Matrix factorization, stabilization and compact generation

§ Recollection

We follow Dyckerhoff's notation

(R, m) regular local ring, $w \in m^2$, $S = R/(w)$

$MF(R,w)$ = dg-category of matrix factorizations $\left(F^1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F^0\right)$ $\quad \varphi\psi = w\,\mathrm{id}_{F^0}$, $\psi\varphi = w\,\mathrm{id}_{F^1}$

$\underline{MF}(R,w)$ = homotopy category

$MCM(S)$ = maximal C-M modules $\subset$ S-Mod

$\underline{MCM}(S)$ = stable category $\quad \underline{\mathrm{Hom}}(M,N) = \mathrm{Hom}_S(M,N)/f$ factor through projectives

$D^b(S)$ = bounded derived category

$\underline{D}^b(S)$ = stabilized derived category $= D^b(S)/D^{perf}(S)$

Thm (Eisenbud) every $N \in \mathrm{Mod}^{fg}(S)$ has minimal resolution that is eventually 2-periodic; if $M \in MCM(S)$ with no free summands the minimal resolution of M is 2-periodic

We have functors

$$\underline{MF}(R,w) \to \underline{MCM}(S) \to \underline{D}^b(S)$$

$X = (F^1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F^0) \mapsto \mathrm{coker}(\varphi) = \mathrm{coker}(X)$ $\qquad M \mapsto$ 2-periodic res. of M

$$\to \bar{F}^0 \xrightarrow{\psi} \bar{F}^1 \xrightarrow{\varphi} \bar{F}^0$$

which are all equivalences of triangulated categories.

Note. given $X, Y \in \underline{D}^b(S)$ $\exists\, i_0 = i_0(X,Y)$ s.t.

$$\mathrm{Hom}_{\underline{D}^b(S)}(X, Y[i]) = \mathrm{Hom}_{D^b(S)}(X, Y[i]) \quad \forall i \geq i_0$$

Explicit stabilization

Let L be an S-module of finite rank. The functor $S\text{-Mod} \to \underline{MF}(R,w)$ is given explicitly by $L \mapsto L^{stab}$; $D^b(S) \to \underline{D}^b(S)$

taking $\Omega^{2n} L$ for $n \gg 0$, which is in $MCM(S)$ and write $\Omega^{2n} L = \mathrm{cok}(L^{stab})$

Suppose $L = S/I$ with $I = (f_1 \dots f_m)$, $f_1 \dots f_m$ regular sequence, and $w \in I$.

We have an explicit R-free resolution of L as follows.

Let $V = \bigoplus^m R e_i \rightsquigarrow K^* : \mathrm{Kos}(V, (f_1, \dots, f_m))$

$$K^i = \Lambda^i V \xrightarrow{\partial} \Lambda^{i-1} V = K^{i-1} \quad \text{by } \sum f_i \frac{\partial}{\partial e_i}$$

Next, $w \in I \Rightarrow w = \sum^m f_i w_i$; $w_i \in R$

$$s_1 = \left(\sum w_i e_i\right) \wedge - : \Lambda^i R^m \to \Lambda^{i+1} R^m, \text{ with}$$

$s_0 s_1 + s_1 s_0 = \times w \, \mathrm{id}_{\bar K}$, $s_0^2 = s_1^2 = 0$ $\rightsquigarrow$ $\boxed{s_0 s_1 = -s_1 s_0 \ \mathrm{mod}\ w}$ and $(s_0+s_1)^2 = \times w$

Then $L^{stab} = \left(\Lambda^{odd} V \underset{s_0+s_1}{\overset{s_0+s_1}{\rightleftarrows}} \Lambda^{ev} V\right) \in \underline{\mathrm{MF}}(R, w)$

Pf Consider the double complex $\bar K_{**}$ in S-mod, augmented over L: $(\bar{\ }) = (-)\otimes_R S$) $\bar K_{a,b} = \bar K^{a-b}$

$$\begin{tikzcd} \bar K_{a,b} \arrow[r, "\bar s_0"] \arrow[d, "\bar s_1"] & \bar K_{a-1,b} \\ \bar K_{a,b-1} \end{tikzcd}$$

$$\begin{tikzcd}
\bar K^m \arrow[r, "\bar s_0"] \arrow[d, "\bar s_1"] & \bar K^{m-1} \arrow[r, "\bar s_0"] \arrow[d, "\bar s_1"] & \bar K^{m-2} \cdots \arrow[r, "\bar s_0"] & \bar K^0 \arrow[d, "\bar s_1"] & & \\
\bar K^m \arrow[r] \arrow[d, "\bar s_1"] & \bar K^{m-1} \cdots \arrow[r] & \bar K^1 \arrow[r, "\bar s_0"] \arrow[d, "\bar s_1"] & \bar K^0 \arrow[d, "s_1"] & & \\
\bar K^m \arrow[r] & \cdots \arrow[r, "\bar s_1"] & \bar K^2 \arrow[r, "\bar s_0"] & \bar K^1 \arrow[r, "\bar s_0"] & \bar K^0 = R \arrow[r, "\varepsilon"] & L = R/I
\end{tikzcd}$$

