsor for a vector bundle over X) of relative dimension d. Then $p^* : \mathrm{CH}_q(X,n) \to \mathrm{CH}_{q+d}(E,n)$ is an isomorphism.
5. For X, X' k-varieties, we have well-defined external products

$$\boxtimes_{X,X'} : z_q(X,*) \otimes_{\mathbb{Z}} z_{q'}(X',*) \to z_{q+q'}(X \times Y,*)$$

in $D^-(\mathbf{Ab})$, inducing external products on homology $\boxtimes_{X,X'} : \mathrm{CH}_q(X,n) \otimes \mathrm{CH}_{q'}(X',n') \to \mathrm{CH}_{q+q'}(X \times X', n+n')$.

For $f : Y \to X$, $f' : Y' \to X'$ we have

$$(f \times f')_* \circ \boxtimes_{Y,Y'} = \boxtimes_{X,X'} \circ (f_* \otimes f'_*)$$

if f, f' are proper.

$$(f \times f')^* \circ \boxtimes_{X,X'} = \boxtimes_{Y,Y'} \circ (f^* \otimes f'^*)$$

if f and f' are flat or if X, X' are smooth and Y, Y' are locally equi-dimensional.
5'. If X is smooth over k, let $\Delta_X : X \to X \times X$ be the diagonal. Then $\cup_X := \Delta^* \circ \boxtimes_{X,X}$ makes $\mathrm{CH}^*(X,*) := \oplus_{q,n\geq 0}\mathrm{CH}^q(X,n)$ a bi-graded $\mathbb{Z}$-algebra, commutative in q and graded-commutative in n. Moreover, if $f : Y \to X$ is a proper morphism in $\mathbf{Sm}_k$, then we have the projection formula

$$f_* \circ (f^* \cup \mathrm{Id}_Y) = \mathrm{Id}_X \cup f_*.$$

6. (Projective bundle formula). Let $V \to X$ be a rank $n+1$ vector bundle over some smooth X, with associated projective space bundle $\mathbb{P}(V) := \mathrm{Proj}(\mathrm{Sym}^*V) \xrightarrow{q} X$. We have $\xi : c_1(\mathcal{O}(1)) \in \mathrm{CH}^1(\mathbb{P}(V))$, and via q^*, $\mathrm{CH}^*(\mathbb{P}(V),*)$ is a bi-graded $\mathrm{CH}^*(X,*)$-module. Then $\mathrm{CH}^*(\mathbb{P}(V),*)$ is a free $\mathrm{CH}^*(X,*)$-module with basis $1, \xi, \ldots, \xi^n$.

Finally, all these structures and properties restrict to the classical ones for $\mathrm{CH}_*(-)$ and $\mathrm{CH}^*(-)$ via the isomorphism (1).

A crucial property of the higher Chow groups is the long exact *localization sequence*. Let X be a k-variety, $i : W \to X$ a closed subvariety with open complement $j : U \to X$. We have the classical right-exact localization sequence

$$\mathrm{CH}_q(W) \xrightarrow{i_*} \mathrm{CH}_q(X) \xrightarrow{j^*} \mathrm{CH}_q(U) \to 0$$

Theorem 1.5 (Bloch [12], 1992)**.** *With $i : W \to X$, $j : U \to X$ as above, the sequence*

$$z_q(W,*) \xrightarrow{i_*} z_q(X,*) \xrightarrow{j^*} z_q(U,*)$$

extends canonically to a distinguished triangle

$$z_q(W,*) \xrightarrow{i_*} z_q(X,*) \xrightarrow{j^*} z_q(U,*) \to z_q(W,*)[1]$$

in $D^-(\mathbf{Ab})$, giving rise to the long exact localization sequence

$$\ldots \to \mathrm{CH}_q(W,n) \xrightarrow{i_*} \mathrm{CH}_q(X,n) \xrightarrow{j^*} \mathrm{CH}_q(U,n) \xrightarrow{\partial_n} \mathrm{CH}_q(W,n-1) \to \ldots$$
$$\to \mathrm{CH}_q(W,0) \xrightarrow{i_*} \mathrm{CH}_q(X,0) \xrightarrow{j^*} \mathrm{CH}_q(U,0) \to 0$$

extending the classical sequence via the isomorphism $\mathrm{CH}_q(-,0) \cong \mathrm{CH}_q(-)$.

Remarks 1.6. 1. The construction of the proper pushforward and flat pullback maps is straightforward. The homotopy property for the projection $\mathbb{A}^1 \times X \to X$ is proven by identifying $(\mathbb{A}^1, i_0, i_1)$ with $(\Delta^1, \delta_1^1, \delta_1^0)$ and using an algebraic version of the standard subdivision of $\Delta^n \times \Delta^1$. One needs to be careful here as the "new" faces introduced in $\Delta^n \times \Delta^1$ to make the subdivision need to be taken into account when defining the suitable cycle groups on $\Delta^n \times \Delta^1 \times X$, and one needs an elementary moving lemma to make the proof of homotopy invariance of homology from topology work in this setting.

The contravariant functoriality for morphisms to a smooth variety relies on a version of the classical Chow moving lemma, in case X is affine or projective. To get this to work in the general smooth case, one needs to use the localization property to yield a Mayer-Vietoris sequence (as discussed in (2)), which reduces to the affine case.

2. By a standard argument, the localization theorem gives a long exact Mayer-Vietoris sequence: For $X = U \cup V$, U, V open, we have $W := X \setminus U = V \setminus (U \cap V)$ and one pieces the two resulting localization triangles together to give a Mayer-Vietoris distinguished triangle

$$z_q(X,*) \xrightarrow{(j_U^*, j_V^*)} z_q(U,*) \oplus z_q(V,*) \xrightarrow{j_{U\cap V}^{V*} - j_{U\cap V}^{U*}} z_q(U \cap V,*) \xrightarrow{\partial} z_q(X,*)[1]$$

The Mayer-Vietoris sequence is then used to extend the basic homotopy invariance, for $X \times \mathbb{A}^1 \to X$, to the general version, and is similarly used to prove the projective bundle theorem, starting from the case of a product (which is proven using localization for the standard cell decomposition of $\mathbb{P}^n$, plus homotopy invariance).

The proof of the localization theorem uses essentially new ideas introduced by Bloch [12]. The basic problem is as follows. Let $i : W \to X$, $j : U \to X$ be as in the statement of Theorem 1.5, and let $z_q(U,*)_j \subset z_q(U,*)$ be the image $j^*(z_q(X,*))$. This gives us the degree-wise exact sequence of complexes

$$0 \to z_q(W,*) \xrightarrow{i_*} z_q(X,*) \xrightarrow{j^*} z_q(U,*)_j \to 0$$

so it suffices to show that the inclusion $z_q(U,*)_j \to z_q(U,*)$ is a quasi-isomorphism (it is easy to construct examples for which this is a proper inclusion). The problem is that a subvariety $W \in z_q(U,n)$ may have closure $\bar{W}$ in $Z_{q+n}(\Delta^n \times X)$ that no longer intersects all faces properly. To deal with this, Bloch shows that after a sequence of blow-ups of faces $U \times F$ in $\Delta^n \times U$, which creates a new "polyhedral" version $\tilde{\Delta}^n \times U$ of $\Delta^n \times U$, the inverse image of W has closure in $\tilde{\Delta}^n \times X$ that intersects all "faces" properly. Then by a clever subdivision construction, Bloch puts this blow-up of W back in $z_q(U,n)$ and shows that it actually lands in $z_q(U,n)_j$. Finally, Bloch shows that this transformation defines a retraction $z_q(U,*) \to z_q(U,*)_j$ in the derived category (at least on finitely generated subcomplexes), which shows that the inclusion $z_q(U,*)_j \to z_q(U,*)$ is a quasi-isomorphism.

Since this is a crucially new idea in intersection theory, let's at least look at a simple case to see what is going on in a bit more detail. For simplicity, we look at a cycle W on $U \times \Delta^2$ intersecting all faces properly, and consider its closure $\bar{W} \subset X \times \Delta^2$. The possible non-proper intersections all involve faces of the form $X \times v$, with v a vertex (0-dimensional face) of Δ^2.

We try to resolve the problem by blowing up such a face $X \times v$. This creates a new codimension one face, the exceptional divisor of the blow-up $\Delta^2_1 \to \Delta^2$, and two new vertices, v', v'', both lying over v. Continuing in this fashion, we have a sequence of blow-ups of vertices,

$$\Delta^2_m \to \ldots \to \Delta^2_1 \to \Delta^2,$$

where Δ^2_m has a "polygon" of $m+3$ codimension one faces, each with two vertices, these being the codimension two faces of Δ^2_m. We continue the process by blowing up one of the vertices on Δ^2_m.

The fact that W intersects faces in $U \times \Delta^2$ properly says that the pullback W_m of W to $U \times \Delta^2_m$ is equal to its proper transform, and that W_m intersects all faces in $U \times \Delta^2_m$ properly. The blow-up part of the story says that for some $m >> 0$, the closure $\bar{W}_m$ of W_m in $X \times \Delta^2_m$ also intersects all faces in $X \times \Delta^2_m$ properly.

Accepting all this, how to we get back to a cycle W' on $U \times \Delta^2$ with closure $\bar{W}'$ in $X \times \Delta^2$ that both intersect faces properly, in such a way that the induced map of complexes $z^q(U,*) \to z^q(U,*)$ sending W to W' is homotopic to the identity? The answer is Bloch's clever subdivision construction, which we now describe.

On Δ^2, each vertex v comes with a canonical choice of coordinates (x_v, y_v), with $v = (0,0)$. Indeed, $\Delta^2 = \operatorname{Spec} k[t_0, t_1, t_2]/\sum_i t_i - 1$. At $t_i = t_j = 0$, $t_k = 1$, we take $x_v = t_i$, $y_v = t_j$ with $i < j$. Now suppose we have coordinates (x_v, y_v) on some open neighborhood U_v of $v \in \Delta^2_m$. On the blow-up $\mu_v : \Delta^2_{m+1} \to \Delta^2_m$ of Δ^2_m at v we have local coordinates $(x_{v'} := x_v, y_{v'} := y_v/x_v)$ at one new vertex v' and $(x_{v''} := x_v/y_v, y_{v''} := y_v)$ at the other. We take $U_{v'} = \mu_v^{-1}(U_v) \setminus \mu_v^{-1}[x_v = 0]$, $U_{v''} = \mu_v^{-1}(U_v) \setminus \mu_v^{-1}[y_v = 0]$, where $\mu_v^{-1}[-]$ denotes the proper transform.

If $w' \in \Delta^2_{m+1}$ is vertex mapping to a vertex $w \in \Delta^2_m$ different from v, we use coordinates $(x(w), y(w))$ on $U_{w'} := U_w \setminus E_v$, where E_v is the exceptional curve of μ_v.

For the subdivision, one chooses a very general point $c = (c_0, c_1, c_2) \in \Delta^2$ (with respect to the chosen cycle W). Since c is not in any face, we have the unique point $c_m \in \Delta^2_m$ mapping down to c. We define new canonical coordinates $(x_{v,c} := x_v/x_v(c), y_{v,c} := y_v/y_v(c))$. This gives the coordinate system $(U_v, x_{v,c}, y_{v,c})$ a cubical structure, with four faces $x_{v,c} = 0, 1$, $y_{v,c} = 0, 1$, and thus an open immersion $j_v : V_v \to U_v$ of an open subset $V_v \subset \operatorname{Spec} k[x,y]$ via $j_v^*(x_{v,c}) = x$, $j^*(y_{v,c}) = y$. By construction we may assume that V_v contains all four vertices $(x,y) = (\epsilon_1, \epsilon_2)$ with $\epsilon_i \in \{0,1\}$. Let $\square^2 := \operatorname{Spec} k[x,y]$, endowed with the four codimension one faces $x = 0, 1$, $y = 0, 1$ and four vertices (codimension 2 faces) $(x,y) = (\epsilon_1, \epsilon_2)$ with $\epsilon_i \in \{0,1\}$.

We now have the crucial lemma.

Lemma 1.7 (see [12, Lemma (1.3.1)])**.** *Let $\mu_m : \Delta^2_m \to \Delta^2$ be the blow-down map. Suppose that c is suitably general. Then*

1. *$\mu_m \circ j_v : V_v \to \Delta^2$ extends to a morphism $f_v : \square^2 \to \Delta^2$.*
2. *$(\mathrm{Id}_U \times f_v)^*(W)$ intersects $U \times F$ properly for all faces F of $\square^2$.*
3. *Suppose that $(\mathrm{Id}_U \times \mu_m)^*(W)$ has closure in $X \times \Delta^2_m$ that intersects all faces $X \times F$ properly. Then the closure of $(\mathrm{Id}_U \times f_v)^*(W)$ in $X \times \square^2$ intersects $X \times F$ properly for all faces F of $\square^2$.*

Finally, one needs to take an appropriately signed sum $f_m := \sum_v \sigma_v f_v^*$ over vertices $v \in \Delta^2_m$, $\sigma_v \in \{\pm 1\}$, giving the cycle $(\mathrm{Id} \times f_m)^*(W)$ on $U \times \square^2$, whose closure in $X \times \square^2$ intersects all faces properly.

It thus becomes useful to replace the simplicially based cycle complex $z^q(-,*)$ with a cubical version $z^q_c(-,*)$. For this, we replace Δ^n with its $n+1$ maximal faces $t_i = 0$ with the *algebraic n-cube* $\square^n = \operatorname{Spec} k[x_1, \ldots, x_n]$, with its 2^n maximal faces $x_i = \epsilon \in \{0,1\}$; let $\iota_i^\epsilon : \square^{n-1} \to \square^n$ be the corresponding inclusion. The differential $d_n := \sum_i (-1)^i (\mathrm{Id} \times \delta_i^n)^*$ for $z^q(-,*)$ is replaced with the differential $\partial_n := \sum_{i=1}^n (-1)^i [(\mathrm{Id} \times \iota_i^1)^* - (\mathrm{Id} \times \iota_i^0)^*]$. The resulting complex of cycles intersecting faces properly, $\tilde{z}^q_c(X,*)$, has some uninteresting homology, arising from the subcomplex of "degenerate" cycles, namely, the cycles in $\tilde{z}^q_c(X,n)$ that are pullbacks from $\tilde{z}^q_c(X,n-1)$ via a projection onto $n-1$ factors $\square^n \to \square^{n-1}$. Taking the quotient by this subcomplex defines the cubical complex $z^q_c(X,*)$.

We have concentrated on the case $z^q(-,2)$ for simplicity, but essentially the same procedure gives a similar result for each finitely generated subcomplex C of

$z^q(U,*)_{*\leq N}$, giving a sequence of blow-ups

$$\Delta^N_m \to \ldots \to \Delta^N$$

and a map of complexes $f^*_{m,N,C} : C \to z^q_c(U,*)$ with image landing in the subcomplex $z^q_c(U,*)_j$ of cycles with closure in $z^q_c(X,*)$; one needs to take the central point c to be suitably general with respect to the generators of C (for this, one needs to assume that k is an infinite field, but for a finite field $\mathbb{F}_q$, the usual trick of passing to an infinite ℓ-primary extension of $\mathbb{F}_q$ for two different primes ℓ allows one to assume this).

One shows rather easily that $z^q_c(U,*)$ and $z^q_c(U,*)_j$ are quasi-isomorphic to the simplicial versions $z^q(U,*)$ and $z^q(U,*)_j$ respectively. To show that the composition

$$C \xrightarrow{f^*_{m,N,C}} z^q_c(U,*)_j \xhookrightarrow{\ i\ } z^q_c(U,*)$$

is homotopic to the inclusion $C \hookrightarrow z^q(U,*)$ followed by a quasi-isomorphism $\theta : z^q(U,*) \to z^q_c(U,*)$, let $f_{0,N,C} : C \to z^q_c(U,*)$ be the map associated to the identity tower $\Delta^N \to \Delta^N$. One makes a sequence of blow-ups of $\Delta^N \times \mathbb{A}^1$ by applying the given sequence to $\Delta^N \times 0 \subset \Delta^N \times \mathbb{A}^1$; a similar cubical decomposition gives a homotopy between $i \circ f_{m,N,C}$ and $f_{0,N,C}$. Degenerating the point c to a vertex shows that $f_{0,N,C}$ is homotopic to the identity. A similar argument shows that the restriction of $f^*_{m,N,C}$ to $C \cap z^q(U,*)_j$ is homotopic to the inclusion $C \cap z^q(U,*)_j \hookrightarrow z^q(U,*)_j$ followed by the quasi-isomorphism $z^q(U,*)_j \to z^q_c(U,*)_j$. Taking a limit over C, this shows that $z^q(U,*)_j \hookrightarrow z^q(U,*)$ is a quasi-isomorphism. which completes the proof of the localization theorem, subject to the assertion that one can always achieve the proper intersection of the closure by blowing up faces.

This is not at all obvious: a version of this, local around a fixed vertex, for the case of codimension 1, is known as "Hironaka's polyhedra game", which was solved by Spivakovsky [87]. Bloch builds on this local result to arrive at a global version. For this, he considers the collection of the vertices of all possible iterated blowups of Δ^n, and the maps between them, as a profinite set, and uses the resulting compactness to reduce to the local case. In other words, given a divisor D on Δ^n, Bloch shows that there is an iterated blow-up $\mu : \tilde{\square}^n \to \Delta^n$ such that the proper transform $\mu^{-1}[D]$ avoids all the vertices on $\tilde{\square}^n$. This becomes the first case in an inductive proof that handles the case of arbitrary codimension; see [12, Theorem (2.1.2)] for details.

1.1.4. *Relations with algebraic K-theory.* For a variety X, the Grothendieck group $G_0(X)$ of coherent sheaves on X is closely related to the Chow groups $\mathrm{CH}_*(X)$ by taking the topological filtration: Let $\mathrm{Coh}_q(X) \subset \mathrm{Coh}(X)$ be the full subcategory consisting of coherent sheaves with support in dimension $\leq q$ and let $F^{\mathrm{top}}_q G_0(X) \subset G_0(X)$ be the image of $K_0(\mathrm{Coh}_q(X))$. A weak form of Grothendieck's Riemann-Roch theorem implies that sending a dimension q subvariety W to $[\mathcal{O}_W] \in F^{\mathrm{top}}_q G_0(X)$ descends to a well-defined homomorphism

$$\mathrm{cl}_q : \mathrm{CH}_q(X) \to \mathrm{gr}^{\mathrm{top}}_q G_0(X)$$

and this map is in fact an isomorphism modulo torsion (at least for X smooth).

Bloch and Lichtenbaum [14] extended this construction to the higher algebraic G-theory, $G_*(X) = G_*(\mathrm{Coh}(X))$ on the one side, and the higher Chow groups on the other, by considering a "topological filtration" of the cosimplicial scheme

$[n] \mapsto \Delta^n_k \times X$. One lets $F_q \operatorname{Coh}(X,n) \subset \operatorname{Coh}(\Delta^n_k \times X)$ be the full subcategory of coherent sheaves $\mathcal{F}$ with support satisfying

$$\dim[\operatorname{supp}(\mathcal{F}) \cap X \times F] \leq n + q - c$$

for each face $F \subset \Delta^n$ of codimension c. Applying Quillen K-theory gives a tower of simplicial spectra

$$\begin{aligned}[n] \mapsto (\ldots \to K(F_p \operatorname{Coh}(X,n)) \to K(F_{p+1} \operatorname{Coh}(X,n)) \to \\ \ldots \to K(F_{\dim X}(\operatorname{Coh}(X,n))) = K(\operatorname{Coh}(\Delta^n_k \times X)))\end{aligned}$$

and associated tower of total spectra

$$\begin{aligned}\ldots \to K(F_p \operatorname{Coh}(X,*)) \to K(F_{p+1} \operatorname{Coh}(X,*)) \to \ldots \\ \to K(F_{\dim X}(\operatorname{Coh}(X,*))) = K(\operatorname{Coh}(\Delta^*_k \times X))\end{aligned}$$

Moreover, the natural map $\operatorname{Coh}(X) \to \operatorname{Coh}(\Delta^*_k \times X)$ induces a weak equivalence

$$G(X) := K(\operatorname{Coh}(X)) \xrightarrow{\sim} G\operatorname{Coh}(\Delta^* \times X)$$

The tricky part is to show that

$$\pi_n(K(F_p \operatorname{Coh}(X,*))/K(F_{p-1} \operatorname{Coh}(X,*))) \cong \operatorname{CH}_p(X,n).$$

The arguments used by Bloch-Lichtenbaum [14] for the case $X = \operatorname{Spec} F$, F a field, are very close to that used to prove the localization property for the higher Chow groups in [12]. Friedlander and Suslin [27] rely on the Bloch-Lichtenbaum result to generalize to the case of arbitrary finite-type X over a field. Levine [57, 58] uses a somewhat different approach to arrive at the same result.

In any case, this tower gives the *Atiyah-Hirzebruch spectral sequence*

$$E^1_{p,q} = \operatorname{CH}_p(X, p+q) \Rightarrow G_{p+q}(X)$$

For X smooth, we have $G_*(X) = K_*(X)$, and one usually reindexes to an E_2 spectral sequence looking like

$$E_2^{p,q} = \operatorname{CH}^{-q}(X, -p-q) \Rightarrow K_{-p-q}(X)$$

We will see later on that Voevodsky's motivic cohomology $H^p(X, \mathbb{Z}(q))$ agrees with Bloch's higher Chow groups after a reindexing: $H^p(X, \mathbb{Z}(q)) = \operatorname{CH}^q(X, 2q-p)$, giving the Atiyah-Hirzebruch spectral sequence in its more familiar form

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X). \tag{1.2}$$

Gillet's theorem of Chern classes for higher K-theory [35] gives Chern class maps (for X smooth)

$$c_{q,n} : K_n(X) \to \operatorname{CH}^q(X,n)$$

extending the classical Chern classes $c_q : K_0(X) \to \operatorname{CH}^q(X)$. Using these, one can show that the AH spectral sequence degenerates rationally.

1.1.5. *Milnor K-theory.* For a field F, the Milnor K-theory of F is the $\mathbb{N}$-graded ring defined as a quotient of the tensor algebra (over $\mathbb{Z}$) on the abelian group of units $F^\times$ modulo the "Steinberg relation":

$$K_*^M(F) = (F^\times)^{\otimes *}/(\{a \otimes (1-a) \mid a \in F \setminus \{0,1\})$$

This was originally constructed by Milnor [66], inspired by Matsumoto's theorem [62] describing the Quillen K_2 of a field:

$$K_2(F) \cong F^\times \otimes F^\times/(\{a \otimes (1-a) \mid a \in F \setminus \{0,1\}).$$

For $x = (x_1, \ldots, x_n) \in (F^\times)^n$, let $\Sigma(x) = \sum_i x_i$. Nestorenko-Suslin [72] showed that the map sending $(F^\times)^n \setminus \{\Sigma(x) \mid x \in (F^\times)^n\}$ to $z^n(F, n)$ by

$$x = (x_1, \ldots, x_n) \mapsto (\frac{-1}{1-\Sigma(x)}, \frac{x_1}{1-\Sigma(x)}, \ldots, \frac{x_n}{1-\Sigma(x)})$$

descends to give an isomorphism $K_n^M(F) \to \mathrm{CH}^n(F, n)$. This fact was also proven by Totaro [91], using the cubical construction of the higher Chow groups described above. These maps for varying n yield an isomorphism of graded rings $K_*^M(F) \cong \oplus_n \mathrm{CH}^n(F, n)$.

1.2. **Triangulated categories of motives.**

1.2.1. *Grothendieck-Chow motives.* Grothendieck constructed a series of categories of motives for smooth projective varieties, depending on a choice of a so-called adequate equivalence relation for algebraic cycles. His ultimate goal was to construct the universal Weil cohomology theory using purely geometric means. This is still an open question. However, using the finest adequate relation, namely, rational equivalence, one arrives at the category of Chow motives, which was later expanded by Voevodsky to form a categorical framework for a good theory of motivic cohomology.

To fit better with Voevodsky's construction, we define a homological version of Chow motives, essentially following [56].

