Applications of compact generation

The main result is to identify $MF^{\infty}(R,w)$ with a category of dg modules over the dg algebra $\mathrm{End}_{MF}(k^{stab})$

§ Homotopy theory of 2-periodic dg category

We have had a review of the model structure on dg-cat_k and a description of the morphism complexes in Hodg. With suitable adjustment this passes to the setting of

2-periodic dg category

Recall a dg category $\mathcal{C}$ is a category enriched in complexes of k-vector spaces $\mathcal{C}(k)$. A 2-periodic dg category is a category enriched in $\frac{\mathbb{Z}}{2}$-graded complexes. Equivalently a dg category over $k[u, u^{-1}]$ with $|u| = 2$, ie

$\mathrm{Hom}(x,y)^{*}$ has an iso $\cdot u : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Y)[2]$

and morphisms are compatible with $\cdot u$.

We have the usual free $\dashv$ forget pair of adjoint functors

$$\mathrm{dgcat}_k \underset{\text{forget}}{\overset{\otimes k[u,u^{-1}]}{\rightleftarrows}} \mathrm{dgcat}_{k[u,u^{-1}]}$$

Model structures via this adjunction the cofibrantly generated model structure on $dgcat_k$ induces one on $dgcat_{k[u,u^{-1}]}$

fib = f s.t. forget(f) is fibration in $dgcat_k$

weq = " weq in "

Moreover the generating (trivial) cofibrations in $dgcat_{k[u,u^{-1}]}$ are the image under $-\otimes k[u,u^{-1}]$ of the generating (trivial) cofibrations in $dgcat_k$

$dgcat_{k[u,u^{-1}]}$ has a closed monoidal structure $-\otimes_{k[u,u^{-1}]}-$, $\underline{Hom}(-,-)$

$\underline{Hom}(T,T')$ = dg cat of $C(k[u,u^{-1}])$ enriched functors
i.e. dg functors with $Hom(-,-)^n$ = dg n transformations natural up to sign wrt $Hom_T(-,-)$

$T\text{-mod} = \underline{Hom}(T, C(k[u,u^{-1}]))$

$x \in T \rightsquigarrow h_x = h_x(y) = T(y,x) \in T^{op}\text{-mod}$

representable $\rightsquigarrow$ $h^x : h^x(y) = T(x,y) \in T\text{-mod}$

corepresentable $\rightsquigarrow$

$h_- : T \to T^{op}\text{-Mod}$, $h^- : T^{op} \to T\text{-mod}$

Standard model structure on $C(k[u,u^{-1}])$ (also same as $C(k)$)

induces (projective) model structure on T-mod fibrations r v.g. defined objectwise

generators (trivial) cof = h^x (generating (trivial) cofibrations)

This gives us the associated homotopy category $Ho(T\text{-Mod})$ with corresponding functor

Def $D(T) = Ho(T\text{-mod}) \in cat$ $\quad [-]: T\text{-Mod} \to Ho(T\text{-Mod})$

$$Int(T\text{-mod}) \subset T\text{-mod}_{fib, cof}$$

We have the functor $H^0: dgcat_k^{???} \to cat$. The main

theorem of model categories says we have a canonical equivalence

$H^0(Int(T\text{-Mod})) \simeq D(T)$ Note that all obj in T^{op}-mod are fibrant and all representables are cofibrant, so we have

$$h_x : T \to Int(T^{op}\text{-mod})$$

Moreover h_x is compact for all $x \in T$

$$T^{op}\text{-mod}$$

Def $\hat{T} = Int(T^{op}\text{-mod})$ so $H^0(\hat{T}) \simeq D(T^{op}) = Ho(T^{op}\text{-Mod})$

$T \subset \hat{T}_{pe}$ = perfect obj in $\hat{T}$: $M \in Int(T^{op}\text{-Mod})$ is perfect if $[M]$ is compact in $H^0(Int(T^{op}\text{-Mod})) = D(T^{op})$

(pre)-triangulated hull of T

Def T is pretriang iff $T \to \hat{T}_{pe}$ is a quasi-equiv

Note $\underset{\simeq\; D(T^{op})}{H^0(\hat{T})}$ is a triang cat with concrete instead from $C(k[h,h^{-1}])$ (9)

$$\underset{\underset{D(T^{op})^c}{\shortparallel}}{H^0(\hat{T})^c} = H^0(\hat{T}_{pe}) \supset H^0(T)$$ is set of compact generators since $H^0(T)^{\perp} = \{0\}$

Thm (Neeman) Let $\mathcal{T}$ be a triangulated category with a set S compact generators. Then the smallest triangulated subcategory of $\mathcal{T}$ containing S and closed under summands is $\mathcal{T}^c$, the full subcategory of compact objects in $\mathcal{T}$.

