§ 1 Statement of main result

For a field F, $\hat{W}(F) = \left\{ \{Q \text{ nondeg quadratic form}/F\} \text{ with } \oplus \otimes \big/ \sim \right\}^+$

additively generated by $\langle a \rangle(x) = ax^2$, $a \in F^\times$. "Grothendieck-Witt ring"

For $F \subset L$ finite sep field extension, one has the additive homomorphism

$\mathrm{Tr}_{L/F} : \hat{W}(L) \to \hat{W}(F)$ $\quad \mathrm{Tr}_{L/F}\langle c \rangle : L \to F$

$\mathrm{Tr}_{L/F}\langle c \rangle(x) = \mathrm{tr}_{L/F}(x^2)$

a "polynomial" map

$\mathrm{Nm}_{L/F} : \hat{W}(L) \to \hat{W}(F)$ $\quad \mathrm{Nm}_{L/F}\langle a \rangle = \langle \mathrm{Nm}_{L/F}\, c \rangle$

Fix X K3 surface with a primitive genus g linear system $\mathcal{C}$

Assume that all rat'l curves in $\mathcal{C}$ are nodal (Chen)

Let $G(X) = \{ g \in \mathcal{C} \mid C_g \text{ is rat'l} \}$

$S(C_g)$ = set of singular pts of C_g

Define

if $p' = \pi^{-1}(p)$ in $\pi : C^N \to C$, $k(p') = k(p)(\sqrt{\alpha_p})$ some $\alpha_p \in k_p^\times$.

$$B_g^{mot}(X) := \sum_{g \in G(X)} \mathrm{Tr}_{k(g)/k} \Big(\prod_{p \in S(C_g)} \big(1 + N_{k(p)/k(g)} (\langle -2 \rangle - \langle 2\alpha_p \rangle) \big) \Big)$$

$\in \hat{W}(k) / \mathbb{J}_g$

= contribution of the node p to $\chi^{mot}(\bar{J}(C_g))$

Here J_g is a certain ideal $\left\{ Q \mid \mathrm{rank}\, Q = 0,\ \mathrm{sign}(Q) = 0,\ \mathrm{disc}\, Q \in \Delta_g\, k^\times / k^{\times 2} \right\}$

$$\mathcal{J} \subset \hat{W}(k)[[t]]$$
$$= \left\{ \sum_{g \geq 1} c_g t^g \mid c_g \in J_g\ \forall g \right\}$$

Then
$$\boxed{1 + \sum_{g \geq 1} B_g^{mot}(X)\, q^g = \prod \left(1 - \chi^{mot}(X^{(m)})\, t^m\right)^{-1}}$$

This recovers the Y-Z formula for $X/\mathbb{C}$ + analogous one by Kharlamov - Răsdeaconu by taking signature for $k = \mathbb{R}$

§ Some additional notation

$\mathbb{H}$ = hyperbolic form $\langle 1 \rangle + \langle -1 \rangle$, $W(k) = \hat{W}(k)/(\mathbb{H})$ Witt

rank : $\hat{W}(k) \to \mathbb{Z}$ rank map, I = ker rank

T = $(2^\wedge)$-torsion subgp of $\hat{W}(k)$ $\quad J = I^2 \cap T$

Euler characteristics (for simplicity assume char $k = 0$)

for $X \in Sm/k$ proper, $M(X) = \pi_{X\#}(1_X) \in SH(k)$ is strongly dualizable

so we have ($\pi_X : X \to \operatorname{Spec} k$)
$$\chi_{SH(k)}(X) = \operatorname{Tr}(\mathrm{id}_{M(X)}) \in \operatorname{End}_{SH(k)}(1_k)$$

Morel's Thm $\widehat{W}(k) \cong \operatorname{End}_{SH(k)}(1_k)$

Thus we get $\chi(X/k) \in \widehat{W}(k)$

This extends to all $U \in Sm/k$

for $Z \in Sch/k$, $\pi_Z : Z \to \operatorname{Spec} k$, $\underbrace{\pi_{Z!}(1_Z)}_{M^c(Z)} \in SH(k)$ is dualizable $\rightsquigarrow \chi_c(Z/k) := \operatorname{Tr}(\mathrm{id}_{M^c(Z)}) \in \widehat{W}(k)$