Definition 1.8. The category of *Chow correspondences* over a field k, $\mathrm{Cor}_{\mathrm{CH}}(k)$, has objects $[X]$ for each smooth projective variety $[X]$ over k and morphism groups (for irreducible X)

$$\mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}(k)}([X],[Y]) := \mathrm{CH}_{\dim X}(X \times Y);$$

in general, write $X = \amalg_i X_i$ as a disjoint union of its irreducible components and define $\mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}(k)}([X],[Y]) = \prod_i \mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}(k)}([X_i],[Y])$.

The composition law is that of *correspondences*: For $W_1 \in \mathrm{CH}_{\dim X}(X \times Y)$ and $W_2 \in \mathrm{CH}_{\dim Y}(Y \times Z)$ define

$$W_2 \circ W_1 := p_{X \times Z*}(p^*_{X \times Y}(W_1) \cdot p^*_{Y \times Z}(W_2))$$

The identity on $[X]$ is given by the diagonal cycle $\Delta_X \in \mathrm{CH}_{\dim X}(X \times X)$.

Note that we need Y to be proper for $p_{X \times Z*}$ to be defined, and we need X, Y and Z to be smooth, and we need to pass from cycles to cycles mod rational equivalence, for the intersection product to be defined (we use projective instead of proper varieties to be able to use classical moving lemmas for the intersection product, but this is not really needed).

We have the functor $\mathbf{SmProj}_k \to \mathrm{Cor}_{\mathrm{CH}}(k)$ sending X to $[X]$ and $f : X \to Y$ to the graph $\Gamma_f \in \mathrm{CH}_{\dim X}(X \times Y)$.

$\mathrm{Cor}_{\mathrm{CH}}(k)$ is an additive category with $\oplus$ induced by disjoint union in $\mathbf{SmProj}_k$. Similarly, product over k makes $\mathrm{Cor}_{\mathrm{CH}}(k)$ a tensor category. The next step is to adjoin summands corresponding to idempotent endomorphisms; this is a formal process where one has objects (X, α) with $\alpha : [X] \to [X]$ in $\mathrm{Cor}_{\mathrm{CH}}(k)([X],[X])$ an idempotent endomorphism. This gives the category of effective Chow motives $\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}$, with

$$\mathrm{Hom}_{\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}}((X,\alpha),(Y,\beta) := \beta \cdot \mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}(k)}([X],[Y]) \cdot \alpha,$$

and with composition induced from $\mathrm{Cor}_{\mathrm{CH}}(k)$. $\mathrm{Cor}_{\mathrm{CH}}(k)$ is embedded in $\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}$ by sending $[X]$ to (X, Id). In $\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}$, we have the *Lefschetz motive* $\mathbb{L}$, this being the summand of $[\mathbb{P}^1]$ given by $0 \times \mathbb{P}^1 \in \mathrm{CH}_1(\mathbb{P}^1 \times \mathbb{P}^1)$; $0 := [1,0] \in \mathbb{P}^1(k)$.

If you've never done this before, it's a nice exercise to show that $0 \times \mathbb{P}^1, \mathbb{P}^1 \times 0 \in \mathrm{CH}_1(\mathbb{P}^1 \times \mathbb{P}^1)$ are both idempotent endomorphisms, and $\Delta_{\mathbb{P}^1} = 0 \times \mathbb{P}^1 + \mathbb{P}^1 \times 0 \in \mathrm{CH}_1(\mathbb{P}^1 \times \mathbb{P}^1)$. $\mathbb{P}^1 \times 0$ is just the graph of the endomorphism $i_0 \circ p$ of $\mathbb{P}^1$, with $i_0 : \mathrm{Spec}\, k \to \mathbb{P}^1$ the inclusion of the point 0 and $p : \mathbb{P}^1 \to \mathrm{Spec}\, k$ the projection, but the other factor $0 \times \mathbb{P}^1$ is not the graph of any endomorphism.

Definition 1.9. The category $\mathrm{Mot}_{\mathrm{CH}}(k)$ of Chow motives is defined by inverting the endofunctor $- \otimes \mathbb{L}$ on $\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}$:

$$\mathrm{Mot}_{\mathrm{CH}}(k) := \mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}[(- \otimes \mathbb{L})^{-1}]$$

Write $M\{n\}$ for $M \otimes \mathbb{L}^{\otimes n}$. Note that we have the *cancellation property*: For $M, N \in \mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}$, the canonical map

$$\mathrm{Hom}_{\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}}(M,N) \to \mathrm{Hom}_{\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}}(M\{1\}, N\{1\})$$

is an isomorphism. Since the objects of $\mathrm{Mot}_{\mathrm{CH}}(k)$ are all of the form $M\{n\}$ for $M \in \mathrm{Mot}^{\mathrm{eff}}_{\mathrm{CH}}(k)$ and $n \in \mathbb{Z}$, and

$$\mathrm{Hom}_{\mathrm{Mot}_{\mathrm{CH}}(k)}(M_1\{n_1\}, M_2\{n_2\}) = \mathrm{colim}_m \mathrm{Hom}_{\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}}}(M_1\{n_1+m\}, M_2\{n_2+m\})$$

the canonical functor $\mathrm{Mot}_{\mathrm{CH}}(k)^{\mathrm{eff}} \to \mathrm{Mot}_{\mathrm{CH}}(k)$ is a fully faithful embedding.

Remark 1.10. The Chow groups $\mathrm{CH}_n(X)$ are represented in $\mathrm{Mot}_{\mathrm{CH}}(k)$ by

$$\mathrm{CH}_n(X) = \mathrm{Hom}_{\mathrm{Mot}_{\mathrm{CH}}(k)}(\mathbb{L}^{\otimes n}, [X])$$

and $\mathrm{CH}^n(X)$ is similarly represented by

$$\mathrm{CH}^n(X) = \mathrm{Hom}_{\mathrm{Mot}_{\mathrm{CH}}(k)}([X], \mathbb{L}^{\otimes n})$$

This follows from the formula for $\mathrm{CH}_*((\mathbb{P}^1)^n)$: For $I = (\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n$, let $\mathbb{P}^I \subset (\mathbb{P}^1)^n$ be the subscheme

$$\mathbb{P}^I = \{(x_1, \ldots, x_n) \in (\mathbb{P}^1)^n \mid x_i = 0 \text{ if } \epsilon_i = 0\}$$

and let $|I| = \sum_i \epsilon_i$. Then

$$\mathrm{CH}_*((\mathbb{P}^1)^n \times X) = \oplus_{I \in \{0,1\}^n} [\mathbb{P}^I] \times \mathrm{CH}_{*-|I|}(X)$$

and

$$\mathrm{CH}^*(X \times (\mathbb{P}^1)^n) = \oplus_{I \in \{0,1\}^n} \mathrm{CH}^{*+|I|-n}(X) \times [\mathbb{P}^I]$$

where one sends $\alpha \in \mathrm{CH}_{*-|I|}(X)$ to $\alpha \times [\mathbb{P}^I] \in \mathrm{CH}_*(X \times (\mathbb{P}^1)^n)$ and similarly for CH^*. Writing $\mathbb{L}^{\otimes n}$ as a summand of $(\mathbb{P}^1)^n$, we see that $\mathrm{Hom}_{\mathrm{Mot}_{\mathrm{CH}}(k)}(\mathbb{L}^{\otimes n}, [X])$ is the summand $[\mathbb{P}^{(0,\ldots,0)}] \times \mathrm{CH}_n(X)$ of $\mathrm{CH}_n((\mathbb{P}^1)^n \times X)$ and $\mathrm{Hom}_{\mathrm{Mot}_{\mathrm{CH}}(k)}([X], \mathbb{L}^{\otimes n})$ is the summand $\mathrm{CH}^n(X) \times [\mathbb{P}^{(1,\ldots,1)}]$ of $\mathrm{CH}^n(X \times (\mathbb{P}^1)^n)$.

$\mathrm{Mot}_{\mathrm{CH}}(k)$ is a tensor category in which each non-zero object has a dual. For instance the dual of $[X]$ is $[X]\{-\dim X\}$ and the dual of $([X],\alpha)$ is $([X]\{-\dim X\},\alpha^t\otimes \mathrm{Id})$, where $\alpha^t=\tau_*(\alpha)$, with $\tau: X\times X\to X\times X$ the symmetry.

To see this, recall that the dual of an object x in a symmetric monoidal category $(\mathcal{C},\otimes,\tau,1)$ is a triple $(x^\vee,\delta,\eta)$ with $\delta: 1\to x\otimes x^\vee$, $\eta: x^\vee\otimes x\to 1$, such that the compositions

$$x\cong 1\otimes x\xrightarrow{\delta\otimes\mathrm{Id}} x\otimes x^\vee\otimes x\xrightarrow{\eta\otimes\mathrm{Id}} 1\otimes x\cong x$$

and

$$x^\vee\cong x^\vee\otimes 1\xrightarrow{\mathrm{Id}\otimes\delta} x^\vee\otimes x\otimes x^\vee\xrightarrow{\mathrm{Id}\otimes\eta} x^\vee\otimes 1\cong x^\vee$$

are the respective identities.

For $p: X\to \mathrm{Spec}\, k$ smooth and projective of dimension d, we have the graph of the diagonal $\Delta_X: X\to X\times X$, giving

$$\Gamma_{\Delta_X}\in \mathrm{Mot}_{\mathrm{CH}}([X],[X]\otimes[X]),\ \Gamma^t_{\Delta_X}\in \mathrm{Mot}_{\mathrm{CH}}([X]\{-d\}\otimes[X],[X]).$$

We also have $X=\mathrm{Spec}\, k\times X=X\times \mathrm{Spec}\, k$, giving

$$\Gamma_p\in \mathrm{Mot}_{\mathrm{CH}}([X],[\mathrm{Spec}\, k]),\ \Gamma^t_p\in \mathrm{Mot}_{\mathrm{CH}}([\mathrm{Spec}\, k],[X]\{-d\}),$$

and we define

$$\delta_X:=\Gamma_{\Delta_X}\{-d\}\circ\Gamma^t_p\in \mathrm{Mot}_{\mathrm{CH}}([\mathrm{Spec}\, k],[X]\otimes[X]\{-d\})$$

$$\eta_X:=\Gamma_p\circ\Gamma^t_{\Delta_X}\in \mathrm{Mot}_{\mathrm{CH}}([X]\{-d\}\otimes[X],[\mathrm{Spec}\, k])$$

The necessary relations follow from the identity

$$p_{13*}((\Delta_X\times X)\cdot(X\times\Delta_X))=\Delta_X$$

1.2.2. *Voevodsky's geometric motives.* We now follow [104] to give a first definition of a triangulated category of motives over a field.

Somewhat in line with Bloch's idea of resolving the Chow groups, Voevodsky defines his triangulated category of geometric motives by replacing cycles modulo rational equivalence with cycles; he also expands the basic objects to all smooth k-varieties, not just the smooth projective ones. To do this, and still have a well-defined composition law using correspondences, one uses the following lemma, which, once stated, is easy to prove (see e.g. [63, Lecture 1]).

Lemma 1.11. *Let X, Y and Z be smooth irreducible k-varieties, and take $W_1\in Z_{\dim X}(X\times Y)$ and $W_2\in Z_{\dim Y}(Y\times Z)$. Suppose that for each irreducible component C_1 of the support of W_1, and for each each irreducible component C_2 of the support of W_2 the projections $C_1\to X$ and $C_2\to Y$ are finite and surjective. Then*

- *Each irreducible component of $p^{-1}_{XY}(W_1)\cap_{XYZ} p^{-1}_{YZ}(W_2)$ has the proper dimension $\dim X$, hence the intersection product $W:=p^*_{XY}(W_1)\cdot p^*_{YZ}(W_2)$ on $X\times Y\times Z$ is a well-defined cycle of dimension $\dim X$.*
- *The support of W is finite over $X\times Z$, so we have a well-defined cycle $p_{XZ*}(W)\in Z_{\dim X}(X\times Z)$*
- *For each irreducible component C of $p_{XZ*}(W)$, the projection $C\to X$ is finite and surjective*

With this lemma, we can follow Voevodsky in defining the category $\mathrm{Cor}(k)$ of *finite correspondences on* $\mathbf{Sm}_k$

Definition 1.12. Let k be a field. The category $\mathrm{Cor}(k)$ has objects $[X]$ for $X \in \mathbf{Sm}_k$ and morphism group $\mathrm{Hom}_{\mathrm{Cor}(k)}([X],[Y])$ the subgroup of $Z_{\dim X}(X \times Y)$ freely generated by subvarieties $W \subset X \times Y$ such that the projection $W \to X$ is finite, and is surjective onto an irreducible component of X. The composition law is given by composition of correspondences: For $W_1 \in \mathrm{Hom}_{\mathrm{Cor}(k)}([X],[Y])$ and $W_2 \in \mathrm{Hom}_{\mathrm{Cor}(k)}([Y],[Z])$ we set

$$W_2 \circ W_1 := p_{XZ*}(p^*_{XY}(W_1) \cdot p^*_{YZ}(W_2))$$

The identity $\mathrm{Id}_{[X]}$ is given by the diagonal on $X \times X$.

Sending $X \in \mathbf{Sm}_k$ to $[X]$ and a morphism $f : X \to Y$ to its graph gives a faithful embedding $[-] : \mathbf{Sm}_k \to \mathrm{Cor}(k)$, by which we consider $\mathbf{Sm}_k$ as a subcategory of $\mathrm{Cor}(k)$.

We now use methods from triangulated categories to promote $\mathrm{Cor}(k)$ to a "motivic" category. $\mathrm{Cor}(k)$ is an additive tensor category, with $\oplus$ induced by disjoint union in $\mathbf{Sm}_k$ and $\otimes$ induced by product over k. We consider the bounded homotopy category $K^b(\mathrm{Cor}(k))$ (i.e. bounded complexes with morphisms chain homotopy classes of maps) and perform a Verdier localization; see [92] or [106, Chap. 10] for details on triangulated categories and localization.

Definition 1.13. The triangulated category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ of effective geometric motives is the localization of $K^b(\mathrm{Cor}(k))$ with respect to the thick subcategory generated by the following two families of complexes, (HI) and (MV):

- (Homotopy invariance) For $X \in \mathbf{Sm}_k$, the complex

$$[X \times \mathbb{A}^1] \xrightarrow{p_X} [X] \tag{HI}$$

- (Mayer-Vietoris) For $X \in \mathbf{Sm}_k$, suppose we have open subschemes $j_U : U \to X$, $j_V : V \to X$ with $X = U \cup V$, giving the inclusions $j^U_{U\cap V} : U \cap V \to U$, $j^V_{U\cap V} : U \cap V \to V$, and the complex

$$[U \cap V] \xrightarrow{(j^U_{U\cap V}, -j^V_{U\cap V})} [U] \oplus [V] \xrightarrow{j_U + j_V} [X] \tag{MV}$$

We let $M^{\mathrm{eff}} : \mathbf{Sm}_k \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ be the functor sending X to the image of $[X]$, concentrated in degree 0, in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

$\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ is a triangulated tensor category with $\otimes$ induced by the tensor structure on $\mathrm{Cor}(k)$, i.e.,

$$M^{\mathrm{eff}}(X) \otimes M^{\mathrm{eff}}(Y) = M^{\mathrm{eff}}(X \times_k Y).$$

We have the *reduced motive* $\tilde{M}^{\mathrm{eff}}(\mathbb{P}^1)$ of $\mathbb{P}^1$, namely, the complex

$$[\mathbb{P}^1] \xrightarrow{p} [\mathrm{Spec}\, k]$$

with $[\mathbb{P}^1]$ in degree zero and $p : \mathbb{P}^1 \to \mathrm{Spec}\, k$ the projection. Define $\mathbb{Z}(1)$ by

$$\mathbb{Z}(1) := \tilde{M}^{\mathrm{eff}}(\mathbb{P}^1)[-2].$$

We define the triangulated tensor category of geometric motives, $\mathrm{DM}_{\mathrm{gm}}(k)$ by

$$\mathrm{DM}_{\mathrm{gm}}(k) := \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)[(- \otimes \mathbb{Z}(1))]^{\natural},$$

where $(-)^{\natural}$ means adjoin summands corresponding to idempotent endomorphisms; this process yields a triangulated category by [14, Theorem 1.5].

The objects $\mathbb{Z}(n) := \mathbb{Z}(1)^{\otimes n}$, $n \in \mathbb{Z}$ are called the pure Tate motives. Concretely, the objects of $\mathrm{DM}_{\mathrm{gm}}(k)$ are of the form $M(m) := M \otimes \mathbb{Z}(m)$ for some $m \in \mathbb{Z}$, and with morphism groups

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(m), N(n))\\ &\qquad := \mathrm{colim}_{r \geq \max(-m,-n)} \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M(r+m), N(r+n))\end{aligned}$$

$\mathbb{Z}(0) := M(\mathrm{Spec}\, k)$ is the unit for the tensor structure.

A fundamental theorem of Voevodsky (the *cancellation theorem*) reduces the study of $\mathrm{DM}_{\mathrm{gm}}(k)$ to $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$

Theorem 1.14 (Voevodsky [104, Chap 5, Theorem 4.3.1], [102])**.** *For all $M, N \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$, the stabilization map*

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M, N) \to \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M(1), N(1))$$

is an isomorphism. In particular, the canonical functor $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \to \mathrm{DM}_{\mathrm{gm}}(k)$ is a fully faithful embedding.

Let $M : \mathbf{Sm}_k \to \mathrm{DM}_{\mathrm{gm}}(k)$ be the composition of M^{eff} with the canonical functor. The reason for inverting $- \otimes \mathbb{Z}(1)$ is the same as for the case of Chow motives: if k has characteristic zero, then $M(X)$ is a dualizable object in $\mathrm{DM}_{\mathrm{gm}}(k)$ for all $X \in \mathbf{Sm}_k$; if k has characteristic $p > 0$, this also holds after inverting p (see [104, Chap. 5, §4.3] for the result in characteristic 0, and [53] for the result in positive characteristic). Just as for $\mathrm{Mot}_{\mathrm{CH}}$, if X is smooth and projective of dimension d, then

$$M(X)^\vee = M(X)(-d)[-2d].$$

We will show in the next lecture that $\mathrm{Mot}_{\mathrm{CH}}(k)$ is a full (additive tensor) subcategory of $\mathrm{DM}_{\mathrm{gm}}(k)$, with Lefschetz motive $\mathbb{L} \in \mathrm{Mot}_{\mathrm{CH}}(k)$ mapping to $\tilde{M}(\mathbb{P}^1) = \mathbb{Z}(1)[2]$.

1.2.3. *Motivic cohomology.* Via $\mathrm{DM}_{\mathrm{gm}}(k)$, we have the categorical construction of motivic cohomology.

Definition 1.15. For $X \in \mathbf{Sm}_k$, define

$$H^p(X, \mathbb{Z}(q)) := \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(X), \mathbb{Z}(q)[p])$$

More generally, for an arbitrary $M \in \mathrm{DM}_{\mathrm{gm}}(k)$, we set

$$H^p(M, \mathbb{Z}(q)) := \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M, \mathbb{Z}(q)[p])$$

Immediate consequences of this construction include:

1. Functoriality. Each morphism $f : M \to N$ in $\mathrm{DM}_{\mathrm{gm}}(k)$ induces $f^* : H^p(N, \mathbb{Z}(q)) \to H^p(M, \mathbb{Z}(q))$. In particular, each $f : Y \to X$ in $\mathbf{Sm}_k$ induces $f^* := M(f)^* : H^p(Y, \mathbb{Z}(q)) \to H^p(X, \mathbb{Z}(q))$.
2. Mayer-Vietoris. Let $X = U \cup V$ be an open cover of $X \in \mathbf{Sm}_k$. Then we have the long exact Mayer-Vietoris sequence

$$\begin{aligned}&\ldots \to H^{p-1}(U \cap V, \mathbb{Z}(q)) \to H^p(X, \mathbb{Z}(q))\\ &\qquad \to H^p(U, \mathbb{Z}(q)) \oplus H^p(V, \mathbb{Z}(q)) \to H^p(U \cap V, \mathbb{Z}(q)) \to \ldots\end{aligned}$$

Homotopy invariance. Let $p : \mathbb{A}^1_k \to \mathrm{Spec}\, k$ be the projection. For each $M \in \mathrm{DM}_{\mathrm{gm}}(k)$, the map $\mathrm{Id}_M \otimes M(p_1) : M \otimes M(\mathbb{A}^1) \to M$ and the induced map $(\mathrm{Id}_M \otimes$

$M(p_1))^* : H^p(M, \mathbb{Z}(q)) \to H^p(M \otimes M(\mathbb{A}^1, \mathbb{Z}(q))$ is an isomorphism. In particular, $p_X^* : H^p(X, \mathbb{Z}(q)) \to H^p(X \times \mathbb{A}^1, \mathbb{Z}(q))$ is an isomorphism for all $X \in \mathbf{Sm}_k$.

Together with the Mayer-Vietoris sequence, this gives the extended homotopy property: For $p : E \to X$ an affine space bundle (torsor for a vector bundle on X), the map $p^* : H^p(X, \mathbb{Z}(q)) \to H^p(E, \mathbb{Z}(q))$ is an isomorphism.

Variants:
Mod n motivic cohomology. For a positive integer n, define $\mathbb{Z}/n(q)$ as the complex

$$\mathbb{Z}(q) \xrightarrow{\times n} \mathbb{Z}(q)$$

concentrated in degrees -1, 0, define

$$H^p(X, \mathbb{Z}/n(q)) := \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(X), \mathbb{Z}/n(q)[p])$$

and define $H^p(M, \mathbb{Z}/n(q))$ for $M \in \mathrm{DM}_{\mathrm{gm}}(k)$ similarly. We thus have the long exact coefficient sequence

$$\ldots \to H^{p-1}(M, \mathbb{Z}/n(q)) \to H^p(M, \mathbb{Z}(q)) \xrightarrow{\times n} H^p(M, \mathbb{Z}(q)) \to H^p(M, \mathbb{Z}/n(q)) \to \ldots$$

and the motivic Milnor sequence

$$0 \to H^p(M, \mathbb{Z}(q))/n \to H^p(M, \mathbb{Z}/n(q)) \to H^{p+1}(M, \mathbb{Z}(q))_{n-\mathrm{torsion}} \to 0$$

Motivic cohomology with support: Let $i : Z \to X$ be a closed subscheme and $j : U \to X$ the open complement, with $X \in \mathbf{Sm}_k$. Define the motive with support $M_Z(X)$ as the complex

$$M(U) \xrightarrow{M(j)} M(X)$$

in degrees -1, 0, and

$$H^p_Z(X, \mathbb{Z}(q)) := H^p(M_Z(X), \mathbb{Z}(q));\ H^p_Z(X, \mathbb{Z}/n(q)) := H^p(M_Z(X), \mathbb{Z}/n(q)).$$

This gives us the distinguished triangle

$$M(U) \xrightarrow{M(j)} M(X) \to M_Z(X) \to M(U)[1];$$

applying $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(-, \mathbb{Z}(q)[*])$ gives the long exact motivic cohomology sequence

$$\ldots \to H^p_Z(X, \mathbb{Z}(q)) \to H^p(X, \mathbb{Z}(q)) \to H^p(U, \mathbb{Z}(q)) \to H^{p+1}_Z(X, \mathbb{Z}(q)) \to \ldots$$

One can also define motivic homology:

$$H_p(X, \mathbb{Z}) := \mathrm{Hom}(\mathbb{Z}(0), M(X)[p]);\ H_p(M, \mathbb{Z}) := \mathrm{Hom}(\mathbb{Z}(0), M[p]).$$

We will revisit this construction later on, in the context of *Suslin homology*.

2. Lecture 2: Triangulated categories of motivic sheaves

2.1. **The category of effective motivic sheaves.**

2.1.1. *Presheaves and sheaves with transfer.* Voevodsky noted the difficulty one usually has in computing morphisms in a localization. To better understand the category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$, he constructed a sheaf-theoretic version, $\mathrm{DM}^{\mathrm{eff}}(k)$, which contains $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ as the full subcategory of compact objects. We follow his treatment, detailed in [104], but with some refinements introduced later on by Cisinski-Déglise [18]; mainly this involves replacing Voevodsky's use of bounded above derived categories D^- with the unbounded versions. Our goal here is not to give a complete treatment, but only to sketch the main points, to give the reader an overview of the construction and its main properties; we refer the reader to [18, 63, **?**] for details. To avoid technical difficulties, we work throughout over a perfect field k.