Consequence $H^0(\hat{T}_{pe})$ is the smallest triangulated subcategory of $\underset{H^0(\hat{T})\,\shortparallel}{D(T^{op})}$ containing $H^0(T)$ and closed under summands. If $H^0(T)$ is already a triangulated subcategory, then $H^0(\hat{T}_{pe})$ is the idempotent completion of $H^0(T)$.

5

Dg derived category

Let W be a set of morphisms in a dg cat T. By Toën $\exists$ a functor

$$l : T \to L_W(T) \quad \text{in } Ho(dgcat_{k[\mathcal{U},\mathcal{U}']})$$

with the universal property: for $T' \in dgcat_{k[\mathcal{U},\mathcal{U}']}$

$$l^* : [L_W(T), T']_{Hodg^{\mathbb{Z}/2}} \to [T, T']_{Hodg^{\mathbb{Z}/2}}$$

is injective, with image the $f : T \to T'$ such that $H^0(f)(W) \subset Iso\, H^0(T')$ (Actually this is a property of $L_W(T)$ but not the universal property that Toën proves)

construction: adjoin K_0 to each $f \in W$

<u>Def</u> Let $W = \{$w equiv in T-Mod$\}$ The dg-derived cat of T is $L_W(T\text{-Mod})$

<u>Note</u>. $H^0(L_W(T\text{-Mod})) \cong D(T) = Ho(T\text{-Mod}) = H^0(Int(T\text{-Mod}))$

<u>Idea</u> for $M \in T\text{-Mod}$ $QM \to M$ is w equiv $\Rightarrow$ $Int(T\text{-Mod})$ maps to iso in $H^0(L_W(T\text{-Mod}))$

$H^0(Int(T\text{-Mod})) \cong H^0(L_W(T\text{-Mod})) \Leftarrow$

- $T \to L_W(T)$ is an iso in $Hodg^{\mathbb{Z}/2}_{?}$: for each $f : T \to T'$ $\{f(w)\} \subset Iso\, T'$ $\Rightarrow$ $[L_W(T), T'] \xrightarrow[l^*]{\sim} [T, T']$

Here is dg version of Neeman's theorem

Recall T is pretriangulated if $T \xrightarrow{h} \hat{T}_{pe}$ is a q-equiv onto its image in Hodg ($\Rightarrow H^0(T) \simeq H^0(\hat{T}_{pe})$)

Thm 5.1 Suppose $T \in dgcat_k^{\mathbb{Z}/2}$ is pretriangulated with (small) coproducts, $S \subset T$ a full subcat of objects that are compact in $Ho(T)$. Suppose the smallest localizing subcat of $Ho(T)$ containing $Ho(S)$ is $Ho(T)$. Then

$$f : T \to L_W(S^{op}\text{-mod}) \quad x \mapsto \ell(\underline{h}_x|S)$$

is an isomorphism in $Ho(dgcat_k^{\mathbb{Z}/2})$

Moreover, f induces an isomorphism $T_{pe} \xrightarrow[\text{(in Hodg)}]{\sim} L_{W_S}(S^{op}\text{-mod})_{pe}$

Here $T_{pe} = \{x \in T \mid [x] \text{ is compact in } H^0(T)\}$

Pf follows Keller. The 2nd assertion is a direct consequence, since $f : T \xrightarrow{\sim} T'$, iso in Hodg, $H^0(f) : H^0(T) \to H^0(T')$ is an equivalence hence induces a equivalence on subcat of compact objects

Notation Let A be $\mathbb{Z}/2$-graded dg alg/k $\leadsto$ $A\text{-Mod} \in dgcat_k^{\mathbb{Z}/2}$

