PhD Seminar SS-2024
Periods and Nori Motives

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Introduction

In the long story of transcendence proofs that started in the late twentieth century with the transcendence of e and π, a certain class of complex numbers has been isolated, which seems to be reasonably treatable for the purposes of number theory and large enough to include almost all numbers that one could possibly care about. These numbers, called *periods*, have a strong connection with geometry, to the point that a deep theory has been developed around them in the hope that geometric insights can shed some light to otherwise unreachable arithmetic statements concerning these numbers.

One possibility to define periods is to consider integrals of algebraic differential forms over some algebraic integration domain. Ideally, one should be able to exploit the geometry of certain varieties and basic manipulations of integral to recover all algebraic relations between periods.

A slightly more refined way to define periods goes through Hodge theory. Given a smooth variety X over a subfield k of $\mathbb{C}$, there is a comparison isomorphism between de Rham and singular cohomology of the form

$$\mathrm{comp}: H^*_{\mathrm{dR}}(X,k) \otimes_k \mathbb{C} \to H^*_{\mathrm{sing}}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C},$$

which, if X is affine, is induced by the perfect pairing

$$H^*_{\mathrm{dR}}(X,k) \otimes_{\mathbb{Q}} H_*^{\mathrm{sing}}(X,\mathbb{Q}) \to \mathbb{C}, \quad (\omega,\gamma) \mapsto \int_\gamma \omega.$$

Curiously enough, this isomorphism is only defined in a functorial way when extending coefficients to $\mathbb{C}$. This means that, fixing a k-basis and a $\mathbb{Q}$-basis of the de Rham and singular cohomology respectively, the matrix representing

this isomorphism has entries which are genuine complex numbers, a priori not algebraic. These numbers are also called *periods*. Despite the similarity of these two definitions in the affine case, it is not at all clear why these two notions of periods should coincide.
Continuing with this second definition, the next step is to upgrade this setting to Hodge structures. A pure Hodge structure is a triple $(H_{\mathrm{B}}, H_{\mathrm{dR}}, \mathrm{comp})$ of a $\mathbb{Q}$-vector space H_{B}, a k-vector space H_{dR} with an exhaustive filtration and an isomorphism comp between the complexifications of these vector spaces in such a way that the induced filtration on $H_B \otimes_{\mathbb{Q}} \mathbb{C}$ satisfies a certain decomposition property. Although this definition may seem rather arbitrary, the cohomology groups of every smooth projective k-variety carry a canonical pure Hodge structure, and by considering certain iterated extensions of these Hodge structures one can extend this theory to all varieties over k. Hodge structures have been intensively studied, and one could hope that results about Hodge structures, which are purely geometric in nature, might lead to arithmetic statements about periods.
There is a third and more sophisticated definition for periods, for which we first need to talk about motives. A category of motives, whose existence at the moment is purely conjectural, should be a category which enjoys the following properties:

- it is abelian, otherwise we do not like it;
- it is a universal cohomology theory, in the sense that every cohomology theory over the category of varieties over k which satisfies certain properties should always factor through this category of motives. In some sense, motives should capture the very essence of cohomology.
- it is Tannakian, which means that it is isomorphic to the category of finite dimensional representations of some group scheme. In particular, there should be a group scheme, usually called *motivic fundamental group*, which acts on the motive of every variety.
- Given a cohomology theory from the category of k-varieties to, say, abelian groups, the induced functor from the category of motives to the one of abelian groups should be a fiber functor. Roughly speaking, this means that taking one of the classical cohomologies instead of motivic cohomology should be the same as forgetting the action of the motivic fundamental group. Basically, instead of considering a representation of a group scheme, we are only able to see the underlying abelian group.

Although a proper category of motives has not been constructed yet, there are several candidates, one of which has been proposed by Nori and will be one of the main themes of the seminar. The category of Nori motives satisfies many of the properties listed above, except that it has a less general universal property. To put it short, in order for a cohomology theory to factor through Nori motives, it needs to be in some sense comparable to singular cohomology. Needless to say, this property will be more than enough for our purposes.

Once we have the gadget of Nori motives in our hands, singular and de Rham cohomology will induce two fiber functors on the category of Nori motives. Isomorphisms between two fiber functors give a torsor $X = \mathrm{Spec}\, A$ under the action of the motivic fundamental group, and we will be able to revisit the comparison isomorphism introduced above as a complex point of this torsor. Thus, periods will appear naturally as the numbers obtained by evaluating algebraic functions of X at this complex point. Once again, it is not so clear why this definition should coincide with the ones given before.

And after all this, what have we gained? Well, we have a big algebra A and an evaluation map $A \to \mathbb{C}$ whose image is precisely the algebra of periods. The coaction of the motivic fundamental group on this algebra should translate into a better understanding of relations among periods. Moreover, this map is conjectured to be injective, in which case all relations among periods would be induced by purely formal operations on integrals. This statement usually goes by the name of *period conjecture* and will be the content of the last part of the seminar.

Although the period conjecture in its general form is out of reach at the moment, a version thereof has been established in the case of curves. For this, we can restrict our attention to a smaller subcategory of Nori motives, the so called 1-*motives*, which is generated by motives of curves. A similar theory to the one of Nori motives can be developed for this smaller subcategory, and one can state an analogous form of period conjecture. This version of the conjecture has been proved by means of the analytic subgroup theorem, which pertains solely to the realm of complex Lie groups. In the last part of the seminar, we will therefore introduce 1-motives and deduce most of their properties from the ones of Nori motives, then we will sketch the proof of the analytic subgroup theorem and finally enjoy some concrete applications of the period conjecture.

There is still one talk left in the seminar, in which I could annoy everybody by talking about motivic multiple zeta values, or in which we could compare the category of Nori motives to other categories of motives, so as to put it in perspective. According to the number of speakers, we could otherwise cancel this very last talk.

Very briefly, the structure of the seminar is the following:

- Talks 0 - 2: introduction to some background material, like de Rham cohomology and Hodge structures;
- Talks 3 - 5: abstract constructions for the diagram category, which are essential to Nori motives;
- Talks 6 - 8: Nori motives and their properties;
- Talks 9 - 12: period conjecture for 1-motives.