We first recall the definition of the Nisnevich topology. For $X \in \mathbf{Sch}_k$ an *elementary Nisnevich square* is a Cartesian diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

with j an open immersion and p an étale map, such that p induces an isomorphism of reduced schemes $\bar{p} : V \setminus p^{-1}(U) \to X \setminus U$. For example, if p is also an open immersion, then $p^{-1}(U) = U \cap V$ and $\{U, V\}$ is a Zariski open cover of X. The Nisnevich topology on $\mathbf{Sch}_k$ is the Grothendieck topology generated by covering families of the form $\{j : U \to X, p : V \to X\}$ for $X \in \mathbf{Sch}_k$, where the maps $p : V \to X$, $j : U \to X$ define an elementary Nisnevich square as above.

The Nisnevich topology is thus finer than the Zariski topology and coarser than the étale topology. Restricting to $\mathbf{Sm}_k$ gives us the Nisnevich topology on $\mathbf{Sm}_k$.

Definition 2.1. 1. A *presheaf with transfers* (PST) on $\mathbf{Sm}_k$ is an additive functor

$$P : \mathrm{Cor}(k)^{\mathrm{op}} \to \mathbf{Ab}$$

We let $\mathrm{PST}(k)$ denote the category of such additive functors.

2. A PST P is a *Nisnevich sheaf with transfers* (NST) if the restriction of P to $\mathbf{Sm}_k \subset \mathrm{Cor}(k)$ is a Nisnevich sheaf.

3. For $X \in \mathbf{Sm}_k$, we let $\mathbb{Z}^{\mathrm{tr}}(X)$ denote the representable functor $\mathrm{Hom}_{\mathrm{Cor}(k)}(-, X)$ on $\mathrm{Cor}(k)$.

Note that $\mathbb{Z}^{\mathrm{tr}}(X)$ is an NST, and we consider the NSTs as a full subcategory $\mathrm{NST}(k)$ of $\mathrm{PST}(k)$. $\mathrm{PST}(k)$ is an abelian category, with exactness determined objectwise: A sequence

$$P' \to P \to P''$$

in $\mathrm{PST}(k)$ is exact if for each $X \in \mathbf{Sm}_k$, the sequence of abelian groups $P'(X) \to P(X) \to P''(X)$ in exact.

A crucial property enjoyed by certain PSTs is that of *homotopy invariance* and for NSTs that of *strict homotopy invariance.*

Definition 2.2. 1. A PST P is *homotopy invariant* if the map $p_X^* : P(X) \to P(X \times \mathbb{A}^1)$ is an isomorphism for all $X \in \mathbf{Sm}_k$.

2. An NST N is *strictly homotopy invariant* if for each n the presheaf $H^n((-)_{\mathrm{Nis}}, N)$ on $\mathbf{Sm}_k$ is homotopy invariant, i.e., for each $X \in \mathbf{Sm}_k$, the map $p_X^* : H^n(X_{\mathrm{Nis}}, N) \to H^n((X \times \mathbb{A}^1)_{\mathrm{Nis}}, N)$ is an isomorphism. Let $\mathrm{HI}(k) \subset \mathrm{NST}(k)$ be the full subcategory of strictly homotopy invariant NSTs.

Here are some basic facts about these categories.

Proposition 2.3. *1. For $P \in \mathrm{PST}(k)$, let $P_{|\mathbf{Sm}_k}$ denote the restriction to a presheaf on $\mathbf{Sm}_k$, and let $P_{|\mathbf{Sm}_k,\mathrm{Nis}}$ be the Nisnevich sheafification. Then $P_{|\mathbf{Sm}_k,\mathrm{Nis}}$ admits a unique extension to a object $P_{\mathrm{Nis}} \in \mathrm{NST}(k)$, with $(P_{\mathrm{Nis}})_{|\mathbf{Sm}_k} = P_{|\mathbf{Sm}_k,\mathrm{Nis}}$. Let $\alpha_{\mathrm{Nis}} : \mathrm{PST}(k) \to \mathrm{NST}(k)$ be the resulting Nisnevich sheafification functor: $\alpha_{\mathrm{Nis}}(P) := P_{\mathrm{Nis}}$.*
2. Defining a sequence $N' \to N \to N''$ in $\mathrm{NST}(k)$ to be exact if and only if the sequence is exact as a sequence of Nisnevich sheaves on $\mathbf{Sm}_k$ makes $\mathrm{NST}(k)$ into an abelian category, for which α_{Nis} is an exact left adjoint to the inclusion $\mathrm{NST}(k) \hookrightarrow \mathrm{PST}(k)$.
3.. Via the Yoneda isomorphism $\mathrm{Hom}_{\mathrm{PST}(k)}(\mathbb{Z}^{tr}(X), P) = P(X)$, each $P \in \mathrm{PST}(k)$ admits the canonical surjection

$$\oplus_{X \in \mathbf{Sm}_k, \alpha \in P(X)} \mathbb{Z}^{tr}(X) \to P.$$

Applying this construction to the kernel of the above map and iterating gives the canonical left resolution

$$\mathcal{L}_\bullet(P) \to P \to 0$$

with each $\mathcal{L}_n(P)$ a direct sum of representable PSTs.
4. Define $\mathbb{Z}^{tr}(X) \otimes^{tr} \mathbb{Z}^{tr}(Y) := \mathbb{Z}^{tr}(X \times Y)$, and extend to arbitrary PSTs by

$$P \otimes^{tr} Q := H_0(\mathcal{L}_\bullet(P) \otimes^{tr} \mathcal{L}_\bullet(Q))$$

This makes $\mathrm{PST}(k)$ into an abelian tensor category. Extend this to $\mathrm{NST}(k)$ by sheafification:

$$N \otimes^{tr}_{\mathrm{Nis}} M := (N \otimes^{tr} M)_{\mathrm{Nis}}$$

This makes $\mathrm{NST}(k)$ into an abelian tensor category.
5. Let N be an NST. Then for each n, the cohomology presheaf $X \mapsto H^n(X_{\mathrm{Nis}}, N)$ has a canonical structure of a PST.
6. Let P be a homotopy invariant PST. Then P_{Nis} is strictly homotopy invariant and the canonical map of presheaves on $\mathbf{Sm}_k$, $P_{\mathrm{Zar}} \to P_{\mathrm{Nis}}$ is an isomorphism.
7. $\mathrm{HI}(k)$ is a full abelian subcategory of $\mathrm{NST}(k)$, closed under extensions.

The property (6) is not at all obvious. (see [104, Chap. V, Theorem 3.1.12] or [63, Proposition 13.4, 13.7] for details). Property (7) follows rather easily from (6).

Parallel to the definition of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$, one defines $\mathrm{DM}^{\mathrm{eff}}(k)$ as a localization. Let $\mathcal{T}$ be a triangulated category admitting arbitrary (set-indexed) direct sums. Recall that a *localizing* subcategory of $\mathcal{T}$ is a full subcategory, closed under arbitrary (set-indexed) direct sums; a localizing subcategory is automatically a thick subcategory in the sense of Verdier, in other words, closed under direct summands.

Definition 2.4. $\mathrm{DM}^{\mathrm{eff}}(k)$ is defined to be the localization of $D(\mathrm{NST}(k))$ by the localizing subcategory generated by objects of the form $\mathrm{Cone}(K_* \otimes \mathbb{Z}^{\mathrm{tr}}(\mathbb{A}^1_k) \xrightarrow{\mathrm{Id}_{K^*} \otimes \mathbb{Z}^{\mathrm{tr}}(p)} K_*)$, where $p : \mathbb{A}^1_k \to \mathrm{Spec}\, k$ is the projection. Let $Q : D(\mathrm{NST}(k)) \to \mathrm{DM}^{\mathrm{eff}}(k)$ be the quotient functor.

The category $D_{\mathbb{A}^1}(\mathrm{NST}(k))$ is defined as the full subcategory of the derived category $D(\mathrm{NST}(k))$ with objects the complexes K^* whose cohomology sheaves $h^n(K^*)_{\mathrm{Nis}}$ are strictly homotopy invariant.

It follows from Proposition 2.3 that $D_{\mathbb{A}^1}(\mathrm{NST}(k))$ is a triangulated subcategory of $D(\mathrm{NST}(k))$, and that the restriction of the usual t-structure on $D(\mathrm{NST}(k))$ defines a t-structure on $D_{\mathbb{A}^1}(\mathrm{NST}(k))$ with heart $\mathrm{HI}(k)$.

2.1.2. *The Suslin complex.* The category $\mathrm{PST}(k)$ has an internal Hom with

$$\mathcal{H}om(P,Q)(X) := \mathrm{Hom}_{\mathrm{PST}(k)}(P \otimes^{\mathrm{tr}} \mathbb{Z}^{\mathrm{tr}}(X), Q)$$

This defines internal Hom functors on $\mathrm{NST}(k)$, $C(\mathrm{PST}(k))$, $C(\mathrm{NST}(k))$ and on $D(\mathrm{NST}(k))$.

Given a smooth cosimplicial scheme $[n] \mapsto D^n$ and $K^* \in C^-(\mathrm{PST}(k))$ this gives us the simplicial object $[n] \mapsto \mathcal{H}om(\mathbb{Z}^{\mathrm{tr}}(D^n), K^*)$, and the associated mapping complex $K^*(D^*) \in C^-(\mathrm{PST}(k))$, with

$$K^*(D^*)^{-n}(X) := \oplus_m K^m(X \times D^{m+n}),$$

and with differential the usual alternating sum of face maps. For $K^* \in C(\mathrm{PST}(k))$, write K^* as a colimit of subcomplexes (using the canonical truncation $\tau_{\leq n}$): $K^* = \mathrm{colim}_{n\to+\infty}\tau_{\leq n}K^*$ and define $K^*(D^*) = \mathrm{colim}_{n\to+\infty}(\tau_{\leq n}K^*)(D^*)$.

Definition 2.5. For $K^* \in C(\mathrm{PST}(k))$, define

$$C^{Sus}(K^*) := K^*(\Delta^*_k).$$

Using the triangulation of $\Delta^n_k \times \Delta^1_k \cong \Delta^n_k \times \mathbb{A}^1_k$ again, one shows that for $K^* \in C(\mathrm{PST}(k))$, the cohomology presheaves $h^i(C^{Sus}(K^*))$ are homotopy invariant.

Definition 2.6. For $K^* \in C(\mathrm{PST}(k))$, the *nth Suslin homology* $H_n^{\mathrm{Sus}}(K^*)$ is defined by

$$H_n^{\mathrm{Sus}}(K^*, \mathbb{Z}) := H_n(C^{Sus}(K^*)(\mathrm{Spec}\, k)).$$

For $X \in \mathbf{Sm}_k$, we write $H_n^{\mathrm{Sus}}(X, \mathbb{Z})$ for $H_n^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X), \mathbb{Z})$.

Here are the central results about homotopy invariant PSTs and the Suslin complex.

Theorem 2.7 ([104, Chap. 3, Theorem 5.9])**.** *1. Let $K^* \in C(\mathrm{PST}(k))$ be a complex in* $\mathrm{PST}(k)$*. Suppose that the Nisnevich sheafification K^*_{Nis} is acyclic. Then the Zariski sheafification of the Suslin complex $C^{Sus}(K^*)_{\mathrm{Zar}}$ is also acyclic.*
2. For $K^* \in C(\mathrm{PST}(k))$, the canonical map $K \to C^{Sus}(K^*)$ induces an isomorphism $Q(K_{\mathrm{Nis}}) \to Q(C^{Sus}(K^*)_{\mathrm{Nis}})$ in $\mathrm{DM}^{\mathrm{eff}}(k)$*.*

We won't say anything about the proofs, except that some of the geometric input is a version of Chow's moving lemma, and the transfer structure is crucial; the analogous properties are not valid for arbitrary presheaves or Nisnevich sheaves.

2.2. **Motivic complexes.** The Suslin complex construction gives us presheaves of complexes $\mathbb{Z}(q)^*$ on $\mathbf{Sm}_k$ that will turn out to be the strictly functorial versions of Bloch's cycle complexes that satisfy (most of) the Beilinson-Lichtenbaum axioms.

Definition 2.8. For $q \geq 0$ be an integer, $\mathbb{Z}(q)^*$ is the presheaf of complexes on $\mathbf{Sm}_k$ defined by

$$\mathbb{Z}(q)^*(X) := C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(q))^*(X);$$

Explicitly, $\mathbb{Z}^{\mathrm{tr}}(q) := \mathbb{Z}^{\mathrm{tr}}(1)^{\otimes^{\mathrm{tr}}_{\mathrm{Nis}} q}$ and

$$\mathbb{Z}(q)^n(X) = \oplus_m \mathbb{Z}^{\mathrm{tr}}(q)^m(\Delta^{m-n} \times X)$$

Remark 2.9. Recall that

$$\mathbb{Z}^{\mathrm{tr}}(q)[2q] = (\mathbb{Z}^{\mathrm{tr}}(\mathbb{P}^1) \to \mathbb{Z}^{\mathrm{tr}}(\operatorname{Spec} k))^{\otimes q} \xleftarrow[\sim]{\text{q-iso}} [\ker(\mathbb{Z}^{\mathrm{tr}}(\mathbb{P}^1) \to \mathbb{Z}^{\mathrm{tr}}(\operatorname{Spec} k))]^{\otimes q},$$

with $\mathbb{Z}^{\mathrm{tr}}(\mathbb{P}^1)$ in degree 0, so $\mathbb{Z}^{\mathrm{tr}}(q)$ is quasi-isomorphic to a complex supported in degree $2q$, and thus $\mathbb{Z}(q)^*$ is quasi-isomorphic to a complex supported in degrees $\leq 2q$. Actually, we shall see that $\mathcal{H}^n(\mathbb{Z}(q)^*) = 0$ for $n > q$.

2.2.1. *The localization theorem and the embedding theorem.* The Suslin complex construction gives an effective way of understanding the localization

$$Q : D(\mathrm{NST}(k)) \to \mathrm{DM}^{\mathrm{eff}}(k).$$

We note that $C^{Sus}(K^*)_{\mathrm{Nis}}$ is in $D_{\mathbb{A}^1}(\mathrm{NST}(k))$ for K^* in $C(\mathrm{PST}(k))$. Indeed, the cohomology presheaves $h^n(C^{Sus}(K^*))$ are homotopy invariant PSTs, so the Nisnevich cohomology sheaves $h^n(C^{Sus}(K^*))_{\mathrm{Nis}}$ are strictly homotopy invariant, by Proposition 2.3.

Theorem 2.10. *Sending $K^* \in C(\mathrm{PST}(k))$ to $C^{Sus}(K^*)_{\mathrm{Nis}} \in D_{\mathbb{A}^1}(\mathrm{NST}(k))$ defines an exact functor*

$$RC^{\mathrm{Sus}} : D(\mathrm{NST}(k)) \to D_{\mathbb{A}^1}(\mathrm{NST}(k))$$

that is left adjoint to the inclusion $D_{\mathbb{A}^1}(\mathrm{NST}(k)) \hookrightarrow D(\mathrm{NST}(k))$. Moreover, RC^{Sus} factors through the localization $Q : D(\mathrm{NST}(k)) \to D_{\mathbb{A}^1}(\mathrm{NST}(k))$ and defines an equivalence of $D_{\mathbb{A}^1}(\mathrm{NST}(k))$ with $\mathrm{DM}^{\mathrm{eff}}(k)$, with

$$RC^{\mathrm{Sus}}(\mathbb{Z}^{tr}(X)) = C^{\mathrm{Sus}}(\mathbb{Z}^{tr}(X))_{\mathrm{Nis}}$$

for all $X \in \mathbf{Sm}_k$.

We henceforth identify $\mathrm{DM}^{\mathrm{eff}}(k)$ with the subcategory $D_{\mathbb{A}^1}(\mathrm{NST}(k))$ of $D(\mathrm{NST}(k))$ via RC^{Sus}.

Via the localization functor $Q = RC^{\mathrm{Sus}} : D(\mathrm{NST}(k)) \to \mathrm{DM}^{\mathrm{eff}}(k)$, the tensor structure $\otimes^{\mathrm{tr}}_{\mathrm{Nis}}$ on $D(\mathrm{NST}(k))$ induces a tensor structure on $\mathrm{DM}^{\mathrm{eff}}(k)$, making $\mathrm{DM}^{\mathrm{eff}}(k)$ a tensor triangulated category, with internal Hom, denoted $\underline{\mathcal{H}om}$.

We let $\mathbb{Z}(n) = C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(n))_{\mathrm{Nis}}$, $\mathbb{Z}(X) := C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))_{\mathrm{Nis}}$ and note that $\mathbb{Z}(0)$ is the unit for the tensor structure on $\mathrm{DM}^{\mathrm{eff}}(k)$. For an object M of $\mathrm{DM}^{\mathrm{eff}}(k)$, we write $M(n)$ for $M \otimes \mathbb{Z}(n)$.

Corollary 2.11. *1. For $X \in \mathbf{Sm}_k$ and $n \in \mathbb{Z}$, we have a canonical isomorphism*

$$H^{\mathrm{Sus}}_n(X, \mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}(0)[n], \mathbb{Z}(X))$$

Moreover $H^{\mathrm{Sus}}_n(X, \mathbb{Z}) = 0$ for $n < 0$.

2. Suppose $X \in \mathbf{Sm}_k$ is a union of open subschemes U, V. Then we have a long exact Mayer-Vietoris sequence

$$\begin{aligned}\ldots \to H^{\mathrm{Sus}}_n(U \cap V, \mathbb{Z}) \to H^{\mathrm{Sus}}_n(U, \mathbb{Z}) \oplus H^{\mathrm{Sus}}_n(V, \mathbb{Z}) \to H^{\mathrm{Sus}}_n(X, \mathbb{Z})&\\ \xrightarrow{\partial} H^{\mathrm{Sus}}_{n-1}(U \cap V, \mathbb{Z}) \to \ldots \to H^{\mathrm{Sus}}_0(X, \mathbb{Z}) \to 0&\end{aligned}$$

Proof. (1) uses the adjoint property:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}(0)[n], \mathbb{Z}(X)) &= \mathrm{Hom}_{D_{\mathbb{A}^1}(\mathrm{NST}(k))}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(\operatorname{Spec} k))[n], C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X)))\\ &= \mathrm{Hom}_{D(\mathrm{NST}(k))}(\mathbb{Z}^{\mathrm{tr}}(\operatorname{Spec} k)[n], C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))_{\mathrm{Nis}})\\ &= H_n(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))(\operatorname{Spec} k))\\ &= H^{\mathrm{Sus}}_n(X, \mathbb{Z})\end{aligned}$$

Since $C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))(\operatorname{Spec} k)$ is concentrated in homological degrees ≥ 0, we have $H_n^{\mathrm{Sus}}(X,\mathbb{Z}) = 0$ for $n < 0$.

For (2), we have the right exact sequence in $\mathrm{PST}(k)$

$$0 \to \mathbb{Z}^{\mathrm{tr}}(U \cap V) \to \mathbb{Z}^{\mathrm{tr}}(U) \oplus \mathbb{Z}^{\mathrm{tr}}(V) \xrightarrow{j_{U*}+j_{V*}} \mathbb{Z}^{\mathrm{tr}}(X)$$

but the last map is not in general a surjective map of presheaves, for example, if X is irreducible and U and V are proper open subsets, then the diagonal in $\mathbb{Z}^{\mathrm{tr}}(X)(X) \subset Z_{\dim X}(X \times X)$ is not in the image. However, $\mathbb{Z}^{\mathrm{tr}}(Y)$ is a Nisnevich sheaf for all $Y \in \mathbf{Sm}_k$ and we claim that the map $\mathbb{Z}^{\mathrm{tr}}(U) \oplus \mathbb{Z}^{\mathrm{tr}}(V) \to \mathbb{Z}^{\mathrm{tr}}(X)$ is a surjective map in $\mathrm{NST}(k)$.

Indeed, the points in the Nisnevich topology are the hensel local rings $\mathcal{O}^h_{Y,y}$ for $y \in Y \in \mathbf{Sm}_k$. Given $Z \in \mathbb{Z}^{\mathrm{tr}}(X)(\mathcal{O}^h_{Y,y})$, we have the restriction $Z_y \in \mathbb{Z}^{\mathrm{tr}}(X)(k(y))$, with support

$$|Z_y| = \{z_1, \ldots, z_s\},$$

where the z_i are closed points in $X_{k(y)}$. We can arrange the z_i so that $z_1, \ldots, z_r$ is in $U_{k(y)}$ and $z_{r+1}, \ldots, z_s$ is in $V_{k(y)}$. Since $\mathcal{O}^h_{Y,y}$ is hensel, we can write the support of Z as a disjoint union

$$|Z| = \amalg_{i=1}^s Z_i$$

with $(Z_i \cap X_{k(y)})_{\mathrm{red}} = z_i$. Since $\mathcal{O}^h_{Y,y}$ is local, this says that $Z_i \subset U_{\mathcal{O}^h_{Y,y}}$ for $i = 1, \ldots, r$ and $Z_i \subset V_{\mathcal{O}^h_{Y,y}}$ for $i = r+1, \ldots, s$, thus Z is in the image of $\mathbb{Z}^{\mathrm{tr}}(U)(\mathcal{O}^h_{Y,y}) \oplus \mathbb{Z}^{\mathrm{tr}}(V)(\mathcal{O}^h_{Y,y}) \to \mathbb{Z}^{\mathrm{tr}}(X)(\mathcal{O}^h_{Y,y})$.

Letting $\overline{\mathbb{Z}}^{\mathrm{tr}}(X)$ denote the presheaf image of $j_{U*} + j_{V*}$, we thus have the exact sequences in $\mathrm{PST}(k)$

$$0 \to \mathbb{Z}^{\mathrm{tr}}(U \cap V) \to \mathbb{Z}^{\mathrm{tr}}(U) \oplus \mathbb{Z}^{\mathrm{tr}}(V) \xrightarrow{j_{U*}+j_{V*}} \overline{\mathbb{Z}}^{\mathrm{tr}}(X) \to 0$$

$$0 \to \overline{\mathbb{Z}}^{\mathrm{tr}}(X) \to \mathbb{Z}^{\mathrm{tr}}(X) \to \mathbb{Z}^{\mathrm{tr}}(X)/\overline{\mathbb{Z}}^{\mathrm{tr}}(X) \to 0,$$

with $(\mathbb{Z}^{\mathrm{tr}}(X)/\overline{\mathbb{Z}}^{\mathrm{tr}}(X))_{\mathrm{Nis}} = 0$.

By Theorem 2.7(3), $C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X)/\overline{\mathbb{Z}}^{\mathrm{tr}}(X))(\operatorname{Spec} k)$ is acyclic, and as $C^{\mathrm{Sus}}(-)$ transforms exact sequences in $\mathrm{PST}(k)$ to termwise exact sequences in $C^-(\mathrm{PST}(k))$, we see that

$$C^{\mathrm{Sus}}(\overline{\mathbb{Z}}^{\mathrm{tr}}(X))(\operatorname{Spec} k) \to C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))(\operatorname{Spec} k)$$

is a quasi-isomorphism. Thus the sequence

$$C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(U \cap V))(\operatorname{Spec} k) \to C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(U))(\operatorname{Spec} k) \oplus C^{\mathrm{Sus}}((\mathbb{Z}^{\mathrm{tr}}(V))(\operatorname{Spec} k)$$
$$\xrightarrow{j_{U*}+j_{V*}} C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))(\operatorname{Spec} k)$$

extends to a distinguished triangle in $D^-(\mathbf{Ab})$, yielding the desired Mayer-Vietoris sequence by taking homology. $\square$

Remark 2.12. The map $\mathbb{Z}^{\mathrm{tr}}(U) \oplus \mathbb{Z}^{\mathrm{tr}}(V) \xrightarrow{j_{U*}+j_{V*}} \mathbb{Z}^{\mathrm{tr}}(X)$ is in general not surjective as a map of Zariski sheaves and the proof of the Mayer-Vietoris sequence given above points out one of the main reasons that the Nisnevich topology, rather than the Zariski topology, is used in the construction of $\mathrm{DM}^{\mathrm{eff}}(k)$.

