Knörrer Periodicity

Main Theorem

$S := k[[x_1,\dots,x_n]]$ $\quad f \in \mathfrak{m}^2 \quad R = S/(f)$

$\cup$

$\mathfrak{m} = (x_1,\dots,x_n)$

We have MCM_R and the stable version $\underline{MCM}_R$

Let $S_2 = k[[x_1,\dots,x_n,u,v]]$ $\quad R_2 = S_2/(f+uv)$

We have $H: MF(f) \to MF(f+uv)$ by

$$(\varphi,\psi) := F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0 \mapsto \left(H(\varphi) = \begin{pmatrix} u\,\mathrm{id}_{F_1} & \psi \\ \varphi & -v\,\mathrm{id}_{F_0} \end{pmatrix}, H(\psi) = \begin{pmatrix} v\,\mathrm{id}_{F_1} & \psi \\ \varphi & -u\,\mathrm{id}_{F_0} \end{pmatrix} \right)$$

$$; (F_1 \oplus F_0)\otimes S_2 \to (F_1 \oplus F_0)\otimes S_2$$

and $(\alpha: F_1 \to F_1', \beta: F_0 \to F_0') \in \mathrm{Hom}^0$

$$\mapsto \left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right)$$

$(\alpha: F_1 \to F_0', \beta: F_0 \to F_1') \in \mathrm{Hom}^1$

$$\mapsto \left(\begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix} \right)$$

Thm (Knörrer) Suppose $k = \bar{k}$, char $k \neq 2$. Then H induces equivalences

$$\underline{MCM}(R) \to \underline{MCM}(R_2)$$

$$\underline{MF}(f) \to \underline{MF}(f+uv) \cong \underline{MF}(f+y^2+z^2)$$

Remark Suppose $f = Q$. Then

$$\underline{MF}(Q) \cong \underline{MCM}(R) \cong C_0(Q)\text{-Mod}$$

$$\underline{MF}(Q+uv) \cong C_0(Q+uv)\text{-Mod}$$

$\leftarrow$ these are isomorphic for all n (char $k \neq 2$)

since $\operatorname{rank} Q \equiv \operatorname{rank} Q+uv \mod 2$

and if $\operatorname{rank} Q$ is even

$$(-1)^{\operatorname{rank} Q/2} \operatorname{disc} Q = (-1)^{\frac{\operatorname{rank}(Q+uv)}{2}} \operatorname{disc}(Q+uv)$$

The proof of the main theorem relies on a two-step process, 1st comparing $\underline{MF}(f)$ with $\underline{MF}(f+y^2)$ then repeating, going from $\underline{MF}(f+y^2)$ to $\underline{MF}(f+y^2+z^2)$.

In some detail let $S_1 = k[[x_1, \dots, x_n, y]]$, $R_1 = S_1/(f+y^2) \supset S$.
Write $C_2 = \langle \sigma \rangle$ ($\sigma^2 = id$) so S_1, R_1 have the σ-action $\sigma(y) = -y$, $\sigma|_S = id$. Let $R_1[\sigma]$ be the twisted group ring

$$(a+b\sigma)(c+d\sigma) = (ac + b\sigma(d)) + (ad + b\sigma(c))\sigma$$

and let $MCM_\sigma(R_1) = R_1[\sigma]$-modules M that are in $MCM(R_1)$ by forgetting the σ-action and associated stable category $\underline{MCM}_\sigma(R_1)$.

$R = S/(f)$. Recall loop functor $\Omega : \underline{MCM}(R) \circlearrowleft$

$\Omega : \underline{MF}_R(f) \circlearrowleft$ (we use $\Omega(\varphi, \psi) = (\psi, \varphi)$)

We have the diagram of functors (non-commuting in general) (3)

$$\begin{tikzcd}
\underline{MF}_{S_1}(f+y^2) \arrow[rr, shift right, "\mathrm{Res}"'] \arrow[d, "\mathrm{cok}"', "\simeq"] & & \underline{MF}_S(f) \arrow[ll, shift right, "G"'] \arrow[dl, "A", "\simeq"'] \arrow[d, "\mathrm{cok}", "\simeq"'] \\
\underline{MCM}(R_1) \arrow[r, shift left, "Fn"] & \underline{MCM}_\sigma(R_1) \arrow[l, shift left, "U"] \arrow[r, "\mathrm{res}"] & MCM(R)
\end{tikzcd}$$

$(*)$

U: forget, $Fn = R_1[\sigma]\otimes_{R_1} -$, Res, res resp. restricting $y \to 0$

for A: given $M \in MCM_\sigma(R_1)$, write $M = M^+ \oplus M^-$ with $\sigma|_{M^+} = id$, $\sigma|_{M^-} = -id$. M^+, M^- are free S-modules