The main reference we will follow is [2], but for the period conjecture we will use [3]. In the next section the program of each talk is briefly discussed, then some more details about the mathematics appearing in the talks are given in the subsequent section.

List of the talks

Talk 0: Introduction. (10/04)
I will go through the pages above and try to make them sound appealing.

Talk 1: De Rham cohomology. (17/04)
Introduce algebraic and holomorphic de Rham cohomology, following chapters 3 and 4 of [2]. Start with algebraic de Rham cohomology, see [2, Section 3.1], skipping Subsection 3.1.5. Of [2, Subsection 3.1.6] we only need to recall that the de Rham cohomology groups are finite dimensional vector spaces. We also need to have an idea of de Rham cohomology for singular varieties, so cover one among Section 3.2, Subsection 3.3.1 and Subsection 3.3.2.
Then turn to holomorphic de Rham cohomology, see [2, Subsection 4.1.1]. Explain Proposition 4.1.7 without proof, then give an idea of the general case for complex analytic spaces as in [2, Section 4.2].

Talk 2: Mixed Hodge structures. (24/04)
Introduce the category of $(k, \mathbb{Q})$-vector spaces as in [2, Section 5.1] and explain the comparison isomorphism as in [2, Section 5.3], also quickly mentioning the case of singular varieties in [2, Section 5.4].
Then give an introduction to pure and mixed Hodge structures, following [1, Section 2.6]. It is important that we get some familiarity with the topic and that the main results are mentioned, but we will not rely on them later.

Talk 3: Nori's diagram category. (8/05)
Give the definitions of diagrams and representations as in [2, Subsection 7.1.1], then construct the diagram category as in [2, Subsection 7.1.2]. If time allows, you could mention the universal property of the diagram category explained in [2, Subsection 7.1.3], but in any case it will be the main topic of the following talk.
In order for us to get familiar with the diagram category, cover the material in [2, Section 7.2]. The main results are Proposition 7.2.3 and 7.2.5, which explain some basic properties of the diagram category. The last lemmas in this sections will be often used in the rest of the seminar, but are very elementary and not so deep.
Finally, describe the diagram category as a category of comodules over a coalgebra, following [2, Section 7.5].

Talk 4: Universal property of the diagram category. (15/05)
The goal of this talk is to prove the universal property of the diagram category, which is covered in [2, Sections 7.3 and 7.4]. The first step is to prove Theorem 7.1.20 as in Section 7.3, then the universal property follows rather formally in Section 7.4.
Subsection 7.3.1 is rather technical and will not be needed elsewhere, so feel free to state the main results without spending too much time on the proofs.
Subsection 7.3.2 is the very heart of the proof, and the main ingredient is Lemma 7.3.16.

Talk 5: Nori's rigidity criterion. (22/05)
The aim is now to give a criterion to establish whether a given diagram category is rigid, so equivalent to the category of representations of some group scheme.
First we need to understand how to define a tensor structure on the diagram category, which can be induced from a product structure on the diagram itself. Go through the definitions of product of diagrams, commutative product structure and graded commutative representation as in [2, Section 8.1] and discuss the tensor product on the diagram category in Proposition 8.1.5.
Next, we realize the diagram category as a category of representations of a monoid scheme, which requires Proposition 8.1.15 and the subsequent corollary. Finally, we can turn to Nori's rigidity criterion, which is discussed in [2, Section 8.3]. Introduce the notions of strong duals and perfect duality at the beginning of this section and prove Proposition 8.3.4.

Talk 6: Nori motives. (29/05)
Before introducing Nori motives, discuss localization of diagrams as in [2, Section 8.2], which is the diagram version of the localization of a tensor category with respect to one object. The properties that we need are summarized in Proposition 8.2.5.
Then pass to [2, Section 9.1] and explain the definition of Nori motives. Also introduce the main results that will be proved in the next talk, namely Theorem 9.1.5 about rigidity and Theorem 9.1.10 for the universal property. It would be nice to have a look at Example 9.1.11, 9.1.12 and 9.1.13, if time allows.
We also start preparing the proof of the theorems of the next talk. Discuss the content of [2, Subsection 9.2.1], and when it appears necessary state the Basic Lemma (Theorem 2.5.2). For the proof of the Basic Lemma, the fastest one is in Section 2.5.2, but also other proofs are presented right after.

Talk 7: Rigidity of Nori motives. (5/06)
The goal of this talk is to prove Theorem 9.1.5 and with it rigidity of Nori motives. This is the only talk in which some basic intersection theory appears.
Give an overview of [2, Subsection 9.2.1], whose results are only needed in the proof of Proposition 9.2.18. The next fundamental step is Theorem 9.2.4, which allows us to replace the diagram of pairs with the one of good pairs. The proof should follow rather flawlessly after Proposition 9.2.18.
Finally, we aim at Theorem 9.3.10, which is equivalent to Theorem 9.1.5. In order to prove rigidity, we need to check the assumptions of Nori's rigidity criterion. First explain the tensor structure on Nori motives, which is straightforward after Theorem 9.2.4. For the remaining assumptions we need Lemma 9.3.8 and 9.3.9, which require some intersection theory.

Talk 8: Equivalence of the definitions of periods. (12/06)
In this talk we go through possible definitions of periods and prove their equivalence using the machinery of Nori motives.
First introduce NC-periods as in [2, Section 11.1], which are integrals of algebraic differential forms over some algebraic integration domain. Then explain the definition of cohomological periods as in [2, Subsection 11.3.1], which are the coefficients of the comparison isomorphism of any variety with respect to rational bases. The next goal is to prove that these definitions are equivalent, which is the content of Theorem 11.4.2.
An ad-hoc proof is given in [2, Section 11.4], but we can embed this result in a deeper theory. Thus, go back to [2, Section 8.4], in which we pick

up again the theory of diagram categories with the aim of comparing two isomorphic representations of the same diagram. Under certain assumptions, the isomorphisms between two representations make up a torsor under the action of the group scheme which comes with a rigid diagram category. This is explained in Proposition 8.4.10, while [2, Subsection 8.4.3] gives an explicit description of this torsor.
Finally, we apply this machinery to Nori motives with the representations induced by singular and de Rham cohomology. Go through [2, Section 13.1] and prove the equivalence of the definition of periods via Corollary 13.1.10. If time permits, you could also state the period conjecture as in [2, Subsection 13.2.1].