Lemma (Eisenbud) $\mathrm{Tot}(\bar K^*)$ is an S-free resolution of L

Pf Use the spectral sequence $E^1_{a,b} = H_{a+b}(\bar K_{*,b}, \bar s_0)$

Note $\bar K^* \otimes_R S \cong K^* \otimes_R (R \xrightarrow{w} R) \cong L \otimes_R (R \xrightarrow{w} R) = L \xrightarrow{0} L$

Apply the horizontal filtration to $\mathrm{Tot}\, \bar K_{**}$ to take $E_1 = gr^F(\bar K_{**})$

got

$$\begin{tikzcd}
\cdots \arrow[r] & L \arrow[r, "0"] & L \arrow[d, "g"] & \\
& L \arrow[r, "0"] & L \arrow[r, "\varepsilon = id"] & L
\end{tikzcd}$$

Claim $g = \mathrm{id}_L$

Pf Let $t \in \mathrm{End}(R \underset{w}{\to} R)^{-1}$ be the map $R \xrightarrow{0} R$, $R \xrightarrow{0} R$ (id)

We have q.iso's

$$\begin{tikzcd} & K^* \otimes_R (R \xrightarrow{w} R) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ K^* \otimes_R S & \xrightarrow{\sim} & L \otimes_R (R \xrightarrow{0} R) \cong L \xrightarrow{0} L \end{tikzcd}$$

with $p_1 \circ (s_1 \otimes 1 + 1 \otimes t)$ and $p_2 \circ (s_1 \otimes 1 + 1 \otimes t)$
$= (s_1 \otimes 1) \circ p_1$ and $= (1 \otimes t) \circ p_2$

Thus $H_*(\bar{K}^*) \xrightarrow{\sim} H^*(L \xrightarrow{0} L)$. The map $\bar{s}_1 : H_0(\bar{K}^*) \to H_1(\bar{K}^*)$ is equal to $t \otimes t = \mathrm{id} : H_0(L \xrightarrow{0} L) \to H_1(L \xrightarrow{0} L)$. Thus

$E^2_{p,q} = \begin{cases} 0 & (p,q) \neq (0,0) \\ L & (p,q) = (0,0) \end{cases} \Rightarrow \varepsilon : \mathrm{Tot}(K_{**}) \to L$ is q.iso. $\square$

<u>Cor</u> $L^{stab} = \left(\Lambda^{odd} R^n \underset{\bar{s}_0 + \bar{s}_1}{\overset{s_0 + s_1}{\rightleftarrows}} \Lambda^{ev} R^n \right) \in MF(R, w)$

Pf The R-free resolution $\mathrm{Tot}(K_{**})$ has 2-periodic part

$\to \Lambda^{ev} R^n \to \Lambda^{odd} R^n \xrightarrow{\bar{s}_0 + \bar{s}_1} \Lambda^{ev} R^n$ $\quad \bar{s}_0 \bar{s}_1 = -\bar{s}_1 \bar{s}_0$ $\square$

$\mathrm{End}^{\mathbb{Z}/2}(L^{stab}) = \mathrm{End}_R\left(\Lambda^{odd} R^n \oplus \Lambda^{ev} R^n, [\bar{s}_0 + \bar{s}_1, -]\right)$

(dg algebra) Writing $\Lambda^* R^n = R(\theta_1, \dots, \theta_n)$, θ_i deg 1

$\mathrm{End}^{\mathbb{Z}/2}(L^{stab}) = $ Alg of diff operators $\sum h_I \frac{\partial^I}{\partial \theta^I}$, $h_I \in R$

with differential $\delta = \sum \delta_i$, $\delta_i = f_i \frac{\partial}{\partial \theta_i} + w_i \theta_i$

$[\delta, -]$

Generators in $MF^{\infty}(R,w)$

In order to apply Bousfield localization (later) it will be necessary to introduce a larger version of $MF(R,w)$

Def $MF^{\infty}(R,w) = \left\{ \left(F^1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F^0\right) \;\middle|\; \begin{array}{l} F^i \text{ free } R\text{-modules} \\ \varphi\psi = w \cdot \mathrm{id}_{F^0} \\ \psi\varphi = w \cdot \mathrm{id}_{F^1} \end{array} \right\}$

as a $\mathbb{Z}/2$ dg category. The only difference is that we do not assume F^1, F^0 have finite rank.