Multiplication by y, resp. $-y$

$$-y = \varphi : M^+ \to M^-,\ y|_{M^-} = \psi : M^- \to M^+,$$

with $\varphi\psi = f\cdot id$, $\psi\varphi = f\cdot id$ so $(\varphi,\psi) \in MF_S(f)$

A has inverse $(F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0) \rightsquigarrow M := F_1 \oplus F_0$ as S-mod

with $y(a,b) = (-\psi(b), \varphi(a))$ $\quad \sigma(a,b) = (a,-b)$

$G : MF_S(f) \to MF_{S_1}(f+y^2)$ sends $(F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0)$ to $(F_1\oplus F_0 \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} F_1 \oplus F_0)$

where $\Phi = \begin{pmatrix} y\,\mathrm{id} & \psi \\ \varphi & -y\,\mathrm{id} \end{pmatrix}$ and $\begin{array}{ccc} F_1 & \rightleftarrows & F_0 \\ \alpha\downarrow & & \downarrow\beta \\ F_1' & \rightleftarrows & F_0' \end{array}$ $=(\alpha,\beta) \in \mathrm{Hom}(\,)^0$ (8)

to $\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\right)$ and $\begin{array}{cc} F_1 & F_0 \\ \alpha\searrow & \nearrow\beta \\ F_1' & F_0' \end{array} \in \mathrm{Hom}(\,)^1$

to $\left(\begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}\right)$.

$\sigma: R_1 \to R_1$ induces $\tilde\sigma: MF_{R_1}(f+y^2)\circlearrowleft$ and $\tilde\sigma: MCM(R_1)\circlearrowleft$

with $\mathrm{cok} \circ \tilde\sigma \cong \tilde\sigma \circ \mathrm{cok}$.

Applying in two-step $MF(f)$, $MF(f+y^2)$, $M(f+y^2+z^2)$
$\cong$
$M(f+uv)$

we have

$$M(f+y^2+z^2) \underset{\mathrm{Res}_2}{\overset{G_2}{\rightleftarrows}} MF(f+y^2) \underset{\mathrm{Res}}{\overset{G_1}{\rightleftarrows}} MF(f)$$

let $\rho = \mathrm{Res}_1 \mathrm{Res}_2$

Some elementary relations (just simple computations)

(3.3) $\quad G_2 \circ G_1 \cong H \oplus \Omega \circ H$

(3.4) $\quad \rho \circ H \cong \mathrm{id} \oplus \Omega$

(3.5) $\quad \Omega \circ H \cong H \circ \Omega$

Note that (3.4) $\Rightarrow$ H is faithful. We show H is a bijection on Iso-classes, relying on:

We call $(\varphi, \psi) \in MF(f)$ non-trivial indecomposable if coker φ is an indecomposable, non-free in $MCM(R)$, same for $MF(f+y^2)$, ect.

Lemma 2.5 (i) $F_N \mathcal{U} \cong (M \mapsto M \oplus M \otimes_{R_1(\sigma)} R_1^{\sigma})$ σ acts by $-\sigma$ on R

and $\mathcal{U} F_N \cong \mathrm{id} \oplus \sigma^*$

(ii) $\mathrm{Res}\, G \cong \mathrm{id} \oplus \Omega$ and $G \circ \mathrm{Res} \cong \mathrm{id} \oplus \sigma$, so $G \circ \mathrm{Res}(Y) \cong Y \oplus \sigma Y$

non-trivial indecomp $Y \in MF(f+y^2)$

Note that the second identity in (ii) $\Rightarrow$ Res is faithful

Note in general finitely gen modules over Hensel rings have the Krull-Schmidt property: M is uniquely a sum of indecomposable + End(M) is local if M is indec.

Lemma 2.6 i) $Y \in MF(f+y^2)$ is in the essential image of $G(\text{-})$ $\Leftrightarrow$ $Y \cong \sigma Y$

ii) if Y is non-trivial indecomposable then $\Omega Y \cong \sigma Y$.

Prop 2.7. (i) Let $X \in MF(f)$ be non-trivial indecomposable. Then $G(X) \in MF(f+y^2)$ is decomposable $\Leftrightarrow$ $X \cong \Omega X$. In this case $G(X) \cong Y \oplus \Omega Y$, for some indecomp nontrivial $Y \in MF(f+y^2)$ with $Y \not\cong \Omega Y$

(ii) Let $Y \in MF(f+y^2)$ nontrivial indecomposable. Then $\mathrm{Res}(Y)$ is decomposable $\Leftrightarrow$ $Y \cong \Omega Y$ (i.e. $Y \cong G(X)$). In this case $\mathrm{Res}(Y) \cong X \oplus \Omega X$ for X non-triv indecomposable with $X \not\cong \Omega X$.