We have the evident map of triangulated categories $K^b(\mathrm{Cor}(k)) \to D(\mathrm{NST}(k))$ sending $[X]$ to $\mathbb{Z}^{\mathrm{tr}}(X)$, and the localization map $q : K^b(\mathrm{Cor}(k)) \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

Theorem 2.13 (Embedding theorem, [104, Chap. 5, Theorem 3.26])**.** *There is a unique exact functor $i : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \to \mathrm{DM}^{\mathrm{eff}}(k)$ sending $M^{\mathrm{eff}}(X)$ to $\mathbb{Z}(X) := C^{\mathrm{Sus}}(\mathbb{Z}^{tr}(X))_{\mathrm{Nis}}$ for each $X \in \mathbf{Sm}_k$, and making the diagram*

$$\begin{tikzcd} K^b(\mathrm{Cor}(k)) \arrow[r] \arrow[d, "q"] & D(\mathrm{NST}(k)) \arrow[d, "RC^{\mathrm{Sus}}"] \\ \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \arrow[r, "i"] & D_{\mathbb{A}^1}(\mathrm{NST}(k)) \end{tikzcd}$$

commute. Moreover i is a fully faithful embedding with dense image.

Proof. For the existence of i, we need to show that the map $X \mapsto C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))_{\mathrm{Nis}}$ sends the two complexes defining the localization q to acyclic complexes in $D_{\mathbb{A}^1}(\mathrm{NST}(k))$. The cone of the map $C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X \times \mathbb{A}^1)) \xrightarrow{p_*} C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))$ is acyclic, by the homotopy invariance of the Suslin complex construction, hence the same holds for the Nisnevich sheafification. The argument used in the proof of Corollary 2.11(2) shows that the Mayer-Vietoris sequence

$$C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(U \cap V))_{\mathrm{Nis}} \to C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(U))_{\mathrm{Nis}} \oplus C^{\mathrm{Sus}}((\mathbb{Z}^{\mathrm{tr}}(V))_{\mathrm{Nis}} \to C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))_{\mathrm{Nis}}$$

gives a quasi-isomorphism of $C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))_{\mathrm{Nis}}$ with the cone of the map $C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(U\cap V))_{\mathrm{Nis}} \to C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(U))_{\mathrm{Nis}} \oplus C^{\mathrm{Sus}}((\mathbb{Z}^{\mathrm{tr}}(V))_{\mathrm{Nis}}$, so the total complex of the Mayer-Vietoris sequence is zero in $D(\mathrm{NST}(k))$. This shows the existence of the exact functor i.

To show that i is fully faithful, one relies on results of Neeman. One considers a triangulated category $\mathcal{T}$ admitting arbitrary small direct sums and its full subcategory $\mathcal{T}_0$ of compact objects. If $\mathcal{L}_0$ is a thick subcategory of $\mathcal{T}_0$, generating a localizing subcategory $\mathcal{L}$ of $\mathcal{T}$, then Neeman shows that the induced exact functor

$$\mathcal{T}_0/\mathcal{L}_0 \to \mathcal{T}/\mathcal{L}$$

is fully faithful with dense image (see [71, Theorem 1.14] for the precise statement). Taking $\mathcal{T} = D(\mathrm{PST}(k))$, we need to consider the localizing subcategory $\mathcal{L}$ generated by complexes $\mathbb{Z}^{\mathrm{tr}}(X \times \mathbb{A}^1) \to \mathbb{Z}^{\mathrm{tr}}(X)$ together with all $P \in \mathrm{PST}(k)$ such that $P_{\mathrm{Nis}} = 0$, and then show that $\mathcal{L}$ is generated by the complexes $\mathbb{Z}^{\mathrm{tr}}(X \times \mathbb{A}^1) \to \mathbb{Z}^{\mathrm{tr}}(X)$ and the "Mayer-Vietoris" complexes

$$\mathbb{Z}^{\mathrm{tr}}(U \cap V) \to \mathbb{Z}^{\mathrm{tr}}(U) \oplus \mathbb{Z}^{\mathrm{tr}}(V) \xrightarrow{j_{U*}+j_{V*}} \bar{\mathbb{Z}}^{\mathrm{tr}}(X)$$

Letting $\mathcal{L}'$ be the localizing subcategory generated by these two types of complexes, we need to show that if $P_{\mathrm{Nis}} = 0$, then P is in $\mathcal{L}'$.

For such a P, consider the functors

$$H^n(X) := \mathrm{Hom}_{D(\mathrm{PST}(k))/\mathcal{L}'}(\mathbb{Z}^{\mathrm{tr}}(X), P[n]).$$

It suffices to show that $H^n(X) = 0$ for all $X \in \mathbf{Sm}_k$ and all n if $P_{\mathrm{Nis}} = 0$.

Since $\mathcal{L}'$ contains all Mayer-Vietoris complexes, the family $\{H^n(X)\}$ admits long exact Mayer-Vietoris sequences for Zariski open covers, so we need only show that the Zariski sheafifications H^n_{Zar} of the presheaves $X \mapsto H^n(X)$ are all zero. But the $H^n(-)$ are PSTs and by the definition of $\mathcal{L}'$, the $H^n(-)$ are homotopy invariant, hence by Proposition 2.3(6), we have

$$H^n_{\mathrm{Zar}} = H^n_{\mathrm{Nis}}$$

An element $\alpha \in H^n(X)$, i.e., a morphism $\alpha : \mathbb{Z}^{\mathrm{tr}}(X) \to P[n]$ in $D(\mathrm{PST}(k))/\mathcal{L}'$, is represented by a diagram in $D(\mathrm{PST}(k))$

$$\begin{tikzcd} \mathbb{Z}^{\mathrm{tr}}(X) \arrow[dr] & & P[n] \arrow[dl, "g"] \\ & K & \end{tikzcd}$$

such that the cone of g is in $\mathcal{L}'$. The proof of homotopy invariance for C^{Sus} shows that the canonical map $K \to C^{\mathrm{Sus}}(K)$ is an isomorphism in $D(\mathrm{PST}(k))/\mathcal{L}'$. Moreover, $P_{\mathrm{Nis}} = 0$ implies that $C^{\mathrm{Sus}}(P)_{\mathrm{Nis}} = 0$ in $D(\mathrm{NST}(k))$ (Theorem 2.7(1)). Since the functor $C^{\mathrm{Sus}}(-)$ sends the elements of $\mathcal{L}'$ to zero, this implies that $C^{\mathrm{Sus}}(K)_{\mathrm{Nis}} = 0$ in $D(\mathrm{NST}(k))$ as well. But then there is a Nisnevich cover $U \to X$ such that the composition

$$\mathbb{Z}^{\mathrm{tr}}(U) \to \mathbb{Z}^{\mathrm{tr}}(X) \xrightarrow{g \circ \alpha} K \to C^{\mathrm{Sus}}(K)$$

is zero, hence the pullback of α to $H^n(U)$ is zero, and thus $H^n_{\mathrm{Nis}} = 0$. $\square$

Corollary 2.14. *For $X \in \mathbf{Sm}_k$, $p, q \in \mathbb{Z}$, $q \geq 0$, we have a canonical isomorphism*

$$H^p(X, \mathbb{Z}(q)) \cong \mathbb{H}^p(X_{\mathrm{Nis}}, \mathbb{Z}(q)^*)$$

natural in X.

Proof. Let $\mathbb{Z}_{\mathrm{Nis}}(X) \in \mathrm{Sh}^{\mathrm{Nis}}(\mathbf{Sm}_k)$ be the sheaf represented by X and let $\mathbb{Z}_X \in \mathrm{Sh}^{\mathrm{Nis}}(X)$ be the constant sheaf on X. We have

$$\begin{aligned} H^p(X, \mathbb{Z}(q)) &:= \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M^{\mathrm{eff}}(X), \mathbb{Z}(q)[p]) \\ &\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}(X), \mathbb{Z}(q)[p]) \\ &\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))_{\mathrm{Nis}}, C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(q))_{\mathrm{Nis}}[p]) \\ &\cong \mathrm{Hom}_{D(\mathrm{NST}(k))}(\mathbb{Z}^{\mathrm{tr}}(X), \mathbb{Z}(q)^*[p]) \\ &\cong \mathrm{Hom}_{D(\mathrm{Sh}^{\mathrm{Nis}}(\mathbf{Sm}_k))}(\mathbb{Z}_{\mathrm{Nis}}(X), \mathbb{Z}(q)^*[p]) \\ &\cong \mathrm{Hom}_{D(\mathrm{Sh}^{\mathrm{Nis}}(X))}(\mathbb{Z}_X, \mathbb{Z}(q)^*_{|X_{\mathrm{Nis}}}[p]) \\ &= \mathbb{H}^p(X_{\mathrm{Nis}}, \mathbb{Z}(q)^*) \end{aligned}$$

$\square$

2.3. **Motivic cohomology and the higher Chow groups.** We have seen an interpretation of Suslin homology as maps in $\mathrm{DM}^{\mathrm{eff}}(k)$. However, Suslin homology is not closely related to Bloch's higher Chow groups; they have quite different functoriality. There is a natural map $H_0^{\mathrm{Sus}}(X, \mathbb{Z}) \to \mathrm{CH}_0(X)$, which is always surjective, and even an isomorphism if X is proper. To connected motivic cohomology with the higher Chow groups, we need to introduce some new NSTs which capture some of these aspects of "non-properness"; these play an important role in duality.

2.3.1. *Equi-dimensional cycles and quasi-finite cycles.*

Definition 2.15. Let X be a finite type k-scheme. For an integer $r \geq 0$ and a $Y \in \mathbf{Sm}_k$, define $z_{\mathrm{equi},r}(X)(Y)$ to be the subgroup of $z_{\dim Y + r}(Y \times X)$ freely generated by integral closed subschemes $W \subset Y \times X$ such that the projection $p_Y : W \to Y$ is equi-dimensional of relative dimension r. Precisely, this means that for each point $y \in Y$ and each irreducible component Z of $W \cap y \times X$, we have $\dim_{k(y)} Z = r$ (this condition is satisfied if $W \cap y \times X = \emptyset$).

We let $z_{\mathrm{qfin}}(X)(Y) := z_{\mathrm{equi},0}(X)(Y)$.

Remarks 2.16. 1. For each integral $W \in z_{\mathrm{equi},r}(X)(Y)$, the projection $p_Y : W \to Y$ is dominant onto an irreducible component of Y. In particular, $z_{\mathrm{qfin}}(X)(Y)$ is the subgroup of $z_{\dim Y}(Y \times X)$ freely generated by integral closed subschemes $W \subset Y \times X$ that are quasi-finite and dominant over an irreducible component of Y.

2. $Y \mapsto z_{\mathrm{equi},r}(X)(Y)$ extends canonically to an NST: For $Z \in \mathrm{Cor}_k(Y', Y)$ we have

$$Z^* : z_{\mathrm{equi},r}(X)(Y) \to z_{\mathrm{equi},r}(X)(Y')$$

defined by the usual formula for correspondences

$$Z^*(W) := p_{Y' \times X*}(p^*_{Y \times X}(W) \cdot p_{Y' \times Y}(Z))$$

This makes sense even for X not smooth, by taking local closed embeddings of X in a smooth k-scheme M, taking the intersection product on $Y' \times Y \times M$ and noting that the resulting cycle is supported in $Y' \times Y \times X$.

Definition 2.17. For $X \in \mathbf{Sm}_k$, define $\mathbb{Z}(X)^c$ to be the image in $\mathrm{DM}^{\mathrm{eff}}(k)$ of the NST $z_{\mathrm{qfin}}(X)$, i.e., $\mathbb{Z}(X)^c := C^{\mathrm{Sus}}(z_{\mathrm{qfin}}(X))_{\mathrm{Nis}}$.

We have the localization distinguished triangle:

Theorem 2.18. *Let $i : W \to X$ a closed immersion in $\mathbf{Sch}_k$, with open complement $j : U \to X$, and let $r \geq 0$ be an integer, giving the right exact sequence in* $\mathrm{NST}(k)$

$$0 \to z_{\mathrm{equi},r}(W) \xrightarrow{i_*} z_{\mathrm{equi},r}(X) \xrightarrow{j^*} z_{\mathrm{equi},r}(U)$$

Then the induced sequence

$$C^{\mathrm{Sus}}(z_{\mathrm{equi},r}(W))_{\mathrm{Nis}} \xrightarrow{i_*} C^{\mathrm{Sus}}(z_{\mathrm{equi},r}(X))_{\mathrm{Nis}} \xrightarrow{j^*} C^{\mathrm{Sus}}(z_{\mathrm{equi},r}(U))_{\mathrm{Nis}}$$

extends canonically to a distinguished triangle in $\mathrm{DM}^{\mathrm{eff}}(k)$ *(after inverting* $\mathrm{char} k$ *if* $\mathrm{char} k > 0$*).*

For the proof, we need the extension of Theorem 2.7(3) to the setting of the *cdh topology.* This is the Grothendieck topology on $\mathbf{Sch}_k$ generated by the Nisnevich topology and the coverings given by “abstract blow-up squares”: a cartesian square

$$\begin{tikzcd} Y' \arrow[r, "i'"] \arrow[d] & Y \arrow[d, "f"] \\ X' \arrow[r, "i"] & X \end{tikzcd}$$

with i and i' closed immersions, f proper and inducing an isomorphism $f_0 : Y \setminus Y' \to X \setminus X'$. For such a square, the map $X' \amalg Y \to X$ is a cdh cover.

We have the fundamental result (see e.g. [104, Chap. 5, Theorem 4.1.2]).

Theorem 2.19. *Take $K \in C(\mathrm{PST})$ and suppose that the cdh-sheafification K_{cdh} is acyclic Then $C^{\mathrm{Sus}}(K)_{\mathrm{Zar}}$ is acyclic (after inverting* $\mathrm{char} k$ *if this is > 0).*

To apply this to the localization theorem, note that for $Y \in \mathbf{Sm}_k$ and $z \in z_{\mathrm{equi},r}(U)(Y) \subset Z_{r+\dim Y}(U \times Y)$. there is a blow-up $Y' \to Y$ of Y (with Y' smooth in characteristic zero) such that the proper transform z' of z to $Z_{r+\dim Y}(U \times Y')$ has closure in $X \times Y'$ that is in $z_{\mathrm{equi},r}(X)(Y')$. In other words $\mathrm{coker}(j^*)_{\mathrm{cdh}} = 0$, which shows that $j^*(C^{\mathrm{Sus}}(z_{\mathrm{equi},r}(X)))_{\mathrm{Nis}}) \hookrightarrow C^{\mathrm{Sus}}(z_{\mathrm{equi},r}(U))_{\mathrm{Nis}}$ is a quasi-isomorphism, as desired. In positive characteristic, one needs to use alterations.

We consider the Suslin complex $C^{\mathrm{Sus}}(z_{\mathrm{equi},r}(X))$. In homological degree n, $C^{\mathrm{Sus}}(z_{\mathrm{equi},r}(X))_n(k)$ is the subgroup of $Z_{n+r}(\Delta^n_k \times X)$ freely generated by integral closed subschemes $W \subset \Delta^n_k \times X$ that are equi-dimensional of dimension r over Δ^n_k. In particular each such W intersects $F \times X$ properly for each face F of Δ^n_k, so we have a natural inclusion of (homological) complexes

$$C^{\mathrm{Sus}}_*(z_{\mathrm{equi},r}(X))(k) \hookrightarrow z_r(X,*).$$

2.3.2. *The moving lemmas of Suslin and Friedlander-Lawson.* The following result is mainly taken from [104, Chap. 5, Proposition 4.2.9], [104, Chap. 5, §4.3] and [104, Chap. 6, Theorem 2.1].

Theorem 2.20. *In the following, we invert* $\mathrm{char} k$ *if* $\mathrm{char} k > 0$.

1. Suppose X is a finite-type k-scheme. Then the inclusion

$$C^{\mathrm{Sus}}_*(z_{\mathrm{equi},r}(X))(\mathrm{Spec}\, k) \hookrightarrow z_r(X,*)$$

is a quasi-isomorphism.

2. $\mathbb{Z}(X)^c \in \mathrm{DM}^{\mathrm{eff}}(k)$ *is in the image of an object* $M^c(X)$ *of* $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

3. There are canonical isomorphisms

$$C^{\mathrm{Sus}}(z_{\mathrm{equi},r}(X)) \cong \underline{\mathcal{H}om}(\mathbb{Z}(r)[2r], \mathbb{Z}(X)^c)$$

in $\mathrm{DM}^{\mathrm{eff}}(k)$.

4. For $X \in \mathbf{Sm}_k$ *of pure dimension d over k, the dual* $M(X)^\vee$ *of* $M(X)$ *in* $\mathrm{DM}_{\mathrm{gm}}(k)$ *is given by*

$$M(X)^\vee = M^c(X)(-d)[-2d].$$

Some comments on (1). The proof is in two parts. For X affine, Suslin [104, Chap. 6] constructs an explicit homotopy, i.e., a geometric moving lemma, that "moves" cycles in $z_r(X,n)$ to cycles in $z_{\mathrm{equi},r}(X)(\Delta^n)$. For this, he takes a closed embedding $X \hookrightarrow \mathbb{A}^N_k$ for some N and reduces to the case $X = \mathbb{A}^N$. Let

$$h_n : \Delta^n_k \to \mathbb{A}^1_k$$

be the function $t_0 \cdot t_1 \cdots t_n$ and let $\partial\Delta^n \subset \Delta^n$ be the union of the codimension one faces, i.e., the divisor on Δ^n_k defined by $h_n = 0$. Suslin constructs, inductively in n, certain maps $\Phi_n : \Delta^n \times \mathbb{A}^N \to \Delta^n \times \mathbb{A}^N$ over $\mathbb{A}^N$ with the property that $\Phi_n \circ (\delta^n_i \times \mathrm{Id}) = (\delta^n_i \times \mathrm{Id}) \circ \Phi_{n-1}$ for all $i = 0, \ldots, n$, $n \geq 1$, starting with $\Phi_0 = \mathrm{Id}_{\mathbb{A}^N}$ This implies that the maps Φ_{n-1} on the codimension one faces of Δ^n fit together to give a map $\partial\Phi_n : \partial\Delta^n \times \mathbb{A}^N \to \partial\Delta^n \times \mathbb{A}^N$ over $\mathbb{A}^N$, which we write as

$$\partial\Phi_n(t,y) = (\partial\phi_n(t,y), y);\ \partial\phi_n(t,y) : \partial\Delta^n \times \mathbb{A}^N \to \partial\Delta^n.$$

Since $\partial\Delta^n \times \mathbb{A}^N$ is a closed subscheme of the affine space $\Delta^n \times \mathbb{A}^N \cong \mathbb{A}^{n+N}$, we can extend $\partial\phi_n$ to a map $\widetilde{\partial\phi}_n : \Delta^n \times \mathbb{A}^N \to \Delta^n$.

Let $\Delta^n_0 \subset \mathbb{A}^{n+1}$ be the hyperplane $\sum_{i=0}^n t_i = 0$, so Δ^n_0 acts on $\Delta^n \subset \mathbb{A}^{n+1}$ by vector addition. Choose a map $p_n : \mathbb{A}^N \to \Delta^n_0$ and define Φ_n by

$$\Phi_n(t,y) := (\widetilde{\partial\phi}_n(t,y) + h_n(t) \cdot p_n(y), y).$$

This continues the inductive construction, subject to the choice of the maps p_n, $n = 1, 2, \ldots$.

Suslin shows that for a given finite set of cycles $\{Z_1, \ldots, Z_s\} \subset z_r(X, n)$, by choosing the $p_n(y) = (1 - \sum_j p_{n,j}(y), p_{n,1}(y), \ldots, p_{n,n}(y))$ with the $p_{n,j}(y)$ sufficiently general polynomials of sufficiently high degree in $y = (y_1, \ldots, y_N)$, the cycles $\Phi_n(t, y)^*(Z_i)$ are all in $z_{\mathrm{equi},r}(X)(\Delta^n)$. Using this, he shows that, for each finitely generated subcomplex $z_r(X,*)_{\mathcal{W}} \subset z_r(X,*)$, there is a map of complexes

$$\Phi^*_{\mathcal{W}} : z_r(X,*)_{\mathcal{W}} \to C^{\mathrm{Sus}}_*(z_{\mathrm{equi},r}(X))(k)$$

such that the composition of $\Phi^*_{\mathcal{W}}$ with the inclusion $C^{\mathrm{Sus}}_*(z_{\mathrm{equi},r}(X))(\mathrm{Spec}\, k) \hookrightarrow z_r(X,*)$ is homotopic to the inclusion $z_r(X,*)_{\mathcal{W}} \hookrightarrow z_r(X,*)$. This shows that $C^{\mathrm{Sus}}_*(z_{\mathrm{equi},r}(X))(\mathrm{Spec}\, k) \hookrightarrow z_r(X,*)$ is a quasi-isomorphism, at least for X affine.

Now take X to be a finite type k-scheme. We prove (1) by noetherian induction, the case dimension zero being trivially true. Let $Y \subset X$ be a proper closed subscheme such that $X \backslash Y$ is affine. Evaluating at $\mathrm{Spec}\, k$, this gives us the commutative diagram in $D(\mathbf{Ab})$

$$\begin{tikzcd}
C^{\mathrm{Sus}}_*(z_{\mathrm{equi},r}(Y))(\mathrm{Spec}\, k) \arrow[r] \arrow[d, "i_*"] & z_r(Y,*) \arrow[d, "i_*"] \\
C^{\mathrm{Sus}}_*(z_{\mathrm{equi},r}(X))(\mathrm{Spec}\, k) \arrow[r] \arrow[d, "j^*"] & z_r(X,*) \arrow[d, "j^*"] \\
C^{\mathrm{Sus}}_*(z_{\mathrm{equi},r}(X \setminus Y))(\mathrm{Spec}\, k) \arrow[r] \arrow[d] & z_r(X \setminus Y,*) \arrow[d] \\
C^{\mathrm{Sus}}_*(z_{\mathrm{equi},r}(Y))(\mathrm{Spec}\, k)[1] \arrow[r] & z_r(Y,*)[1]
\end{tikzcd}$$

The right-hand column is a distinguished triangle by Bloch's moving lemma and the left-hand column is distinguished by Theorem 2.18. Induction shows the 1st and 4th horizontal arrows are quasi-isomorphisms and the affine case shows that the 3rd horizontal arrow is a quasi-isomorphism as well. This proves (1).

For (3), this relies on the Friedlander-Voevodsky extension of the Friedlander-Lawson moving lemma (see [104, Chap. 4, §6] and [26]). For $X \in \mathbf{Sch}_k$, $Y, U \in \mathbf{Sm}_k$, with U of dimension u, we have the map

$$z_{\mathrm{equi},s}(X)(U \times Y) \to z_{\mathrm{equi},s+u}(X \times U)(Y)$$

by noting that if $W \subset X \times U \times Y$ is equi-dimensional of relative dimension s over $Y \times U$, then W is equi-dimensional of relative dimension $s+u$ over Y. The induced map

$$C^{\mathrm{Sus}}(z_{\mathrm{equi},s}(X))(U \times Y) \to C^{\mathrm{Sus}}(z_{\mathrm{equi},s+u}(X \times U))(Y)$$

thus gives the map of complexes of presheaves

$$\underline{\mathcal{H}om}(\mathbb{Z}^{\mathrm{tr}}(U), C^{\mathrm{Sus}}(z_{\mathrm{equi},s}(X))) \to C^{\mathrm{Sus}}(z_{\mathrm{equi},s+u}(X \times U)) \tag{2.1}$$

Theorem 2.21 ([104, Chap. 4, Theorem 7.4])**.** *The map* (2.1) *defines a quasi-isomorphism in* $C(\mathrm{NST}(k))$.

We won't give a proof of this, except to note that the original result of Friedlander-Lawson on "moving families of algebraic cycles of bounded degree" [26, Theorem 3.7] says as a special case that, given smooth (irreducible) varieties $X, Y \in \mathbf{Sm}_k$ with Y projective, and $W \in Z_n(X \times Y)$ a dimension n cycle with $n \geq \dim Y$, then

W is rationally equivalent to some $W' \in z_{equi,n-\dim Y}(X \times Y) \subset Z_n(X \times Y)$. This fact is the starting point of the proof of Theorem 2.21; the entire proof is quite involved, occupying most of [104, Chap. 4].