Talk 9: 1-Motives. (19/06)
In this talk we start to have a look at the period conjecture for curves and introduce 1-motives. The goal is to explain the definition of 1-motives and their basic properties, which is the content of [3, Chapter 8]. The definition requires some notions from [3, Chapter 4], which can be treated only to a minimal extent in order to understand the definition of 1-motives. For this talk, some familiarity with algebraic groups might be useful.
The key link between 1-motives and Nori motives is explained in [3, Appendix A], in particular Theorem A.7. All the properties of Nori motives quoted here follow immediately from the results of the previous talks. I will try to cut down as much as possible on 1-motives and see to what extent we can deduce properties of 1-motives from the ones that we already know of Nori motives.

Talk 10: The analytic subgroup theorem. (26/06)
The main tool in the proof of the period conjecture for curves is the analytic subgroup theorem, which is Theorem 6.2 in [3, Chapter 6]. Go through the necessary prerequisites in [3, Chapter 5] regarding Lie algebras, Lie groups and the exponential map. Some preliminary familiarity with Lie groups and Lie algebras might help you in dealing much faster with the topic.
Next, explain the singular and de Rham realizations of 1-motives as in [3, Section 8.1]. The singualr realization is rather straightforward, while for the de Rham one you have to lift universal vector extensions to the motivic setting. After this present theorem 6.1, the so-called analytic subgroup theorem, and the proof of Theorem 6.2.

Talk 11: The period conjecture for 1-motives. (3/07)
In this talk we state and prove the period conjecture for curves. Explain periods of 1-motives as in [3, Chapter 9], especially section 9.1 and 9.2.

This should actually be oa special case of the more general definitions of periods that we have already given. The main result that we need is Theorem 9.10, which translates the analytic subgroup theorem into the language of 1-motives. For this, first go through Proposition 8.21, which is the last missing piece, then present the proof of Thorem 9.10.
Finally, go to [3, Chapter 13] and prove Theorem 13.3. Notice that thanks to the machinery of Nori motives this result becomes an immediate corollary of Theorem 9.10.

Talk 12: t.b.a. (10/07)

Proposal session for the next term. (17/07)

Some more details about each talk

In this section, we let R be a noetherian commutative ring and denote by $R-\mathrm{Mod}$ the category of finitely generated R-modules.

Talk 1: Algebraic and holomorphic de Rham cohomology.
The goal of this talk is to recall the main aspects of de Rham cohomology, with the idea of covering chapters 3 and 4 of [2]. The amount of material is very large, but these concepts should be more or less familiar to everybody. We can skip almost all details and proofs and focus on the broad picture instead.
More in detail, we start with algebraic de Rham cohomology for smooth varieties, see [2, Section 3.1]. We should refresh our memory about the definition of the complex of differential forms, of de Rham cohomology, functoriality, Künneth formula and so on. We will not need the comparison with the étale topology, which is in [2, Subsection 3.1.5], while of [2, Subsection 3.1.6] we only need to recall that the de Rham cohomology groups are finite dimensional vector spaces.
Next, we need to extend de Rham cohomology to possibly singular varieties and the book [2] proposes several equivalent definitions. I guess that the fastest option is to follow [2, Subsection 3.2] and introduce the h-topology. We have been working with the étale topology during the previous semester and there are no new entries to the group of PhD students, so I suppose nobody is afraid of Grothendieck topologies. Otherwise feel free to explore [2, Subsection 3.3] to find out your favorite definition of algebraic de Rham

cohomology for singular varieties.
Let us now turn to holomorphic de Rham cohomology. Of [2, Subsection 4.1.1] the Poincaré Lemma explained in Proposition 4.1.3 has a great relevance for us, since it will be a piece of the comparison isomorphism. Then [2, Subsection 4.1.2] is not needed, and of [2, Subsection 4.1.3] we only need Proposition 4.1.7, which can be taken as a black-box.
Finally, we need to mention that these constructions carry through to singular complex analytic spaces, and for this we need [2, Subsection 4.2.1]. All properties that we need are summarized in Lemma 4.2.6 and Proposition 4.2.10, which are the analogs of the smooth counterpart.
Also [1] deals with these topics in detail, see Section 2.2, so you could have a look at these notes as well. The main problem is that they do not treat the case of singular varieties, which is necessary for the rest of the seminar.

Talk 2: Mixed Hodge structures.
The goal of this talk is to compare singular and de Rham cohomology and endow them with a refined algebraic structure. We start by introducing the category of $(k, \mathbb{Q})$-vector spaces as in [2, Section 5.1] and the comparison isomorphism as in [2, Section 5.3]. The latter should be rather straightforward to introduce, since it is merely a matter of putting together the results of the previous talks.
The idea is the following. Given a smooth variety X over a field k, there is a comparison isomorphism

$$\mathrm{comp} : H^*_{\mathrm{dR}}(X, k) \otimes_k \mathbb{C} \to H^*_{\mathrm{sing}}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C},$$

which, if X is affine, is induced by the perfect pairing

$$H^*_{\mathrm{dR}}(X, k) \otimes_{\mathbb{Q}} H_*^{\mathrm{sing}}(X, \mathbb{Q}) \to \mathbb{C}, \quad (\omega, \gamma) \mapsto \int_\gamma \omega.$$

