Unstable and stable motivic homotopy theories

Clémentine Lemarié−−Rieusset
Based on Asok, Bachmann and Hopkins' 2023 preprint

May 21, 2024

Notation : schemes over a field

Throughout:

- k is a field
- Sch_k is the category of finite-type separated k-schemes
- Sm_k is the full subcategory of Sch_k whose objects are the objects of Sch_k which are smooth k-schemes
- Sch_k^{qp} is the full subcategory of Sch_k whose objects are the objects of Sch_k which are quasi-projective k-schemes

Notation : simplicial presheaves / presheaves of spaces

If $S_k \in \{\mathrm{Sch}_k, \mathrm{Sm}_k, \mathrm{Sch}_k^{qp}\}$ then we denote by $\mathrm{P}(S_k)$:

- (in a model-categorical setting) the category of simplicial presheaves over S_k, i.e. $\mathrm{Fun}(S_k^{\mathrm{op}}, \mathrm{sSet})$ with sSet the category of simplicial sets
- (in an ∞-categorical setting) the ∞-category (better: ∞-topos) of presheaves of spaces over S_k, i.e. $\mathrm{Fun}(S_k^{\mathrm{op}}, \mathcal{S})$ with $\mathcal{S}$ the ∞-category of spaces (i.e. the coherent nerve of the simplicial category of Kan complexes)

Notation : simplicial presheaves / presheaves of spaces

If $S_k \in \{\mathrm{Sch}_k, \mathrm{Sm}_k, \mathrm{Sch}_k^{qp}\}$ then we denote by $\mathrm{P}(S_k)$:

- (in a model-categorical setting) the category of simplicial presheaves over S_k, i.e. $\mathrm{Fun}(S_k^{\mathrm{op}}, \mathrm{sSet})$ with sSet the category of simplicial sets
- (in an ∞-categorical setting) the ∞-category (better: ∞-topos) of presheaves of spaces over S_k, i.e. $\mathrm{Fun}(S_k^{\mathrm{op}}, \mathcal{S})$ with $\mathcal{S}$ the ∞-category of spaces (i.e. the coherent nerve of the simplicial category of Kan complexes)

If τ is a Grothendieck topology on S_k then we denote by $\mathrm{Shv}_\tau(S_k)$ the full subcategory of S_k whose objects are the objects of S_k which are τ-sheaves.

Notation : simplicial presheaves / presheaves of spaces

If $S_k \in \{\mathrm{Sch}_k, \mathrm{Sm}_k, \mathrm{Sch}_k^{qp}\}$ then we denote by $\mathrm{P}(S_k)$:

- (in a model-categorical setting) the category of simplicial presheaves over S_k, i.e. $\mathrm{Fun}(S_k^{\mathrm{op}}, \mathrm{sSet})$ with sSet the category of simplicial sets
- (in an ∞-categorical setting) the ∞-category (better: ∞-topos) of presheaves of spaces over S_k, i.e. $\mathrm{Fun}(S_k^{\mathrm{op}}, \mathcal{S})$ with $\mathcal{S}$ the ∞-category of spaces (i.e. the coherent nerve of the simplicial category of Kan complexes)

If τ is a Grothendieck topology on S_k then we denote by $\mathrm{Shv}_\tau(S_k)$ the full subcategory of S_k whose objects are the objects of S_k which are τ-sheaves. Versions with pointed spaces: $\mathrm{P}(S_k)_*$ and $\mathrm{Shv}_\tau(S_k)_*$.

Contents

Motivic spaces

Definition

A presheaf $\mathscr{X} \in \mathrm{P}(\mathrm{Sm}_k)$ is $\mathbb{A}^1$-invariant if for every $U \in \mathrm{Sm}_k$ the projection $U \times_k \mathbb{A}^1_k \to U$ induces a weak equivalence / an equivalence $\mathscr{X}(U) \to \mathscr{X}(U \times_k \mathbb{A}^1_k)$ (hence an isomorphism in the homotopy category).

Motivic spaces

Definition

A presheaf $\mathscr{X} \in \mathrm{P}(\mathrm{Sm}_k)$ is $\mathbb{A}^1$-invariant if for every $U \in \mathrm{Sm}_k$ the projection $U \times_k \mathbb{A}^1_k \to U$ induces a weak equivalence / an equivalence $\mathscr{X}(U) \to \mathscr{X}(U \times_k \mathbb{A}^1_k)$ (hence an isomorphism in the homotopy category).

Definition

The category / ∞-category of motivic spaces, denoted $\mathrm{Spc}(k)$, is the full subcategory of $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ whose objects are the objects of $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ which are $\mathbb{A}^1$-invariant.

Motivic spaces

Definition

A presheaf $\mathscr{X} \in \mathrm{P}(\mathrm{Sm}_k)$ is $\mathbb{A}^1$-invariant if for every $U \in \mathrm{Sm}_k$ the projection $U \times_k \mathbb{A}^1_k \to U$ induces a weak equivalence / an equivalence $\mathscr{X}(U) \to \mathscr{X}(U \times_k \mathbb{A}^1_k)$ (hence an isomorphism in the homotopy category).

Definition

The category / ∞-category of motivic spaces, denoted $\mathrm{Spc}(k)$, is the full subcategory of $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ whose objects are the objects of $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ which are $\mathbb{A}^1$-invariant.