Taking $U = (\mathbb{P}^1)^r$, and understanding $\mathbb{Z}^{\mathrm{tr}}(r)[2r]$ as a summand of $\mathbb{Z}^{\mathrm{tr}}((\mathbb{P}^1)^r)$ gives the isomorphism in $DM^{\mathrm{eff}}(k)$

$$\underline{\mathcal{H}om}(\mathbb{Z}(r)[2r], C^{\mathrm{Sus}}(z_{\mathrm{equi},s}(X))) \cong C^{\mathrm{Sus}}(z_{\mathrm{equi},r+s}(X)))$$

proving (3).

The proof of (4) in case X projective is just by noting that $\mathbb{Z}(X)^c = \mathbb{Z}(X)$ for projective X, and using duality in $\mathrm{DM}_{\mathrm{gm}}(k)$. The general result (assuming resolution of singularities) follows by taking a smooth projective compactification of X with normal crossing divisor as complement, using the localization and Gysin triangles (see below) and induction on dimension.

2.3.3. *Gysin triangle, the projective bundle formula and Chern classes.* Let $i : W \to X$ be a closed immersion in $\mathbf{Sch}_k$ with open complement $j : U \to X$. Taking $r = 0$ in Theorem 2.18 gives the distinguished localization triangle in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$

$$M^c(W) \xrightarrow{i_*} M^c(X) \xrightarrow{j^*} M^c(U) \xrightarrow{\partial} M^c(W)[1].$$

If X and W are smooth, and W has codimension c in X, we apply duality and Theorem 2.20(4), giving the Gysin distinguished triangle

$$M(U) \xrightarrow{j_*} M(X) \xrightarrow{i^*} M(W)(c)[2c] \to M(U)[1]$$

The map i^* gives us the map (Gysin pushforward)

$$i_* : H^p(W, \mathbb{Z}(q)) \to H^{p+2c}(X, \mathbb{Z}(q+c))$$

by applying i^* to $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(-, \mathbb{Z}(q+c)[p+2c])$.

We also have first Chern classes for line bundles, the projective bundle formula and the resulting theory of Chern classes for vector bundles: For $L \to Y$ a line bundle on some $Y \in \mathbf{Sm}_k$, we have $c_1(L) \in H^2(Y, \mathbb{Z}(1)) = \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M(Y), \mathbb{Z}(1)[2])$ defined by

$$c_1(L) := s^* s_*(1_Y)$$

where $s : Y \to L$ is the 0-section and $1_Y \in H^0(Y, \mathbb{Z}(0))$ is the class of the map $M(Y) \to M(\mathrm{Spec}\, k) = Z(0)$ induced by the structure map $p : Y \to \mathrm{Spec}\, k$.

For $V \to X$ a rank $n+1$ vector bundle over $X \in \mathbf{Sm}_k$, with associated projective space bundle $q : \mathbb{P}(V) \to X$ and tautological quotient line bundle $\mathcal{O}(1)$, we have

$$M(\mathbb{P}(V)) \cong \oplus_{i=0}^n M(X)(i)[2i]$$

where one maps $M(\mathbb{P}(V))$ to $M(X)(i)[2i]$ by

$$M(\mathbb{P}(V)) \xrightarrow{\times c_1(\mathcal{O}(1))^i} M(\mathbb{P}(V))(i)[2i] \xrightarrow{q_*} M(X)(i)[2i]$$

This induces the usual isomorphism

$$H^p(\mathbb{P}(V), \mathbb{Z}(q)) \cong \oplus_{i=0}^n H^{p-2i}(X, \mathbb{Z}(q-i))$$

This extends the pushforward i_* for closed immersions to pushforward for projective morphisms $f : Y \to X$ in $\mathbf{Sm}_k$ by factoring f (of relative dimension d) as

$$Y \xrightarrow{i} \mathbb{P}^n \times X \xrightarrow{p} X$$

with i a closed immersion and p the projection. Then

$$f_*: H^p(Y,\mathbb{Z}(q)) \to H^{p-2d}(X,\mathbb{Z}(q-d))$$

is the composition of the Gysin map

$$i_*: H^p(Y,\mathbb{Z}(q)) \to H^{p+2n-2d}(\mathbb{P}^n \times X, \mathbb{Z}(q+n-d))$$

with the projective bundle formula isomorphism

$$H^{p+2n-2d}(\mathbb{P}^n \times X, \mathbb{Z}(q+n-d)) \cong \oplus_{i=0}^n H^{p+2n-2d-2i}(X,\mathbb{Z}(q+n-d-i))$$

and projection on the factor $i=n$,

$$\oplus_{i=0}^n H^{p+2n-2d-2i}(X,\mathbb{Z}(q+n-d-i)) \to H^{p-2d}(X,\mathbb{Z}(q-d)).$$

One shows that this is independent of the choice of factorization and is functorial for projective f.

Finally, one has the blow-up formula: for $Z \subset X$ codimension c, with X, Z smooth, let $N_{Z/X} \to Z$ be the normal bundle, let $\mathrm{Bl}_Z X \to X$ be the blow-up of X along Z and let $E \cong \mathbb{P}(N_{Z/X})$ be the exceptional divisor. Then

$$M(\mathrm{Bl}_Z X) \cong M(X) \oplus_{i=1}^{c-1} M(Z)(i)[2i].$$

This follows by comparing the Gysin sequence for $Z \subset X$ and $E \subset \mathrm{Bl}_Z X$, and using the projective bundle formula for $M(E)$.

2.3.4. *Motivic cohomology and higher Chow groups.*

Corollary 2.22. *For $X \in \mathbf{Sm}_k$, we have a natural isomorphism $H^p(X,\mathbb{Z}(q)) \cong \mathrm{CH}^q(X, 2q-p)$.*

Proof. Suppose X is integral of dimension d over k and that $q \le d$. Then

$$\begin{aligned}
H^p(X,\mathbb{Z}(q)) &:= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(X), \mathbb{Z}(q)[p]) \\
&= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(\mathbb{Z}(0), M(X)^\vee \otimes \mathbb{Z}(q)[p]) \\
&\cong \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(\mathbb{Z}(0), M^c(X) \otimes \mathbb{Z}(q-d)[p-2d]) \\
&\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(\mathbb{Z}(d-q)[2d-2q], M^c(X)[p-2q]) \\
&\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}(0), \underline{\mathcal{H}om}(\mathbb{Z}(d-q)[2d-2q], M^c(X))[p-2q]) \\
&\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}(0), C^{\mathrm{Sus}}(z_{\mathrm{equi},d-q}(X))[p-2q]) \\
&= \mathrm{Hom}_{D(\mathrm{NST}(k))}(\mathbb{Z}^{\mathrm{tr}}(\mathrm{Spec}\, k), C^{\mathrm{Sus}}(z_{\mathrm{equi},d-q}(X))[p-2q]) \\
&= H_{2q-p}(C^{\mathrm{Sus}}(z_{\mathrm{equi},d-q}(X))(k)) \\
&= H_{2q-p}(z^q(X,*)) = \mathrm{CH}^q(X, 2q-p).
\end{aligned}$$

If $q > d$, we replace X with $X \times \mathbb{A}^{q-d}$ and use homotopy invariance for the higher Chow groups. $\square$

Corollary 2.23. *1. For $X \in \mathbf{Sm}_k$ and $q \ge 0$ an integer, $H^{2q+i}(X,\mathbb{Z}(q)) = 0$ for $i > 0$.*
2. For $X \in \mathbf{Sm}_k$, $H^p(X,\mathbb{Z}(0)) = 0$ for $p \neq 0$ and $H^0(X,\mathbb{Z}(0)) = H^0(X,\mathbb{Z}_{\mathrm{Zar}})$
3. For $X \in \mathbf{Sm}_k$,

$$H^p(X,\mathbb{Z}(1)) = \begin{cases} \mathrm{Pic}(X) & \text{for } p=2 \\ \Gamma(X, \mathcal{O}_X^\times) & \text{for } p=1 \\ 0 & \text{else.} \end{cases}$$

4. For $X \in \mathbf{Sm}_k$ and $q > 0$ an integer, $H^p(X, \mathbb{Z}(-q)) = 0$ for all $p \in \mathbb{Z}$
5. For F a field, $H^{n+i}(F, \mathbb{Z}(n)) = 0$ for $i > 0$

Proof. (1) $H^{2q+i}(X, \mathbb{Z}(q)) = \mathrm{CH}^q(X, -i) = 0$ for $i > 0$.
(2) $H^p(X, \mathbb{Z}(0)) = \mathrm{CH}^0(X, -p)$. For X integral, the complex $z^0(X, *)$ is $\mathbb{Z}$ in every degree with differentials alternating between 0 and the identity, so $\mathrm{CH}^0(X, -p) = 0$ except for $p = 0$ and $\mathrm{CH}^0(X, 0) = \mathbb{Z}$.
(3) $H^p(X, \mathbb{Z}(1)) = \mathrm{CH}^1(X, 2-p)$. Bloch [11, §6] shows that

$$\mathrm{CH}^1(X, n) = \begin{cases} \mathrm{CH}^1(X) = \mathrm{Pic}(X) & \text{for } n = 0 \\ \Gamma(X, \mathcal{O}_X^\times) & \text{for } n = 1 \\ 0 & \text{else.} \end{cases}$$

(4) We may assume X is integral. Then

$$\begin{aligned} H^p(X, \mathbb{Z}(-q)) &= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(X), \mathbb{Z}(-q)[p]) \\ &= \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M^{\mathrm{eff}}(X)(q)[2q], \mathbb{Z}(0)[p+2q]]). \end{aligned}$$

Note that $M^{\mathrm{eff}}(X)(q)[2q]$ is a summand of $M^{\mathrm{eff}}(X \times \mathbb{P}^q) = \oplus_{i=0}^q M(X)(i)[2i]$, and thus $H^p(X, \mathbb{Z}(-q))$ is the corresponding summand of $H^{p+2q}(X \times \mathbb{P}^q, \mathbb{Z}(0))$. This latter group is zero for $p+2q \neq 0$, and the projection $M^{\mathrm{eff}}(X \times \mathbb{P}^q) \to M^{\mathrm{eff}}(X)(0)[0]$ induces the isomorphism

$$p_X^* : \mathbb{Z} = H^0(X, \mathbb{Z}(0)) \to H^0(X \times \mathbb{P}^q, \mathbb{Z}(0)) = \mathbb{Z}$$

Thus $\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M^{\mathrm{eff}}(X)(q)[2q], \mathbb{Z}(0)) = 0$ if $q > 0$.
(5) Changing k to F, we have

$$H^{n+i}(F, \mathbb{Z}(n)) = \mathrm{CH}^n(\mathrm{Spec}\, F, n-i)$$

Note that $z^n(\mathrm{Spec}\, F, n-i)$ is a subgroup of $Z^n(\Delta_F^{n-i})$. But since Δ_F^{n-i} has dimension $n-i$ over F, there are no subvarieties of codimension n on Δ_F^{n-i} if $i > 0$, so $z^n(\mathrm{Spec}\, F, n-i) = 0$ and hence $\mathrm{CH}^n(\mathrm{Spec}\, F, n-i) = 0$ for $i > 0$. □

2.3.5. *Chow motives and Voevodsky motives.*

Corollary 2.24 ([104, Chap. 5, §2.2, pg. 197])**.** *For X, Y smooth and projective over k, we have natural isomorphisms*

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M(X), M(Y)[i]) \cong H^{2\dim Y + i}(X \times Y, \mathbb{Z}(\dim Y)) \cong \mathrm{CH}_{\dim X}(X \times Y, -i)$$

In particular, for $i > 0$, $\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M(X), M(Y)[i]) = 0$. For $i = 0$,

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M(X), M(Y)) \cong \mathrm{CH}_{\dim X}(X \times Y, 0) = \mathrm{CH}_{\dim X}(X \times Y)$$

and composition in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ transforms to composition of correspondences.

Proof. Since Y is projective, we have $z_{\mathrm{qfin}}(Y) = \mathbb{Z}^{\mathrm{tr}}(Y)$, so $M^c(Y) = M(Y)$ and thus $M(Y)^\vee = M(Y)(-d)[-2d]$, where d is the dimension of Y over k. Thus

$$\begin{aligned} \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M(X), M(Y)[i]) &= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(Y)^\vee \otimes M(X), \mathbb{Z}(0)[i]) \\ &= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(X \times Y), \mathbb{Z}(d)[2d+i]) \\ &= H^{2d+i}(X \times Y, \mathbb{Z}(d)) \\ &= \mathrm{CH}^d(X \times Y, -i) \end{aligned}$$

To show that composition in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ corresponds to composition of correspondences (in cycles mod rational equivalence) one uses the Friedlander-Lawson moving lemma [26] to show that $\mathrm{CH}_{\dim X}(X \times Y)$ is generated by cycles each component of which is finite over X. For such cycles, the composition as correspondences is exactly the composition in $\mathrm{Cor}(k)$, hence in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$. $\square$

Corollary 2.25. *Sending a smooth projective X to $M(X) \in \mathrm{DM}_{\mathrm{gm}}(k)$ extends to a fully faithful embedding*

$$\mathrm{Mot}_{\mathrm{CH}}(k) \to \mathrm{DM}_{\mathrm{gm}}(k)$$

2.4. **Realizations.** Realizations form an important tool for the study of motives and their use in arithmetic. Here we sketch the construction of the de Rham and Betti realizations, and say a word about étale realizations.

For the de Rham realization, we need to show that the sheaf $\Omega^p_{-/k}$ on $\mathbf{Sm}_k$ extends to a Nisnevich sheaf with transfers. For simplicity, we work in characteristic zero. The main point is the following result. For a normal k-scheme Y, we let $\Omega^{p**}_{Y/k}$ denote the double dual of $\Omega^p_{Y/k}$; of course if Y is smooth over k, $\Omega^{p**}_{Y/k} = \Omega^p_{Y/k}$.

Lemma 2.26. *Let $f : W \to X$ be a finite Galois cover of normal schemes, with Galois group G. Then the map $f^* : \Omega^{p**}_{X/k} \to f_*\Omega^{p**}_{W/k}$ identifies $\Omega^{p**}_{X/k}$ with the G-invariants $(f_*\Omega^{p**}_{W/k})^G$.*

For a general finite map of integral k-schemes $f : W \to X$, with X smooth, this gives rise to a transfer map of sheaves

$$\mathrm{Tr}_{W/X} : f_*\Omega^p_{W/k} \to \Omega^p_{X/k}$$

as follows: Let $g : W^* \to W$ be the (normal) Galois closure over X of the normalization W^N of W, with induced map $h : W^* \to X$, let d denote the degree of $W^* \to W^N$ and let G be the Galois group of W^* over X. Define $\mathrm{Tr}_{W/X}$ as the composition

$$f_*\Omega^p_{W/k} \xrightarrow{g^*} h_*\Omega^{p**}_{W^*/k} \xrightarrow{(1/d)\mathrm{Tr}_G} (h_*\Omega^{p**}_{W^*/k})^G \cong \Omega^p_{X/k}$$

Here Tr_G is the map $\eta \mapsto \sum_{g\in G} g^*\eta$.

For $W \subset X \times Y$ an integral closed subscheme, finite over X, we thus have

$$W_* : \Omega^p_{Y/k}(Y) \to \Omega^p_{X/k}(X)$$

sending $\eta \in \Omega^p_{Y/k}(Y)$ to $\mathrm{Tr}_{W/X}(p_Y^*\eta)$. One then shows that this makes the de Rham complex $X \mapsto (\Omega^*_{X/k}, d)$ into a complex in $\mathrm{PST}(k)$, and the well-known properties of de Rham cohomology make this into an object of $D_{\mathbb{A}^1}(\mathrm{NST}(k))$, i.e., an object Ω^*/k of $\mathrm{DM}^{\mathrm{eff}}(k)$, representing de Rham cohomology via

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(M^{\mathrm{eff}}(X), \Omega^*/k[n]) \cong \mathbb{H}^n(X_{\mathrm{Nis}}, \Omega^*_{X/k}) =: H^n_{dR}(X/k).$$

Similarly, sending $p_X : X \to \operatorname{Spec} k$ in $\mathbf{Sm}_k$ to the derived pushforward $Rp_{X*}\Omega^*_{X/k}$ extends to a functor

$$\Re_{dR} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)^{\mathrm{op}} \to D(k\text{-}\mathbf{Vec})$$

and then extends further to

$$\Re_{dR} : \mathrm{DM}_{\mathrm{gm}}(k)^{\mathrm{op}} \to D(k\text{-}\mathbf{Vec})$$

noting that $\Re_{dR}(M^{\mathrm{eff}}(X) \otimes \mathbb{Z}(1)) \cong \Re_{dR}(M^{\mathrm{eff}}(X))$ (there is a shift in the Hodge filtration, but that's another story).

The Betti realization is done similarly, using a Čech complex for the constant sheaf $\mathbb{Z}$. The point is that for $p : W \to Y$ finite and surjective with W, Y integral and Y smooth (all over $\mathbb{C}$), one can take a Leray open cover $\mathcal{U}$ of $Y(\mathbb{C})$ that pulls back to a Leray open cover of $W(\mathbb{C})$. Moreover, for each $U_I = \cap_{i \in I} U_i$ the locus over which $p^{-1}(U_I) \to U_I$ is not étale has codimension ≥ 1 in U_I, so the complement is connected in $W(\mathbb{C})$, and thus any section of the constant sheaf on the "étale" locus extends uniquely to a section over $U_I(\mathbb{C})$.

Finally, there are versions of étale realizations. One can repeat the basic idea for the Betti realization using the étale topology. Alternatively, one can repeat the construction of $\mathrm{DM}^{\mathrm{eff}}(k)$, replacing the Nisnevich topology with the étale topology. One can also have a theory in sheaves of $\mathbb{Z}/n$-modules. For k of finite n-cohomological dimension, with n prime to the characteristic of k if this is positive, the resulting category $\mathrm{DM}_{\text{ét}}(k; \mathbb{Z}/n)$ is equivalent to the derived category of n-torsion étale sheaves on the small étale site of $\operatorname{Spec} k$, while the étale theory with $\mathbb{Q}$-coefficients is equivalent to $\mathrm{DM}(k; \mathbb{Q})$ (see e.g. [104, Chap. 5, §3.3]). In particular, the change of topology functor gives the mod n étale realization map

$$H^p(X, \mathbb{Z}/n(q)) \to H^p(X, \mathbb{Z}/n(q)_{\text{ét}}) \cong H^p_{\text{ét}}(X, \mu_n^{\otimes q})$$

Huber [48] has given a quite general construction of realization functors, which extend the constructions sketched above. In particular, she constructs exact tensor realization functors corresponding to rational mixed Hodge structures and continuous $\mathbb{Q}_\ell$-étale cohomology from $\mathrm{DM}_{\mathrm{gm}}(k)$ (for the MHS, one needs to be given an embedding $k \hookrightarrow \mathbb{C}$):

$$\Re_{MHS} : \mathrm{DM}_{\mathrm{gm}}(k) \to D_{MHS}(\mathbb{Q})$$

$$\Re_{\text{ét},\ell} : \mathrm{DM}_{\mathrm{gm}}(k) \to D_{\text{ét},ctn}(\mathbb{Q}_\ell, k)$$

Here $D_{MHS}(\mathbb{Q})$ is Beilinson's triangulated tensor category of $\mathbb{Q}$-mixed Hodge complexes and $D_{\text{ét},ctn}(\mathbb{Q}_\ell, k)$ is Ekedahl's triangulated tensor category of constructible complexes of $\mathbb{Q}_\ell$-étale sheaves on $\operatorname{Spec}(k)$.

For $X \in \mathbf{Sm}_k$, these induce natural maps

$$\Re^{p,q}_{MHS}(X) : H^p(X, \mathbb{Q}(q)) \to H^p_{MHS}(X_{\mathbb{C}}, \mathbb{Q}(q)),$$

with $H^p_{MHS}(X, \mathbb{Q}(q))$ the $\mathbb{Q}$-mixed Hodge structure on the cohomology of the $\mathbb{C}$-scheme $X_{\mathbb{C}}$, and

$$\Re^{p,q}_{\text{ét},\ell}(X) : H^p(X, \mathbb{Q}(q)) \to H^p_{\text{ctn},\text{ét}}(X, \mathbb{Q}_\ell(q))$$

with $H^p_{\text{ctn},\text{ét}}(X, \mathbb{Q}_\ell(q))$ continuous ℓ-adic étale cohomology. These functors are compatible with the various structures described above, e.g., products, projective push-forward, Gysin sequences, Chern classes, etc.

3. Lecture 3: Motivic cohomology and motivic stable homotopy

3.1. **The Beilinson-Lichtenbaum conjectures.** The Beilinson-Lichtenbaum conjectures are what started off the whole search for a motivic cohomology. All but one of them (the Beilinson-Soulé vanishing conjecture) have been verified. Beilinson's vision of a category of mixed motivic sheaves over a given base-scheme, endowed with a six-functor formalism, has not been realized, but a triangulated (or infinity categorical) version has been constructed and will be discussed below.

The most difficult part of Beilinson-Lichtenbaum conjectures (aside from the vanishing conjecture) concerns the comparison map from motivic to étale cohomology

$$H^p(X, \mathbb{Z}/n(q)) \to H^p_{\text{ét}}(X, \mu_n^{\otimes q})$$

for the case $X = \operatorname{Spec} F$, F a field, and n prime to the characteristic of F, and asserts that this map is an isomorphism for $p \leq q$. This part of the overall set of conjectures of Beilinson and Lichtenbaum is often referred to as "the" Beilinson-Lichtenbaum conjecture.

Note that the isomorphism $H^q(F, \mathbb{Z}(q)) \cong K^M_q(F)$ (described in §1.1.5) and the fact that $H^{q+1}(F, \mathbb{Z}(q)) = 0$ shows that $H^q(F, \mathbb{Z}/n(q)) = K^M_q(F)/n$, so the case $p = q$ of the Beilinson-Lichtenbaum conjecture asserts that the map (the Galois symbol)

$$K^M_q(F)/n \to H^q_{\text{ét}}(F, \mu_n^{\otimes q}) \tag{3.1}$$

is an isomorphism for all F and for all n prime to char$F$. This map is defined concretely by first noting that Kummer theory gives the isomorphism $F^\times/(F^\times)^n \cong H^1_{\text{ét}}(F, \mu_n)$. An argument due to Tate[2] [90, Theorem 3.1] shows the map $F^\times \otimes F^\times \to H^2_{\text{ét}}(F, \mu_n^{\otimes 2})$ sends elements $a \otimes (1-a)$ to zero, so using products again gives the map (3.1).

The assertion that (3.1) is an isomorphism (for all F and all n prime to char$F$) is now known as the Bloch-Kato conjecture, although it is really just part of the Beilinson-Lichtenbaum conjecture. Bloch and Kato [13] considered the map (3.1), and showed this to be an isomorphism for a henselian discretely valued field F of characteristic zero, with residue field of positive characteristic p, for all q, and for n a power of p [13, Theorem 5.12]. They did not formulate the statement that the map (3.1) is an isomorphism in general as a conjecture, merely stating [13, pg. 118] that "The cohomological symbol defined by Tate [90] defines a map (3.1), which one conjectures to be an isomorphism quite generally." Nevertheless, we will conform with current custom and refer to this conjecture as the Bloch-Kato conjecture.

In fact, just the surjectivity part of the Bloch-Kato conjecture implies the Beilinson-Lichtenbaum conjecture (and thus the full Bloch-Kato conjecture):

Theorem 3.1 (Suslin-Voevodsky [89], Geisser-Levine [33])**.** *Let F_0 be the prime field and let ℓ be a prime $\neq$ charF_0. Suppose the Galois symbol $K^M_n(F) \to H^n_{ét}(F, \mu_\ell^{\otimes n})$ is surjective for all fields $F \supset F_0$. Then the change of topology map*

$$H^p(F, \mathbb{Z}/n(q)) \to H^p_{ét}(F, \mathbb{Z}/n(q))$$

is an isomorphism for all fields $F \subset F_0$, and all p, q with $0 \leq q \leq n$ and $p \leq q$.