This is the beginning of the story of periods: this isomorphism does not respect rational structures, so the coefficients of its matrix with respect to any choice of rational bases are not algebraic in general.
Since we are not content with smooth varieties only, we need to deal with the singular varieties as well. This is the content of [2, Section 5.4], which basically tells us that everything works as expected.
Next, we introduce Hodge structures. A detailed account on the subject is given in [1], which starts defining pure Hodge structures in Subsection 2.6.1. Classically, a pure Hodge structure of weight n on a complex vector space H consists of a decomposition

$$H = \bigoplus_{p+q=n} H^{p,q}$$

of H into subspaces $H^{p,q}$ which satisfy $\overline{H^{p,q}} = H^{q,p}$. One then shows that the n-t de Rham cohomology group of a smooth projective variety over $\mathbb{C}$ carries a pure Hodge structure of weight n.
However, since we all love number theory, we had better remember that de Rham cohomology is actually a k-vector space whenever our variety is defined over a number field k. Besides, we are interested in studying periods, so we cannot forget that, after tensoring with $\mathbb{C}$, our beloved de Rham cohomology turns into singular cohomology via an isomorphism which is not rational. In this setting, even birational geometers will agree that it is a pity to define Hodge structures starting with complex vector spaces, which is the reason why [1] considers a slightly more refined version of Hodge structures.
A pure Hodge structure of weight n for the rest of the seminar will be a triple $(H_B, H_{\mathrm{dR}}, \mathrm{comp})$ consisting of

- a finite dimensional $\mathbb{Q}$-vector space H_B;
- a finite dimensional k-vector space H_{dR} together with an exhaustive decreasing filtration $F^* H_{\mathrm{dR}}$;
- a $\mathbb{C}$-linear isomorphism $\mathrm{comp} : H_{\mathrm{dR}} \otimes_k \mathbb{C} \to H_B \otimes_{\mathbb{Q}} \mathbb{C}$;

such that the induced filtration on $H_{\mathbb{C}} = H_B \otimes_{\mathbb{Q}} \mathbb{C}$ satisfies

$$H_{\mathbb{C}} = F^p H_{\mathbb{C}} \otimes \overline{H_{\mathbb{C}}^{n+1-p}}.$$

This condition on the filtration is equivalent to the more classical decomposition written above, but is probably more practical for our purposes.
We next turn to mixed Hodge structures, which can be thought of as iterated extensions of pure Hodge structures, see [1, Subsection 2.6.2]. The idea would be to give a broad overview of these concepts and state some results, for example the fact that cohomology groups of smooth varieties carry a mixed Hodge structure and that the category of mixed Hodge structures is abelian. We will not need these results explicitly, so feel free to go through these topics as you prefer.

Talk 3: Nori's diagram category.
It is time to introduce the main concept that will play a fundamental role in the definition of Nori motives: the diagram category. First, we need to introduce the notion of diagrams and representations, which are covered in [2, Subsection 7.1.1]. Roughly speaking, a *diagram with identities* is a directed graph D with one distinguished loop for every edge or, equivalently, a category in which composition of morphisms is not defined. A representation

of a diagram D into $R-\mathrm{Mod}$ is a functor $T : D \to R-\mathrm{Mod}$ on which we do not require any constraint about composition of morphisms. Given a representation T of D, one defines the ring of endomorphisms of T, denoted by $\mathrm{End}(T)$.
At this point [2, Subsection 7.1.2] explains the definition of Nori's diagram category. Take a diagram D and a representation $T : D \to R-\mathrm{Mod}$. If D is finite, we define the diagram category of T as

$$\mathcal{C}(D,T) \coloneqq \mathrm{End}(T) - \mathrm{Mod}.$$

Nothing weird so far, just notice that the ring $\mathrm{End}(T)$ is not commutative in general, so we consider left modules over this ring. If on the other hand the diagram D is not finite, we set

$$\mathcal{C}(D,T) \coloneqq 2 - \mathrm{colim}_F \ \left(\mathrm{End}(T|_F) - \mathrm{Mod}\right),$$

the 2-colimit ranging over all finite subdiagrams of D. You don't like this 2-colimit? Me neither, but luckily [2, Section 7.2] takes some time to explain in detail how it works.
Let us then turn to [2, Section 7.2]. The first two lemmas, 7.2.1 and 7.2.2, are just some commutative algebra and show that the ring $\mathrm{End}(T)$ is not too bad when D is finite. Proposition 7.2.3 tells us that for finite diagrams the representation T factors through the diagram category as follows:

$$\begin{tikzcd} & \mathcal{C}(D,T) \arrow[dr, "f_T"] & \\ D \arrow[ur, "\tilde{T}"] \arrow[rr, "T"] & & R-\mathrm{Mod} \end{tikzcd}$$

Although this proposition is rather elementary, it is of great relevance for us: the functor f_T will be the analogue of a fiber functor in the Tannakian case. The analogy with Tannakian formalism starts to appear: we have an abelian category $\mathcal{C}(D,T)$ together with a functor $\mathcal{C}(D,T) \to R-\mathrm{Mod}$... we still miss the tensor structure, rigidity and so on, but don't worry, we will slowly put all pieces at the correct place.
Finally, Corollary 7.2.4 and Proposition 7.2.5 explain the aforementioned 2-colim. At this point, we should have gained enough familiarity with diagram categories to be able to believe the final lemmas of [2, Section 7.2]. These deal with functoriality in D and base change with flat R-algebras. They are not particularly deep and very elementary, so they are probably the most reasonable topic to skip in case of lack of time.
Last but not least, we give one final description of the diagram category. We

go back to [2, Section 7.1.2] and have a look at theorem 7.1.12. Under certain assumptions, the R-linear dual of $\mathrm{End}(T|_F)$ is a coalgebra and we can define the coalgebra

$$A(D,T) \coloneqq \mathrm{colim}_F \, \mathrm{End}(T|_F)^\vee.$$

Theorem 7.1.12 tells us that $C(D,T)$ is equivalent to the category of comodules over $A(D,T)$. This is great news: instead of a big colimit of categories of modules over some rings, the diagram category is nothing but comodules over some big coalgebra!
The proof of this statement is to be found in [2, Section 7.5]. The main ingredient is Lemma 7.5.4, which is purely a statement of commutative algebra; it is obtained by duality, with the only caveat that we need to work with finitely generated projective R-modules in order for the dual to commute with tensor product. Then Corollary 7.5.7, which proves Theorem 7.1.12, follows essentially by a formal categorical argument.