Proposition-Definition

The inclusion $\mathrm{Spc}(k) \subset \mathrm{P}(\mathrm{Sm}_k)$ admits a left adjoint $\mathrm{L}_{mot} : \mathrm{P}(\mathrm{Sm}_k) \to \mathrm{Spc}(k)$ which is called the motivic localization functor.

Motivic localization and connectivity

Theorem (Morel-Voevodsky, Paragraph 2, Cor. 3.22)

Let $\mathscr{X} \in \mathrm{Spc}(k)$. The morphism $\pi_0(\mathscr{X}) \to \pi_0(\mathrm{L}_{mot}(\mathscr{X}))$ is an epimorphism. In particular, if $\mathscr{X} \in \mathrm{Spc}(k)$ is connected then $\mathrm{L}_{mot}(\mathscr{X})$ is connected.

Motivic localization and connectivity

Theorem (Morel-Voevodsky, Paragraph 2, Cor. 3.22)

Let $\mathscr{X} \in \mathrm{Spc}(k)$. The morphism $\pi_0(\mathscr{X}) \to \pi_0(\mathrm{L}_{mot}(\mathscr{X}))$ is an epimorphism. In particular, if $\mathscr{X} \in \mathrm{Spc}(k)$ is connected then $\mathrm{L}_{mot}(\mathscr{X})$ is connected.

Theorem (Unstable connectivity theorem (Morel, Thm 6.38); for a detailed proof see: Asok, Wickelgren, Williams 2017, Thm 2.2.12)

If $\mathscr{X} \in \mathrm{Spc}(k)$ is pulled back from a perfect subfield of k and is n-connected (for some $n \geq 1$) then $\mathrm{L}_{mot}(\mathscr{X})$ is n-connected.

Quotients of strongly $\mathbb{A}^1$-invariant sheaves

Definition

A sheaf $\mathscr{X} \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ pointed by $x : \mathrm{Spec}(k) \to \mathscr{X}$ together with a base-point preserving morphism $m : \mathscr{X} \times_k \mathscr{X} \to \mathscr{X}$ is an H-group with inverse morphism $i : \mathscr{X} \to \mathscr{X}$ if in the homotopy category of $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ the following diagrams commute:

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{x \times \mathrm{Id}} & \mathscr{X} \times_k \mathscr{X} \\ {\scriptstyle \mathrm{Id} \times x}\downarrow & \overset{\mathrm{Id}}{\searrow} & \downarrow{\scriptstyle m} \\ \mathscr{X} \times_k \mathscr{X} & \xrightarrow[m]{} & \mathscr{X} \end{array} \qquad \begin{array}{ccc} \mathscr{X} & \xrightarrow{i \times \mathrm{Id}} & \mathscr{X} \times_k \mathscr{X} \\ {\scriptstyle \mathrm{Id} \times i}\downarrow & \overset{c_x}{\searrow} & \downarrow{\scriptstyle m} \\ \mathscr{X} \times_k \mathscr{X} & \xrightarrow[m]{} & \mathscr{X} \end{array}$$

$$\begin{array}{ccc} \mathscr{X} \times_k \mathscr{X} \times_k \mathscr{X} & \xrightarrow{m \times \mathrm{Id}} & \mathscr{X} \times_k \mathscr{X} \\ {\scriptstyle \mathrm{Id} \times m}\downarrow & & \downarrow{\scriptstyle m} \\ \mathscr{X} \times_k \mathscr{X} & \xrightarrow[m]{} & \mathscr{X} \end{array}$$

Lemma

If k is perfect and $\mathscr{X} \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ is an H-group such that $\pi_0(\mathscr{X})$ receives a surjection from a strongly $\mathbb{A}^1$-invariant sheaf then $\pi_0(\mathrm{L}_{mot}(\mathscr{X}))$ is strongly $\mathbb{A}^1$-invariant.

Since the functor L_{mot} preserves finite products, $\mathrm{L}_{mot}(\mathscr{X})$ is still an H-group, hence $\pi_0(\mathrm{L}_{mot}(\mathscr{X}))$ is $\mathbb{A}^1$-invariant by [Choudhury 2014 (*Connectivity of motivic H-spaces*), Theorem 4.18] and by the same theorem, it is strongly $\mathbb{A}^1$-invariant since it receives a surjection from a strongly $\mathbb{A}^1$-invariant sheaf (via the surjection $\pi_0(\mathscr{X}) \to \pi_0(\mathrm{L}_{mot}(\mathscr{X}))$).

Lemma

If k is perfect and $\mathscr{X} \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ is an H-group such that $\pi_0(\mathscr{X})$ receives a surjection from a strongly $\mathbb{A}^1$-invariant sheaf then $\pi_0(\mathrm{L}_{mot}(\mathscr{X}))$ is strongly $\mathbb{A}^1$-invariant.

Since the functor L_{mot} preserves finite products, $\mathrm{L}_{mot}(\mathscr{X})$ is still an H-group, hence $\pi_0(\mathrm{L}_{mot}(\mathscr{X}))$ is $\mathbb{A}^1$-invariant by [Choudhury 2014 (*Connectivity of motivic H-spaces*), Theorem 4.18] and by the same theorem, it is strongly $\mathbb{A}^1$-invariant since it receives a surjection from a strongly $\mathbb{A}^1$-invariant sheaf (via the surjection $\pi_0(\mathscr{X}) \to \pi_0(\mathrm{L}_{mot}(\mathscr{X}))$).