Properties (1) $U \in Sm/k$ of dimension $d \Rightarrow \chi_c(U/k) = (-1)^d \chi(U/k)$

$X \in Sm/k$ proper $\Rightarrow \chi_c(X/k) = \chi(X/k)$

(2) $F \to E \to B$ Nisnevich loc trivial bundle with fiber F

$\Rightarrow \chi_c(E/k) = \chi_c(F/k)\chi_c(B/k)$. In particular

+ same for $\chi(-/k)$ if all are smooth $\quad \chi_{(c)}(X \times Y/k) = \chi_{(c)}(X/k)\,\chi_{(c)}(Y/k)$

(3) $Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U = X \setminus Z \Rightarrow \chi_c(X/k) = \chi_c(Z/k) + \chi_c(U/k)$

(2) + (3) $\Rightarrow$

Prop sends X smooth & proper/k to $\chi(X/k) \in \hat{W}(k)$ extends to a ring homomorphism (motivic measure)

$$\chi^{mot} : K_0(Var_k) \to \hat{W}(k)$$

with $\chi^{mot}(Z) = \underbrace{\chi_c(Z/k)}_{\chi(X/k)}$ $\forall Z \in Sch/k$
$\forall X \in Sm/k$ proper/k

(4) define $e(X) = \sum (-1)^i \dim_{\mathbb{Q}_\ell} H^i_{et,c}(X_{\bar{k}}, \mathbb{Q}_\ell) \in \mathbb{Z}$

since $e : K_0(Var_k) \to \mathbb{Z}$ "compactly supported Euler char"

Then $e = rk \circ \chi^{mot}$

(5) let $\sigma : k \hookrightarrow \mathbb{R}$ be an embedding

$$e_\sigma(X) = \sum (-1)^i \dim_{\mathbb{Q}} H^i_c(X(\mathbb{R}), \mathbb{Q})$$

Then $e_\sigma = sign_\sigma \circ \chi^{mot}$ $\quad \hat{W}(k) \xrightarrow{\sigma_*} \hat{W}(\mathbb{R}) \xrightarrow{sign} \mathbb{Z}$

(more generally for σ an ordering on k ⇝ $k \hookrightarrow_\sigma \mathbb{R}_\sigma$ = real closure) $sign_\sigma$

Sketch for (4), (5).

(4) (assuming $k \subset \mathbb{C}$) $X \mapsto C^{sing}_*(X(\mathbb{C}), \mathbb{Q})$ extends

to symmetric monoidal functor

$$HRe_{\mathbb{Q}} : SH(k) \to D(\mathbb{Q})$$
$$M(X) \mapsto C^{sing}_*(X(\mathbb{C}), \mathbb{Q})$$

so $$HRe_{\mathbb{Q}}(Tr(id_{M(X)})) = Tr(id_{C^{sing}_*(X(\mathbb{C}),\mathbb{Q})}) \in End_{D(\mathbb{Q})}(\mathbb{Q}[0]) \cong \mathbb{Q}$$
$$= \sum (-1)^i \dim_{\mathbb{Q}} H^i(X(\mathbb{C}),\mathbb{Q})$$

And induces $rk : \widehat{W}(k) \to \mathbb{Z} \subset \mathbb{Q}$

since $\langle a \rangle \in \widehat{W}(k) \mapsto [x_0, x_1] \mapsto [x_0, ax_1] \in Aut(\mathbb{P}^1)$

$\downarrow Re_{\mathbb{C}}$

same map on $\mathbb{CP}^1 = S^2$

$\simeq id$ stably

$1 \in End(\mathbb{Q})$

Then take X smooth proj

so get $rk \circ \chi^{mot}(X) = e(X)$

$\Rightarrow$ true for $cl(Z) \in K_0(Var)$

pf of 1.5) b the same except use $C^{sing}_*(X(\mathbb{R}), \mathbb{Q})$

and $(x_0, x_1) \mapsto [x_0, ax_1]$ on $\mathbb{RP}^1 = S^1$ is $\simeq id$ for $\sigma(a) > 0$