Prop 3.6 H induces a bijection $\mathrm{Iso}(\underline{\mathrm{MF}}(f)) \xrightarrow{\sim} \mathrm{Iso}(\underline{\mathrm{MF}}(f+uv))$

in fact H induces a bijection $\{$non trivial indec $\mathrm{MF}(f)\}/_\sim \xrightarrow{\sim}$
$\{$non triv indec $\mathrm{MF}(f+uv)\}$)

Pf Take $X \in \underline{\mathrm{MF}}(f)$ indecomposable. Then by Prop 2.7(1)

$G_2 \circ G_1(X) = Y \oplus Y'$, Y, Y' non-trivial indec.
$\;\; \| \; (3.9)$
$H(X) \oplus \Omega H(X)$

$\Downarrow$

$H(X)$ is indecomposable, non-trivial

Suppose we have $X' \in \mathrm{MF}(f)$, non-triv, indec, with $H(X) \cong H(X')$. Then by (3.9)

$$X \oplus \Omega X \cong \rho H(X) \cong \rho H(X') \cong X' \oplus \Omega X'$$

$\Rightarrow$ $\underset{(a)}{X \cong X'}$ or $\underset{(b)}{X \cong \Omega X'}$ In case (a) we are done
?

If $X \cong \Omega X' \not\cong X' \Rightarrow X \not\cong \Omega X \underset{\text{Prop 2.7}}{\Rightarrow} G_1(X)$ is indecomposable
so $\Omega X \cong X'$

and since $G_2 G_1(X) = H(X) \oplus \Omega H(X) \leftarrow$ decomposable, we have again by Prop 2.7(2), $H(X) \not\cong \Omega H(X)$. But

$\Omega H(X) \cong H(\Omega X) \cong H(X') \cong H(X)$ a contradiction.

Thus H is injective on iso classes of non-trivial indecomposable

For surjectivity, take $Y \in \mathrm{MF}(\mathfrak{f} + y^2 + z^2)$ non-trivial, indec. (7)

Then $G_2 G_1(\rho Y) \underset{(7.1)}{=} H(\rho Y) \oplus S H(\rho Y) \underset{(3.5)}{=} H(\rho Y) + H(\Omega_\rho Y)$

and $G_2 G_1(\rho Y) \cong G_2 G_1(\mathrm{Res}\,\mathrm{Res}\, Y)$

$\underset{\text{Lemma 5(ii)}}{\cong} G_2(\mathrm{Res}_2 Y \oplus \Omega\, \mathrm{Res}_2 Y)$

$\cong Y \oplus \Omega Y \oplus \Omega Y \oplus Y \cong Y^{\oplus 2} \oplus \Omega Y^{\oplus 2}$

so Y is a summand of an obj in $\mathrm{Im}\, H$

since $H(\text{indec})$ is indec $\Rightarrow Y \cong H(X)$ some indec X.

To show H is fully faithful: we have already seen that H is faithful. For fullness, we use the following construction. Start with

$X = (F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0)$, $X' = (F_1' \underset{\psi'}{\overset{\varphi'}{\rightleftarrows}} F_0')$, so $(\tilde{\ } = S_2 \oplus \_\,)$

$HX = (\tilde{F}_1 \oplus \tilde{F}_0 \underset{\begin{pmatrix} u, & \psi \\ \varphi, & -u \end{pmatrix}}{\overset{\begin{pmatrix} u, & \psi \\ \varphi, & -u \end{pmatrix}}{\rightleftarrows}} \tilde{F}_1 \oplus \tilde{F}_0)$,

$\Big\downarrow (A,B) = \begin{pmatrix} \alpha_1, & \alpha_2 \\ \alpha_3, & \alpha_4 \end{pmatrix} \Big\downarrow \qquad \Big\downarrow \begin{pmatrix} \beta_1, & \beta_2 \\ \beta_3, & \beta_4 \end{pmatrix}$

$HX' = (\tilde{F}_1' \oplus \tilde{F}_0' \rightleftarrows \tilde{F}_1' \oplus \tilde{F}_0')$