Note We will see that $MF^{\infty}(R,w)$ is a model for dg Mod $MF(R,w)$

Def $\mathcal{T}$ a triangulated category admitting (small) coproducts

- $X \in \mathcal{T}$ is compact if $\mathrm{Hom}(X,-)$ commutes with small coproducts
- X is a generator if the localizing subcategory of $\mathcal{T}$ generated by X (smallest triangulated subcategory containing X & closed under coproducts) is $\mathcal{T}$ itself
- The right orthogonal complement $X^{\perp}$ is the full subcategory of $\mathcal{T}$ of objects Y such that $\mathrm{Hom}_{\mathcal{T}}(X, Y[i]) = 0 \;\forall i$

Note: Using Bousfield localization one shows

X is generator $\Leftrightarrow X^{\perp} = 0$

- $\underline{MF}^{\infty}(R, W)$ is triangulated with arbitrary (small) coproducts
- $\underline{MF}(R, W) \subset \underline{MF}^{\infty}(R, W)$ consists of compact objects

Hmm Dyckerhoff Theorem

Thm 4.7 Assume $V(W) \subset \operatorname{Spec} R$ has isolated singularities. Let $k_m = R/m$, considered as S-module. Then k^{stab} is a compact generator of $\underline{MF}^{\infty}(R, W)$

- $\operatorname{End}(k^{stab})$: Write $m = (x_1 \dots x_n)$ so $k = R/(x_1, \dots x_n)$

and $V = R^n = \oplus R\theta_i$, $w = \sum x_i w_i$, $s_0 = \sum x_i \frac{\partial}{\partial \theta_i}$

$k^{stab} = (\Lambda^{odd} V \underset{s_0 + s_1}{\overset{s_0 + s_1}{\rightleftarrows}} \Lambda^{ev} V) \in MF(R, W)$, $s_1 = (\sum w_i \theta_i) \wedge$

$$MF(k^{stab}, k^{stab}) = (A, [\delta, -]) \quad \text{as dg}$$

$A = \{\sum h_I \frac{\partial^I}{\partial \theta^I}\}$

$\delta = \sum \delta_i$, $\delta_i = f_i \frac{\partial}{\partial \theta_i} + w_i \theta_i$

§ The stable category $\underline{MCM}^\infty(S)$

We have the stable cat. of arbitrary S-modules $\underline{Mod}(S)$

Sending $(F^1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F^0)$ to $\mathrm{coker}(\varphi) = \mathrm{coker}(F)$ identifies

$\underline{MF}^\infty(R,w)$ with a subcategory $\underline{MCM}^\infty(S)$

of $\underline{Mod}(S)$, containing $\underline{MCM}(S)$.

If $\mathrm{coker}(F) = \Omega^N L$ some $L \in \underline{Mod}(S)$ and some even N, we write $F = L^{stab}$

<u>Lemma 4.2</u> Take $X \in MF(R,w)$, $Y \in MF^\infty(R,w)$ and assume $Y = L^{stab}$ for some $L \in \mathrm{Mod}(S)$, let $A = \mathrm{End}_{MF}(X)$

Then $MF(X,Y) \cong \mathrm{Hom}_R(X, L)$ in $D(A^{op})$

Pf. Write $X = (X^1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} X^0)$, $Y = (Y^1 \underset{\psi'}{\overset{\varphi'}{\rightleftarrows}} Y^0)$