$\simeq -id$ for $\sigma(a) < 0$

so get $sgn_\sigma \circ \chi^{mot} = e_\sigma$

(6) For X smooth and projective we have deRham cohomology $H^*_{dR}(X/k)$, finite dim'l k-VS, supported in $*\in[0, 2\dim X]$

on $H^{2*}_{dR}(X/k)$ have the quadratic form $[H^{2*}_{dR}(X/k)]$

$$x \mapsto tr_{X/k}(x^2) \qquad tr_{X/k} : H^{2n}_{dR}(X/k) \to k$$

$$H^{2n}_{dR}(X/k) = H^n(X, \Omega^n_{X/k}) \quad \text{Serre duality}$$

Def $\chi^{dR}(X) = [H^{2*}_{dR}(X/k)] - m \cdot H \in \widehat{W}(k)$, $m = \frac{1}{2}\dim_k H^{odd}_{dR}(X/k)$ ↑ always even

Thm (L.-Raksit)

for X smooth proj $/k$ $\chi^{dR}(X) = \chi^{mot}(X)$

A reformulation Let $b^- = \sum_{i<n} \dim_k H^i_{dR}(X/k)$

$n = \dim_k X$

$b^+ = \sum_{i<n} (-1)^i \dim_k H^i_{dR}(X/k)$

for n even let $[H^n_{dR}(X/k)] =$ rest. of $[H^{2*}_{dR}(X/k)]$

Then for n even $\chi^{dR}(X) = [H^n_{dR}(X/k)] + b^+ \cdot H$

n odd $\chi^{dR}(X) = (b^+ - \frac{1}{2}\dim_k H^n_{dR}(X/k)) H$

$= \frac{1}{2} e(X) \cdot H$

To see this for n even $b^+ = m + \sum_{2i<n} \dim_k H^{2i}(X/k)$

$m = \frac{1}{2}\dim H^{odd}_{dR}$

The pairing $H^i_{dR} \times H^j_{dR} \to H^{2n}_{dR}$

pairs $H^{2i}_{dR} \times H^{2n-2i}_{dR}$ so these contribute hyperbolic factors

$\chi^{dR}(X) = [H^{2*}_{dR}(X/k)] - m \cdot \mathbb{H}$

$= [H^n_{dR}(X/k)] + b^+ \cdot \mathbb{H}$

For n odd $b^+ = -m + \sum_{2i<n} \dim_k H^{2i}(X/k) + \frac{1}{2}\dim_k H^n_{dR}(X/k)$

& $[H^{2*}_{dR}(X/k)] = \left(\sum_{i<n} \dim_k H^{2i}(X/k)\right) \cdot \mathbb{H}$

so $\chi^{dR}(X/k) = \left(\sum_{i<n} \dim_k H^{2i}(X/k) - m\right) \mathbb{H}$

$= \left(b^+ - \frac{1}{2}\dim_k H^n_{dR}(X/k)\right) \cdot \mathbb{H}$

§ Discriminant and Saito's theorem

Def $Q \leftrightarrow (Q_{ij})$ symmetric matrix. Then $\operatorname{disc}(Q) = \det(Q_{ij}) \in k^\times / k^{\times 2}$

Isometry $(\mathbb{Q}_{\ell\omega}) \to {}^t S(\mathbb{Q}_{\ell\omega}) S$ so disc is a well-defined map $\mathrm{disc} : \hat{W}(k) \to k^\times / k^{\times 2}$

For a Galois rep n

$$\rho : \mathrm{Gal}_k \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_n) \quad \text{finite dim'l } \mathbb{Q}_\ell \text{ vspace}$$

we have the character $\det \rho : \mathrm{Gal}_k \to \mathbb{Q}_\ell^\times = \mathrm{Aut}_{\mathbb{Q}_\ell}(\mathbb{Q}_\ell)$

For $X \in \mathrm{Sch}/k$ we have the

virtual repn $R\Gamma_c(X, \mathbb{Q}_\ell) = \bigoplus (-1)^i [H^i_c(X_{\bar{k}}, \mathbb{Q}_\ell)]$ $\circlearrowleft \mathrm{Gal}_k$

giving the character

$$\det R\Gamma_c(X, \mathbb{Q}_\ell) = \bigotimes_{i=0}^{2\dim X} \det H^i_c(X_{\bar{k}}, \mathbb{Q}_\ell)^{\otimes (-1)^i}$$