Write $\hat{A}$ for $\widehat{A\text{-mod}}$, so $H^0(\hat{A}) = D(A^{op})$

Application to matrix factorization categories

<u>Theorem 5.2</u> (R, w) with isolated singularities and let $A = \mathrm{End}_{\mathrm{MF}}(k^{stab})$

Then we have iso's in $\mathrm{Ho}(\mathrm{dgcat}_k^{\mathbb{Z}/2})$

(1) $\mathrm{MF}^{\infty}(R,w) \cong \widehat{\mathrm{MF}(R,w)}$ $(= \mathrm{Int}(\mathrm{MF}(R,w)^{op}\text{-mod}))$

(2) $\mathrm{MF}^{\infty}(R,w) \cong \widehat{A}$ $(= \mathrm{Int}(A^{op}\text{-mod}))$

(3) $\widehat{\mathrm{MF}(R,w)}_{pe} \cong \tilde{A}_{pe}$

pf By Thm 4.1, k^{stab} is compact generator for $[\mathrm{MF}^{\infty}(R,w)]$

(2) Take $S = \{k^{stab}\}$ so $S^{op}\text{-Mod} = A^{op}\text{-Mod}$.

We have

$$\mathrm{MF}^{\infty}(R,w) \xrightarrow{\alpha} L_W(A^{op}\text{-Mod})$$

$x \mapsto \mathrm{MF}(k^{stab}, x) \in A^{op}\text{-Mod}$

$$\uparrow l$$

$$\widehat{A} = \mathrm{Int}\,A^{op}\text{-Mod} \overset{i}{\hookrightarrow} A^{op}\text{-Mod}$$

α is an iso in $\mathrm{Ho}\,\mathrm{dg}_k^{\mathbb{Z}/2}$ by Thm 5.1

l is an iso in $\mathrm{Ho}\,\mathrm{dg}_k^{\mathbb{Z}/2}$ by "Note" i is a quasi-eq modulo homo iso in $\mathrm{Ho}\,\mathrm{dg}_k^{\mathbb{Z}/2}$

(1) Take $S = MF(R,w) \ni k^{stab}$, We have a similar diagram ⑧

$$MF^{\infty}(R,w) \xrightarrow{\alpha} L_w(MF(R,w)^{op}\text{-Mod})$$

$$\uparrow \iota$$

$$\widehat{MF(R,w)} = \mathrm{Int}\,MF(R,w)^{op}\text{-Mod} \underset{i}{\hookrightarrow} MF(R,w)^{op}\text{-Mod}$$

(3) follows from (1) & (2) & 2nd part of Thm 5.1 □

<u>Cor 5.3</u> $H^0(\widehat{MF(R,w)}_{pe}) \subset Ho(MF^{\infty}(R,w))$ is the smallest triangulated subcategory containing k^{stab} and closed under summands

Pf k^{stab} is a compact generator of $Ho(MF^{\infty}(R,w))$, and

$$H^0(\widehat{MF(R,w)}_{pe}) = Ho(\widehat{MF(R,w)})^c \quad Ho(\widehat{MF(R,w)})$$

Then apply Neeman's theorem □.

<u>Cor 5.4</u> Take $X, Y \in MF^{\infty}(R,w)$ and $f \in Z^0 \mathrm{Hom}_{MF}(X,Y)$

Then f induces an iso in $Ho(MF^{\infty}(R,w)) \iff$ $\mathrm{id}_k \otimes f : k \otimes_R X \to k \otimes_R Y$ is a quasi isomorphism (of $\mathbb{Z}/2$-graded complexes)

Pf By Theorem 5.2 f is an iso in $Ho(MF^{st}(R,w))$ ⑨

$\iff f_* = MF(k^{stab}, f): MF(k^{stab}, x) \to MF(k^{stab}, y)$ is a quasi iso. By Prop 4.9 $(n = \dim R)$

$$H_{n+i}(MF(k^{stab}, x)) = H_i(k \otimes_S \tilde{x}) = \begin{cases} \ker(\bar\varphi)/\operatorname{im}(\bar\psi) & i \text{ odd} \\ \ker(\bar\psi)/\operatorname{im}(\bar\varphi) & i \text{ even} \end{cases}$$