Definition

A sifted diagram is a diagram D such that for every finite set S and every representation $F : D \times S \to \mathrm{Set}$ the canonical morphism $\mathrm{colim}_{d\in D} \prod_{s\in S} F(d,s) \to \prod_{s\in S} \mathrm{colim}_{d\in D} F(d,s)$ is an isomorphism.
A sifted colimit is a colimit over a sifted diagram.

Lemma

If k is perfect then the forgetful functor from strongly $\mathbb{A}^1$-invariant sheaves of groups to $\mathbb{A}^1$-invariant sheaves of groups preserves sifted colimits.

By [Lurie 2009 (HTT), Corollary 5.5.8.17], it suffices to treat filtered colimits (okay since cohomology commutes with these) and geometric realizations. If $G_\bullet$ is a simplicial diagram ($G_0 \rightleftarrows G_1 \cdots$) of strongly $\mathbb{A}^1$-invariant sheaves of groups and $|G_\bullet|^{\mathrm{Nis}}$ is its colimit in $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$, then $\pi_0(|G_\bullet|^{\mathrm{Nis}})$ is a quotient of the strongly $\mathbb{A}^1$-invariant sheaf $\pi_0(G_0)$ hence, by the previous lemma, $\pi_0(\mathrm{L}_{mot}(|G_\bullet|^{\mathrm{Nis}}))$ is grouplike and strongly $\mathbb{A}^1$-invariant, hence the colimit of $G_\bullet$ in both $\mathbb{A}^1$-invariant sheaves of groups and strongly $\mathbb{A}^1$-invariant sheaves of groups.

Corollary

If k is perfect and $\mathscr{X}_\bullet$ is a sifted diagram of H-groups in $\mathrm{Spc}(k)$ such that each $\pi_0(\mathscr{X}_\bullet)$ is grouplike and strongly $\mathbb{A}^1$-invariant then $\pi_0(\operatorname{colim} \mathscr{X}_\bullet)$ is grouplike and strongly $\mathbb{A}^1$-invariant.

Follows from [Choudhury 2014, Theorem 4.18] and the previous lemma.

Corollary

If k is perfect and $\mathscr{X}_\bullet$ is a sifted diagram of H-groups in Spc(k) such that each $\pi_0(\mathscr{X}_\bullet)$ is grouplike and strongly $\mathbb{A}^1$-invariant then $\pi_0(\text{colim}\, \mathscr{X}_\bullet)$ is grouplike and strongly $\mathbb{A}^1$-invariant.

Follows from [Choudhury 2014, Theorem 4.18] and the previous lemma.

Corollary

If k is perfect and $\mathscr{X}_\bullet$ is a simplicial diagram of H-groups in Spc(k) such that $\pi_0(\mathscr{X}_0)$ is grouplike and strongly $\mathbb{A}^1$-invariant then $\pi_0(\text{colim}\, \mathscr{X}_\bullet)$ is grouplike and strongly $\mathbb{A}^1$-invariant.

Follows from [Choudhury 2014, Theorem 4.18] and the fact that $\pi_0(\mathscr{X}_0) \to \pi_0(\text{colim}\, \mathscr{X}_\bullet)$ is surjective.

Solvability

Definition

Let G be a strongly $\mathbb{A}^1$-invariant sheaf of groups which acts on itself by conjugation and $[G, G]_{\mathbb{A}^1}$ be the (strongly $\mathbb{A}^1$-invariant) kernel of the quotient map. The $\mathbb{A}^1$-derived series of G is defined by $G^{(0)}_{\mathbb{A}^1} = G$ and for all $i \geq 0$, $G^{(i+1)}_{\mathbb{A}^1} = [G^{(i)}_{\mathbb{A}^1}, G^{(i)}_{\mathbb{A}^1}]_{\mathbb{A}^1}$. A strongly $\mathbb{A}^1$-invariant sheaf of groups is $\mathbb{A}^1$-solvable if its $\mathbb{A}^1$-derived series terminates after finitely many steps. A pointed connected motivic space $\mathscr{X}$ is solvable if $\pi_1(\mathscr{X})$ is $\mathbb{A}^1$-solvable.

Important remark

Nilpotent motivic spaces are solvable.

Proposition

If $\mathscr{X}$ is a solvable motivic space then $\mathscr{X}$ admits a functorial Whitehead tower. More precisely, there exist a weakly increasing sequence of integers $1 = n_0 \leq n_1 \leq n_2 \leq \dots$ such that $n_i \to +\infty$, a sequence of strictly $\mathbb{A}^1$-invariant sheaves of abelian groups $(A_i(\mathscr{X}))_{i \geq 0}$ and a sequence of motivic spaces $(\mathscr{X}\langle i \rangle)_{i \geq 0}$ such that $\mathscr{X}\langle 0 \rangle = \mathscr{X}$ and for each $i \geq 0$, $\mathscr{X}\langle i \rangle$ is n_i-connective (i.e. $(n_i - 1)$-connected), fitting into the fiber sequences

$$\mathscr{X}\langle i+1 \rangle \to \mathscr{X}\langle i \rangle \to K(A_i(\mathscr{X}), n_i)$$

One defines the $\mathscr{X}\langle i \rangle)$ inductively (and the $A_i(\mathscr{X})$ to have these fiber sequences and the n_i maximal to have n_i-connectivity of $\mathscr{X}\langle i \rangle$ (except possibly $n_0 := 1$)) as follows: if $n_i = 1$ then $\mathscr{X}\langle i+1 \rangle)$ is the fiber of $\mathscr{X}\langle i \rangle \to B\pi_1(\mathscr{X}\langle i \rangle) \to B\pi_1(\mathscr{X})^{ab}_{\mathbb{A}^1}$, otherwise the fiber of the corresponding k-invariant in the Postnikov tower. The fact that $n_i \to +\infty$ comes from the solvability of $\mathscr{X}$.