We have the cyclotomic character $\mathbb{Q}_\ell(1) = (\varprojlim_\nu \mu_{\ell^\nu}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$

i.e. a 1-dim'l $\mathbb{Q}_\ell$ vspace with Galois action $\mathbb{Q}_\ell(1) : \mathrm{Gal}_k \to \mathbb{Q}_\ell^\times$ (sometimes written χ_{cyc})

and $\mathbb{Q}_\ell(n) = \mathbb{Q}_\ell(1)^{\otimes n}$, $n \in \mathbb{Z}$.

Lemma 1 X smooth projective scheme of dim n.
$\text{/ } R\Gamma = R\Gamma_c$

For n odd

$$\det R\Gamma(X, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-\tfrac{1}{2} n e(X))$$

For n even, there is a (unique) character $\kappa(X) : G_k \to \{\pm 1\}$
with

$$\det R\Gamma(X, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-\tfrac{1}{2} n e(X)) \otimes \kappa(X)$$

We define $\kappa(X) = 1$ for n odd to give

$$\det R\Gamma(X, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-\tfrac{1}{2} n e(X)) \otimes \kappa(X) \quad \text{for } X \text{ smooth proj dim } n/k$$

Pf This follows from Poincaré duality : perfect pairing
of G_k modules $H^i(X, \mathbb{Q}_\ell) \times H^{2n-i}(X, \mathbb{Q}_\ell) \to H^{2n}(X, \mathbb{Q}_\ell) \xrightarrow{\operatorname{tr}} \mathbb{Q}_\ell(-n)$

ie $H^i(X, \mathbb{Q}_\ell) \cong H^{2n-i}(X, \mathbb{Q}_\ell)^\vee \otimes \mathbb{Q}(-n)$

$\Rightarrow \det H^i(X, \mathbb{Q}_\ell) \cong \det H^{2n-i}(X, \mathbb{Q}_\ell)^{\vee} \otimes \mathbb{Q}_\ell(-b_i n)$

$\Rightarrow \det H^i(X, \mathbb{Q}_\ell) \otimes \det H^{2n-i}(X, \mathbb{Q}_\ell) \cong \mathbb{Q}(-b_i n)$

For n odd, $\langle , \rangle : H^n \times H^n \to \mathbb{Q}_\ell(-n)$ is alternating, write
$\omega = \langle , \rangle \in \operatorname{Hom}(\Lambda^2 H^n, \mathbb{Q}_\ell(-n))$. Then $\omega^{b_n/2} \in \operatorname{Hom}(\det H^n, \mathbb{Q}_\ell(-\tfrac{b_n}{2} n))$

defines an iso $\det H^n \xrightarrow[\omega^n]{} \mathbb{Q}(-n b_n/2)$

$\Rightarrow \det R\Gamma(X, \mathbb{Q}_\ell) = \mathbb{Q}_\ell\left(-\sum_{i<n} (-1)^i b_i n - \frac{b_n}{2} n\right) = \mathbb{Q}_\ell\left(-\frac{e(X)}{2} n\right)$.

For n even $H^n \otimes H^n \to \mathbb{Q}(-n)$ is symmetric

$\det H^{n \otimes 2} = \mathbb{Q}(-n b_n) \Rightarrow \left(\det H^n \otimes \mathbb{Q}\left(\frac{n}{2} b_n\right)\right)^{\otimes 2} \cong \mathbb{Q}_\ell(0)$

$\Rightarrow \det H^n \otimes \mathbb{Q}\left(\frac{n}{2} b_n\right) = \kappa(X)$, for some $\kappa(X) : \mathrm{Gal}_k \to \{\pm 1\}$

$\Rightarrow$ (as for n odd) $\det R\Gamma(X, \mathbb{Q}_\ell) = \kappa(X) \otimes \mathbb{Q}\left(-\frac{n e(X)}{2}\right)$. $\square$