Apply ρ:

$$\begin{array}{ccc}
\rho H X = \Big(F_1 \oplus f_0 & \underset{\begin{pmatrix} 0 & \psi \\ \varphi & 0 \end{pmatrix}}{\overset{\begin{pmatrix} 0 & \psi \\ \varphi & 0 \end{pmatrix}}{\rightleftarrows}} & F_1 \oplus f_0 \Big) \\
\rho(A,B) \Big\downarrow \begin{pmatrix} \bar\alpha_1, \bar\alpha_2 \\ \bar\alpha_3, \bar\alpha_4 \end{pmatrix} & & \Big\downarrow \begin{pmatrix} \bar\beta_1, \bar\beta_2 \\ \bar\beta_3, \bar\beta_4 \end{pmatrix} \\
\rho H X' = \Big(F_1' \oplus f_0' & \underset{\begin{pmatrix} 0 & \psi' \\ \varphi' & 0 \end{pmatrix}}{\overset{\begin{pmatrix} 0 & \psi' \\ \varphi' & 0 \end{pmatrix}}{\rightleftarrows}} & F_1' \oplus f_0' \Big)
\end{array}$$

Lemma 3.8 (i) $(\bar\alpha_3, \bar\beta_2) : X \to \Omega X'$ and $(\bar\alpha_3, \bar\beta_2) : \Omega X \to X'$ are both ~ 0

(ii) $(\bar\alpha_1, \bar\beta_4) \sim (\bar\beta_1, \bar\alpha_4) : X \to X'$

Pf (i) $(A,B) : HX \to HX' \Rightarrow \varphi' \alpha_1 - v\alpha_2 = u\beta_3 + \beta_4 \varphi$

and $u\alpha_2 + \psi' \alpha_4 = \beta_1 \psi - v\beta_2$

expanding $\alpha_1, \beta_1, \alpha_4, \beta_4$ in a power series in (u, v) we have $\gamma_2, \delta_2, \gamma_3, \delta_3$

$$\bar\beta_2 = \psi' \gamma_2 + \delta_2 \psi, \quad \bar\beta_3 = \varphi' \gamma_3 + \delta_3 \varphi$$

ie
$$\begin{array}{ll}
\gamma_2 : f_0 \to f_0' & \gamma_3 : F_1 \to F_1' \\
\delta_2 : F_1 \to F_1' & \delta_3 : f_0 \to f_0' \\
RH & LH
\end{array}$$

Thus the pair $(\delta_2, \gamma_3) \in \mathrm{Hom}(X, \Omega X')^1$ gives the homotopy $(\bar\alpha_3, \bar\beta_2) \sim 0$ the pair $(\delta_3, \gamma_3) \in \mathrm{Hom}(\Omega X, X')^1$ gives the homotopy $(\bar\alpha_2, \bar\beta_3) \sim 0$

(note: since $\varphi, \psi, \varphi', \psi'$ are injective one only need to check that $\bar\beta_3, \bar\beta_4$ gets sent to 0 by the homotopy) 9

ii) is similar, we have

$\varphi' \alpha_2 - v\alpha_4 = \beta_3 \psi - v\beta_4$, so we have maps $\gamma : F_1 \to F_0'$, $\delta : F_0 \to F_1'$

defining elements $(\gamma, \delta) \in \mathrm{Hom}(X, X')^1$, $(\delta, \gamma) \in \mathrm{Hom}(\Omega X, \Omega X')^1$

with $(\bar\alpha_1 - \bar\beta_1, \bar\beta_4 - \bar\alpha_4) = d(\gamma, \delta)$, $(\bar\beta_4 - \bar\alpha_4, \bar\alpha_1 - \bar\beta_1) = d(\delta, \gamma)$

Given $(A, B) : HX \to HX'$ as above, consider

$$(A,B) - H(\bar\beta_1, \bar\alpha_4) = \left(\begin{pmatrix} \alpha_1 - \bar\beta_1 & \alpha_2 \\ \alpha_3 & \alpha_4 - \bar\alpha_4 \end{pmatrix}, \begin{pmatrix} \beta_1 - \bar\beta_1 & \beta_2 \\ \beta_3 & \beta_4 - \bar\alpha_4 \end{pmatrix} \right)$$

then $\rho(\quad\downarrow\quad) \overset{\text{lemma 2.2}}{\sim} 0$ but ρ is faithful

so $(A,B) \sim H(\bar\beta_1, \bar\alpha_4)$ and H is full

We now prove Prop 2.7, Lemma 2.6 + Lemma 2.5

We return to the setting of $R = S/(f)$, $R_1 = S_1/f + y^2$, $R_1[\sigma]$, etc and the diagram $(*)$

(2.4) Some natural isomorphisms

(i) $\mathrm{Res} \circ \mathrm{coh} \cong \mathrm{coh} \circ \mathrm{Res}$

(ii) $\mathrm{coh} \circ A \circ F_n \cong \mathrm{Res}$

(iii) $\mathrm{coh} \circ G \cong 1 \circ A^{-1}$

(iv) $\qquad G G \cong G$

Proofs are mainly elementary computations + some chase of words

Moreover

$$\mathrm{Res}\ \Omega \cong \Omega \circ \mathrm{Res},\ \Omega \circ G \cong G \circ \Omega,\quad \Omega \circ G \cong G$$

<u>Lemma 1.2</u> for $X \in \mathrm{Mf}(f)$, $X \cong \Omega X \iff X \cong (\mathcal{S}_0, \mathcal{S}_1)$

proof uses $k = \bar{k}$ need to take $(-)^{1/2}$ in $GL_p(S)$.