Form the product total complex Z of the double complex

$$\begin{array}{ccccccc}
\downarrow & & & & & & \\
\mathrm{Hom}(\bar{X}^0,\bar{Y}^0) \to \mathrm{Hom}(\bar{X}^1,\bar{Y}^0) & \xrightarrow{+(-\partial\bar{\psi})} & \mathrm{Hom}(\bar{X}^0,\bar{Y}^0) & \xrightarrow{-(-\partial\bar{\varphi})} & \mathrm{Hom}(\bar{X}^1,\bar{Y}^0) & \xrightarrow{(-\partial\bar{\psi})} & \\
\downarrow \;\; +(\bar{\psi}'\circ -) \downarrow & & \downarrow (\bar{\psi}'\circ -) & & \downarrow \cdot(\bar{\psi}'\circ -) & & \\
\mathrm{Hom}(\bar{X}^1,\bar{Y}^1) \to \mathrm{Hom}(\bar{X}^1,\bar{Y}^1) & \xrightarrow{-(-\partial\bar{\psi})} & \mathrm{Hom}(\bar{X}^0,\bar{Y}^1) & \xrightarrow{+(-\partial\bar{\varphi})} & \mathrm{Hom}(\bar{X}^1,\bar{Y}^1) & & \\
\downarrow \;\; +(\bar{\psi}'\circ -) \downarrow & & \downarrow +(\bar{\psi}'\circ -) & & \downarrow (\bar{\psi}'\circ -) & & \\
\mathrm{Hom}(\bar{X}^0,\bar{Y}^0) \to \mathrm{Hom}(\bar{X}^1,\bar{Y}^0) & \xrightarrow{+(-\partial\bar{\psi})} & \mathrm{Hom}(\bar{X}^0,\bar{Y}^0) & \xrightarrow{-(-\partial\bar{\varphi})} & \mathrm{Hom}(\bar{X}^1,\bar{Y}^0) & \to &
\end{array}$$

we can consid Z as $Z^{\mathbb{Z}/2}$ ($\mathbb{Z}/2$-graded)

$$Z^0 = \prod_{i\geq 0} \mathrm{Hom}(\bar{X}^1,\bar{Y}^1) \times \mathrm{Hom}(\bar{X}^0,\bar{Y}^0)$$

$$Z^0 \underset{d^1}{\overset{d^0}{\rightleftarrows}} Z^1$$

$$Z^1 = \prod_{i\geq 0} \mathrm{Hom}(\bar{X}^1,\bar{Y}^0) \times \mathrm{Hom}(\bar{X}^0,\bar{Y}^1)$$

we have $\quad MF(X,Y) \to Z^{\mathbb{Z}/2}$

by $\quad (f^1,f^0) \to \prod_{i\geq 0}(\bar{f}^1,\bar{f}^0)$

Claim 1 $MF(X,Y) \to Z^{\mathbb{Z}/2}$ is a quasi-iso

Pf: surjective on H^0: given
$$\begin{array}{ccccccc}
\to \bar{X}^0 & \to & \bar{X}^1 & \to & \bar{X}^0 & & \\
\downarrow \bar{f}^0 & & \downarrow \bar{f}^1 & & \downarrow \bar{f}^0 & ; & \bar{f}' \\
\to \bar{Y}^0 & \to & \bar{Y}^1 & \to & \bar{Y}^0 & &
\end{array}$$

$\bar f^\circ$ induces $\tilde f : \mathrm{coh}(X) \to \mathrm{coh}(Y)$ and $\bar f$ is determined up to homotopy by $\tilde f$. But $\tilde f$ arises from a map $f' : X \to Y$ in $MF^\infty(R,w)$ with $\tilde f = \mathrm{coh}(f')$ and we can extend f' to $\bar f'$ by 2-periodicity with $\tilde{\bar f}' = \tilde f \Rightarrow \bar f' \sim \bar f^\circ$

Injective on H^0: given $f : X \to Y$ in $MF^\infty(R,w)$ take the two periodic extension $\bar f^\circ$

$$\begin{tikzcd}
\cdots \arrow[r] & \bar X^0 \arrow[r] \arrow[d, "\bar f^0"] & \bar X^1 \arrow[r] \arrow[d, "\bar f^1"] \arrow[dl, "\bar g^1"'] & \bar X^0 \arrow[d, "\bar f^0"] \arrow[dl, "\bar g^0"'] \\
\cdots \arrow[r] & \bar Y^0 \arrow[r] & \bar Y^1 \arrow[r] & \bar Y^0
\end{tikzcd}$$

if $\bar f^\circ = d\bar g^\circ$ in $Z^{\mathbb{Z}/2}$, then lift $\bar g^1, \bar g^0$ to g^1, g^0 which changes f to $f + d(g^1, g^0) = f'$. Then $f' \equiv 0 \bmod w$ so factors through

$$\begin{tikzcd}
X \arrow[d, "\cdot w"] \\
X \arrow[d, "\varphi' = \frac{1}{w} f'"] \\
Y
\end{tikzcd} \Rightarrow f' \sim 0$$

$$\begin{tikzcd}
X^1 \arrow[r] \arrow[d, "\cdot w"'] & X^0 \arrow[d, "\cdot w"] \arrow[dl, "\varphi"'] \\
X^1 \arrow[r, "g"'] & X^0
\end{tikzcd}$$