For $q = 2$, the Bloch-Kato conjecture is the theorem of Merkurjev-Suslin theorem [65] (proven about two years before the Beilinson-Lichtenbaum conjectures arrived). An essential part of their argument considers the Severi-Brauer variety $SB(a, b, \zeta)$ for $a, b \in F^\times$ and ζ a primitive ℓth root of 1 (one can safely assume that ζ is in F). This is a twisted form of a projective space $\mathbb{P}^{\ell-1}$, representing the functor of maximal left ideals in a certain central simple F-algebra $A(a, b, \zeta)$. Here $A(a, b, \zeta)$ is defined as the free associative algebra $F\langle X, Y\rangle$ over F, modulo the relations $X^\ell = a$, $Y^\ell = b$, $XY = \zeta YX$.

[2] The case $n = 2$ is to be found in Milnor's paper [66, Lemma 6.1], attributed to Bass-Tate [?] (but before *loc. cit.* appeared in publication).The result of Tate referred to here is a stronger one, showing that $a \otimes (1-a)$ goes to zero in the continuous ℓ-adic cohomology $H^2(F, \mathbb{Z}_\ell(2))$.

Example 3.2. For $\ell = 2$, $A(a, b, -1)$ is a quaternion algebra over F and $SB(a, b, -1)$ is the conic in $\mathbb{P}^2_F$ defined by $X_0^2 - aX_1^2 - bX_2^2 = 0$.

The main point is to show that the kernel of the map $K_2^M(F)/\ell \to K_2^M(F(SB(a, b, \zeta)))\ell$ is exactly the $\mathbb{Z}/\ell$-span of the symbol $\{a, b\}$. This being before the advent of motivic cohomology, they use Quillen K-theory instead, via Matsumoto's theorem [62]: $K_2^M(F) = K_2(F)$. Quillen [75, §8, Theorem 4.1] computed the K-theory of $SB(a, b, \zeta)$ as

$$K_*(SB(a, b, \zeta)) = \oplus_{i=0}^{\ell-1} K_*(A(a, b, \zeta)^{\otimes_F i}$$

so one needs to pass from $SB(a, b, \zeta)$ to its function field. For this, Gillet's Riemann-Roch theorem for higher K-theory [35] allows them to compare the relatively easy to understand $K_2(SB(a, b, \zeta))$ with $K_2(F(SB(a, b, \zeta))$.

A few years later, Merkurjev-Suslin [64] and independently Levine [59] used a "relativization" method to extend this to give an isomorphism

$$H^1(F, \mathbb{Z}/n(2)) \cong H^1_{\text{ét}}(F, \mu_n^{\otimes 2})$$

(their result was phrased in terms of the so-called "indecomposable K_3", as motivic cohomology was not yet around). This was before the general Bloch-Kato $\Rightarrow$ Beilinson-Lichtenbaum result mentioned above was proven and was in a sense an early precursor.

For $q = 3$, Rost [83] extended the Merkurjev-Suslin method to prove the Bloch-Kato conjecture in weight 3, for the prime $\ell = 2$. After a long development, Voevodsky put together his work on DM together with his construction of motivic Steenrod operations plus results of Rost and others to prove the Beilinson-Lichtenbaum conjectures in general, first for n a power of 2, and then the general case.

The case $\ell = 2$ was handled first (by Voevodsky [96], relying on results of Rost). The proof is again based (at least in part) on the method used by Merkurjev-Suslin for $q = 2$, but is more complicated. The Severi-Brauer varieties that play a central role as splitting varieties for a symbol $\{a, b\}$ mod ℓ in the proof of the Merkurjev-Suslin are replaced by the Pfister neighbor quadrics $Q_{\underline{a}}$ associated to an element $\underline{a} = (a_1, \ldots, a_n) \in (k^\times)^n$. This is an explicit quadric hypersurface in $\mathbb{P}_F^{2^{n-1}}$ defined as follows.

For units $a_1, \ldots, a_r \in k^\times$, one has the rank r form $\langle a_1, \ldots, a_r\rangle$, defined by $\langle a_1, \ldots, a_r\rangle(x_1, \ldots, x_r) = \sum_{i=1}^r a_i x_i^2$. For quadratic forms q, q', one has the sum $q \perp q'$ and tensor product $q \otimes q'$, with $\langle a_1, \ldots, a_r\rangle \perp \langle b_1, \ldots, b_s\rangle = \langle a_1, \ldots, a_r, b_1, \ldots, b_s\rangle$ and $\langle a_1, \ldots, a_r\rangle \otimes \langle b_1, \ldots, b_s\rangle = \langle \ldots, a_i b_j, \ldots\rangle$. The *n-fold Pfister form* $\langle\langle a_1, \ldots, a_n\rangle\rangle$ is defined by

$$\langle\langle a_1, \ldots, a_n\rangle\rangle := \langle 1, -a_1\rangle \otimes \ldots \otimes \langle 1, -a_n\rangle$$

The quadric $Q_{\underline{a}} \subset \mathbb{P}^{2^{n-1}}$ is then defined by the quadratic polynomial in $2^{n-1} + 1$ variables $\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle \perp \langle -a_n\rangle$.

Example 3.3. For $n = 2$, $(a, b) \in (k^\times)^2$, $Q_{(a,b)} \subset \mathbb{P}^2_k$ is the conic defined by $X_0^2 - aX_1^2 - bX_2^2$, that is $Q_{(a,b)}$ is the Severi-Brauer variety $SB(a, b, -1)$.

Voevodsky streamlines the argument by using the 2-local étale version of motivic cohomology, *Lichtenbaum motivic cohomology*

$$H_L^{p,q}(-, \mathbb{Z}_{(2)}) := \mathbb{H}^p_{\text{ét}}(-, \mathbb{Z}_{(2)}(q)^*_{\text{ét}})$$

where $\mathbb{Z}(q)^*_{\text{ét}}$ is the étale sheafification of motivic complex $\mathbb{Z}(q)^*$ discussed in §2.2. Theorem 3.1 reduces the Bloch-Kato conjecture in weight n to showing that $H_L^{n+1,n}(F, \mathbb{Z}_{(2)}) =$

0 for all fields F. We use the notation $H^{p,q}(-,\mathbb{Z}_{(2)})$ for our usual 2-local motivic cohomology $H^p(-,\mathbb{Z}_{(2)}(q))$.

The Merkurjev-Suslin arguments involving the Severi-Brauer varieties turn into showing that, for $\underline{a}=(a_1,\ldots,a_n)\in(k^\times)^n$, one has:
(3.2)

(1) The symbol $\{a_1,\ldots,a_n\}\in K_n^M(k)$ goes to zero in $K_n^M(k(Q_{\underline{a}}))/2$.
(2) Let $k_1\subset k_2$ be a finite field extension of odd degree. Then $H_L^{n+1,n}(k_1,\mathbb{Z}_{(2)})\to H_L^{n+1,n}(k_2,\mathbb{Z}_{(2)})$ is injective.
(3) The map $H_L^{n+1,n}(k,\mathbb{Z}_{(2)})\to H_L^{n+1,n}(k(Q_{\underline{a}}),\mathbb{Z}_{(2)})$ is injective.

Having these facts at hand, the argument is just as for the Merkurjev-Suslin theorem: Starting with k, one takes the compositum k' of all odd degree extensions of k, and then take the compositum of k' with all fields $k(Q_{\underline{a}})$ for $\underline{a}=(a_1,\ldots,a_n)$ with $\{a_1,\ldots,a_n\}\neq 0$ in $K_n^M(k)/2$, forming the field k_1. By (2), $H_L^{n+1,n}(k,\mathbb{Z}_{(2)})\to H_L^{n+1,n}(k',\mathbb{Z}_{(2)})$ is injective, and by (3), $H_L^{n+1,n}(k',\mathbb{Z}_{(2)})\to H_L^{n+1,n}(k_1,\mathbb{Z}_{(2)})$ is injective as well.

Then repeat this construction infinitely often, giving the tower

$$k\subset k_1\subset k_2\subset\ldots\subset k_\infty:=\cup_m k_m.$$

Then k_∞ has no non-trivial odd degree extensions, $K_n^M(k_\infty)/2=0$, and the map $H_L^{n+1,n}(k,\mathbb{Z}_{(2)})\to H_L^{n+1,n}(k_\infty,\mathbb{Z}_{(2)})$ is injective. An induction in n is used to show that $H_L^{n+1,n}(F,\mathbb{Z}_{(2)})=0$ if $K_n^M(F)/2=0$ and F has no non-trivial odd degree extension, and thus $H_L^{n+1,n}(k,\mathbb{Z}_{(2)})=0$.

The existence of pushforward maps for finite extensions $k_1\subset k_2$ easily proves (2), and the proof of (1) involves some fairly straightforward arguments in Galois cohomology. However, the injectivity in (3) is quite hard to prove, and relies on an intricate application of the *motivic Steenrod operations* (see [95, 97]), and some deep results of Rost on the motives of the quadrics $Q_{\underline{a}}$, including the injectivity of the pushforward map $H^{2n-1,n}(Q_{\underline{a}},\mathbb{Z})\to H^{1,1}(k,\mathbb{Z})=k^\times$ (see [80, 81, 82]).

In a bit more detail, one main technical point is to show that $H^{n+1,n}(\mathcal{X}_{\underline{a}},\mathbb{Z}_{(2)})=0$, where $\mathcal{X}_{\underline{a}}$ is the Čech simplicial scheme $[n]\mapsto Q_{\underline{a}}^{n+1}$ associated to $Q_{\underline{a}}$. Letting $\tilde{\mathcal{X}}_{\underline{a}}$ be the "reduced" version of $\mathcal{X}_{\underline{a}}$ (cofiber of the map $\mathcal{X}_{\underline{a}},\to\operatorname{Spec}k$), this is the same as the vanishing of $H^{n+2,n}(\tilde{\mathcal{X}}_{\underline{a}},\mathbb{Z}_{(2)})$. A sequence of Milnor operations in the motivic Steenrod algebra maps this group (injectively!) to $H^{2^n,2^{n-1}}(\tilde{\mathcal{X}}_{\underline{a}},\mathbb{Z}_{(2)})=H^{2^n-1,2^{n-1}}(\mathcal{X}_{\underline{a}},\mathbb{Z}_{(2)})$. This latter group is a subgroup of $H^{2^n-1,2^{n-1}}(Q_{\underline{a}},\mathbb{Z}_{(2)})$ and Rost's injectivity theorem [81, Theorem 6], [82, Proposition 2] show that pushforward by the structure map defines injection $H^{2^n-1,2^{n-1}}(Q_{\underline{a}},\mathbb{Z})\hookrightarrow H^1(k,\mathbb{Z}(1))=k^\times$.

But now this says the base-change to $\bar{k}$ defines an injection

$$H^{2^n-1,2^{n-1}}(Q_{\underline{a}},\mathbb{Z}_{(2)})\hookrightarrow H^{2^n-1,2^{n-1}}(Q_{\underline{a}}\times_k\bar{k},\mathbb{Z}_{(2)})$$

Running the construction in reverse, this says that

$$H^{n+1,n}(\mathcal{X}_{\underline{a}},\mathbb{Z}_{(2)})\to H^{n+1,n}(\mathcal{X}_{\underline{a}}\times_k\bar{k},\mathbb{Z}_{(2)})$$

is injective.

An elementary property of $\mathcal{X}_{\underline{a}}$ is that the structure map $\mathcal{X}_{\underline{a}}\times_k F\to\operatorname{Spec}F$ is a weak equivalence of simplicial schemes if $Q_{\underline{a}}(F)\neq\emptyset$, and since $Q_{\underline{a}}(\bar{k})\neq\emptyset$,

$\mathcal{X}_{\underline{a}} \times_k \bar{k} \sim \operatorname{Spec} \bar{k}$. Thus

$$H^{n+1,n}(\mathcal{X}_{\underline{a}} \times_k \bar{k}, \mathbb{Z}_{(2)}) \cong H^{n+1,n}(\bar{k}, \mathbb{Z}_{(2)}) \cong \mathrm{CH}^n(\bar{k}, n-1) \otimes \mathbb{Z}_{(2)} = 0.$$

this last identity following from the simple fact that $\Delta^{n-1}_{\bar{k}}$ has no codimension n points. Thus $H^{n+1,n}(\mathcal{X}_{\underline{a}}, \mathbb{Z}_{(2)}) = 0$.

The proof that $H^{n+1,n}_L(k, \mathbb{Z}_{(2)}) \to H^{n+1,n}_L(k(Q_{\underline{a}}), \mathbb{Z}_{(2)})$ is injective follows from this and the fundamental distinguished triangle (see [96, Theorem 4.4]) relating the "Rost motive" $M_{\underline{a}}$ (a certain summand of the motive of $Q_{\underline{a}}$) with the motives $M(\mathcal{X}_{\underline{a}})$, and $M(\mathcal{X}_{\underline{a}})((2^{n-1}-1)[2^n-1]$:

$$M(\mathcal{X}_{\underline{a}})((2^{n-1}-1)[2^n-2] \to M_{\underline{a}} \to M(\mathcal{X}_{\underline{a}}) \to M(\mathcal{X}_{\underline{a}})((2^{n-1}-1)[2^n-1].$$

Rost's construction of $M_{\underline{a}}$, along with his important *nilpotence theorem*, was presented in his preprint [80]. Karpenko [52] has given a somewhat simpler construction of the Rost motive, relying on the nilpotence theorem. Brosnan [16] has written an alternative proof of Rost's nilpotence theorem.

The analog of the result of Merkurjev-Suslin, that the kernel of the map $K_2(k)/\ell \to K_2(k(SB(a, b, \zeta))/\ell$ is exactly $\mathbb{Z}/\ell \cdot \{a, b\}$, was proven by Orlov-Vishik-Voevodsky [73] (for fields of characteristic zero):

Theorem 3.4 ([73, Theorem 2.1]). *Let k be a field of characteristic 0. Then for $\underline{a} = (a_1, \ldots, a_n) \in (k^\times)^n$, the kernel of $K^M_n(k)/2 \to K^M_n(k(Q_{\underline{a}}))$ is $\mathbb{Z}/2 \cdot \{a_1, \ldots, a_n\}$.*

Using this, they prove the Milnor conjecture for fields of characteristic zero. For a field k of characteristic $\neq 2$, let $\mathrm{GW}(k)$ denote the Grothendieck-Witt ring of isometry classes of (virtual) non-degenerate quadratic forms. We have the Witt ring $W(k) := \mathrm{GW}(k)/(h)$, where $h = x^2 - y^2$ is the class of the hyperbolic form. The rank homomorphism $\mathrm{GW}(k) \to \mathbb{Z}$ induces the mod 2 rank map $\overline{\mathrm{rank}} : W(k) \to \mathbb{Z}(2)$; let $I(k) \subset W(k)$ be the kernel of $\overline{\mathrm{rank}}$. For $a \in k^\times$, the class of the form $\langle 1, -a\rangle$ is in $I(k)$, so the classes of n-fold Pfister forms $\langle\langle a_1, \ldots, a_n\rangle\rangle = \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ are in $I(k)^n$. Milnor [66, Theorem 4.1] showed that sending $(a_1, \ldots, a_n)$ to the class of $\langle\langle a_1, \ldots, a_n\rangle\rangle$ in $I(k)^n/I(k)^{n+1}$ gives a well-defined, surjective group homomorphism

$$s_n(k) : K^M_n(k)/2 \to I(k)^n/I(k)^{n+1}.$$

He showed that $s_1(k)$ and $s_2(k)$ are always isomorphisms and he asked [66, Question 4.3] if $s_n(k)$ is an isomorphism for all k and n. This question is often known as the Milnor conjecture.

Corollary 3.5 ([73, Theorem 4.1]). *Let k be a field of characteristic 0. Then sending $(a_1, \ldots, a_n) \in (k^\times)^n$ to the n-fold Pfister form $\langle\langle a_1, \ldots, a_n\rangle\rangle \in I(k)^n$ descends to an isomorphism*

$$K^M_n(k)/2 \cong I(k)^n/I(k)^{n+1}.$$

Proof. We follow the argument of [73, Theorem 4.1].

We need only show that $s_n(k)$ is injective. We prove by induction on $m \geq 0$ that for all fields F of characteristic zero, if $s_n(F)(\sum_{i=0}^m \{a_{1,i}, \ldots, a_{n,i}\}) = 0$ in $I(F)^n/I(F)^{n+1}$, then $\sum_{i=0}^m \{a_{1,i}, \ldots, a_{n,i}\} = 0$ in $K^M_n(F)/2$.

For the case $m = 0$, suppose $s_n(F)(\{a_1, \ldots, a_n\}) = 0$. It follows from a result of Elman-Lam [24, Theorem 3.2] that $\{a_1, \ldots, a_n\} = 0$ in $K^M_n(F)/2$.

Now take $m > 0$ and assume the result for $m - 1$. We suppose that we have $(a_{1,i}, \ldots, a_{n,i}) \in (F^\times)^n$, $i = 0, \ldots, m$, with $s_n(F)(\sum_{i=0}^m \{a_{1,i}, \ldots, a_{n,i}\}) = 0$ in

$I(F)^n/I(F)^{n+1}$. By (3.2)(1), $\{a_{1,m},\ldots,a_{n,m}\}$ goes to zero in $K^M_n(F(Q_{\underline{a}_m}))/2$, so we have $\sum_{i=0}^m\{a_{1,i},\ldots,a_{n,i}\}=\sum_{i=0}^{m-1}\{a_{1,i},\ldots,a_{n,i}\}$ in $K^M_n(F(Q_{\underline{a}_m}))/2$. Since the map $s_n(F)$ is natural in F, we have

$$0=s_n(F(Q_{\underline{a}_m}))(\sum_{i=0}^m\{a_{1,i},\ldots,a_{n,i}\})=s_n(F(Q_{\underline{a}_m}))(\sum_{i=0}^{m-1}\{a_{1,i},\ldots,a_{n,i}\})$$

so by our induction hypothesis, $\sum_{i=0}^{m-1}\{a_{1,i},\ldots,a_{n,i}\}=0$ in $K^M_n(F(Q_{\underline{a}_m}))/2$. By Theorem 3.4,

$$\sum_{i=0}^{m-1}\{a_{1,i},\ldots,a_{n,i}\}=\epsilon\cdot\{a_{1,m},\ldots,a_{n,m}\}\in K^M_n(F)/2$$

for some $\epsilon\in\mathbb{Z}/2$, so $\sum_{i=0}^m\{a_{1,i},\ldots,a_{n,i}\}=(1+\epsilon)\{a_{1,m},\ldots,a_{n,m}\}$, and we reduce back to the known case $m=0$. □

Remark 3.6. Other proofs of the Milnor conjecture in characteristic zero have appeared in work of Kahn-Sujatha [51, Remark 3.3] and Morel [68, Theorem 1]. Morel [67, Theorem 1.1] gives a proof of the Milnor conjecture for all fields k of characteristic $\neq 2$.

The Bloch-Kato conjecture for an odd prime ℓ is treated in rough outline the same as for $\ell=2$, but numerous technical problems arise, for instance, there is no nice collection of smooth projective varieties that play the role of the Severi-Brauer varieties for weight two, and the Pfister neighbor quadrics in higher weight for $\ell=2$. Voevodsky [94], relying on work of Haesemeyer-Weibel [39], Rost [77, 78, 79], Suslin-Joukhovitski [88], and Weibel [105], overcame these difficulties to prove the Bloch-Kato conjecture in general. Besides these original sources, we refer the reader to the book [38] for a detailed treatment of this topic.

The construction of the Steenrod operations, an integral part of the proof, requires the introduction of the motivic stable homotopy category and understanding its relation with the triangulated category of motives, our next topic.

3.2. **The motivic stable homotopy category.**

3.2.1. *The unstable and stable motivic homotopy categories.* Subsequent to Voevodsky's construction of the category $\mathrm{DM}^{\mathrm{eff}}(k)$, Morel and Voevodsky [70] developed a parallel $\mathbb{A}^1$ *homotopy theory.* In essence, this is much simpler, although this requires more input from the theory of model categories to carry out the construction; later treatments use the setting of infinity categories. We will suppress these technical foundations in our overview.

First of all, correspondences no longer play a role, and parallel to classical homotopy theory, one relies on presheaves of simplicial sets rather than presheaves of complexes of abelian groups. One can work over an arbitrary noetherian base-scheme S of finite Krull dimension. Formally, one has the category $\mathrm{Psh}^{\mathbf{sSets}}(\mathbf{Sm}_S)$ of presheaves of simplicial sets, **sSets**, on $\mathbf{Sm}_S$, the category of smooth separated S-schemes of finite type. This is called the category of *spaces over* S, $\mathbf{Spc}(S)$. Working in the Nisnevich topology again, one inverts morphisms $P\to Q$ that are weak equivalences (i.e. induce bijections on π_0 and all homotopy groups $\pi_n(-,x)$ for $n\geq 1$ and all choice of base-point) of simplicial sets on all Nisnevich stalks $P_x\to Q_x$, where a Nisnevich point is given by $x\in X\in\mathbf{Sm}_k$, and the stalk P_x is the colimit of $P(U)$ over all $(U,u) \to (X,x)$ Nisnevich neighborhoods of x. Formally, one can write $P_x = P(\mathcal{O}^h_{X,x})$ where $\mathcal{O}^h_{X,x}$ is the henselization of the local ring $\mathcal{O}_{X,x}$,

$$\mathcal{O}^h_{X,x} = \mathrm{colim}_{(U,u)\to(X,x)}\mathcal{O}_{U,u}.$$

In addition, one inverts all morphisms $p^* : P \to P^{\mathbb{A}^1}$, where for a presheaf $P : \mathbf{Sm}_S^{\mathrm{op}} \to \mathbf{sSets}$, $P^{\mathbb{A}^1}$ is the presheaf $X \mapsto P(X \times \mathbb{A}^1)$, and p^* is the collection of maps $p^*_X : P(X) \to P(X \times \mathbb{A}^1)$. This gives us the *unstable* $\mathbb{A}^1$ *homotopy category* $\mathcal{H}(S)$.

We have the Yoneda functor $\mathbf{Sm}_S \to \mathbf{Spc}(S)$, sending $X \in \mathbf{Sm}_S$ to the presheaf $X(-) : \mathbf{Sm}_S^{\mathrm{op}} \to \mathbf{sSets}$, with $X(Y)$ the constant simplicial set on the set $\mathrm{Hom}_{\mathbf{Sm}_S}(Y, X)$. We also have the constant presheaf functor $c : \mathbf{sSets} \to \mathbf{Spc}(S)$. Thus, the category $\mathbf{Spc}(S)$ and its localization $\mathcal{H}(S)$ allow the mixing of algebraic geometry and classical homotopy theory: algebraic geometry enters via the Yoneda functor while classical homotopy theory enters via the constant presheaf functor. Moreover, the presheaf category $\mathbf{Spc}(S)$ inherits all the usual constructions of topology, including limits, colimits, products and internal Homs, by operating objectwise on the presheaves, and these all pass to the corresponding constructions in $\mathcal{H}(S)$.

Replacing $\mathbf{sSets}$ with pointed simplicial sets $\mathbf{sSets}_\bullet$, one has the category of pointed spaces over S, $\mathbf{Spc}_\bullet(S) := \mathrm{Psh}^{\mathbf{sSets}_\bullet}(\mathbf{Sm}_S)$ and a parallel localization defines the pointed unstable $\mathbb{A}^1$ homotopy category, $\mathcal{H}_*(S)$, with parallel structures to those in $\mathbf{Spc}(S)$. For instance, disjoint union is replaced with pointed union, product gets replaced with smash product $(\mathcal{X}, x) \wedge (\mathcal{Y}, y) := \mathcal{X} \times \mathcal{Y}/x \times \mathcal{Y} \vee \mathcal{X} \times y$, a smooth S-scheme $X \in \mathbf{Sm}_S$ defines the object X_+ associated to the "pointed" S-scheme $X \amalg S$, and we have the constant presheaf functor $c : \mathbf{sSets}_\bullet \to \mathbf{Spc}_\bullet(S)$.