Prop 2.7 i) Take $X \in MF^{red}(f)$ with $M = \operatorname{cok} X$ indecomposable not free (11)

Then $G(X)$ is decomposable $\iff \Omega X \cong X$, and in this case

$G(X) = Y \oplus \Omega Y$ with Y indecomposable + $Y \not\cong \Omega Y$

ii) Take $Y \in MF(f+y^2)$ indecomposable then

$\operatorname{Res}(Y)$ is decomposable $\iff \Omega Y \cong Y$ and in this case

$\operatorname{Res}(Y) \cong X \oplus \Omega X$, X indecomposable with $\Omega X \not\cong X$.

Pf (i) Show for $X \in MF(f)$ indec $G(X)$ decomposes $\iff \Omega X \cong X$

if $X \cong \Omega X$ we can write $X = (\varphi, \varphi)$ $\varphi^2 = f \cdot id$ (Lemma 12)

Then $G(X) = \left(\begin{pmatrix} y & \varphi \\ \varphi & -y \end{pmatrix}, \begin{pmatrix} y & \varphi \\ \varphi & -y \end{pmatrix} \right)$

$$\cong \left(\begin{pmatrix} \varphi + iy\,id & 0 \\ 0 & \varphi - iy\,id \end{pmatrix}, \begin{pmatrix} \varphi - iy\,id & 0 \\ 0 & \varphi + iy\,id \end{pmatrix} \right)$$

$$= (\varphi + iy, \varphi - iy) \oplus \Omega(\varphi + iy, \varphi - iy)$$

Since $\operatorname{Res} G(X) = X \oplus \Omega X$ has 2 indec factors we have $(\varphi + iy, \varphi - iy)$ is indecomposable

If $\omega(X) = Y \oplus Y'$ with Y indec, non-trivial, $Y' \neq 0$ not free

$\mathrm{Res}\,\omega(X) = X \oplus \Omega X \Rightarrow \mathrm{Res}\,Y' = X$ or ΩX

$\,\,\,\,\,\| \quad (2.5(ii))$ Replace X with ΩX if needed

$\mathrm{Res}\,Y \oplus \mathrm{Res}\,Y'$

we can assume $\mathrm{Res}\,Y = X \Rightarrow \omega X = \omega\,\mathrm{Res}\,Y$

$\cong Y \oplus \sigma Y = Y \oplus \Omega Y$

$(2.5\ ii)$

Then $\mathrm{Res}\,\omega X = X \oplus \Omega X \Rightarrow \mathrm{Res}\,\sigma Y = \Omega X$

$\quad\quad\,\|\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \|$ i.e. $X \cong \Omega X$

$\mathrm{Res}\,Y \oplus \mathrm{Res}\,\sigma Y \quad\quad\quad \mathrm{Res}\,Y = X$

ii) Take $Y \in \mathrm{Mf}(f_{+3}^{\circ})$ indecomposable. If $Y \cong \Omega Y \cong \sigma Y$

then $Y = \omega X$ and $\mathrm{Res}\,Y \cong X \oplus \Omega X$ is decomposable

(2.6 i)

if $\mathrm{Res}\,Y = X \oplus X'$ with X indec $X' \neq 0$ nontriv

then ωX is a summand of $\omega\,\mathrm{Res}\,Y \cong Y \oplus \sigma Y \cong Y \oplus \Omega Y$ (Claim 2.6 (iii))

May assume $\omega X = Y$. Then $\Omega Y = \sigma Y = \sigma\omega X \cong \omega X = Y$

Thus $\mathrm{Res}\,Y$ decomposes $\Leftrightarrow Y \cong \Omega Y$

To complete the proof of (i) suppose $X \not\cong \Omega X$, then

$$G(X) = Y \oplus \Omega Y \Rightarrow \underset{\underset{\mathrm{Res}\,Y \oplus \Omega\,\mathrm{Res}\,Y}{\shortparallel}}{\mathrm{Res}\,G(X)} = X \oplus \Omega X$$

$\Rightarrow \mathrm{Res}\,Y = X$ or ΩX is indec $\Rightarrow Y$ is indec and $\mathrm{Res}\,Y$ is also indec.