<u>Saito's theorem</u> Let X be smooth projective of even dimension n. Then

$$\kappa(X) = (-1)^{\frac{n}{2} e(X) + b^-} \cdot \mathrm{disc}\left[H^n_{dR}(X/k)\right]$$

To interpret this $\mathrm{Hom}(\mathrm{Gal}_k, \{\pm 1\}) = H^1(k, \mu_2) = k^\times/k^{\times 2}$
so we may consider the equation as an identity in $k^\times/k^{\times 2}$

<u>Theorem 2.27 [PP]</u> Let X be a smooth proj $/k$ of dim n. Then

$$\mathrm{disc}(\chi^{mot}(X)) = (-1)^{\frac{n e(X)}{2}} \cdot \kappa(X) \quad \left(\text{in } k^\times/k^{\times 2}\right)$$

Pf (n odd) Then $\chi^{mot}(X) = \frac{1}{2} e(X) \cdot \mathbb{H} \Rightarrow \mathrm{disc} = (-1)^{\frac{e(X)}{2}} = (-1)^{\frac{ne(X)}{2}}$

(n even) Then $\chi^{mot}(X) = \chi^{dR}(X) = [H^n_{dR}(X/k)] + b^+ \cdot \mathbb{H}$

so $\mathrm{disc}\, \chi^{mot}(X) = \mathrm{disc}\, [H^n_{dR}(X/k)] \cdot (-1)^{b^+}$

$$\text{(Saito)} = \kappa(X) (-1)^{\frac{ne(X)}{2}} (-1)^{b^+ + b^-}$$

$$= \kappa(X) (-1)^{\frac{ne(X)}{2}} \quad \left(b^+ + b^- = \sum_{i<n} 2 \dim H^{2i} \right)$$

$\square$

<u>Cor</u> X smooth proj dim n over k. Then

$$\mathrm{disc}\, \chi^{mot}(X) = (-1)^{\frac{ne(X)}{2}} Q\left(\frac{ne(X)}{2}\right) \det R\Gamma(X, \mathbb{Q}_\ell)$$

$\in \mathrm{Hom}(G_{al_k}, \mathbb{Q}_\ell^\times)$

Pf $\mathrm{disc}\, \chi^{mot}(X) = (-1)^{\frac{ne(X)}{2}} \cdot \kappa(X)$ (Thm 2.21)

$\det R\Gamma(X, \mathbb{Q}_\ell) = \kappa(X) \cdot Q\left(\frac{-ne(X)}{2}\right)$ $\square$

(Lemma 4)

§ Extending to $K_0(\mathrm{Var}_k)$ We want to extend the identity in Cor 1 to an identity of maps $K_0(\mathrm{Var}_k) \to \mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times)$

Lemma 2 Sending $X \in \mathrm{Sch}/k$ to $\det R\Gamma_c(X_{\bar{k}}, \mathbb{Q}_\ell)$ (as Gal_k character) extends to an additive homomorphism

$$\det_\ell : K_0(\mathrm{Var}_k) \to \mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times)$$

Pf check cut and paste: given $Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U = X \setminus Z$

we have $0 \to j_! \mathbb{Q}_\ell|_{U_{\bar{k}}} \to \mathbb{Q}_\ell|X_{\bar{k}} \to i_* \mathbb{Q}_\ell|Z_{\bar{k}} \to 0$

since $R\Gamma_c(W_{\bar{k}}, \mathbb{Q}_\ell) = R\mathrm{Hom}(\mathbb{Q}_\ell, Rp_{W!}\mathbb{Q}_\ell|W_{\bar{k}})$ we apply $Rp_!$ & get long exact seq in $H^i_c(-, \mathbb{Q}_\ell) \Rightarrow$

$$\det R\Gamma_c(X_{\bar{k}}, \mathbb{Q}_\ell) = \det R\Gamma_c(Z_{\bar{k}}, \mathbb{Q}_\ell) \cdot \det R\Gamma_c(U_{\bar{k}}, \mathbb{Q}_\ell) \quad \square$$

Lemma 3 Sending smooth projective X to $b_i(X)$ extends uniquely to $b_i : K_0(\mathrm{Var}_k) \to \mathbb{Z}$, with $b_i(Z) = 0$ for $i \geq i_Z$ $\forall Z \in K_0(\mathrm{Var}_k)$

Pf use Hodge polynomial / weight polynomial

$$b_i = \sum_{p+q=i} h^{p,q}$$

& fact that $K_0(\mathrm{Var}_k)$ is gen by SmProj/k.