Motivic localization, colimits and limits

Proposition

The loop functor $\Omega : \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)_{*,conn} \to H - \mathrm{Grp}$ and the classifying space functor $B : H - \mathrm{Grp} \to \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)_{*,conn}$ form an equivalence of categories.

Proposition

If k is perfect then a pointed connected space $\mathscr{X} \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ is $\mathbb{A}^1$-invariant (i.e. a motivic space) if and only if for every $i \geq 1$ the homotopy sheaf $\pi_i(\mathscr{X})$ is strongly $\mathbb{A}^1$-invariant.

Corollary

If k is perfect then the loop functor $\Omega : \mathrm{Spc}(k)_{*,conn} \to H - \mathrm{Grp}_{\pi_0 \text{ strongly } \mathbb{A}^1\text{-invariant}}$ and the classifying space functor $B : H - \mathrm{Grp}_{\pi_0 \text{ strongly } \mathbb{A}^1\text{-invariant}} \to \mathrm{Spc}(k)_{*,conn}$ form an equivalence of categories.

Proposition

If $\mathscr{B} \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)_{*,conn}$ then there is an equivalence of categories between $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)_{/\mathscr{B}}$ and $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)_{\Omega\mathscr{B}-\mathrm{action}}$. This equivalence sends $\mathscr{E} \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)_{/\mathscr{B}}$ to $\mathrm{fib}(\mathscr{E} \to \mathscr{B})$ with the canonical action and $\mathscr{F} \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)_{\Omega\mathscr{B}-\mathrm{action}}$ to $\mathscr{F}//\Omega\mathscr{B}$ (with the canonical morphism $\mathscr{F}//\Omega\mathscr{B} \to *//\Omega\mathscr{B} \simeq \mathscr{B}$), where $\mathscr{F}//\Omega\mathscr{B}$ is the (homotopy) quotient (a.k.a. bar construction), i.e. the geometric realization of

$$\mathscr{F} \leftleftarrows \mathscr{F} \times_k \Omega\mathscr{B} \overset{\longleftarrow}{\leftleftarrows} \dots$$

Corollary

If k is perfect and $\mathscr{B} \in \mathrm{Spc}(k)_{*,conn}$ then the previous equivalence of categories restricts to an equivalence of categories between $\mathrm{Spc}(k)_{/\mathscr{B}}$ and $\mathrm{Spc}(k)_{\Omega\mathscr{B}-\mathrm{action}}$.

If $\mathscr{E} \in \mathrm{Spc}(k)_{/\mathscr{B}}$ then $\mathrm{fib}(\mathscr{E} \to \mathscr{B})$ is $\mathbb{A}^1$-invariant (since $\mathscr{E}$ and $\mathscr{B}$ are $\mathbb{A}^1$-invariant and $\mathscr{B}$ is connected). If $\mathscr{F} \in \mathrm{Spc}(k)_{\Omega\mathscr{B}-\mathrm{action}}$ then $\mathscr{F}//\Omega\mathscr{B}$ is $\mathbb{A}^1$-invariant since we have the fiber sequence in $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$

$$\mathscr{F} \longrightarrow \mathscr{F}//\Omega\mathscr{B} \longrightarrow \mathscr{B}$$

(note that $\mathscr{F}$ and $\mathscr{B}$ are $\mathbb{A}^1$-invariant and $\mathscr{B}$ is connected).

Theorem

If k is perfect, $\mathscr{X}_{1,1}$ is a pointed connected Nisnevich sheaf such that $\pi_0(\mathsf{L}_{mot}(\Omega\mathscr{X}_{1,1}))$ is strongly $\mathbb{A}^1$-invariant and the following is a cartesian square in $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)_*$:

$$\begin{array}{ccc} \mathscr{X}_{0,0} & \xrightarrow{g_0} & \mathscr{X}_{1,0} \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ \mathscr{X}_{0,1} & \xrightarrow[g_1]{} & \mathscr{X}_{1,1} \end{array}$$

then the following square is cartesian:

$$\begin{array}{ccc} \mathsf{L}_{mot}(\mathscr{X}_{0,0}) & \xrightarrow{\mathsf{L}_{mot}(g_0)} & \mathsf{L}_{mot}(\mathscr{X}_{1,0}) \\ {\scriptstyle \mathsf{L}_{mot}(f_0)}\downarrow & & \downarrow{\scriptstyle \mathsf{L}_{mot}(f_1)} \\ \mathsf{L}_{mot}(\mathscr{X}_{0,1}) & \xrightarrow[\mathsf{L}_{mot}(g_1)]{} & \mathsf{L}_{mot}(\mathscr{X}_{1,1}) \end{array}$$