But if $Y \cong \Omega Y$ then $\mathrm{Res}\,Y$ is decomposable by the part of (ii) proven above, so $Y \not\cong \Omega Y$

To complete the proof of (ii) if $\mathrm{Res}\,Y$ is decomposable then $Y = GX$, $Y \cong \Omega Y$ Then

$\mathrm{Res}\,Y = \mathrm{Res}\,G(X) = X \oplus \Omega X$. If X is decomposable then $Y = GX$ is also decomposable $\Rightarrow X$ is indecomposable. If $X \cong \Omega X$ then $Y = GX$ is decomposable by (i)

$\square$

<u>Lemma 26</u> i) $Y \in MF(f + y^2)$ is in the essential image of $G(\mathcal{F})$ iff $Y \cong Y$

ii) if Y is non-trivial indecomposable then $\Omega Y \cong \sigma Y$.

ty $G(\varphi,\psi) = \begin{pmatrix} g & \psi \\ \varphi & -g \end{pmatrix}$ so $\sigma G(\varphi,\psi) = \begin{pmatrix} -g & \psi \\ \varphi & g \end{pmatrix}$

and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g & \psi \\ \varphi & -g \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -g & -\psi \\ \varphi & -g \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -g & \psi \\ \varphi & g \end{pmatrix}$

so $\sigma G \cong G$. If $\sigma(\varphi,\psi) \cong (\varphi,\psi)$

$G(f,1) = \begin{pmatrix} g & 1 \\ f & -g \end{pmatrix} : S_1^n \oplus S_1^n \to S_0^n \oplus S_0^n$

$\int$ ||

$(g^2+f, 1)$ so we can assume (φ,ψ) is reduced, and then indecomposable, let $M = \operatorname{cok}\varphi$. The isomorphism $(\varphi,\psi) \cong \sigma(\varphi,\psi)$ induces an isomorphism $\Phi : M \to \sigma^* M$.

As an S-module M is free $M \cong S^r$, and the R_1-module structure is given by representing xy as $A \in M_r(S)$ $S[y]/y^2+f$ with $A^2 = -f\,\mathrm{id}$: $M = (S^r, A)$. Then $\sigma^* M = (S^r, -A)$ and Φ is represented by $B \in GL_r(S)$ with $BA = -AB$

$(S^r, A) \xrightarrow{B} (S^r, -A) \xrightarrow{B} (S^r, \ )$. Then B^2 represents an endomorphism of M

$M \xrightarrow{\Psi} \sigma^* M \xrightarrow{\Psi} M$

Lemma 2.5 (i) $F_N \mathcal{U} \cong (M \mapsto M \oplus M \otimes_{R_1[\sigma]} R_1^{-})$ σ acts $b_3 \to -\sigma$ on R (15)

and $\mathcal{U} F_N \cong id \oplus \sigma^*$

(ii) $\operatorname{Res} G \cong id \oplus \Sigma$ and $G \circ \operatorname{Res} \cong id \oplus \sigma$, so $G \circ \operatorname{Res}(Y) \cong Y \oplus Y$

non-trivial indecomp $Y \in MF(f+y^2)$

Note that the second identity in (ii) $\Rightarrow$ Res is faithful

$R_1^{-} = R_1$ as R_1 mod $\quad \sigma(a + b_3) = -a + b_3$

Pf $(F_N \circ \mathcal{U})(M) = R_1[\sigma] \otimes_{R_1} M$ $\quad$ Map $M \oplus M \otimes_{R_1} R_1^{-}$ $\downarrow$ $R_1[\sigma] \otimes_{R_1} M$, M with $-\sigma$ action

$M \in \operatorname{Mod}(R_1[\sigma])$ $\quad = M \oplus \sigma \otimes M$ $\quad (m_1, m_2) \downarrow$ $\quad = M \oplus \sigma \otimes M$

$\sigma(m) \quad \cong \sigma^* M$ $\quad \sigma$ $\quad ((m_1 + m_2), \sigma \otimes (\sigma(m_1) - \sigma(m_2)))$

$(\sigma(m_1), -\sigma(m_2))$

$\searrow (\sigma(m_1) - \sigma(m_2), \sigma \otimes (\sigma(m_1) + \sigma(m_2)))$ σ

$(\mathcal{U} \circ F_N)(M) = R_1[\sigma] \otimes_{R_1} M$ as R_1-module

$= M \oplus \sigma \otimes M = M \oplus \sigma^* M$

(ii) $\operatorname{Res} G(\varphi,\psi) = \operatorname{Res}(\Phi,\Phi) \quad \Phi = \begin{pmatrix} y & \psi \\ \varphi & -y \end{pmatrix}$

$$\cong \left(\begin{pmatrix} 0, & \psi \\ \varphi & 0 \end{pmatrix}, \begin{pmatrix} 0 & \psi \\ \varphi, & 0 \end{pmatrix} \right)$$

$$\cong \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \psi \\ \varphi & 0 \end{pmatrix}, \begin{pmatrix} 0 & \psi \\ \varphi & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix}, \begin{pmatrix} \psi & 0 \\ 0 & \varphi \end{pmatrix} \right) = (\varphi,\psi) \oplus (\psi,\varphi)$$