$\varphi g = w$, $\cdot w = g\varphi$, so $(w, w) = d(\varphi, 0)$

By apply Ω we see this map is an iso on H^1

Next enlarge Z by filling in the full S-free resolution of L with 2-periodic tail φ

$\to \bar Y^0 \to \bar Y^{-1} \to \dots \to \bar Y^{-r} \to L \to 0$, & let W^{*} be the total product complex of $\{W_{a,b} = \mathrm{Hom}(\bar X^a, \bar Y^b)\} = W_{**}$

We have the projection (for each $b \geq 0$) $W \xrightarrow{p} Z$

<u>Claim 2</u> p is a quasi iso

Pf. on H^0: need to show that a map of complexes

$$\begin{tikzcd} \cdots \arrow[r] & \bar X^2 \arrow[r] \arrow[d, "f^2"] & \bar X^1 \arrow[r] \arrow[d, "f^1"] & \bar X^0 \arrow[d, "f^0"] \\ \cdots \arrow[r] & \bar Y^2 \arrow[r] & \bar Y^1 \arrow[r] & \bar Y^0 \end{tikzcd}$$

extends uniquely up to homotopy to a map of complexes

$$\begin{tikzcd} \cdots \arrow[r] & \bar X^2 \arrow[r] \arrow[d, "f^2"] & \bar X^1 \arrow[r] \arrow[d, "f^1"] & \bar X^0 \arrow[r] \arrow[d, "f^0"] & \bar X^{-1} \arrow[r] \arrow[d, "f^{-1}"] & \cdots \arrow[r] & \bar X^{-r} \arrow[d, "f^{-r}"] \\ \cdots \arrow[r] & \bar Y^2 \arrow[r] & \bar Y^1 \arrow[r] & \bar Y^0 \arrow[r] & \bar Y^{-1} \arrow[r] & \cdots \arrow[r] & \bar Y^{-r} \end{tikzcd}$$

To extend to f^{-1}: consider

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "?"] & \bar X^{-1} \arrow[r] & \Omega M \arrow[r] & 0 \\ \bar X^0 \arrow[ur] \arrow[d, "f^0"] \\ \bar Y^0 \arrow[r] & \bar Y^{-1} \end{tikzcd}$$

$\Omega M \subset \bar X^0$

Take the pushout

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & \bar X^{-1} \arrow[r] \arrow[d] & \Omega M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \bar Y^{-1} \arrow[r] & \tilde X \arrow[r] & \Omega M \arrow[r] & 0 \end{tikzcd}$$

which splits because $\mathrm{Ext}^1_S(\Omega M, \bar Y^{-1}) = 0$, so we have the desired extension. Given two extensions $f^{-1}, \tilde f^{-1}$ let $g = \tilde f^{-1} - f^{-1}$

$$\begin{array}{l} M \to \bar X^{-1} \to \Omega M \hookrightarrow \bar X^0 \to M \to 0 \\ \quad \searrow \; g \downarrow \;\swarrow \tilde g \\ \quad 0 \in \bar Y^{-1} \end{array}$$

and $\tilde g$ extends to $h^{-2} : \bar X^0 \to \bar Y^{-1}$ after because $\mathrm{Ext}^1_S(M, \bar Y^{-1}) = 0$

These two extensions differ by a homotopy

Continuing in this way proves the claim $\square$

We have the augmentation $\to \bar Y^{-\bullet} \xrightarrow{\theta} L \to 0$ induces an augmentation $W^{\mathbb{Z}/2} \to \mathrm{Hom}_S(\bar X^\bullet, L) \cong \mathrm{Hom}_R(X^1 \oplus X^0, L)$

$\mathbb{Z}/2$-graded

<u>Claim:</u> This is a quasi-iso

Pf: The "vertical" filtration

$$(F^p W)^{a,b} = \begin{cases} W^{a,b} & \text{for } a \geq p \\ 0 & \text{else} \end{cases}$$

is complete: $W = \varprojlim W/F^pW$ and exhaust. $W = \bigcup F^p W$

& we have a similar filtration on $\mathrm{Hom}_S(\bar X^\bullet, L)$ (complete, exhaustive)

Moreover the map $E_1^{p,q} W = H^{p+q}(\mathrm{Hom}_S(\bar X^p, \bar Y^\bullet)) \to H^{p+q}\mathrm{Hom}(\bar X^p, L)$

is an iso $\Rightarrow$ $W \to \mathrm{Hom}_S(\bar X^\bullet, L)$ is a quasi-iso

(convergence)