Lemma 4 For X smooth and proj of dim n/k,

$$\sum_{i\in\mathbb{N}} (-1)^i \frac{i}{2} b_i = \frac{n e(X)}{2} \in \mathbb{Z}.$$

Pf
$$\sum (-1)^i \frac{i}{2} b_i = \frac{(-1)^n n b_n}{2} + \sum_{i<n} (-1)^i \frac{i}{2} b_i + (-1)^{2n-i} \frac{(2n-i)}{2} b_{2n-i}$$

with $b_{2n-i} = b_i$

$$= \frac{(-1)^n n b_n}{2} + \sum_{i<n} (-1)^i \frac{n}{2} \cdot 2b_i$$

$$= \frac{n}{2} e(X).$$

$\frac{n}{2} e(X) \in \mathbb{Z}$ since $e(X) \in 2\mathbb{Z}$ for n odd □

Prop The map $X \mapsto \frac{n}{2} e(X) \in \mathbb{Z}$ for X smooth + proj extends to an additive homomorphism

$$\omega : K_0(\mathrm{Var}_k) \to \mathbb{Z}$$

Pf let $\omega(Z) = \sum_{i\geq 0} (-1)^i \frac{i}{2} b_i(Z) \in \frac{1}{2}\mathbb{Z}$

By Lemma 3 ω is an additive hom $\omega: K_0(\mathrm{Var}_k) \to \frac{1}{2}\mathbb{Z}$

but by Lemma 4 $\omega(X) \in \mathbb{Z}$ for X smooth proj. As these generate $K_0(\mathrm{Var}_k)$, $\omega(Z) \in \mathbb{Z}$ $\forall Z$.

Theorem 2 For all $Z \in \mathrm{Sch}/k$ we have

$$\mathrm{disc}(\chi^{mot}(Z)) = \det_\ell(Z) \cdot Q_\ell(\omega(Z)) \cdot (-1)^{\omega(Z)}$$

in $\mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times)$

Note $\mathrm{disc}(X^{mot}(\mathbb{Z}))$ & $(-1)^{w(\mathbb{Z})}$ are in $k^\times/k^{\times 2} = \mathrm{Hom}(\mathrm{Gal}_k, \{\pm 1\}) \subset \mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times)$

Pf $\mathrm{disc}(X^{mot}(-))$, $\det_\ell$, $\mathbb{Q}_\ell(w(-))$ and $(-1)^{w(-)}$ [$\mathbb{Q}_\ell(1)^{\otimes w(-)}$]

are all additive homomorphism $K_0(\mathrm{Var}_k) \to \mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}_\ell^\times)$

and the identity holds for Z smooth + proj $/k$ by Cor 1. $\square$

<u>Cor 2.28</u> Assume k is finitely generated over $\mathbb{Q}$. For $X \in \mathrm{Sch}/k$

$$\det_\ell(X) = 1 \iff w(X) = 0 \text{ and } \mathrm{disc}(X^{mot}(\mathbb{Z})) = 1$$

Pf ($\Leftarrow$) follows from Thm 2.27.

($\Rightarrow$) suff to show $\det_\ell(X) = 1 \Rightarrow w(X) = 0$ (use thm 2.27 again)

For X smooth + proj $\det_\ell(X)^2 = \mathbb{Q}_\ell(-2w(X))$

Then $\det_\ell(X)^2 = \mathbb{Q}_\ell(-2w(X))$ $\forall X \in \mathrm{Sch}/k$ since SmPrs$/k$ generates $K_0(\mathrm{Var}_k)$

so $\det_\ell(X) = 1 \Rightarrow \mathbb{Q}_\ell(-2w(X))$ is the trivial

character. But k fin gen$/\mathbb{Q}$ " $\mathbb{Q}_\ell(-w(X))^2$

$\Rightarrow \mathbb{Q}_\ell(1)$ has infinite order $\Rightarrow 2w(X) = 0 \Rightarrow w(X) = 0$ $\square$

We deduce further information about X^{mot} from $\det_\ell$ using the following construction

Def An augmented ring is a commutative ring R with a ring homomorphism $\varepsilon : R \to \mathbb{Z}$ s.t. $\mathbb{Z} \to R \to \mathbb{Z}$ is the identity.