Talk 4: Universal property of the diagram category.
The goal of this talk is to prove the universal property of the diagram category, which is Theorem 7.1.13 in [2, Subsection 7.1.3]. This property is kind of tedious to write down, so let us say that, given a diagram D and a representation $T : D \to R - \mathrm{Mod}$, the diagram category $\mathcal{C}(D,T)$ is the universal R-linear abelian category $\mathcal{A}$ for which the representation T factors as

$$\begin{tikzcd} & \mathcal{A} \arrow[dr, "f"] & \\ D \arrow[ur, "F"] \arrow[rr, "T"] & & R - \mathrm{Mod} \end{tikzcd}$$

with the correct adjectives for the functors F and f.
The universal property of the diagram category is a non-trivial result and we divide the proof into two parts. First, we prove Theorem 7.1.20, whose proof is the content of [2, Section 7.3]; after this, we prove Theorem 7.1.13 following [2, Section 7.4].
Let us have a look at Theorem 7.1.20. As a choice for a diagram D we could consider an abelian category $\mathcal{A}$ in which we forget composition of functors. We then take a representation $T : \mathcal{A} \to R - \mathrm{Mod}$ (just keep in mind that composition of morphisms need not be preserved) and construct the diagram category $\mathcal{C}(\mathcal{A},T)$, which is also abelian. What is the relation between $\mathcal{A}$ and $\mathcal{C}(\mathcal{A},T)$? Under some mild assumptions on T, Theorem 7.1.20 tells us that these two categories are equivalent.
Before constructing the equivalence, we need to go through some general categorical nonsense in [2, Subsection 7.3.1]. Given an object p of $\mathcal{A}$ which is

a right E-module for some R-algebra E, in the sense that there is a morphism $E^{\mathrm{op}} \to \mathrm{End}_{\mathcal{A}}(p)$, we can view the functor $\mathrm{Hom}_{\mathcal{A}}(p,-) : \mathcal{A} \to R - \mathrm{Mod}$ as a functor

$$\mathrm{Hom}_{\mathcal{A}}(p,-) : \mathcal{A} \to E - \mathrm{Mod}.$$

Proposition 7.3.5 constructs a left adjoint to this functor, denoted by

$$p \otimes_E - : E - \mathrm{Mod} \to \mathcal{A}.$$

Lemma 7.3.8 and Proposition 7.3.9 show that this right adjoint behaves exactly as a usual tensor product, in a suitable sense. The last proposition of this section repeates the same construction starting with the contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-,p)$. In order for you to understand how much time should be dedicated to these exciting constructions, I will just say that they will never be used again in the other talks...

It is then time to have a look at the very heart of the proof of Theorem 7.1.20, namely [2, Subsection 7.3.2]. The main ingredient is Lemma 7.3.16, in which the most difficult task is the construction of the object denoted by $X(p)$ in the proof. Once we have seen this, the equivalence is proved soon after Proposition 7.3.18. Let us briefly explain the idea.

Recall that

$$\mathcal{C}(\mathcal{A},T) := 2 - \mathrm{colim}_F \ (\mathrm{End}(T|_F) - \mathrm{Mod}),$$

the 2-colimit ranging over all finite subdiagrams of $\mathcal{A}$. Since we have taken $D = \mathcal{A}$ as an abelian category, we can replace the huge inductive system of finite subdiagrams of $\mathcal{A}$ by a much smaller one: we consider the system of all subcategories of the form $\langle p \rangle^{\mathrm{psab}}$ for all $p \in \mathcal{A}$ (roughly speaking, the pseudo-abelian subcategory of $\mathcal{A}$ generated by p). The advantage is that

$$\mathcal{C}(\langle p \rangle^{\mathrm{psab}}, T) = \mathcal{C}(\{p\}, T),$$

and the right-hand side is much more controllable. The key Lemma 7.3.16 allows us to use this observation to construct a functor $\mathcal{C}(\mathcal{A},T) \to \mathcal{A}$: tensoring by the object $X(p)$ gives a functor $\mathcal{C}(\{p\},T) \to \mathcal{A}$, and all these functors for each $p \in \mathcal{A}$ are compatible with transition maps, so they induce a functor $\mathcal{C}(\mathcal{A},T) \to \mathcal{A}$. Proving that it is an equivalence should be rather straightforward.

Now we are ready to prove the universal property of the diagram category, which is in [2, Section 7.4]. We start with an abelian category $\mathcal{A}$ with a factorization of T as above and we want to construct a suitable functor $L(F) : \mathcal{C}(D,T) \to \mathcal{A}$. The idea is to replace the category $\mathcal{A}$ by $\mathcal{C}(\mathcal{A}, f)$ and then use functoriality of the diagram category, see Proposition 7.4.1. Notice that there is a typo at the beginning of page 166, in which the notation for f and $T_{\mathcal{A}}$ is confused. Finally, uniqueness follows with classical arguments.