(use induced filtr. on $\mathrm{cone}(W \to \mathrm{Hom}(\bar X^\bullet, L))$ which is complete, exhaustive & regular hence convergent)

We have thus constructed quasi-isos of $\mathbb{Z}/2$-graded complexes

$MF(X, L^{stab}) \to Z^{\mathbb{Z}/2} \hookrightarrow N^{\mathbb{Z}/2} \to \mathrm{Hom}(\bar{X}, L) \cong \mathrm{Hom}^{\mathbb{Z}/2}_R(X^1 \rightleftarrows X^0, L)$

also of dg A^{op} module $(A = \mathrm{End}_{MF}(L^{stab}))$ ⊡

We can describe this quo explicitly if $L = S/(f_1, \dots f_n)$. Then

$L^{stab} = \left(\bigoplus_{i=0}^{n} \Lambda^i \oplus R\theta_i , s_0 + s_1 \right) \in MF(R, w)$ $\quad s_0 = \sum f_i \frac{\partial}{\partial \theta_i}$

$s_1 = (\sum w_j \theta_j) \wedge$

with $w = \sum w_j f_j$

let g be the composition

$$\bigoplus \Lambda^i (\oplus R\theta_j) \xrightarrow[\pi]{} \Lambda^0 (\oplus R\theta_j) = R \to R/I = L$$

Then the quo above is given by

$$MF(X, L^{stab}) \xrightarrow[f_*]{} \mathrm{Hom}^{\mathbb{Z}/2}_R(X^1 \rightleftarrows X^0, L)$$

~~If~~ we may take $Y^{-i} = \begin{cases} \bigoplus_{j=0}^{(n-i)/2} \Lambda^{2j} \oplus R\theta_e & i \text{ even} \\ \bigoplus_{j=0}^{(n-i-1)/2} \Lambda^{2j+1} \oplus R\theta_e & i \text{ odd} \end{cases}$

let $\pi_i : \Lambda^{\substack{odd \\ even}} V \to Y^{-i}$ be the projection

starting with $X \xrightarrow{\varphi} MF(R, w)$ we have the 2-periodic obj
L^{stab}

$f^* : \bar{X} \to \bar{Y}$ correspond to $f : X \to Y$

and then we can take $f^{-i} = \begin{cases} \pi_i f^{odd} & i \text{ odd} \\ \pi_i f^{even} & i \text{ even} \end{cases}$
$i > 0$

so $f^{-n} = \pi$ and εf^{-n} is the given map ⊡

A reduction

for $X = (X^1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} X^0) \in MF(R,w)$ we have

$$X^\vee = \mathrm{Hom}^{\mathbb{Z}/2}(X, R) = \left(X^{1\vee} \underset{\varphi^*}{\overset{\psi^*}{\rightleftarrows}} X^{0\vee}\right) \in MF(R,-w)$$

Prop 4.4) Take $Y \in \underline{MF}^\infty(R,w)$, let $N = \mathrm{coker}(\varphi)$
$(Y^1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} Y^0)$ $\in \underline{MCM}^\infty(S)$

$n = \dim R$

Then
$$\underline{MF}^\infty(k^{stab}[n+2], Y) \cong \mathrm{Tor}_i^S(k, N)$$

Pf from lem 4.2 we have

$$MF(k^{stab}[n+2], Y) \cong \mathrm{Hom}_R^{\mathbb{Z}/2}(k[n+2], N)$$

$$\mathrm{Hom}_R^{\mathbb{Z}/2}(k^{stab}[n+2], N) \cong (k^{stab}[n+2])^\vee \otimes_S N$$

Since $k = R/(x_1, \dots, x_n)$ we can take $k^{stab} = \left(\Lambda^{odd} R^n \underset{\delta+\delta_1}{\overset{\delta+\delta_1}{\rightleftarrows}} \Lambda^{ev} R^n\right)$