& same for y. □

Explicit expression for $\operatorname{End}(k^{stab})$

for $(R, m = (x_1 \ldots x_n))$ we recall $k^{stab} = \Lambda^{odd}(\bigoplus_1^n Re_i) \underset{s_0+s_1}{\overset{s_0+s_1}{\rightleftarrows}} \Lambda^{ev}(\bigoplus_1^n Re_i)$

The dg-algebra $\operatorname{End}(k^{stab})$ has the following explicit presentation (assume char $k \neq 2$)

(*) $\Lambda^*(\bigoplus Re_i) = R\langle \theta_1 \ldots \theta_n \rangle$ with $\deg \theta_i = 1$

Let $\partial_i = \dfrac{\partial}{\partial \theta_i}$ acting on $R\langle \theta_1 \ldots \theta_n \rangle$ via super Leibnitz rule

$\partial_i(ab) = \partial_i a \cdot b + (-1)^{|a|} a\, \partial_i b$, i.e. $\deg \partial_i = 1$

Let $R\langle \theta_1 \ldots \theta_n, \partial_1 \ldots \partial_n \rangle$ be the corresp. $\mathbb{Z}_2$ (super) Weyl algebra of polynomial differential operators on $R\langle \theta_1 \ldots \theta_n \rangle$

so $[\partial_i, \theta_j]^{gr} = \begin{cases} 0 & i \neq j \\ \mathrm{id} & i = j \end{cases}$

Then via (*) we have $R\langle \theta_1 \ldots \theta_n, \partial_1 \ldots \partial_n \rangle \xrightarrow{\sim} \operatorname{End}(k^{stab})$ (dg alg iso)

with differential $[\delta, -]$, $\delta = \sum x_i \partial_i + w_i \theta_i$

§ Knörrer periodicity

We use non-local extension of the above results, assume $V(w)$ has only isolated singularities (or pass always to the localization at $V(w)_{sing}$)

Theorem (Knörrer periodicity, local version)

Take (R, w) with $V(w)_{sing} = \operatorname{Spec}(k)$. Then

$$MF^{\infty}(R, w) \quad \text{and} \quad MF^{\infty}(R[x,y], w + xy)$$

are weakly equivalent.

Pf. Consider $MF^{\infty}(k[x,y], xy)$, $k(0)^{stab}$ is a compact generator of $Ho(\ )$.

Let

$$X := k[x,y] \underset{y}{\overset{x}{\rightleftarrows}} k[x,y]$$

Then $k(0)^{stab} \cong X \oplus X[1]$ in $MF(k[x,y], xy)$

$$0 \to k[x,y]e_1 e_2 \xrightarrow{s_0} \begin{matrix} k[x,y]e_1 \\ \oplus \\ k[x,y]e_2 \end{matrix} \xrightarrow{s_0} k[x,y] \to k(0)$$

$s_1 \quad \wedge\left(\frac{1}{2} y e_1 + \frac{1}{2} x e_2\right) \cdot e_2 \qquad (y e_1) \wedge \quad \text{or} \quad (x e_2) \wedge$

k^{stab}

$$\begin{matrix} k[x,y]e_1 \\ \oplus \\ k[x,y]e_2 \end{matrix} \underset{\psi = s_0 + s_1}{\overset{\varphi = s_0 + s_1}{\rightleftarrows}} \begin{matrix} k[x,y] \\ \oplus \\ k[x,y]e_1 e_2 \end{matrix} \qquad \varphi = \begin{pmatrix} x & y \\ +\frac{1}{2}x & -\frac{1}{2}y \end{pmatrix} \qquad \psi = \begin{pmatrix} \frac{1}{2}y & +y \\ \frac{1}{2}x & -x \end{pmatrix}$$

$$\begin{matrix} k[x,y]e_1 \\ \oplus \\ k[x,y]e_2 \end{matrix} \underset{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}}{\overset{\begin{pmatrix} x & 0 \\ 0 & -y \end{pmatrix}}{\rightleftarrows}} \begin{matrix} k[x,y] \\ \oplus \\ k[x,y]e_1 e_2 \end{matrix} \qquad \uparrow \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