This gives us the usual suspension and loops functors, $\Sigma_{S^1}, \Omega_{S^1}$, via $\mathcal{X} \mapsto S^1 \wedge \mathcal{X}$, $\mathcal{X} \mapsto \mathcal{H}om_{\mathbf{Spc}_\bullet(S)}(S^1, \mathcal{X})$. We have the Nisnevich sheaves of "connected components" $\pi_0^{\mathrm{Nis}}(\mathcal{X})$ and $\pi_0^{\mathbb{A}^1}(X)$, the first being the sheaf associated to the presheaf $U \mapsto \pi_0(\mathcal{X}(U))$ and the second the sheaf associated to the presheaf $U \mapsto [U, \mathcal{X}]_{\mathcal{H}(S)}$. Higher homotopy sheaves are defined similarly, with identities for $\mathcal{X} \in \mathbf{Spc}_\bullet(S)$

$$\pi_n^{\mathbb{A}^1}(\mathcal{X}) = \pi_0^{\mathbb{A}^1}(\Omega^n_{S^1}\mathcal{X});\ \pi_n^{\mathrm{Nis}}(\mathcal{X}) = \pi_0^{\mathrm{Nis}}(\Omega^n_{S^1}\mathcal{X}).$$

One uses the theory of model categories to show in the first place that such a localization exists, and to yield cofibrant and fibrant models in $\mathbf{Spc}(S)$, with the property that homotopy classes of map $P \to Q$, for P cofibrant and Q fibrant, compute the morphisms $[P, Q]_{\mathcal{H}(k)}$. One particular choice of model structure has the representable presheaves X for $X \in \mathbf{Sm}_S$ being cofibrant, but the fibrant models are much more difficult to understand.

Another notable feature of $\mathcal{H}_\bullet(S)$ is the two-parameter family of spheres. Let $\mathbb{G}_m = (\mathbb{A}^1 \setminus \{0\}, \{1\})$ be the "Tate circle" and for $a \geq b \geq 0$, let $S^{a,b} := S^{a-b} \wedge \mathbb{G}_m^{\wedge b}$. We have corresponding suspension functors $\mathcal{X} \mapsto \Sigma^{a,b}\mathcal{X} := \mathcal{X} \wedge S^{a,b}$ and loops functors $\mathcal{X} \mapsto \Omega^{a,b}\mathcal{X} := \mathcal{H}om(S^{a,b}, \mathcal{X})$. We also have the canonical isomorphism $(\mathbb{P}^1, \infty) \cong S^{2,1}$ in $\mathcal{H}_\bullet(S)$, giving the natural isomorphism $\Sigma^n_{\mathbb{P}^1} \cong \Sigma^{2n,n}$. This gives us as well the bigraded family of homotopy sheaves

$$\pi^{\mathbb{A}^1}_{a,b}(\mathcal{X}) := \pi_0^{\mathbb{A}^1}(\Omega^{a,b}\mathcal{X})$$

and the similarly defined Nisnevich version $\pi^{\mathrm{Nis}}_{a,b}(\mathcal{X})$.

The stable theory, introduced by Voevodsky [103], is modeled on the classical case of suspension spectra of spaces, except that we replace S^1-suspension with

$\mathbb{P}^1$-suspension, yielding the category of $\mathbb{P}^1$-spectra over S, $\mathrm{Sp}_{\mathbb{P}^1}(S)$. This is the category of sequences $E_* := (E_0, E_1, \ldots)$, $E_n \in \mathbf{Spc}_\bullet(S)$, together with bonding maps $\epsilon_n : \Sigma_{\mathbb{P}^1} E_n \to E_{n+1}$, where $\Sigma_{\mathbb{P}^1} E_n := E_n \wedge (\mathbb{P}^1, \infty)$. A morphism $E_* \to F_*$ in $\mathrm{Sp}_{\mathbb{P}^1}(S)$ is a collection of maps $f_n : E_n \to F_n$ in $\mathbf{Spc}_\bullet(S)$ that commute with the respective bonding maps. A map $f : E_* \to F_*$ is an $\mathbb{A}^1$ *stable weak equivalence* if the f_n induce isomorphisms of $\mathbb{A}^1$ stable homotopy sheaves

$$\mathrm{colim}_n f_{n*} : \mathrm{colim}_n \pi^{\mathbb{A}^1}_{a+2n,b+n}(E_n) \to \mathrm{colim}_n \pi^{\mathbb{A}^1}_{a+2n,b+n}(F_n)$$

for all $a, b \in \mathbb{Z}$. Here the inductive system $\{\pi^{\mathbb{A}^1}_{a+2n,b+n}(E_n)\}$ has transition maps

$$\pi^{\mathbb{A}^1}_{a+2n,b+n}(E_n) \cong \pi^{\mathbb{A}^1}_{a+2n+2,b+n+1}(\Sigma_{\mathbb{P}^1} E_n) \xrightarrow{\epsilon_{n*}} \pi^{\mathbb{A}^1}_{a+2n+2,b+n+1}(E_{n+1})$$

and is defined for all n sufficiently large (depending on a, b): $n \geq \max(0, -b, b-a)$. The $\mathbb{A}^1$ stable homotopy category is then defined by inverting stable weak equivalences in $\mathrm{Sp}_{\mathbb{P}^1}(S)$. This gives a triangulated tensor category, with translation functor induced by Σ_{S^1} and with the suspension functors $\Sigma^{a,b}$ defined and invertible for all $a, b \in \mathbb{Z}$. As in the classical case one has the adjoint pair of infinite $\mathbb{P}^1$-suspension/infinite $\mathbb{P}^1$-loops functors

$$\Sigma^\infty_{\mathbb{P}^1} : \mathcal{H}_\bullet(S) \rightleftarrows \mathrm{SH}(S) : \Omega^\infty_{\mathbb{P}^1}$$

For details, we refer the reader to [50, 18, 45].

3.2.2. *The category* $\mathrm{DM}(k)$. One might ask, what about inverting $\mathbb{Z}(1)$ in $\mathrm{DM}^{\mathrm{eff}}(k)$? Here the naive Gabriel-Zisman localization $\mathrm{DM}^{\mathrm{eff}}(k)[(-\otimes\mathbb{Z}(1))^{-1}]$ is not really what one wants, as this category lacks arbitrary homotopy limits and colimits. A better construction is to be guided by homotopy theory, in forming the category of $\mathbb{Z}(1)[2]$-spectra, as above. Replacing $\mathbf{Spc}_\bullet(S)$ with $C(\mathrm{NST}(k))$, $\mathcal{H}_\bullet(S)$ with $\mathrm{DM}^{\mathrm{eff}}(k)$, and $\Sigma_{\mathbb{P}^1}$ with $-\otimes^{\mathrm{tr}} \mathbb{Z}^{\mathrm{tr}}(1)[2]$, we arrive at the triangulated tensor category $\mathrm{DM}(k)$, the localization of $\mathbb{Z}(1)[2]$-spectra in $C(\mathrm{NST}(k))$ with respect to stable $\mathbb{A}^1$-weak equivalence. The functor $-\otimes \mathbb{Z}(1)[2]$ on $\mathrm{DM}(k)$ is invertible and we have the adjoint pair of exact functors

$$\Sigma^\infty_{\mathbb{Z}(1)[2]} : \mathrm{DM}^{\mathrm{eff}}(k) \rightleftarrows \mathrm{DM}(k) : \Omega^\infty_{\mathbb{Z}(1)[2]}$$

This is a bit different from $\mathbb{A}^1$-homotopy theory, in that the translation functor $M \mapsto M[1]$ is already invertible on triangulated category $\mathrm{DM}^{\mathrm{eff}}(k)$, while $\mathcal{H}_\bullet(S)$ does not have a triangulated structure and Σ_{S^1} is not invertible there.

Voevodsky's embedding theorem (Theorem 2.13) extends to show that

$$\mathrm{DM}_{\mathrm{gm}}(k) \to \mathrm{DM}(k)$$

is an exact, fully faithful embedding with dense image, identifying $\mathrm{DM}_{\mathrm{gm}}(k)$ with the subcategory of compact objects in $\mathrm{DM}(k)$.

Remarks 3.7. 1. $\mathrm{DM}(k)$ is a triangulated category admitting small coproducts. Using the theory of *symmetric spectra* gives $\mathrm{DM}(k)$ the structure of tensor triangulated category.

2. A general theory of stabilization due to Hovey [43] uses a different definition of stable weak equivalence, but in the case of $\mathrm{DM}^{\mathrm{eff}}(k)$, Jardine [50] (and also Voevodsky) shows that this agrees with the "naive" notion of stable weak equivalence described above.

3.2.3. $\mathrm{DM}(k)$ *and the category of* $H_{\mathrm{mot}}\mathbb{Z}$*-modules.* The objects of $\mathrm{SH}(S)$ represent bi-graded cohomology theories on $\mathbf{Sm}_S$ in the following way: Given $E \in \mathrm{SH}(S)$ and $U \in \mathbf{Sm}_S$, define

$$E^{a,b}(U) := [\Sigma^\infty_{\mathbb{P}^1} U_+, \Sigma^{a,b} E]_{\mathrm{SH}(S)}$$

Properties built into $\mathrm{SH}(S)$ give contravariant functoriality, Mayer-Vietoris properties and homotopy invariance to this bi-graded family of abelian groups. If E admits the structure of a commutative monoid in $\mathrm{SH}(S)$, then $E^{**}(U)$ has a bi-graded ring structure, with a certain form of graded commutativity.

Conversely, given a bi-graded "cohomology theory" $U \mapsto H^{**}(U) := \oplus_{a,b} H^{a,b}(U)$ on $\mathbf{Sm}_S$, one says that H^{**} is represented by some $E \in \mathrm{SH}(S)$ if there is a natural isomorphism of functors $H^{**} \cong E^{**}$.

For a perfect field k, motivic cohomology on $\mathbf{Sm}_k$ is in fact represented by a certain object $H_{\mathrm{mot}}\mathbb{Z} \in \mathrm{SH}(k)$, constructed as the sequence

$$H_{\mathrm{mot}}\mathbb{Z} := (\mathrm{EM}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(0)), \mathrm{EM}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(1)[2]), \ldots, \mathrm{EM}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(n)[2n]), \ldots)$$

Here EM is the Eilenberg-MacLane functor from $C(\mathbf{Ab})$ to usual suspension spectra, and the bonding maps are defined by applying EM to the maps of complexes

$$C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(n)[2n]) \otimes^{\mathrm{tr}} \mathbb{Z}(1)[2] \to C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(n+1)[2n+2])$$

induced by the natural map $\mathcal{H}om(X, A) \otimes^{\mathrm{tr}} B \to \mathcal{H}om(X, A \otimes^{\mathrm{tr}} B)$. One also uses the natural map (graph) of the representable functor $Y \mapsto \mathrm{Hom}_{\mathbf{Sm}_k}(Y, X)$ to $\mathbb{Z}^{\mathrm{tr}}(X)$ to define the map

$$\mathrm{EM}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(n)[2n])) \wedge (\mathbb{P}^1, \infty) \to \mathrm{EM}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(n)[2n]) \otimes^{\mathrm{tr}} \mathbb{Z}(1)[2]);$$

putting these together gives the bonding map

$$\Sigma_{\mathbb{P}^1} \mathrm{EM}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(n)[2n])) \to \mathrm{EM}(C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(n+1)[2n+2]))$$

One can give $H_{\mathrm{mot}}\mathbb{Z}$ the structure of an E_∞ object in $\mathrm{Sp}_{\mathbb{P}^1}(k)$, which gives us the homotopy category of $H_{\mathrm{mot}}\mathbb{Z}$-modules, $H_{\mathrm{mot}}\mathbb{Z} - \mathbf{Mod}$, and the free-forget adjunction

$$H_{\mathrm{mot}}\mathbb{Z} \wedge - : \mathrm{SH}(k) \rightleftarrows H_{\mathrm{mot}}\mathbb{Z} - \mathbf{Mod} : \mathrm{EM}_{\mathrm{mot}}$$

In fact, we have the following fundamental theorem

Theorem 3.8 (Röndigs-Østvær [76])**.** *Suppose that k has characteristic zero. Then there is a natural isomorphism of tensor triangulated categories $H_{\mathrm{mot}}\mathbb{Z} - \mathbf{Mod} \cong \mathrm{DM}(k)$.*

This has been extended to characteristic $p > 0$, after inverting p, by Hoyois-Kelly-Østvær [47, Theorem 5.8].

This connection of $\mathrm{DM}(k)$ with $\mathrm{SH}(k)$ has opened the way to a more "homotopical" approach to motives; we give a few examples.

3.2.4. *The motivic Steenrod algebra.* We have already mentioned the motivic Steenrod algebra and its role in the proof of the Beilinson-Lichtenbaum/Bloch-Kato conjectures. In classical homotopy theory, the mod ℓ Steenrod algebra $\mathcal{A}^*$ is simply the (graded) endomorphism ring of the spectrum $H\mathbb{Z}/\ell$ representing mod ℓ singular cohomology in the stable homotopy category SH, that is

$$\mathcal{A}^* = H\mathbb{Z}/\ell^*(H\mathbb{Z}/\ell) = \oplus_n [H\mathbb{Z}/\ell, \Sigma^n H\mathbb{Z}/\ell]_{\mathrm{SH}}$$

The dual Steenrod algebra $\mathcal{A}_*$ is the mod ℓ homology of $H\mathbb{Z}/\ell$, that is

$$\mathcal{A}_* = H\mathbb{Z}/\ell_*(H\mathbb{Z}/\ell) = \oplus_n \pi_n^s(H\mathbb{Z}/\ell \wedge H\mathbb{Z}/\ell)$$

where π_n^s is stable homotopy: $\pi_n^s(E) := [\Sigma^n\mathbb{S}, E]_{\mathrm{SH}}$ and $\mathbb{S}$ is the sphere spectrum, $\mathbb{S} := \Sigma^\infty S^0$.

Making the obvious changes yields the motivic version: The motivic sphere spectrum over S is $\mathbb{S}_S : \Sigma^\infty_{\mathbb{P}^1} S^0_S$, where $S^0_S = S_+$, i.e., the base-scheme S with a disjoint copy of S added as a base-point. E-cohomology of a spectrum $F \in \mathrm{SH}(S)$ is $E^{a,b}(F) := [F, \Sigma^{a,b}E]_{\mathrm{SH}(S)}$, E-homology is $E_{a,b}(F) := [\Sigma^{a,b}\mathbb{S}_S, E \wedge F]_{\mathrm{SH}(S)}$, giving us the mod ℓ motivic Steenrod algebra

$$\mathcal{A}^{*,*} := H_{\mathrm{mot}}\mathbb{Z}/\ell^{**}(H_{\mathrm{mot}}\mathbb{Z}/\ell)$$

and the dual Steenrod algebra

$$\mathcal{A}_{*,*} := H_{\mathrm{mot}}\mathbb{Z}/\ell_{**}(H_{\mathrm{mot}}\mathbb{Z}/\ell).$$

Voevodsky [97] was able to construct explicit natural operations on mod ℓ motivic cohomology in case $S = \operatorname{Spec} k$, k a field of characteristic zero, and show that the algebra of operations they generate is remarkably similar to the classical case. He later showed in [95] that this construction gives a description of the full algebra of operations and that this is also isomorphic to the algebra $H_{\mathrm{mot}}\mathbb{Z}/\ell^{**}(H_{\mathrm{mot}}\mathbb{Z}/\ell)$.

One main difference from the classical case that the classical Steenrod algebra is an algebra over $H\mathbb{Z}/\ell^*(pt) = \mathbb{Z}/\ell$ (concentrated in degree 0), whereas the motivic version is a ring with both a left and right module structure over $H_{\mathrm{mot}}\mathbb{Z}/\ell^*(k) = \oplus_{a,b} H^a(\operatorname{Spec} k, \mathbb{Z}/\ell(b))$; these structures in general are not the same. The classical version acts trivially on the cohomology of a point, while (in general) the motivic version acts non-trivially on the motivic cohomology of k, which accounts for the two different module structures. However, Voevodsky's generators correspond directly to the standard classical generators, and fulfill essentially the same relations (the Adem relations). The most notable difference occurs at $\ell = 2$. Here $-1 \in k^\times$ shows up in two different places, one as the element $\tau \in H^0(k, \mathbb{Z}/2(1)) = \mu_2(k) = \{\pm 1\}$ and a second time as $\rho \in H^1(k, \mathbb{Z}/2(1)) = k^\times/k^{\times 2}$ as the class of -1 modulo squares. τ behaves differently from $1 \in H^0(pt, \mathbb{Z}/2)$ because, if k does not contain $\sqrt{-1}$, the map $H^0(k, \mathbb{Z}/4(1)) \to H^0(k, \mathbb{Z}/2(1))$ is the trivial map, so the Bockstein of τ is non-zero. Similarly $H^1(pt, \mathbb{Z}/2) = 0$ but if k does not contain $\sqrt{-1}$, then $\rho \neq 0$, so we have this additional "-1" to consider (in fact, the Bockstein of τ is ρ).

This was all extended to the positive characteristic case, at least for $\ell \neq \mathrm{char} k$, by Hoyois-Kelly-Østvær [47]. The motivic Steenrod algebra shows up in many other foundational computations, for instance, in the theorem of Hopkins-Morel-Hoyois [44], describing the relationship of $H_{\mathrm{mot}}\mathbb{Z}$ with Voevodsky's algebraic cobordism spectrum MGL.

Frankland and Spitzweck [25] have shown that the characteristic zero version of the mod p motivic Steenrod algebra is a summand of the actual motivic Steenrod algebra over a field of characteristic p. The lack of a complete understanding of the mod p motivic Steenrod algebra in characteristic p is a significant hinderance to our understanding of motivic homotopy theory in positive or mixed characteristic.

3.2.5. *Voevodsky's slice tower.* Besides motivic cohomology, algebraic K-theory is also represented in $\mathrm{SH}(k)$. One of the main results of Morel-Voevodsky [70, Theorem 3.13] about the unstable category $\mathcal{H}(k)$ is that the infinite Grassmannian

$\mathrm{Gr}(\infty,\infty) := \mathrm{colim}_{m,n}\mathrm{Gr}(m, n+m)$ represents Quillen's algebraic K-theory for $X \in \mathbf{Sm}_k$:

$$K_n(X) \cong [\Sigma^n_{S^1} X_+, \mathrm{Gr}(\infty,\infty) \times \mathbb{Z}]_{\mathcal{H}(k)}$$

Voevodsky [103] promotes this to representability in $\mathrm{SH}(k)$ by the $\mathbb{P}^1$-spectrum KGL

$$\mathrm{KGL} := (\mathrm{Gr}(\infty,\infty) \times \mathbb{Z}, \mathrm{Gr}(\infty,\infty) \times \mathbb{Z}, \ldots),$$

with bonding map given by the maps

$$\mathrm{Gr}(m,\infty) \wedge (\mathbb{P}^1,\infty) \to \mathrm{Gr}(m,\infty)$$

classifying the virtual bundle $p_1^*E_m \otimes p_2^*\mathcal{O}(1) - p_1^*E_m - p_2^*\mathcal{O}(1) + \mathcal{O}$ on $\mathrm{Gr}(m,\infty) \times \mathbb{P}^1$. His idea, described in [99, 100], is to define a version of the classical Postnikov tower, replacing usual connectivity with $\mathbb{P}^1$-connectivity.

More precisely, let $\mathrm{SH}^{\mathrm{eff}}(k)$ be the localizing subcategory of $\mathrm{SH}(k)$ generated by suspension spectra $\Sigma^\infty_{\mathbb{P}^1} X_+$, $X \in \mathbf{Sm}_k$, and for $n \in \mathbb{Z}$, let $\Sigma^n_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(k)$ denote the translate of $\mathrm{SH}^{\mathrm{eff}}(k)$ by the n-fold $\mathbb{P}^1$-suspension functor. This gives the filtration of $\mathrm{SH}(k)$ by localizing subcategories

$$\ldots \subset \Sigma^{n+1}_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(k) \subset \Sigma^n_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(k) \subset \ldots \subset \mathrm{SH}(k).$$

The inclusion $i_n : \Sigma^n_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(k) \to \mathrm{SH}(k)$ admits the right adjoint $r_n : \mathrm{SH}(k) \to \Sigma^n_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(k)$, defining the truncation functor $f_n := i_n r_n : \mathrm{SH}(k) \to \mathrm{SH}(k)$; using the above tower gives the natural transformations $f_{n+1} \to f_n \to \mathrm{Id}_{\mathrm{SH}(k)}$, giving the tower of endofunctors on $\mathrm{SH}(k)$

$$\ldots \to f_{n+1} \to f_n \to \ldots \to \mathrm{Id}_{\mathrm{SH}(k)}.$$

Taking the layers in this tower gives the distinguished triangles

$$f_{n+1} \to f_n \to s_n \to f_{n+1}[1]$$

Voevodsky calls s_n the *nth slice*. Applying this to an $E \in \mathrm{SH}(k)$ gives the tower in $\mathrm{SH}(k)$

$$\ldots \to f_{n+1}E \to f_nE \to \ldots \to E$$

and the distinguished triangles

$$f_{n+1}E \to f_nE \to s_nE \to f_{n+1}E[1]$$

The tower is called the *slice tower* for E, and gives rise to the *slice spectral sequence*

$$E_2^{p,q}(n)(\mathcal{X}) := (s_{-q}E)^{p+q,n}(\mathcal{X}) \Rightarrow E^{p+q,n}(\mathcal{X})$$

In general, this tower does not have good convergence properties, due in part to the fact that the filtration $\Sigma^*_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(k)$ of $\mathrm{SH}(k)$ is neither exhaustive nor separated, so in using the slice spectral sequence, one needs to address convergence.

The classical Postnikov tower in the classical stable homotopy category SH can be constructed in the same way, replacing $\Sigma^n_{\mathbb{P}^1}$ with $\Sigma^n_{S^1}$ and taking $\mathrm{SH}^{\mathrm{eff}}$ to be the localizing subcategory generated by $\Sigma^\infty T_+$, for T an arbitrary simplicial set. This is also the subcategory of -1-connected spectra, i.e. spectra E such that $\pi^s_m E = 0$ for $m < 0$, and $\Sigma^n_{S^1}\mathrm{SH}^{\mathrm{eff}}$ is the subcategory of $n-1$-connected spectra ($\pi^s_m E = 0$ for $m < n$). The corresponding nth slice of E is the shifted Eilenberg-MacLane spectrum $\Sigma^n\mathrm{EM}(\pi_n E)$, characterized by

$$\pi^s_m \Sigma^n \mathrm{EM}(\pi_n E) = \begin{cases} \pi_n E & \text{for } m = n \\ 0 & \text{else.} \end{cases}$$

and the resulting spectral sequence is

$$E_2^{p,q}(X) := H^p(X, \pi_{-q}E) \Rightarrow E^{p+q}(X)$$

This is the classical *Atiyah-Hirzebruch* spectral sequence, so one often calls Voevodsky's version the motivic Atiyah-Hirzebruch spectral sequence. In the classical case, the filtration is separated, so one has much better convergence properties.

Some basic results regarding the slice tower are

Theorem 3.9 ([57, 99])**.** *1. $s_n H_{\mathrm{mot}}\mathbb{Z} = 0$ for $n \neq 0$ and $s_0 H_{\mathrm{mot}}\mathbb{Z} = H_{\mathrm{mot}}\mathbb{Z}$.*
2. $s_n \mathrm{KGL} = \Sigma^n_{\mathbb{P}^1} H_{\mathrm{mot}}\mathbb{Z}$

Relying on the multiplicative properties of the slice tower proven by Pelaez [74], the first result can be promoted to

Corollary 3.10 ([74])**.** *For each $E \in \mathrm{SH}(k)$, $s_n E$ has a canonical structure of an $H_{\mathrm{mot}}\mathbb{Z}$-module*

Thus, we have the *homotopy motive* $\pi_{\mathrm{mot},n}E \in \mathrm{DM}^{\mathrm{eff}}(k)$, with

$$s_n E = \Sigma^n_{\mathbb{P}^1} \mathrm{EM}_{\mathrm{mot}}(\pi_{\mathrm{mot},n}E) = \mathrm{EM}_{\mathrm{mot}}(\pi_{\mathrm{mot},n}E \otimes \mathbb{Z}(n)[2n])$$

and we can rewrite the slice spectral sequence as

$$E_2^{p,q}(n)(X) := H^{p-q}(X, \pi_{\mathrm{mot},-q}E(n)) \Rightarrow E^{p+q,n}(\mathcal{X})$$

For $E = \mathrm{KGL}$ and $n = 0$, this gives

$$E_2^{p,q}(X) = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow \mathrm{KGL}^{p+q,0}(X) = K_{2q-p}(X).$$

Via the isomorphism $H^{p-q}(X, \mathbb{Z}(-q)) \cong \mathrm{CH}^{-q}(X, -p-q)$, this agrees with the E_2-reindexed Bloch-Lichtenbaum/Friedlander-Suslin spectral sequence (1.2) described in §1.1.4.