Talk 5: Nori's rigidity criterion.
Take a diagram D and a representation $T : D \to R - \text{Mod}$. So far we have seen that the diagram category is abelian, it satisfies a certain universal property and under some assumptions it can be written as a category of comodules over some coalgebra. The goal of this talk is to give a criterion (Proposition 8.3.4) for the diagram category to be rigid, so as to replicate Tannakian formalism. However, we cannot muse about rigidity if we have not defined a tensor structure yet.
Morally, the tensor structure on the diagram category $\mathcal{C}(D,T)$ should be induced by a product structure on the diagram D. We therefore need to define the notion of product of diagrams, of commutative product structure on a diagram and of graded multiplicative representation. This is the content of [2, Section 8.1]. If D comes equipped with all this extra information, it is possible to define a tensor structure on $\mathcal{C}(D,T)$, as discussed in Proposition 8.1.5.
Recall that $\mathcal{C}(D,T) = A(D,T) - \text{Comod}$ for some big coalgebra $A(D,T)$. If R is a field or a Dedekind domain, the presence of a tensor structure on $\mathcal{C}(D,T)$ implies that $A(D,T)$ is also a commutative bialgebra. We can then consider $M = \text{Spec } A(D,T)$, which is a monoid scheme. By some general duality statement, $A(D,T) - \text{Comod}$ is isomorphic to the category of representations of M. This sounds very Tannakian: we just need to make sure that M is also a group scheme, rather than just a monoid scheme, and then... Well, not so fast. We first need to take care of an annoying assumption in Proposition 8.1.5, namely that the representation T takes values in (finitely generated) projective R-modules. If this is not the case, duals and tensor product do not commute, and good bye tensor structure on $\mathcal{C}(D,T)$. On the other hand, for constructing Nori motives, we really need to get rid of this assumption with projective modules. If R is a field, we are very lucky because every finitely generated R-module is projective, even free. In this case we have nothing to worry about, see Example 8.1.8. The rest of [2, Section 8.1] is devoted to establish a similar result in the case when R is a Dedekind domain.
Summing up, under certain assumptions $\mathcal{C}(D,T)$ is the category of representations of a certain monoid scheme M. Establishing rigidity is then equivalent to ask whether M is also a group scheme. This is discussed in [2, Section 8.3]. The first definitions mimic the existence of duals in a Tannakian category, although they are tailor-made for our setting. The criterion we are interested in is Proposition 8.3.4, which is proved in a way which is similar to what we did in the previous talk for constructing the equivalence $\mathcal{A} \cong \mathcal{C}(\mathcal{A},T)$. First, we prove that the monoid scheme which arises from a very small subcategory of $\mathcal{C}(D,T)$ is a group scheme, then we deduce that also M is a group scheme.

Talk 6: Nori motives.

The goal of this talk is to introduce the category of Nori motives and start proving rigidity. Before we get into the main topic, we need to learn how to localize diagrams, which is written in [2, Section 8.2]. The idea is to translate the localization of a tensor category with respect to one object into the language of diagrams. We start with a diagram D^{eff}, we choose our favorite edge v_0 to invert and we construct a new diagram D which is called the *localization of D^{eff} at v_0*. Lemma 8.2.4 shows how to extend a representation of D^{eff} to D and Proposition 8.2.5 explains the relation between the diagram categories of D^{eff} and D.

It is finally time to introduce Nori motives, whose definition can be found in [2, Subsection 9.1.1]. First, we define the diagram $\text{Pairs}^{\text{eff}}$ of *effective pairs* over a subfield k of $\mathbb{C}$. Its vertices are given by triples (X, Y, i) where X is a k-variety, Y a closed subvariety of X and i an integer. We can consider the representation H^* of $\text{Pairs}^{\text{eff}}$ with values in abelian groups given by singular cohomology. On vertices:

$$H^* : \text{Pairs}^{\text{eff}} \to \mathbb{Z} - \text{Mod}, \quad (X, Y, i) \mapsto H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Z}).$$

The category of effective mixed Nori motives is defined as the diagram category

$$\mathcal{MM}^{\text{eff}}_{\text{Nori}}(k) := \mathcal{C}(\text{Pairs}^{\text{eff}}, H^*).$$

In order to remove the adjective *effective*, we invert the vertex $(\mathbb{G}_m, \{1\}, 1)$. These are all fancy definitions, but not very useful if we do not prove something about them. The main results that we need are Theorem 9.1.5 and Theorem 9.1.10: the former is about rigidity of $\mathcal{MM}_{\text{Nori}}(k)$, the latter about its universal property. If all the work we have done in the previous talks was not in vain, we should be able to obtain both statements as a byproduct of the abstract machinery of diagram categories. The path to follow is quite clear: we define a product on $\text{Pairs}^{\text{eff}}$ to get a tensor structure on $\mathcal{MM}_{\text{Nori}}(k)$, then we check the assumptions of Nori's rigidity criterion. However, there is one little problem...

Defining a product on the diagram $\text{Pairs}^{\text{eff}}$ is not complicated, since we can induce it from taking products of varieties. However, for this product to yield a tensor structure on the diagram category, our representation H^* should satisfies the property

$$H^{i+j}(X_1 \times X_2, \mathbb{Z}) = H^i(X_1, \mathbb{Z}) \otimes H^j(X_2, \mathbb{Z})$$

(this is the constraint on the pairs (X_1, Y_1, i) and (X_2, Y_2, j) with $Y_1 = Y_2 = \emptyset$ for simplicity). But the singular cohomology of the product of two varieties

is described by the Künneth formula, which is definitely not that simple in general! This is a structural problem in the whole construction, and Nori came up with a smart idea to circumvent it.
We turn to [2, Subsection 9.2.1], where the key concept of this idea is clarified. We consider a subdiagram Good^{eff} of $\text{Pairs}^{\text{eff}}$ consisting of all those pairs (X, Y, i) whose cohomology is concentrated in degree i. For all these pairs, the Künneth formula takes the simplified shape written above, so we can run our beloved machinery of diagram categories for $\mathcal{C}(\text{Good}^{\text{eff}}, H^*)$. Then, one proves that the categories $\mathcal{C}(\text{Good}^{\text{eff}}, H^*)$ and $\mathcal{C}(\text{Pairs}^{\text{eff}}, H^*)$ are equivalent (Theorem 9.2.22) and all our problems suddenly disappear.
Since the whole argument is very involved, for the moment we content ourselves with introducing good pairs and good filtrations, [2, Subsection 9.2.1]. The main ingredient for this subsection is the so called *Basic Lemma*, which is Theorem 2.5.2. The book is so passionate about this result that they give three different proofs for it. Allegedly the first one should be enough for our purposes and is explained in [2, Subsection 2.5.2]. If you prefer to go through some more enhanced versions using weakly constructible sheaves or perverse sheaves, I am looking forward to learning some more high-tech stuff.
A couple of words about this lemma. Since we want to replace $\mathcal{C}(\text{Pairs}^{\text{eff}}, H^*)$ with $\mathcal{C}(\text{Good}^{\text{eff}}, H^*)$, our intuition suggests that the subdiagram Good^{eff} should be relatively big inside $\text{Pairs}^{\text{eff}}$. An inductive application of the Basic Lemma implies that we can always find a filtration of an affine variety X whose intermediate pieces consist of good pairs. The next step is to understand how to relate this property to the diagram category, but you do not need to worry about this, since it is a problem of the speaker coming after you.