If $\ker(\varepsilon)^2 = 0$, say (R, ε) is elementary.

Notation $W(k) = \hat{W}(k)/I^2$ so rank: $\hat{W}(k) \to \mathbb{Z}$ gives an elementary augmentation $\varepsilon : W(k) \to \mathbb{Z}$

Also $\mathrm{disc}|_{I^2}$ is the 1-map so we have $\mathrm{disc} : W(k) \to k^\times/k^{\times 2}$

and $\mathrm{disc} : I/I^2 \to k^\times/k^{\times 2}$ is a gp hom.

$\mathrm{Hom}(\mathrm{Gal}_k, \{\pm 1\})$

Ex for M an abelian gp let $E(M) = \mathbb{Z} \oplus M$ with ring structure

$$(a, m) \cdot (b, n) = (ab, an + bm)$$

$p_1 = \varepsilon : E(M) \to \mathbb{Z}$ is an elementary augmentation

Conversely for (R, ε) elementary we have $\varphi : R \to E(\ker(\varepsilon))$

$R = \mathbb{Z} \oplus \ker(\varepsilon)$ by sending r to $(\varepsilon(r), r - \varepsilon(r) \cdot 1)$

this is a ring iso $R \to E(\ker(\varepsilon))$

in particular

$$\begin{tikzcd} W(k) \arrow[r, "\cong"] & E(\mathrm{Hom}(\mathrm{Gal}_k, \{\pm 1\})) \\ \hat{W}(k) \arrow[u] & Q_k \arrow[u] \end{tikzcd}$$

Def Let
$$\chi_\ell : K_0(\mathrm{Var}_k) \to E(\mathrm{Hom}(G_k, \mathbb{Q}_\ell^\times))$$
$$[X] \mapsto (e(X), \det_\ell(X))$$
by

and
$$\chi_w : K_0(\mathrm{Var}_k) \to E(\mathbb{Z})$$
$$[X] \mapsto (e(X), w(X))$$

Prop 2.35 χ_ℓ + χ_w are ring homomorphism

~~Pf~~ Suffice to check $\chi_?(X \sqcup Y) = \chi_?(X) + \chi_?(Y)$ (+)

$$\chi_?(X \times Y) = \chi_?(X) \cdot \chi_?(Y) \quad (\times)$$

(+) follows since $X \mapsto e(X)$, $\det_\ell(X)$ + $w(X)$ are all additive (Lemma 3, Lemma 4)

$(\times)$ is an easy computation using the Künneth formula, which implies $R\Gamma(X \times Y, \mathbb{Q}_\ell) = R\Gamma(X, \mathbb{Q}_\ell) \otimes R\Gamma(Y, \mathbb{Q}_\ell)$ + $e(X \times Y) = e(X)e(Y)$ □

Prop 2.36 Assume k is finitely generated over $\mathbb{Q}$

Take $X, Y, Z \in K_0(\mathrm{Var}_k)$ such that
$$\chi_\ell(X) = \chi_\ell(Y) \cdot \chi_\ell(Z)$$

Then
$$\chi^{mot}(X) = \chi^{mot}(Y) \cdot \chi^{mot}(Z) \mod I^2$$

and
$$\chi_w(X) = \chi_w(Y)\, \chi_w(Z)$$

Pf $\chi_\ell(X) = \chi_\ell(Y)\cdot\chi_\ell(Z) \Rightarrow e(X) = e(Y)e(Z)$ (2)

(1)