Denoting $\mathscr{G} := \Omega\mathscr{X}_{1,1}$, under the equivalence between $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)_{/\mathscr{X}_{1,1}}$ and $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)_{\mathscr{G}-\mathrm{action}}$ described earlier, $\mathscr{X}_{0,1}$ corresponds to $\mathscr{F} := \mathrm{fib}(\mathscr{X}_{0,1} \to \mathscr{X}_{1,1})$, $\mathscr{X}_{1,0}$ corresponds to $\mathscr{H} := \mathrm{fib}(\mathscr{X}_{1,0} \to \mathscr{X}_{1,1})$ and $\mathscr{X}_{0,0}$ corresponds to $\mathscr{F} \times_k \mathscr{H}$. Since $\mathscr{X}_{1,1}$ is connected and $\pi_0(\mathrm{L}_{mot}(\Omega\mathscr{X}_{1,1}))$ is strongly $\mathbb{A}^1$-invariant, by [AWW17, Theorem 2.3.3] L_{mot} preserves the fiber sequences $\mathscr{G} \to * \to \mathscr{X}_{1,1}$, $\mathscr{F} \to \mathscr{X}_{0,1} \to \mathscr{X}_{1,1}$ and $\mathscr{H} \to \mathscr{X}_{1,0} \to \mathscr{X}_{1,1}$, so that $\mathrm{L}_{mot}(\mathscr{G}) \simeq \Omega\,\mathrm{L}_{mot}(\mathscr{X}_{1,1})$ and (since $\mathrm{L}_{mot}(\mathscr{X}_{1,1})$ is connected) under the equivalence between $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)_{/\,\mathrm{L}_{mot}(\mathscr{X}_{1,1})}$ and $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)_{\mathrm{L}_{mot}(\mathscr{G})-\mathrm{action}}$ described earlier, the cartesian square

$$\begin{array}{ccc} \mathrm{L}_{mot}(\mathscr{F} \times_k \mathscr{H}) \simeq \mathrm{L}_{mot}(\mathscr{F}) \times_k \mathrm{L}_{mot}(\mathscr{H}) & \xrightarrow{\mathrm{L}_{mot}(g_0)} & \mathrm{L}_{mot}(\mathscr{H}) \\ \downarrow{\scriptstyle \mathrm{L}_{mot}(f_0)} & & \downarrow{\scriptstyle \mathrm{L}_{mot}(f_1)} \\ \mathrm{L}_{mot}(\mathscr{F}) & \xrightarrow[\mathrm{L}_{mot}(g_1)]{} & \mathrm{L}_{mot}(\mathscr{G}) \end{array}$$

translates into the announced cartesian square.

Proposition (Realization fibrations)

If k is perfect, $\mathcal{I}$ is a sifted diagram and $\mathscr{X}_{0,0}$, $\mathscr{X}_{1,0}$, $\mathscr{X}_{0,1}$ and $\mathscr{X}_{1,1}$ are representations of $\mathcal{I}$ in $\mathrm{Spc}(k)_*$, with $\mathscr{X}_{1,1}$ objectwise connected, fitting in the following cartesian square:

$$\begin{array}{ccc} \mathscr{X}_{0,0} & \xrightarrow{g_0} & \mathscr{X}_{1,0} \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ \mathscr{X}_{0,1} & \xrightarrow[g_1]{} & \mathscr{X}_{1,1} \end{array}$$

then, denoting for each $i, j \in \{0, 1\}$ by $|\mathscr{X}_{i,j}|_{\mathcal{I}}$ the colimit in $\mathrm{Spc}(k)_*$ of $\mathscr{X}_{i,j}$, the following is a cartesian square in $\mathrm{Spc}(k)_*$:

$$\begin{array}{ccc} |\mathscr{X}_{0,0}|_{\mathcal{I}} & \xrightarrow{|g_0|_{\mathcal{I}}} & |\mathscr{X}_{1,0}|_{\mathcal{I}} \\ {\scriptstyle |f_0|_{\mathcal{I}}}\downarrow & & \downarrow{\scriptstyle |f_1|_{\mathcal{I}}} \\ |\mathscr{X}_{0,1}|_{\mathcal{I}} & \xrightarrow[|g_1|_{\mathcal{I}}]{} & |\mathscr{X}_{1,1}|_{\mathcal{I}} \end{array}$$

The corresponding result in $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)_*$ is [J. Lurie (Higher Algebra) 2017, Lemma 5.5.6.17] and the previous theorem allows us to conclude from this, since $\pi_0(\mathrm{L}_{mot}(\Omega|\mathscr{X}_{1,1}|^{\mathrm{Nis}}_{\mathcal{I}})) \simeq \pi_0(\mathrm{L}_{mot}(|\Omega\mathscr{X}_{1,1}|^{\mathrm{Nis}}_{\mathcal{I}}))$ is strongly $\mathbb{A}^1$-invariant by a previous result (and the argument for $\Omega|\mathscr{X}_{1,1}|^{\mathrm{Nis}}_{\mathcal{I}} \simeq |\Omega\mathscr{X}_{1,1}|^{\mathrm{Nis}}_{\mathcal{I}}$ is similar to the argument in [Lurie (Higher Algebra) 2017, Lemma 5.5.6.17]).

Proposition

If k is perfect then the functor $\Omega : \mathrm{Spc}(k)_* \to \mathrm{Spc}(k)_*$ preserves sifted colimits of connected spaces.