Talk 7: Rigidity of Nori motives.
Now you can start worrying about the problem of the last paragraph, since it is the main topic of this talk. Our main goal is to prove rigidity of $\mathcal{MM}_{\text{Nori}}(k)$, which is Theorem 9.3.10, or equivalently Theorem 9.1.5. We will do this in two steps: first, we show that $\mathcal{C}(\text{Pairs}^{\text{eff}}, H^*)$ is equivalent to $\mathcal{C}(\text{Good}^{\text{eff}}, H^*)$, then we check the assumptions of Nori's rigidity criterion for $\mathcal{C}(\text{Good}, H^*)$.
For the first part, by the abstract machinery of diagram categories, it is easy to convince ourselves that we only need to construct a representation

$$T : \text{Pairs}^{\text{eff}} \to \mathcal{C}(\text{Good}^{\text{eff}}, H^*)$$

satisfying a few minor properties. How do we cook this up? We go the long way round and define T by the following diagram:

$$\begin{tikzcd}
\mathrm{Pairs}^{\mathrm{eff}} \arrow[r, "T"] \arrow[d, "\varphi"] & \mathcal{C}(\mathrm{Good}^{\mathrm{eff}}, H^*) \\
C_b(\mathbb{Z}[\mathrm{Var}]) \times \mathbb{Z} \arrow[r, "R"] & D^b(\mathcal{C}(\mathrm{Good}^{\mathrm{eff}}, H^*)) \times \mathbb{Z} \arrow[u, "H^*"]
\end{tikzcd}$$

Alright, this looks incomprehensible, so one thing at a time. First, the book does everything with a "V" in front of Good, but the discussion here is already quite long, so we will just forget about this "V". The vertical arrow on the left sends (X, Y, i) to the cone of the closed immersion $Y \to X$ and there is a copy of $\mathbb{Z}$ to remember i. The vertical arrow on the right takes i-th cohomology, where i goes according to the $\mathbb{Z}$-component. Finally, there is this mysterious functor R which we need to construct. Before doing so, please notice that this diagram really works only on objects, as there are some subtleties with morphisms; anyway, I hope this gives an idea of what is going on.

The definition of R is carried out in Proposition 9.2.18 and it uses the results of [2, Subsection 9.2.2]. Let us try to construct this R by hands. Please forgive how imprecise I am, but, roughly speaking, we need to map some (complex of) variety X to a complex of good pairs. If X is affine, the Basic Lemma already gives us the recipe: we can find a filtration of X, say

$$\emptyset = F_0X \subseteq F_1X \subseteq \cdots \subseteq F_nX = X$$

such that all intermediate pairs $(F_iX, F_{i-1}X, i)$ are good. We can then construct a complex of objects in $\mathcal{C}(\mathrm{Good}^{\mathrm{eff}}, H^*)$ by passing through the long exact sequence for relative cohomology.

However, it seems that not all varieties are affine, so we need to extend this definition by replacing a variety with a Čech cover. The problem is that this construction is not functorial, so one has to introduce rigidified Čech covers, which in any case have nothing to do with rigid geometry. The book therefore spends Subsection 9.2.2 to explain this in detail, so that the definition of R can be extended to all varieties, and then to complexes of varieties. This discussion is only used to construct the functor R, for the rest we will not need it elsewhere.

After this business, we should be convinced that we can replace $\mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*)$ with $\mathcal{C}(\mathrm{Good}^{\mathrm{eff}}, H^*)$. In order to prove rigidity, we need to check the assumptions of Nori's rigidity criterion, see [2, Section 9.3]. The first thing to do is to define a tensor product on $\mathcal{C}(\mathrm{Good}^{\mathrm{eff}}, H^*)$, which is very straightforward for good pairs. The second assumption to be verified, namely the existence of strong duals, follows from Lemma 9.3.8 and 9.3.9. Here is where some intersection theory shows up. I fear I cannot say much more about the proof of these lemmas, since my head starts spinning as soon as I read the words "deformation to the normal cone", but I promise it will stop spinning before we arrive at this point in the seminar.

Talk 8: Equivalence of the definitions of periods.
What do we do now that we have this gadget of Nori motives in our hands? The goal of this talk is to prove that the several definitions of periods are equivalent and to develop a deeper theory of periods.
In [2, Chapter 11] the possible definitions of periods are presented. Section 11.1 deals with the naive definition of periods as proper integrals of algebraic differential forms over some algebraic integration domain. These are called *NC-periods* and they form a set $\mathbb{P}^{\text{eff}}_{nc}(k)$, which is actually a k-algebra, see Proposition 11.1.7.
Then we turn to cohomological periods, as defined in [2, Subsection 11.3.1]. These come from the image of the period pairing of the mixed Hodge structure coming from a cohomology group of a k-variety relative to a simple normal crossing divisor. They form a k-algebra which we denote by $\mathbb{P}^{\text{eff}}(k)$. Our goal is to prove that $\mathbb{P}^{\text{eff}}_{nc}(k) = \mathbb{P}^{\text{eff}}(k)$, which is the content of Theorem 11.4.2. An ad-hoc proof is given right after the statement in [2, Section 11.4], but why should we drive a Twingo if we have a Ferrari at our disposal? We go back to [2, Section 8.4] to revisit this story from a motivic point of view, with the aim of proving Theorem 11.4.2 through the discussion presented in [2, Section 13.1].
In [2, Section 8.4] we go back to our abstract setting of diagram categories. Fix a diagram D and suppose you are so lucky that you are given not just one, but even two representations of D into $R-$Proj, say T_1 and T_2. We obtain two diagram categories with two coalgebras $A_1 = A(D, T_1)$ and $A_2 = A(D, T_2)$. In order to compare these two, one can construct a big R-module