Assuming (2), we have

i) (1) $\Leftrightarrow \det_\ell(X) = \det_\ell(Y)^{\otimes e(Z)} \otimes \det_\ell(Z)^{\otimes e(Y)}$

ii) $\chi_w(X) = \chi_w(Y)\chi_w(Z) \Leftrightarrow w(X) = e(Z)w(Y) + e(Y)w(Z)$

iii) $\chi^{mot}(X) = \chi^{mot}(Y)\chi^{mot}(Z) \mod I^2$

$\Leftrightarrow \operatorname{disc}\chi^{mot}(X) = \operatorname{disc}(\chi^{mot}(Y))^{e(Z)} \cdot \operatorname{disc}_0(\chi^{mot}(Z))^{e(Y)}$

since $GW(k) \cong E(k^\times/k^{\times 2})$ by (rnk, disc)

But $\operatorname{disc}\chi^{mot}(-) = \det_\ell(-)\cdot Q_\ell(w(-))(-1)^{w(-)}$ (Thm 2.27)

so suffice to show (1) $\Rightarrow \chi_w(X) = \chi_w(Y)\chi_w(Z)$

(1) $\Rightarrow \det_\ell(X)\cdot Q_\ell(w(X))\cdot(-1)^{w(X)}$

$= [\det_\ell(Y)Q_\ell(w(Y))(-1)^{w(Y)}]^{e(Z)} \cdot [\det_\ell(Z)Q_\ell(w(Z))(-1)^{w(Z)}]^{e(Y)}$

$\Rightarrow Q_\ell(w(X))(-1)^{w(X)} = [Q_\ell(w(Y))(-1)^{w(Y)}]^{e(Z)} \cdot [Q_\ell(w(Z))(-1)^{w(Z)}]^{e(Y)}$

square both side $\Rightarrow Q_\ell(2w(X)) = Q_\ell(2w(Y)e(Z) + 2w(Z)e(Y))$

since k is finitely gen $\Rightarrow 2w(X) = 2[w(Y)e(Z) + w(Z)e(Y)]$

then divide by 2 $\square$

Conclude with : $J = (\text{torsion } \hat{W}(\mathbb{Z})) \cap \mathbb{I}^3$, $\text{sign} = \prod_{\sigma \text{ order an } m} \text{sig}_\sigma$

<u>Lemma 2.37</u> For $[q] \in \hat{W}(\mathbb{Z})$, $[q]$ is in J $\Leftrightarrow$

$\text{rk}(q) = 0$, $\text{sign}(q) = 0$, $\text{disc}(q) = 1$

$$\begin{pmatrix} \partial^2 f/\partial x_0^2 & \dfrac{\partial^2 f}{\partial x_0 \partial x_1} & \dfrac{\partial^2 f}{\partial x_0 \partial x_2} & \dfrac{\partial^2 f}{\partial x_0 \partial x_3} \\ \partial^2 f/\partial x_1 \partial x_0 & \partial^2 f/\partial x_1^2 & & \\ \partial^2 f/\partial x_2 \partial x_0 & & \ddots & \\ \partial^2 f/\partial x_3 \partial x_0 & & & \partial^2 f/\partial x_3^2 \end{pmatrix}$$

$$\sum \frac{\partial^2 f}{\partial x_i \partial x_1} x_i = (d-1)\frac{\partial f}{\partial x_1}$$

$$\sum_{j=1}^{3} a_j \frac{\partial^2 f}{\partial x_0 \partial x_j}(1, p_1, \dots p_3)$$

$$= \sum_j a_j (d-1) \frac{\partial f}{\partial x_j}(1, p_1, \dots p_3)$$

$$= \sum_j \sum_{i=1}^{3} p_i \frac{\partial^2 f}{\partial x_i \partial x_j} a_j$$

$$\sum \frac{\partial^2 f}{\partial x_i \partial x_j} \cdot x_j = (d-1)\frac{\partial f}{\partial x_i}$$

$= (d-1) \sum a_j \frac{\partial f}{\partial x_j}(1, p_1, \dots p_3)$ so when take $(1, a_1, a_2, a_3) \in T_p X$ $= 0$

$$F = \sum x_i^d$$

$$d(d-1) \begin{pmatrix} x_0^{d-2} & & 0 \\ & x_1^{d-2} & \\ 0 & & \ddots \\ & & & x_3^{d-2} \end{pmatrix} \rightsquigarrow \det = (x_0 x_1 x_2 x_3)^{d-2} d(d-1)$$