If $\mathcal{I}$ is a sifted diagram and $\mathscr{X}$ is a representation of $\mathcal{I}$ in $\mathrm{Spc}(k)_{*,conn}$ then there is a cartesian square of the form:

$$\begin{array}{ccc} \Omega\mathscr{X} & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathscr{X} \end{array}$$

The result follows from the previous proposition (and the fact that sifted simplicial sets are weakly contractible [Lurie (HTT) 2009, Prop. 5.5.8.7]).

Contents

Notation

- $\mathrm{SH}^{S^1}(k) := \mathrm{Spc}(k)_*[(S^1)^{-1}]$
- $\mathrm{Spc}(k)_* \underset{\Omega^\infty_{S^1}}{\overset{\Sigma^\infty_{S^1}}{\rightleftarrows}} \mathrm{SH}^{S^1}(k)$
- $\mathrm{SH}(k) := \mathrm{Spc}(k)_*[(\mathbb{P}^1)^{-1}]$
- $\mathrm{Spc}(k)_* \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathrm{SH}(k)$

Note that $\Sigma^\infty_{S^1}$ (resp. Σ^∞) is symmetric monoidal and that $\mathrm{SH}^{S^1}(k)$ (resp. $\mathrm{SH}(k)$) is a stable ∞-category which is generated under sifted colimits by S^1 (resp. $\mathbb{P}^1$)-desuspensions of suspension spectra of smooth schemes. (See [Hoyois (The six operations in equivariant motivic homotopy theory) 2017, Proposition 6.4] for further details.)

Similarly, the passage from S^1-spectra to $\mathbb{P}^1$-spectra also fits into an adjunction (with analogous properties to those described earlier):

$$\mathrm{SH}^{S^1}(k) \underset{\omega^\infty}{\overset{\sigma^\infty}{\rightleftarrows}} \mathrm{SH}(k)$$

Indeed, since $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$, we have that $\mathrm{SH}(k) \simeq \mathrm{SH}^{S^1}(k)[(\mathbb{G}_m)^{-1}]$.

Notation

If A is a strictly $\mathbb{A}^1$-invariant sheaf of abelian groups, HA denotes its Eilenberg–Mac Lane S^1-spectrum.

- $S^{p,q} := S^{p-q} \wedge \mathbb{G}_m^{\wedge q}$
- $\Sigma^{p,q}$ is the suspension by $S^{p,q}$
- $\Omega^{p,q}$ is the desuspension by $S^{p,q}$ (a.k.a. (p,q)-fold loops)

Proposition

1. The functor $\Omega^{\infty}_{S^1} : \mathrm{SH}^{S^1}(k)_{\geq 0} \to \mathrm{Spc}(k)_*$ preserves sifted colimits (the objects of $\mathrm{SH}^{S^1}(k)_{\geq 0}$ being the connective spectra).
2. If k is perfect then the functor $\Omega^{1,1} : \mathrm{Spc}(k)_* \to \mathrm{Spc}(k)_*$ preserves sifted colimits of 1-connected spaces.
3. If k is perfect then the functor $\Omega^{2,1} : \mathrm{Spc}(k)_* \to \mathrm{Spc}(k)_*$ preserves sifted colimits of 1-connected spaces and the functor $\Omega^{2,1}\Sigma^{2,1} : \mathrm{Spc}(k)_* \to \mathrm{Spc}(k)_*$ preserves sifted colimits of connected spaces.

1. Since the subcategory of $\mathbb{A}^1$-invariant Nisnevich sheaves of spectra is closed under colimits in the category of Nisnevich sheaves of spectra, it suffices to prove this for Nisnevich sheaves of spectra and we can check this stalkwise, i.e. on spectra: see [Lurie (Higher Algebra) 2017, Prop. 1.4.3.9].
2. Long proof which uses (among other things) the proposition on realization fibrations we saw earlier.

3 Since $\Omega^{2,1} = \Omega\Omega^{1,1}$, the fact that $\Omega^{2,1} : \mathrm{Spc}(k)_* \to \mathrm{Spc}(k)_*$ preserves sifted colimits of 1-connected spaces follows from the analogous fact for $\Omega^{1,1}$ and from the fact that (as we saw earlier) the functor $\Omega : \mathrm{Spc}(k)_* \to \mathrm{Spc}(k)_*$ preserves sifted colimits of connected spaces. Since $\Sigma^{2,1}$ sends connected spaces to 1-connected spaces (by the unstable connectivity theorem) and preserves colimits (as it is a left adjoint functor), it follows that $\Omega^{2,1}\Sigma^{2,1}$ preserves sifted colimits of connected spaces.

Notation

Let $E \in \mathrm{SH}^{S^1}(k)$ and $i, j \in \mathbb{Z}$. We denote by $\pi_i^{S^1}(E)_j$ the Nisnevich sheaf associated to the presheaf $U \mapsto [\Sigma^\infty_{S^1}(U_+) \wedge S^i, E \wedge \mathbb{G}_m^{\wedge j}]_{\mathrm{SH}^{S^1}(k)}$.

Notation

Let $E \in \mathrm{SH}(k)$ and $i, j \in \mathbb{Z}$. We denote by $\pi_i(E)_j$ the Nisnevich sheaf associated to the presheaf $U \mapsto [\Sigma^\infty(U_+) \wedge S^i, E \wedge \mathbb{G}_m^{\wedge j}]_{\mathrm{SH}(k)}$.