3.2.6. *The algebraic Hopf map.* We assume k has characteristic $\neq 2$. The classical stable Hopf map is the element of $\pi_1^s(\mathbb{S})$ induced by the generator $\eta_{\mathrm{top}} : S^3 \to S^2$ of $\pi_3(S^2)$. η_{top} has a purely algebraic representative, as the quotient map

$$(\mathbb{C}^2 \setminus \{0\}, \{(0,1)\} \to (\mathbb{CP}^1, \infty)$$

identifying the Riemann sphere $\mathbb{CP}^1$ with $\mathbb{C}^2 \setminus \{0\}/\mathbb{C}^\times$. We let $\eta : (\mathbb{A}^2 \setminus \{0\}, (0,1)) \to (\mathbb{P}^1, \infty)$ be the corresponding map in $\mathcal{H}_\bullet(k)$. Noting that $(\mathbb{A}^2 \setminus \{0\}, (0,1)) \cong S^{3,2}$, $(\mathbb{P}^1, \infty) \cong S^{2,1}$, this gives us the stable version $\eta \in \pi^{\mathbb{A}^1}_{1,1}(\mathbb{S}_k)$, the *stable algebraic Hopf map.* η is closely related to the automorphism $\tau : \mathbb{S}_k \to \mathbb{S}_k$ induced by the exchange-of-factors symmetry $\tau_{\mathbb{P}^1,\mathbb{P}^1} : \mathbb{P}^1 \wedge \mathbb{P}^1 \to \mathbb{P}^1 \wedge \mathbb{P}^1$, by the identity

$$\tau = \mathrm{Id} + \eta \circ \rho = \mathrm{Id} + \rho \circ \eta,$$

where $\rho : S^0_k \to \mathbb{G}_m$ is the map sending the non-base point of S^0_k to -1. In addition, we have

$$\eta(1 + \tau) = 0$$

These two identities are explained in [69, Remark 6.3.5].

After inverting 2, we can decompose $\mathbb{S}_k$ into the τ $+1$ and -1 eigenspaces, via the idempotents $(1-\tau)/2$, $(1+\tau)/2$. Since $\mathbb{S}_k$ is the unit for the monoidal structure on $\mathrm{SH}(k)$, this decomposes $\mathrm{SH}(k)[1/2]$ as $\mathrm{SH}(k)[1/2] = \mathrm{SH}(k)^+ \times \mathrm{SH}(k)^-$, with

$$\mathrm{SH}(k)^+ := \ker((1-\tau)/2 : \mathrm{SH}(k)[1/2] \to \mathrm{SH}(k)[1/2]);$$

$$\mathrm{SH}(k)^- = \ker((1+\tau)/2 : \mathrm{SH}(k)[1/2] \to \mathrm{SH}(k)[1/2])$$

Thus $\mathrm{SH}(k)^+ \subset \mathrm{SH}(k)[1/2]$ is the η-torsion subcategory, and $\mathrm{SH}(k)^- \subset \mathrm{SH}(k)[1/2]$ is the η-local subcategory $\mathrm{SH}(k)[1/2, \eta^{-1}]$.

For $k \subset \mathbb{C}$, we have the topological $\mathbb{C}$-realization

$$\Re_{\mathbb{C}} : \mathrm{SH}(k) \to \mathrm{SH}$$

sending $\Sigma^\infty_{\mathbb{P}^1} X_+$ to $\Sigma^\infty_{S^1} X(\mathbb{C})_+$. Clearly $\Re_{\mathbb{C}}(\eta)$ is the usual Hopf map in $\pi_1^s(\mathbb{S}) \cong \mathbb{Z}/2$. However, if $k \subset \mathbb{R}$, we have a corresponding real realization

$$\Re_{\mathbb{R}} : \mathrm{SH}(k) \to \mathrm{SH}$$

sending $\Sigma^\infty_{\mathbb{P}^1} X_+$ to $\Sigma^\infty_{S^1} X(\mathbb{R})_+$, and $\Re_{\mathbb{R}}(\eta)$ is induced by the map $\times 2 : S^1 \to S^1$. Thus, after inverting 2 everywhere, we see that $\Re_{\mathbb{C}}$ factors through the projection $\mathrm{SH}(k)[1/2] \to \mathrm{SH}(k)^+$ and $\Re_{\mathbb{R}}$ factors through the projection $\mathrm{SH}(k)[1/2] \to \mathrm{SH}(k)^-$.

Morel remarks that this behavior of the $\pm$ decomposition of $\mathrm{SH}(k)[1/2]$ with respect to $\Re_{\mathbb{C}}$ and $\Re_{\mathbb{R}}$ gives a view of $\mathrm{SH}(k)$ as a category yielding invariants that simultaneously reflect the stable homotopy type of $X(\mathbb{C})$ and that of $X(\mathbb{R})$, for all complex and real embeddings of k and for all smooth X over k.

Returning to motivic cohomology, the Hopf map in $\mathrm{DM}(k)$ becomes a morphism $\mathbb{Z}(2)[3] \to \mathbb{Z}(1)[2]$, that is, an element of $H^{-1}(k, \mathbb{Z}(-1)) = 0$. Thus $H_{\mathrm{mot}}\mathbb{Z}[1/2]$ lives in $\mathrm{SH}(k)^+$. This says that the slice tower on $\mathrm{SH}(k)^-$ is the constant tower of identity maps. This says that the slice tower yields no information on objects in $\mathrm{SH}(k)^-$. Another way to see this is that $\Sigma_{\mathbb{P}^1} = \Sigma^{2,1} = \Sigma_{\mathbb{G}_m} \circ \Sigma_{S^1}$. Since $\mathrm{SH}^{\mathrm{eff}}(k)$ is triangulated, we have $\Sigma^n_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(k) = \Sigma^n_{\mathbb{G}_m}\mathrm{SH}^{\mathrm{eff}}(k)$, and in $\mathrm{SH}(k)^-$, $\rho : \mathbb{S} \to \Sigma_{\mathbb{G}_m}\mathbb{S}$ and $\eta : \Sigma_{\mathbb{G}_m}\mathbb{S} \to \mathbb{S}$ are inverse isomorphisms. Thus $\Sigma^n_{\mathbb{G}_m}\mathrm{SH}^{\mathrm{eff}}(k)^- = \mathrm{SH}^{\mathrm{eff}}(k)^- = \mathrm{SH}(k)^-$.

3.3. **Six functors and motives over a base.** As we mentioned at the beginning of these lectures, Beilinson [8, §0.3] envisaged an abelian category of mixed motivic sheaves on each scheme X, $\mathrm{Sh}^{\mathrm{Mot}}_X$, with the Grothendieck six operations: the adjoint pair of derived pushforward and pullback functors for each morphism $f : Y \to X$

$$f^* : D(\mathrm{Sh}^{\mathrm{Mot}}_X) \rightleftarrows D(\mathrm{Sh}^{\mathrm{Mot}}_Y) : f_*$$

the adjoint pair of exceptional functors

$$f_! : D(\mathrm{Sh}^{\mathrm{Mot}}_Y) \rightleftarrows D(\mathrm{Sh}^{\mathrm{Mot}}_X) : f^!$$

internal Hom and tensor product, satisfying the "usual" relations, e.g., smooth and proper base-change isomorphisms, and a natural transformation $f_! \to f_*$ that is an isomorphism for proper f.

As we have seen, the lack of a Beilinson-Soulé vanishing theorem puts this beyond the realm of the current technology, but one could hope for this type of setup for triangulated categories of motives over a base-scheme X, $\mathrm{DM}(X)$. There are a number of approaches for this, many of which are based on having the Grothendieck six operations for the motivic stable homotopy categories $X \mapsto \mathrm{SH}(X)$.

Without going into detail, the structure of the Grothendieck six operations for $X \mapsto \mathrm{SH}(X)$, $X \in \mathbf{Sch}_B$, with B a fixed noetherian base-scheme of finite Krull dimension, has been constructed by Ayoub [3]. Ayoub's construction has been extended to a larger class of schemes by work of Cisinski-Déglise [18] and extended to the equivariant setting for a "tame" group G by Hoyois [45]. In fact, these constructions have been extended to a theory of six functors on a fairly general subcategory of algebraic stacks by Khan-Ravi [54], and with a similar theory by Chowdhury [17].

As we have seen, one can construct $\mathrm{DM}(k)$ as a homotopy category of $H\mathbb{Z}$-modules, so one could ask if there is a reasonable object $H\mathbb{Z}_S \in \mathrm{SH}(S)$ for which the category of $H\mathbb{Z}_S$-modules would be a reasonable choice for $\mathrm{DM}(S)$. This is in fact the case, and there are now two such constructions, one by Markus Spitzweck, one by Marc Hoyois, which both yield the same family of representing spectra $H\mathbb{Z}_S \in \mathrm{SH}(S)$ and resulting module categories.

3.3.1. *Motivic Borel-Moore homology over a base.* We have seen in §2.3 that the higher Chow groups agree with the theory given by the sheaves $z_{\mathrm{equi},r}(X)$ via the inclusion/quasi-isomorphism $C^{\mathrm{Sus}}_*(z_{\mathrm{equi},r}(X))(k) \subset z_r(X,*)$ for $r \geq 0$. Recalling that $C^{\mathrm{Sus}}_*(z_{\mathrm{equi},r}(X)) \cong \underline{\mathcal{H}om}(\mathbb{Z}(r)[2r], \mathbb{Z}(X)^c)$, we saw that

$$\begin{aligned}\mathrm{CH}_r(X,n) &\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}(0)[n], C^{\mathrm{Sus}}(z_{\mathrm{equi},r}(X)))\\ &\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}(r)[2r+n], \mathbb{Z}(X)^c).\end{aligned}$$

This suggests that it would have been better to consider $C^{\mathrm{Sus}}(z_{\mathrm{qfin}}(X))_{\mathrm{Nis}}$ as the Borel-Moore motive of X, define $\mathbb{Z}_{\mathrm{B.M.}}(X) := C^{\mathrm{Sus}}(z_{\mathrm{qfin}}(X))_{\mathrm{Nis}}$ and define

$$H^{\mathrm{B.M.}}_p(X, \mathbb{Z}(q)) := \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}(p)[q], \mathbb{Z}_{\mathrm{B.M.}}(X))) \cong \mathrm{CH}_q(X, p-2q).$$

In fact, for X of finite type over a Dedekind scheme B, one can give a reasonable extension of the definition we gave for a field to yield a cycle complex $z_r(X/B,*)$ and a good definition of motivic Borel-Moore homology

$$H^{\mathrm{B.M.}}_p(X/B, \mathbb{Z}(q)) := \mathbb{H}_{p-2q}(B_{\mathrm{Zar}}, p_{X*}z_q(*))$$

Here $p_{X*}z_q(*)$ is the Zariski sheaf on B associated to the presheaf $U \mapsto z_q(p_X^{-1}(U)/B,*)$. The elementary functorial properties of motivic Borel-Moore homology, namely proper pushforward and flat pullback, are rather easy to verify, the difficult part is the localization sequence, which was proven in [58].

However, a product structure is lacking even for X smooth over B, due to the fact that cycles in $z_r(X/B,n)$ are not in general flat over B, so one does not have an evident construction of external product

$$z_r(X/B,*) \times z_s(Y/B,*) \to z_{r+s}(X \times_B Y/B,*).$$

3.3.2. *Beilinson motivic cohomology.* Homotopy invariant algebraic K-theory is represented in $\mathrm{SH}(S)$ by Voevodsky's algebraic K-theory spectrum KGL_S. Cisinski-Déglise [18] note that KGL_S admits Adams operations Ψ_k and $\mathrm{KGL}_{S\mathbb{Q}}$ breaks up into the k^q-eigenspectra:

$$\mathrm{KGL}_{S\mathbb{Q}} = \oplus_i \mathrm{KGL}^{(i)}_S,$$

with $\mathrm{KGL}^{(i)}_S$ representing the ith graded piece of (rational) K-theory for the γ-filtration (assuming S is regular). This gives them a commutative monoid object (i.e. commutative ring spectrum) $H^{\mathrm{B}}_S := \mathrm{KGL}^{(0)}_S \in \mathrm{SH}(S)_{\mathbb{Q}}$, whose module category $H^{\mathrm{B}}_S - \mathbf{Mod}$ they call the category of *Beilinson motives over S*. This construction is cartesian, that is, for $f : Y \to X$ a morphism of schemes, one has a canonical isomorphism $f^*H^{\mathrm{B}}_X \cong H^{\mathrm{B}}_Y$, which is essentially what one needs to induce a six-functor formalism on $S \mapsto H^{\mathrm{B}}_S - \mathbf{Mod}$ from $\mathrm{SH}(-)$.

3.3.3. *Spitzweck's motivic cohomology.* Spitzweck [86] has defined an integral motivic cohomology theory over an arbitrary base-scheme by gluing together several "approximations" that each have good multiplicative properties. The result inherits the multiplicative structure from the individual components, and the gluing data yield a good motivic cohomology.

The main components are given by the Bloch cycle complex, truncated ℓ-adic étale cohomology, and logarithmic de Rham-Witt sheaves. As discussed above, the Bloch cycle complexes give rise to a general version of Bloch's higher Chow groups for finite type schemes over a Dedekind domain, which has nice localization properties but has poor functoriality and lacks a multiplicative structure. Using the Bloch-Kato conjectures, as established by Voevodsky [94], the ℓ-completed higher Chow groups are recognized as a truncated ℓ-adic étale cohomology, for ℓ prime to all residue characteristics. The theorem of Geisser-Levine [34] describes the p-completed higher Chow groups in characteristic $p > 0$ in terms of logarithmic de Rham-Witt sheaves. Finally, there is the good theory with $\mathbb{Q}$-coefficients given by Beilinson motivic cohomology of Cisinski-Déglise, as described above.

Each of these three theories: ℓ-adic étale cohomology, the cohomology of the logarithmic de Rham-Witt sheaves, and rational Beilinson motivic cohomology, have good functoriality and multiplicative properties. Gluing the ℓ-adic, p-adic and rational theories together via their respective comparisons with the Bloch cycle complex, Spitzweck constructs a theory with good functoriality and multiplicative properties, and which is described by a presheaf of complexes (equivalently, Eilenberg-MacLane spectra) on smooth schemes over a given Dedekind domain as base-scheme. The corresponding theory agrees with Voevodsky's motivic cohomology for smooth schemes over a perfect field, and is agrees with the hypercohomology of the Bloch complex for smooth schemes over a Dedekind domain (even in mixed characteristic), as described above, ignoring the multiplicative structure. Spitzweck then promotes this to yield an E_∞-object $M\mathbb{Z}_X$ in $\mathbb{P}^1$-spectra over X, for X smooth over a Dedekind ring.

Taking the base-scheme to be $\operatorname{Spec}\mathbb{Z}$, Spitzweck's construction yields a representing object $M\mathbb{Z}_\mathbb{Z}$ in $\mathrm{SH}(\mathbb{Z})$ and one can thus define absolute motivic cohomology for smooth schemes over an arbitrary base-scheme X by pulling back $M\mathbb{Z}_\mathbb{Z}$ to $M\mathbb{Z}_X \in \mathrm{SH}(X)$. In case X is smooth over a Dedekind ring, the two constructions of $M\mathbb{Z}_X$ agree. The resulting motivic cohomology agrees with Voevodsky's for smooth schemes of finite type over an arbitrary perfect base-field, and with the hypercohomology of the Bloch cycle complex for smooth finite type schemes over an arbitrary Dedekind domain. One can then define a triangulated category of motives $\mathrm{DM}_{\mathrm{Sp}}(X)$ over a base-scheme X as the homotopy category of $M\mathbb{Z}_X$-modules; the functor $X \mapsto \mathrm{DM}_{\mathrm{Sp}}(X)$ inherits a Grothendieck six-functor formalism from $X \mapsto \mathrm{SH}(X)$.

3.3.4. *Hoyois' motivic cohomology.* Spitzweck's construction gives a solution to the problem of constructing a triangulated category of motives over an arbitrary base, admitting a six-functor formalism and thus yielding a good theory of motivic cohomology. However, for a scheme that is not representable as a smooth scheme over a Dedekind domain, one does not really have a concrete description of the resulting motivic cohomology.

Hoyois [46] has constructed a theory of motivic cohomology over an arbitrary base-scheme S that is directly defined for each base-scheme S, rather than indirectly via gluing and then pulling back from $\operatorname{Spec}\mathbb{Z}$, as in Spitzweck's version. Hoyois uses a recent breakthrough, giving a really new construction of the motivic stable homotopy categories $\mathrm{SH}(S)$ relying on the notion of *framed correspondences*, more in line with Voevodsky's construction of $\mathrm{DM}(k)$. The basic idea is sketched in notes of Voevodsky [93], which were realized in a series of works by various subsets of Ananyevskiy, Garkusha, Panin, Neshitov [1, 2, 29, 30, 31, 32]. Building on these works, Elmanto, Hoyois, Khan, Sosnilo and Yakerson [21, 22, 23] construct an infinity category of framed correspondences, and use the basic program of Voevodsky's construction of $\mathrm{DM}(k)$ to realize $\mathrm{SH}(S)$ as arising from presheaves of spectra *with framed transfers*, just as objects of $\mathrm{DM}(k)$ arise from presheaves of complexes of sheaves with transfers for finite correspondences. We give here a very brief sketch of this construction.

An integral closed subscheme $Z \subset X \times Y$ that defines a finite correspondence from X to Y can be thought of a special type of a *span* via the two projections

$$\begin{tikzcd} & Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$

For X and Y smooth and finite type over a given base-scheme S, a framed correspondence from X to Y is also a span,

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

satisfying some conditions, together with some additional data (the framing); importantly, one does not insist that Z be a closed subscheme of $X \times_S Y$. For simplicity, assume that X is connected. The morphism p is required to be a finite, flat, local complete intersection (lci) morphism, referred to as a finite *syntomic* morphism. The lci condition means that p factors (locally over X) as closed immersion $i : Z \to P$ followed by a smooth morphism $f : P \to X$, and the closed subscheme $i(Z)$ of P is locally defined by exactly $\dim_X P - \dim_X Z$ equations that form a regular sequence. The morphism p factored in this way has a relative cotangent complex $\mathbb{L}_p$ admitting a simple description, namely

$$\mathbb{L}_p = [\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} i^*\Omega_{P/X}];$$

the conditions on i and p say that both $\mathcal{I}_Z/\mathcal{I}_Z^2$ and $i^*\Omega_{P/X}$ are locally free coherent sheaves on Z of rank $\dim_X P - \dim_X Z$ and $\dim_X P$, respectively. The perfect complex $\mathbb{L}_p$ defines a point $\{\mathbb{L}_p\}$ in the K-theory space $\mathcal{K}(Z)$ of virtual rank $\dim_X Z$; in the case of a finite syntomic morphism the virtual rank is zero.

A *framing* for a syntomic map $p : Z \to X$ is a path $\gamma : [0,1] \to \mathcal{K}(Z)$ connecting $\{\mathbb{L}_p\}$ with the base-point $0 \in \mathcal{K}(Z)$. For a framing to exist, the class $[\mathbb{L}_p] \in K_0(Z)$ must be zero, but the choice of γ is additional data. The morphism $q : Z \to Y$ is arbitrary.

One has the usual notion of a composition of spans:

$$\begin{tikzcd}[column sep=small] & Z' \arrow[dl, "p'"'] \arrow[dr, "q'"] & \\ Y & & W \end{tikzcd} \circ \begin{tikzcd}[column sep=small] & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd} := \begin{tikzcd}[column sep=small] & Z \times_Y Z' \arrow[dl, "p\circ p_1"'] \arrow[dr, "q'\circ p_2"] & \\ X & & W \end{tikzcd}$$

which preserves the finite syntomic condition; since one is not identifying isomorphic spans, one needs an ∞-categorical structure to take care of the fact that the fiber products are only defined up to a (contractible) choice of isomorphisms, which leads to higher homotopies for the necessary coherences for associativity of composition (see [6, Section 3]). The composition of paths also requires an ∞-categorical framework, since we are dealing here with actual paths, not paths up to homotopy, but one also needs some kind of higher isomorphism of the data arising from homotopic paths. In the end, this produces an infinity category $\mathbf{Corr}^{fr}(\mathbf{Sm}_S)$ of framed correspondences on smooth S-schemes, rather than a category.

Via the infinity category $\mathbf{Corr}^{fr}(\mathbf{Sm}_S)$, we have the infinity category of framed motivic spaces, $\mathbf{H}^{fr}(S)$, this being the infinity category of $\mathbb{A}^1$-invariant, Nisnevich sheaves of spaces on $\mathbf{Corr}^{fr}(\mathbf{Sm}_S)$. There is a stable version, $\mathbf{SH}^{fr}(S)$, an infinite suspension functor $\Sigma^\infty_{fr} : \mathbf{H}^{fr}(S) \to \mathbf{SH}^{fr}(S)$, and an equivalence of infinity categories $\gamma_* : \mathbf{SH}^{fr}(S) \to \mathbf{SH}(S)$, where $\mathbf{SH}(S)$ is the infinity category version of the triangulated category $\mathrm{SH}(S)$. In other words, the homotopy category of $\mathbf{SH}(S)$ is $\mathrm{SH}(S)$, and the functor γ_* gives a description of $\mathrm{SH}(S)$ as the homotopy category of $\mathbf{SH}^{fr}(S)$.

The equivalence γ_* can be thought of as a motivic version of the construction of infinite loop spaces from Segal's Γ-spaces, where a framed correspondence $X \leftarrow Z \to S$ of degree n over X is to be considered as a generalization of the map $[n]_+ \to [0]_+$ in Γ^{op}.

Here is a rough idea of Hoyois' construction of the spectrum representing motivic cohomology over S. Hoyois' starts with spans:

$$X \xleftarrow{p} Z \xrightarrow{q} Y,$$

$X, Y \in \mathbf{Sm}_S$, with $p : Z \to X$ a finite morphism such that $p_*\mathcal{O}_Z$ is a locally free $\mathcal{O}_X$-module (this condition is satisfied if p is a syntomic morphism, but is in general a weaker condition). These spans form an infinity category $\mathbf{Corr}^{flf}(\mathbf{Sm}_S)$ under span composition and forgetting the paths γ defines a functor of infinity categories $\pi_{ad} : \mathbf{Corr}^{fr}(\mathbf{Sm}_S) \to \mathbf{Corr}^{flf}(\mathbf{Sm}_S)$.

Given a commutative monoid A, the constant Nisnevich sheaf on $\mathbf{Sm}_S$ with value A extends to a functor

$$A_S : (\mathbf{Corr}^{flf})^{\mathrm{op}} \to \mathbf{Ab},$$

where pullback from Y to X by $X \xleftarrow{p} Z \xrightarrow{q} Y$ is given by multiplication by $\mathrm{rank}_{\mathcal{O}_X}\mathcal{O}_Z$, if X and Y are connected; one extends to general smooth X and Y by additivity. This gives us the presheaf of abelian monoids with framed transfers $A_S^{fr} := A_S \circ \pi_{ad}^{\mathrm{op}}$, and the machinery of Elmanto *et al.* converts this into the motivic spectrum $\gamma_*\Sigma^\infty_{fr}A_S^{fr} \in \mathbf{SH}(S)$. Hoyois shows that this construction produces a cartesian family, and that taking $A = \mathbb{Z}$ recovers Spitzweck's family $S \mapsto M\mathbb{Z}_S$.

This gives us a conceptually simple construction of a motivic Eilenberg-MacLane spectrum, and the corresponding motivic category $\mathrm{DM}_H(S)$, much in the spirit of Voevodsky original construction of DM(k) and the Röndigs-Østvær theorem identifying DM(k) with the homotopy category of EM($\mathbb{Z}(0)$)-modules.

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