$\Rightarrow \operatorname{Res} G = \mathrm{id} \oplus \Omega$

for $G \circ R_x$:

$$\begin{tikzcd}
\underline{\mathrm{MF}}(f+y^2) \arrow[r, shift left, "\operatorname{Res}"] \arrow[dd, "\operatorname{cok}"', "\wr"] & \underline{\mathrm{MF}}(f) \arrow[l, shift left, "G"] \arrow[dd, "\wr", "\operatorname{cok}"'] \\
 & \mathrm{MCM}^{iso}_{\varphi}(R_1) \arrow[u, "A", "\wr"'] \arrow[dl, shift left, "A"] & \\
\underline{\mathrm{MCM}}(R_1)_{iso} \arrow[ur, shift left, "Fn"] \arrow[r, "\operatorname{res}"] & \underline{\mathrm{MCM}}(B)
\end{tikzcd}$$

$\operatorname{cok} \operatorname{Res} \cong \operatorname{res} \circ \operatorname{cok}$ (2.4)(i)

$U\,Fn \cong \mathrm{id} \oplus G^*$

$\operatorname{cok} G \cong U A^{-1}$ (2.4)(iii)

so suffice to show

$$\operatorname{cok} G \circ \operatorname{Res} \circ \operatorname{cok}^{-1} = \mathrm{id} \oplus G^*$$

for this $\operatorname{cok} G \operatorname{Res} \operatorname{cok}^{-1} = \operatorname{cok} G \operatorname{cok}^{-1} \operatorname{res}$

$= U A^{-1} \operatorname{cok} \operatorname{res}$

$= U A^{-1} \operatorname{cok}^{-1} \operatorname{cok} A\, Fn$ (2.4)(ii)

$= U\, Fn$

$= \mathrm{id} \oplus G^*$ $\square$

Cor 2.9 Res : $\underline{MF}(f+y^2) \to \underline{MF}(f)$ is faithful

Pf $\mathrm{coh}\, A F_n \cong \mathrm{res}$ (2.4 ii) $\Rightarrow A F_n\, \mathrm{coh} \cong \mathrm{Res}$, coh & A are equiv and $\mathcal{U} F_n \cong \mathrm{id} \oplus \Omega^*$

$\Rightarrow F_n$ is faithful.

Lemma 1.3 $(\varphi,\psi) \in MF(f)$ with $\Omega(\varphi,\psi) \cong (\varphi,\psi)$. Then $(\varphi,\psi) \cong (\varphi_0,\varphi_0)$, $\varphi_0^2 = f$

Pf $\Omega(\varphi,\psi) = (-\psi,-\varphi) \cong (\psi,\varphi) \xleftarrow[(\alpha,\beta)]{} (\varphi,\psi)$ Moreover (φ,ψ) indecomposable

$$\begin{tikzcd}
S^n \arrow[r, shift left, "\varphi"] \arrow[d, "\alpha"'] \arrow[dd, bend right=60, "\beta\alpha"'] & S^n \arrow[l, shift left, "\psi"] \arrow[d, "\beta"] \arrow[dd, bend left=60, "\alpha\beta"] \\
S^n \arrow[r, shift left, "\psi"] \arrow[d, "\beta"'] & S^n \arrow[l, shift left, "\varphi"] \arrow[d, "\alpha"] \\
S^n \arrow[r, shift left, "\varphi"] & S^n \arrow[l, shift left, "\psi"]
\end{tikzcd}$$

so $(\beta\alpha, \alpha\beta) \in \mathrm{End}(\varphi,\psi)$

$\Rightarrow (\beta\alpha, \alpha\beta) = (\mathrm{id},\mathrm{id}) + (\rho_1,\rho_2)$, $\rho_i \in \mathrm{rad}\,\mathrm{End}$

so $\beta\alpha = \mathrm{id} + \rho_1 \Rightarrow \beta = \alpha^{-1} + \rho_1\alpha^{-1}$

$\alpha\beta = \mathrm{id} + \rho_2 \Rightarrow \beta = \alpha^{-1} + \alpha^{-1}\rho_2 \Rightarrow \alpha\rho_1 = \rho_2\alpha$

and $\beta\rho_2 = \rho_1\beta$

Let $\alpha' = \alpha(\mathrm{id}+\rho_1)^{-1/2} = (\mathrm{id}+\rho_2)^{-1/2}\alpha$