Note that $(\Lambda^{odd} R^n)^\vee \cong \Lambda^{odd+n} R^n$
$(\Lambda^{ev} R^n)^\vee \cong \Lambda^{ev+n} R^n$ $\Rightarrow$ $(k^{stab}[n])^\vee \cong k^{stab}$
(in $MF(R,w)$)

so

$$H^0\left(\left(\widetilde{k^{stab}_{[n+i]}}\right)^\vee \otimes_S N\right) = H^0\left(\left(\widetilde{k^{stab}_{[n]}}\right)^\vee [-i] \otimes_S N\right)$$
$$= H_i\left(\widetilde{k}^{stab} \otimes_S N\right)$$

but $\widetilde{k}^{stab}$ is a R-free resolution of M = stable syzygy of k

$$\Rightarrow \quad H^0\left(\left(\widetilde{k^{stab}_{[n+i]}}\right)^\vee \otimes_S N\right) = \mathrm{Tor}_i^S(M, N)$$
$$\cong \mathrm{Tor}_i^S(k, N) \text{ for } i > 0$$

But we can also compute $\mathrm{Tor}_i^S(k,N)$ by taking an S-free resolution of N; since one such is 2-periodic we have

$$\mathrm{Tor}_i^S(k,N) \cong H^0\left(MF\left(k^{stab}_{[n+i]}, \mathcal{W}\right)\right)$$
for all $i > 0$

Thus Theorem 4.1 reduces to

Suppose $V(w)$ has an isolated singularity. Then

(*) $N \in MCM^\infty(S)$ is free $\Leftrightarrow$
$\mathrm{Tor}_i(k, N) = 0 \quad \forall\, i > 0$

Indeed; $\quad Y \in \left(k^{stab}\right)^{\perp} \iff$

$H^0(\mathrm{MF}(k^{stab}[n+i], Y)) = 0$ for $i = 1, 2$ (2 periodic)

$\Rightarrow \mathrm{Tor}_i(k, N) = 0$ for all $i > 0$ (2-periodic)

$\overset{(*)}{\Rightarrow}$ N is free $\Rightarrow Y = 0$ in $\underline{\mathrm{MF}}^{\infty}(R, w)$

(*) is the homological Nakayama lemma

Thm

for $Y \in \mathrm{MF}^{\infty}(R, w)$ let $N \in \mathrm{MCM}^{\infty}(S) = \mathrm{coker}(\varphi_1)$

Then M is a free S-module $\iff \mathrm{Tor}_i^S(k, N) = 0$

$\forall i > 0$.

Proof of (*) (Homological Nakayama Lemma)

Fix $X \in MF(R,\omega)$, $M = \operatorname{coker}(X)$ $\quad m = (x_1, \dots, x_n)$

Write $\omega = \sum_{i=1}^{n} \omega_i x_i$ set $\partial_k \omega = \omega_k$, $\Omega_\omega = S/(\partial_1\omega, \dots, \partial_n\omega)$ $\left(= R/(\partial_1\omega, \dots, \partial_n\omega)\right)$

Lemma 9.5 for $N \in \mathrm{Mod}(S)$, $\partial_k\omega$ annihilates $\mathrm{Tor}_i^S(M,N)$, all $i>0$

Pf $\omega = \varphi\psi \Rightarrow \partial_k\omega = (\partial_k\varphi)\psi + \varphi\cdot(\partial_k\psi)$

which says that $\partial_k\omega$ induces 0 on $H_i(\bar X^\bullet \otimes N)$ $i>0$

$$\begin{tikzcd} \arrow[r, "\varphi"] & \bar X^0 \arrow[r, "\bar\psi"] \arrow[d, "\partial_k\omega"'] \arrow[dl, "\partial_k\bar\varphi"] & \bar X^1 \arrow[r, "\bar\varphi"] \arrow[d, "\partial_k\omega"'] \arrow[dl, "\partial_k\bar\psi"] & \bar X^0 \arrow[d, "\partial_k\omega"] \arrow[dl, "\partial_k\bar\varphi"] & \otimes N \\ \arrow[r] & \bar X^0 \arrow[r, "\bar\psi"'] & \bar X^1 \arrow[r, "\bar\varphi"'] & \bar X^0 & \end{tikzcd}$$

Thm 9.6 (Gruson–Jensen) if N is a flat S-module the $\mathrm{pd}_S N \leq n-1$.
ex 7.2

Thm 9.7 (Homological Nakayama Lemma) $M = \operatorname{coker}(X)$, TFAE

(1) M is a free S-module
(2) M is a flat S-module
(3) $\mathrm{Tor}_i^S(M,N) = 0$ $\forall$ fg S-modules N, $i>0$

If $V(\omega)$ has an isolated singularity then (1),(2),(3) are equivalent to

(4) $\mathrm{Tor}_i^S(M,N) = 0$ for N fg Ω_ω modules $i>0$
(5) $\mathrm{Tor}_i^S(M,k) = 0$, $i>0$

pf (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) is clear

In case $V(w)$ has an isolated singularity Support$(\Omega_w) = \mathrm{Spec}(R/w)$