Homotopy t-structure on S^1-spectra

The category $\mathrm{SH}^{S^1}(k)$ can be equipped with the following homotopy t-structure:

- $\mathrm{SH}^{S^1}(k)_{\leq 0}$ is the full subcategory of coconnective spectra (i.e. $\pi_i^{S^1} = 0$ for all $i > 0$),
- $\mathrm{SH}^{S^1}(k)_{\geq 0}$ is the full subcategory of connective spectra (i.e. $\pi_i^{S^1} = 0$ for all $i < 0$) and is generated under colimits and extensions (even better: under colimits; better still: under sifted colimits) by the $\Sigma^\infty_{S^1} X_+$ for $X \in \mathrm{Sm}_k$,
- the functor $\pi_0^{S^1}$ is an equivalence (of inverse H) from $\mathrm{SH}^{S^1}(k)^\heartsuit := \mathrm{SH}^{S^1}(k)_{\leq 0} \cap \mathrm{SH}^{S^1}(k)_{\geq 0}$ to the category of strictly $\mathbb{A}^1$-invariant sheaves of abelian groups. See [Morel (The stable $\mathbb{A}^1$-connectivity theorems) 2005, Lemmas 6.2.11 and 6.2.13].

S^1-spectra and monoids

Proposition; see [Elmanto, Hoyois, Khan, Sosnilo, Yakerson (Motivic infinite loop spaces) 2021, Proposition 3.1.13 and Corollary 3.1.15]

If k is perfect then the functor $\Omega^{\infty}_{S^1} : \mathrm{SH}^{S^1}(k)_{\geq 0} \to H - \mathrm{Grp}_{abelian}$ is fully faithful of image the abelian H-groups of strongly $\mathbb{A}^1$-invariant π_0.

Proposition

If k is perfect then the full subcategory of abelian H-groups of strongly $\mathbb{A}^1$-invariant π_0 in $H - \mathrm{Grp}_{abelian}$ is closed under colimits.

By [Lurie (HTT) 2009, Lemma 5.5.8.13], it suffices to prove closure under finite coproducts (okay), filtered colim. and geometric real. (both done).

Corollary

If k is perfect then the functor $\Omega^{\infty}_{S^1} : \mathrm{SH}^{S^1}(k)_{\geq 0} \to H - \mathrm{Grp}_{abelian}$ preserves colimits.

Corollary

If k is perfect and the following is a cocartesian square in $H - \mathrm{Grp}_{abelian}$

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

with $\pi_0(B)$ and $\pi_0(C)$ strongly $\mathbb{A}^1$-invariant then $\pi_0(D)$ is strongly $\mathbb{A}^1$-invariant and, denoting by L an inverse of $\Omega^\infty_{S^1}$, $D \simeq \Omega^\infty_{S^1}(LC \coprod_{LA} LB)$.

By the earlier statement that if k is perfect and $\mathscr{X}_\bullet$ is a simplicial diagram of H-groups in $\mathrm{Spc}(k)$ such that $\pi_0(\mathscr{X}_0)$ is strongly $\mathbb{A}^1$-invariant then $\pi_0(\operatorname{colim} \mathscr{X}_\bullet)$ is strongly $\mathbb{A}^1$-invariant, the fact that $\pi_0(D)$ is strongly $\mathbb{A}^1$-invariant follows from the fact that $\pi_0(C \oplus B)$ is strongly $\mathbb{A}^1$-invariant. The last assertion follows from the previous corollary and the second to last proposition.

Homotopy t-structure on $\mathbb{P}^1$-spectra

The category $\mathrm{SH}(k)$ can be equipped with the following homotopy t-structure:

- $\mathrm{SH}(k)_{\leq 0}$ is the full subcategory of coconnective spectra (i.e. $\pi_i = 0$ for all $i > 0$),
- $\mathrm{SH}(k)_{\geq 0}$ is the full subcategory of connective spectra (i.e. $\pi_i = 0$ for all $i < 0$) and is generated under colimits and extensions by the $\Sigma^{p,q}\Sigma^\infty X_+$ for $X \in \mathrm{Sm}_k$ and $p \geq q$,
- the functor π_0 is an equivalence from $\mathrm{SH}(k)^\heartsuit := \mathrm{SH}(k)_{\leq 0} \cap \mathrm{SH}(k)_{\geq 0}$ to the category of homotopy modules (strictly $\mathbb{A}^1$-invariant sheaves M of abelian groups with isomorphisms $\mu_n : M_n \to (M_{n+1})_{-1}$ where $(M_{n+1})_{-1}(U)$ is the kernel of $M_{n+1}(U \times_k \mathbb{G}_m) \xrightarrow{\mathrm{Id} \times_k 1} M_{n+1}(U)$; these are $\pi_0(S^0)$-modules, i.e. $\underline{K}_*^{\mathrm{MW}}$-modules). See [Morel (An introduction to $\mathbb{A}^1$-homotopy theory) 2004, Theorems 5.2.3, 5.2.6 and 6.4.1] and [Hoyois (From algebraic cobordism to motivic cohomology) 2015, Paragraph 2.1], as well as [Morel 2012, Theorem 6.40].