$\beta' = \beta(\mathrm{id}+\rho_2)^{-1/2} = (\mathrm{id}+\rho_1)^{-1/2}\beta$

so $(\alpha',\beta') : (\varphi,\psi) \to (\psi,\varphi)$ w/ $\alpha'\beta' = \beta'\alpha' = \mathrm{id}$

Take γ with $\gamma^2 = \alpha'$, then $\varphi_0 = \gamma\psi\gamma = \gamma^{-1}\varphi\gamma^{-1}$ has $(\varphi_0,\varphi_0) \cong (\varphi,\psi)$ □

(2.4) Some nat isomorphi

(i) $\mathrm{res} \circ \mathrm{coh} \cong \mathrm{coh} \circ \mathrm{Res}$: $Z_0 MF(f + y^2) \to MCM(R)$

(ii) $\mathrm{coh} \circ A \circ F_\sim \cong \mathrm{res}$: $MCM(R_1) \to MCM(R)$

(iii) $\mathrm{coh} \circ G \cong U \circ A^{-1}$: $Z^0 MF(f) \to MCM(R_1)$

(iv) $\sigma G \cong G$

Pf (i) clear

(ii) $\mathrm{coh} \circ A$ sends M to $M^- / yM^+ = M / R_1 M^+$

so $\mathrm{coh} \circ A(R_1[\sigma] \otimes_{R_1} N) = \mathrm{coh} \circ A(N \oplus \sigma N) \cong \dfrac{(1-\sigma)N}{y(1+\sigma)N} = (1-\sigma)N / yN \cong N / \sigma N$

(iii) Given $S^n \underset{\psi}{\overset{\varphi}{\rightleftarrows}} S^n \in MF(f)$

let $M_{(\varphi,\psi)} = S^n \oplus S^n$ with y-action $y(a,b) = (-\psi b, \varphi a)$

and $\sigma(a,b) = (-a, b)$

$A(M) = \left(M^+ \underset{-y}{\overset{y}{\rightleftarrows}} M^-\right) = \left(S^n \underset{\psi}{\overset{\varphi}{\rightleftarrows}} S^n\right)$

then is $A^{-1}(\varphi,\psi) = M_{(\varphi,\psi)}$

and $UA^{-1}(\varphi,\psi) = UM_{(\varphi,\psi)} = S^n \oplus S^n$ with y acting $y = \begin{pmatrix} -\psi & 0 \\ 0 & \varphi \end{pmatrix}$

$G(\varphi,\psi) = (\Phi, \Phi)$

$\Phi = \begin{pmatrix} y & \psi \\ \varphi & -y \end{pmatrix} : (S^n \oplus S^n) \otimes S^1 \to$

$S_1^n \oplus S_1^n \to$

$NA^{-1}(\varphi,\psi) = S^n_1 \oplus S^n_1 \big/ (a, \varphi b) \sim (-\psi b, \varphi a)$, so $\{(a+\psi b, \varphi b - \varphi a) \mid (a,b) \in S^n \oplus S^n\}$

$\operatorname{coker} \Phi = S^n_1 \oplus S^n_1 \big/ \{(\varphi a + \psi b, \varphi a - \varphi b) \mid (a,b) \in S^n \oplus S^n\}$

so $S^n_1 \oplus S^n_1 \circlearrowleft (id, -id)$

gives an iso $NA^{-1}(\varphi,\psi) \cong \operatorname{coker} \Gamma(\varphi,\psi)$

$\Gamma\Omega(\varphi,\psi) = \left(\begin{pmatrix} -\varphi & \psi \\ \varphi & \varphi \end{pmatrix}, \begin{pmatrix} -\varphi, & \psi \\ \varphi, & b \end{pmatrix} \right)$

$$\begin{array}{ccc}
S^n_1 \oplus S^n_1 & \underset{\longleftarrow}{\overset{\begin{pmatrix} -\varphi & \psi \\ \varphi & \varphi \end{pmatrix}}{\longrightarrow}} & S^n_1 \oplus S^n_1 \\
\Big\downarrow (id, -id) & \begin{pmatrix} -\varphi & \psi \\ \varphi & \varphi \end{pmatrix} & \Big\downarrow (-id, id) \\
S^n_1 \oplus S^n_1 & \underset{\longleftarrow}{\overset{\begin{pmatrix} \varphi & \psi \\ \varphi & -\varphi \end{pmatrix}}{\longrightarrow}} & S^n_1 \oplus S^n_1 \\
& \begin{pmatrix} \varphi & \psi \\ \varphi & -b \end{pmatrix} &
\end{array}$$

gives iso $\Gamma\Omega \cong \Omega$.