so X is also a cpt generator of $MF^{\infty}(k[x,y], xy)$ 11

let $B = \mathrm{End}(X)$

$$B^0 = \begin{array}{cc} 1 & 0 \end{array}$$
$$B^0 = \begin{pmatrix} k[x,y] & 0 \\ 0 & k[x,y] \end{pmatrix} \qquad B^1 = \begin{pmatrix} 0 & k[x,y] \\ k[x,y] & 0 \end{pmatrix}$$

$$d : B^0 \to B^1 \qquad B^1 \xrightarrow{d} B$$

$$d\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} 0 & yg - yf \\ xf - gx & 0 \end{pmatrix} = 0 \Rightarrow f = g$$

$$d\begin{pmatrix} 0 & g \\ f & 0 \end{pmatrix} = \begin{pmatrix} fx + yg & 0 \\ 0 & xg + yf \end{pmatrix} = 0 \Rightarrow f = hy,\ g = -xh$$
and $(y^2 - x^2)h = 0 \Rightarrow h = 0$

so
$$\begin{tikzcd} k \arrow[r, "\lambda \mapsto \begin{pmatrix}\lambda & 0 \\ 0 & \lambda\end{pmatrix}"] \arrow[d] & B^0 \arrow[d, "d"] \\ 0 \arrow[r] & B^1 \end{tikzcd}$$
is a iso of dg algebras

$$\Rightarrow MF^{\infty}(k[x,y], xy) \underset{Mo}{\simeq} C^{\mathbb{Z}/2}(k)$$

Now $k_w^{stab\,\mathbb{Z}/2} \otimes_k k(c)^{stab}_{xy} \simeq k^{stab}_{w+xy}$ (look at Koszul complexes)

$$k_w^{stab\,\mathbb{Z}/2} \otimes (X \oplus X(1)) \Rightarrow k_w^{stab\,\mathbb{Z}/2} \otimes_k X \text{ is a cpt generator of } MF^{\infty}(R, w+xy)$$

Then $\mathrm{End}(k_w^{stab} \otimes X) = \mathrm{End}(k_w^{stab}) \otimes_k \mathrm{End}(X)$

$\underset{qis}{\simeq} \mathrm{End}(k_w^{stab}) \otimes_k k = \mathrm{End}(k_w^{stab})$

so

$$MF^\infty(R[[x,y]], w+xy) \simeq \mathrm{End}(\widehat{k^{stab}_{w+xy}}) \simeq \mathrm{End}(\widehat{k^{stab}_w}) \simeq MF^\infty(R,w)$$

and

$$\widehat{MF(R[[x,y]], w+xy)}_{pe} \simeq \widehat{MF(R,w)}_{pe}$$

Formal completion

We take (R,m) local with m-adic completion $(\hat{R}, \hat{m})$

Theorem 5.7 $\quad \widehat{MF(R,w)}_{pe} \underset{\uparrow}{\simeq} MF(\hat{R}, w)$

in Hodg

and the idempotent completion of $H^0(MF(R,w))$ in $H^0(\widehat{MF(R,w)})$ is equivalent to $MF(\hat{R}, w)$

proof relies on

<u>Lemma 5.6</u> The Yoneda embedding $MF(\hat{R}, w) \to \widehat{MF(\hat{R}, w)}_{pe}$ is an iso in Hodg, so $MF(\hat{R}, w)$ is triangulated and $H^0(MF(\hat{R}, w))$ is idempotent complete

Pf $H^0(MF(\hat{R}, w))$ under Yoneda is a set of cpt generators for $H^0(\widehat{MF(\hat{R}, w)})$ so by Neeman's theorem $H^0(\widehat{MF(\hat{R}, w)}_{pe})$ is the smallest triangulated subcategory of $H^0(\widehat{MF(\hat{R}, w)})$ containing $H^0(MF(\hat{R}, w))$ and closed under summands

So: suffices to show $H^0(MF(\hat{R}, w))$ is closed under summands.