Contents

Definition

Let A be a ring. The real spectrum of $\mathrm{Spec}(A)$, denoted $\mathrm{Spec}(A)_r$, is the topological space whose underlying set is the set of couples $(\mathfrak{p}, \leq)$ with $\mathfrak{p} \in \mathrm{Spec}(A)$ and $\leq$ a field ordering of $\kappa(p)$ (i.e. a total ordering of $\kappa(p)$ such that $a \leq b \Rightarrow \forall c \in F\ a + c \leq b + c$ and $0 \leq a, 0 \leq b \Rightarrow 0 \leq ab$) and whose opens are the unions of finite intersections of the $(D(a) := \{(\mathfrak{p}, \leq) \in \mathrm{Spec}(A)_r \mid a > 0 \text{ in the real closure of } (\mathfrak{p}, \leq)\})_{a \in A}$.
Let X be a scheme. The real spectrum of X, denoted X_r, is obtained by gluing together the real spectra of affine opens in X which cover X.

Definition

Let A be a ring. The real spectrum of $\mathrm{Spec}(A)$, denoted $\mathrm{Spec}(A)_r$, is the topological space whose underlying set is the set of couples $(\mathfrak{p}, \leq)$ with $\mathfrak{p} \in \mathrm{Spec}(A)$ and $\leq$ a field ordering of $\kappa(p)$ (i.e. a total ordering of $\kappa(p)$ such that $a \leq b \Rightarrow \forall c \in F\ a + c \leq b + c$ and $0 \leq a, 0 \leq b \Rightarrow 0 \leq ab$) and whose opens are the unions of finite intersections of the $(D(a) := \{(\mathfrak{p}, \leq) \in \mathrm{Spec}(A)_r \mid a > 0 \text{ in the real closure of } (\mathfrak{p}, \leq)\})_{a \in A}$.
Let X be a scheme. The real spectrum of X, denoted X_r, is obtained by gluing together the real spectra of affine opens in X which cover X.

Definition

A family $(f_i : U_i \to U)_{i \in I}$ of morphisms of k-schemes is real surjective if it is surjective on real spectra, i.e. $U_r = \bigcup_{i \in I} (f_i)_r((U_i)_r)$. These are the coverings of a pretopology on Sm_k whose induced topology on Sm_k is called the real-étale topology and denoted *ret*.

- $\mathrm{Spc}_{ret}(k)$ denotes the ∞-category of real-étale motivic spaces
- L_{ret} denotes the corresponding localization functor
- $\rho = [-1]$ is the morphism $S^0 \to \mathbb{G}_m$ associated to -1 (the convention $\rho = -[-1]$ is also usual but this won't matter)

Proposition

If k is perfect and $\mathscr{X} \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)_{*,conn}$ and $n \geq 0$ verify that for each $0 \leq i \leq n$, the Nisnevich homotopy sheaf $\pi_i(\mathscr{X})$ is a strictly $\mathbb{A}^1$-invariant real-étale sheaf of abelian groups then for each $j \geq 0$, $\pi_j(\mathrm{L}_{ret}(\mathscr{X}))$ is a real-étale sheaf and if $j \leq n$ then $\pi_j(\mathrm{L}_{ret}(\mathscr{X})) \simeq \pi_j(\mathscr{X})$. In particular, if $\mathscr{X}$ is n-connected then $\mathrm{L}_{ret}(\mathscr{X})$ is n-connected.

L_{ret} can be obtained as a countably infinite composition of L_{mot} and of a_{ret} (the real-étale sheafification) and we already studied L_{mot} so it is sufficient to study a_{ret}.

The real-étale sheafification a_{ret} preserves n-connected spaces as well as the fiber sequences $\mathscr{X}_{>n} \to \mathscr{X} \to \mathscr{X}_{\leq n}$, so that it suffices to show that $a_{ret}(\mathscr{X}_{\leq n}) \simeq \mathscr{X}_{\leq n}$. This can be shown by induction on the Moore-Postnikov tower and the fact that if A is a real-étale sheaf of abelian groups then for each integer i, $K_{\mathrm{Nis}}(A, i)$ satisfies real-étale descent, which follows from the fact that $H^*_{ret}(\_, A) \simeq H^*_{\mathrm{Nis}}(\_, A)$ which is a consequence of [Scheiderer (Real and étale cohomology) 1994, Proposition 19.2.1] which states that if R is a semilocal ring and F is a sheaf of abelian groups on the topological space $\mathrm{Spec}(R)_r$ then for each $j \geq 1$, $H^j(\mathrm{Spec}(R)_r, F) = 0$.

Proposition

If k is perfect and $\mathscr{X} \in \mathsf{Spc}(k)_*$ is simply connected then $\mathscr{X}$ satisfies real-étale descent if and only if the morphism $\rho_{\mathscr{X}} : \mathscr{X} \to \Sigma^{1,1}\mathscr{X}$ induced by ρ is an equivalence. In this case, $\rho^*_{\mathscr{X}} : \mathscr{X} \to \Omega^{1,1}\mathscr{X}$ is an equivalence.

The proof uses (among other things) the previous proposition.

Proposition

If k is perfect and $\mathscr{X} \in \mathrm{Spc}(k)_*$ is simply connected then $\mathscr{X}$ satisfies real-étale descent if and only if the morphism $\rho_{\mathscr{X}} : \mathscr{X} \to \Sigma^{1,1}\mathscr{X}$ induced by ρ is an equivalence. In this case, $\rho^*_{\mathscr{X}} : \mathscr{X} \to \Omega^{1,1}\mathscr{X}$ is an equivalence.

The proof uses (among other things) the previous proposition.

Thanks for your attention!