⇒ each f.g. Ω_w module N has a finite filtration $F^\cdot N$ with $gr^F_i N \cong k$ so (5) ⇒ (4)

(2) ⇒ (1) By Gruson–Jensen $pd_S M \leq n+1$ so $\Omega_\ell M$ is projective for $\ell \gg 0$. But $\Omega_\ell M \cong M$ for ℓ even, so M is projective. Since S is local M is free (Raynaud/Kaplansky)

(3) ⇒ (2) is a standard criterion for flatness

(4) ⇒ (3) Show by induction that $Tor_i^S(M,N)=0$ $i>0$ all f.g. $S/(\partial_1 w, \dots, \partial_{v+1} w)$-module N ⇒ $Tor_i^S(M,N)=0$ $i>0$, $\forall$ f.g. $R/(\partial_1 w, \dots, \partial_v w)$-module N. Note that $V(w)$ isolated singular ⇒ $\partial_1 w, \dots, \partial_n w$ is a regular seq in R

Do the case $v=0$.

Compute $H_*(R/\partial_1 w \otimes^L_R M \otimes^L_R N)$ in two ways

Represent $R/\partial_1 w \otimes^L_R M \otimes^L_R N$ as the total complex of

$$\begin{tikzcd}
\downarrow & \downarrow \\
X^0 \otimes_S N \arrow[d] & X^0 \otimes_S N \arrow[l, "\times \partial_1 w"'] \arrow[d] \\
X^1 \otimes_S N \arrow[d] & X^1 \otimes_S N \arrow[l, "\times \partial_1 w"'] \arrow[d] \\
X^0 \otimes_S N & X^0 \otimes_S N \arrow[l, "\times \partial_1 w"']
\end{tikzcd}$$

The spectral seq for the vertical filtration has E^1 page

$$\begin{array}{ccc}
\vdots & & \vdots \\
\mathrm{Tor}_2^S(M,N) & \xleftarrow{\cdot \partial w} & \mathrm{Tor}_2^S(M,N) \\
\mathrm{Tor}_1^S(M,N) & \xleftarrow{\cdot \partial w} & \mathrm{Tor}_1^S(M,N) \\
M \otimes_S N & \xleftarrow{\cdot \partial w} & M \otimes_S N
\end{array}$$

By Lemma 4.5 $\partial w = 0$ on Tor_i for $i > 0$

so $E^2_{p,q} = \mathrm{Tor}_q^S(M,N)$ $\quad p = 0, 1 \quad p+q \geq 2$

& seq degenerates at E^2. Then $\mathrm{Tor}_i^S(M,N) = 0 \ \forall i > 0$

$\iff H_j(\qquad) = 0 \quad j \geq 2.$

Now use the horizontal filtration. The E^1 page is

the E^2 page is then

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\downarrow & \downarrow & & \\
\bar{X}^0 \otimes_S N/\partial w & \bar{X}^0 \otimes_S \mathrm{Tor}_1^R(R/\partial w, N) & \mathrm{Tor}_2^S(M, N/\partial w) & \mathrm{Tor}_2^S(M, \mathrm{Tor}_1^R(R/\partial w, N)) \\
\downarrow & \downarrow & & \\
\bar{X}^1 \otimes_S N/\partial w & \bar{X}^1 \otimes_S \mathrm{Tor}_1^R(R/\partial w, N) & \mathrm{Tor}_1^S(M, N/\partial w) & \mathrm{Tor}_1^S(M, \mathrm{Tor}_1^R(R/\partial w, N)) \\
\downarrow & \downarrow & & \\
\bar{X}^0 \otimes_S N/\partial w & \bar{X}^0 \otimes_S \mathrm{Tor}_1^R(R/\partial w, N) & M \otimes_S N/\partial w & M \otimes_S \mathrm{Tor}_1^R(R/\partial w, N)
\end{array}$$

Now N/a_1w & $\mathrm{Tor}_1^R(R/a_1w, N)$ are fgen S/a_1w modules, so

$H_i(\quad) = 0 \ \forall i \geq 2$ if $\mathrm{Tor}_i^S(M, -)$ vanishes on fgen S/a_1w-modules

so $\mathrm{Tor}_i^S(M, N) = 0 \quad i > 0$, N fgen S/a_1w-module

$\Rightarrow \mathrm{Tor}_i^S(M, N) = 0, \ i > 0 \quad N$ fgen S-module

Repeat for $a_1 v, a_2 v, \ldots a_n w$. $\square$