We know that $\underline{MF}(\hat{R}, w)$ is equivalent to $\underline{MCM}(\hat{R}/w)$

By a result of Swan, if $M \in Mod(\hat{R})$ is indecomposable then $End(M)$ is local: x, y nonunits $\Rightarrow x+y$ is a nonunit

Take $e \in End(M)$ s.t. the image $\bar{e} \in \underline{End}(M)$ is a nontrivial idempotent. Then $\bar{e}$ and $1-\bar{e}$ are non units, hence e & $1-e$ are non unit in $End(M)$

and $e * (1-e) = 1$, a contradiction. Thus $\underline{\mathrm{End}}(M)$ has no (14)
non-trivial idempotents $\Rightarrow$ $\underline{MCM}(\hat{R}/w)$ is idempotent complete

proof of thm 5.7 Let $A_{(R,w)} = \mathrm{End}(k^{stab}_{(R,w)})$, $A_{(\hat{R},w)} =$
By 5.6 and Thm 5.2 we have $\mathrm{End}(k^{stab}_{(\hat{R},w)})$

$$MF(\hat{R},w) \cong \widehat{MF(R,w)}_{pe}$$
$$\cong \left(\hat{A}_{(\hat{R},w)}\right)_{pe}$$

By Thm 5.2 we have

$$\widehat{MF(R,w)} \cong \left(\hat{A}_{(R,w)}\right)_{pe}$$

However,
$$K_{\infty}(R,b) \otimes_R \hat{R} = K_{\infty}(\hat{R},b)$$
We have the explicit description of $A_{(R,w)}$ as
$$R\langle \theta_1 \ldots \theta_n, \frac{\partial}{\partial\theta_1} \ldots \frac{\partial}{\partial\theta_n}\rangle \text{ with different } [\delta,-]^{gr}$$
$$\delta = \sum x_i \frac{\partial}{\partial\theta_i} + w_i\theta_i \quad \text{where } w = \sum w_i x_i$$
and similar for $A_{(\hat{R},w)}$ We claim the inclusion

$A_{(R,w)} \hookrightarrow A^{a}_{(R,w)}$ is a qiso of dg algebras

Filter $= R(\theta_1 \dots, \frac{\partial}{\partial\theta_1} \dots)$ by total deg in $\frac{\partial}{\partial\theta_i}$

then $\mathrm{gr}^F_{A_{(R,w)}} A_{(R,w)}$ is $R(\theta_1 \dots \theta_n) \otimes_R R(\frac{\partial}{\partial\theta_1} \dots \frac{\partial}{\partial\theta_n})$

with 0 differential in $\frac{\partial}{\partial\theta_i}$ and the Koszul differential in the θ_i

the E_1 term is then $k\langle\frac{\partial}{\partial\theta_1}, \dots \frac{\partial}{\partial\theta_n}\rangle$ with differentials $[\delta, -]$

the same holds for $A_{(\hat{R},w)}$ & $A_{(R,w)} \to A_{(\hat{R},w)}$

is compatible with the filtration $\square$

Quadratic hypersurfaces

Take $w = \sum_1^n a_i x_i^2 \qquad \prod a_i \neq 0$

$\qquad = \sum x_i w_i \qquad w_i = a_i x_i$

Then $A = k(\theta_1 \dots \theta_n, \frac{\partial}{\partial\theta_1} \dots \frac{\partial}{\partial\theta_n})$ with

$d\theta_i = x_i \quad d(\frac{\partial}{\partial\theta_i}) = a_i x_i \qquad$ so $\tilde{\partial}_i = \frac{\partial}{\partial\theta_i} - a_i\theta_i$

are cycles and generate $H^*(A)$ as k-algebra

We have $\bar\partial_2^2 = -a_i$ $\quad \bar\partial_i \bar\partial_j = -\bar\partial_j \bar\partial_i$ for $i \neq j$

so $H^\partial(A) = Cl(w)$

and $(Cl(w), 0) \hookrightarrow (A^{op}, d)$ is a quasi A_∞-functor

Then

$$\underset{\sim}{\mathrm{MF}}(\hat{R}, w) \cong H^0\left(\mathrm{Mod}^{\mathbb{Z}/2}_{Cl(w)}\right)$$

$$\bar\partial_i \cdot \bar\partial_i = (\partial_i - a_i\theta_i)(\partial_i - a_i\theta_i)$$
$$= -a_i\theta_i\partial_i - a_i\partial_i\theta_i + a_i^2\theta_i^2 + \partial_i^2$$

Thom Sebastiani

$$A_{(R \hat{\otimes} (x,y)),\, w(x) + w'(b))} = A_w \otimes^{\mathbb{Z}/2}_{k} A_{w'}$$