$$A_{1,2} = \operatorname{colim}_F \operatorname{Hom}(T_1|_F, T_2|_F),$$

exactly in the same spirit as we did for End(T). If D comes with a product structure which is respected by T_1 and T_2, then $A_{1,2}$ becomes even a commutative R-algebra. If we look at the scheme $X_{1,2} = \operatorname{Spec} A_{1,2}$ and take any faithfully flat R-algebra S, an S-point of $X_{1,2}$ corresponds to a morphism of representations $T_1 \otimes S \to T_2 \otimes S$. Moreover, we know that $G_1 = \operatorname{Spec} A_1$ and $G_2 = \operatorname{Spec} A_2$ are monoid schemes, and they also act naturally on $X_{1,2}$ on the left and right respectively. Theorem 8.4.10 tells us that, if T_1 and T_2 are

isomorphic and their diagram category is rigid, then G_1 and G_2 are group schemes and $X_{1,2}$ is a left G_1- and right G_2-torsor. An explicit description of this torsor is given in [2, Subsection 8.4.3].
These are all nice words, but what do we do in practice? The diagram $\mathrm{Pairs}^{\mathrm{eff}}$ comes not only with our favourite representation H^* given by singular cohomology, but actually also with a representation H^*_{dR} given by de Rham cohomology. There is an isomorphism between these two representations after base change to $\mathbb{C}$, which is induced by the comparison isomorphism. Hence, we obtain a torsor $X_{1,2} = \operatorname{Spec} A_{1,2}$ under the motivic fundamental group. The period isomorphism corresponds to a complex point of this torsor, and evaluation at this point yields a map $ev : A_{1,2} \to \mathbb{P}(k)$. This is essentially the content of Theorem 13.1.4, then Corollary 13.1.10 uses some properties of $\mathcal{MM}_{\mathrm{Nori}}(k)$ to deduce immediately that $\mathbb{P}(k) = \mathbb{P}_{nc}(k)$.

Talk 9: 1-Motives.
It is time to say goodbye to Nori motives for some time and welcome on the stage of the seminar another member of the motivic family, namely 1-motives. The goal of this talk is to introduce 1-motives by giving their definition, and nothing more than that.
As a prerequisite for this, we need the discussion in [3, Theorem 4], which deals with certain properties of the category of commutative algebraic groups. This category is abelian, and its simple objects are given by $\mathbb{G}_a$, $\mathbb{G}_m$ and simple abelian varieties. Every object is an iterated extension of these building blocks in a canonical way, as pointed out by [3, Theorem 4.3]. Then we need to survey certain properties of extensions groups in this category, which are explained in [3, Section 4.2].
The main characters of this theory are semi-abelian varieties, which are extensions of an abelian variety by a torus. The two main results that we need are Corollry 4.10 and Corollary 4.12. The discussions in [3, Section 4.3] and [3, Section 4.4] build up on these two corollaries; although we will need these results in the sequel, try to see how much of the proofs is reasonable to go through during the talk.
Now pass to the definition of 1-motives in [3, Section 8.1] - and their definition only, as the rest will be a problem for the next speaker. Let $k = \overline{\mathbb{Q}}$ or $k = \mathbb{C}$. A 1-motive $M = [L \to G]$ is the datum of

- a semi-abelian variety G (which we have just become familiar with);
- a free abelian group L of finite rank (which we are familiar with since our Bachelor studies);
- a group homomorphism $L \to G(k)$ (so nothing scary).

Morphisms are the natural ones, tensored with $\mathbb{Q}$.
So far so good, but what is the point of spending eight talks on Nori motives to end up with this definition? In order to avoid an uprising, please save the future of the seminar by stating [3, Theorem A.7]. After constructing our dear $\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}}(k)$, we can restrict to the thick abelian subcategory generated by motives which come from effective pairs of the form (X, Y, i) with $i \in \{0, 1\}$. This is actually equivalent to the diagram category associated with the subdiagram of effective pairs of the form (X, Y, i) in which X has dimension 0 or 1. This means that this subcategory is itself a diagram category, and all the machinery that has been developed so far can be fruitfully applied. Moreover, one can show that the category of 1-motives is anti-equivalent to said subcategory, which means that 1-motives are nothing but a more explicit description of Nori motives for curves.
I fear we will not have time to explain this anti-equivalence in the seminar, so we will assume it as a black-box. I will try to understand how feasible it would be to at least describe this equivalence and insert it somewhere. In any case, I guess the main ingredient should be to construct the Jacobian of a curve...

Talk 10: The analytic subgroup theorem.
This talk is divided into two main parts: first, we introduce singular and de Rham realization of the category of 1-motives, then we go through the analytic subgroup theorem.
As a prerequisite to both of these topics, we need to define the exponential map of a Lie group, which is the content of [3, Chapter 5]. Let G be a connected commutative algebraic group defined over $\overline{\mathbb{Q}}$. The tangent space $\mathfrak{g}$ of G at the identity has a structure of Lie algebra. If you are scared of the word "Lie algebra", fear not: since G is commutative, the Lie brackets on $\mathfrak{g}$ are trivial, which means that you can translate "Lie algebra" into "vector space", "Lie subalgebra" into "vector subspace" and so on. Let G^{an} be the analytification of G, which is a complex Lie group and let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$.
The exponential map $\exp_G : \mathfrak{g}_{\mathbb{C}} \to G^{\mathrm{an}}$ realizes $\mathfrak{g}_{\mathbb{C}}$ as the universal cover of G^{an}, see [3, Section 5.2]. Thus, the kernel of this map is discrete, and [3, Section 5.2] explains how to identify this kernel with $H_1^{\mathrm{sing}}(G^{\mathrm{an}}, \mathbb{Z})$. The exponential map induces a correspondence between Lie subalgebras of $\mathfrak{g}_{\mathbb{C}}$ and Lie subgroups of G^{an}.
Next, we turn to [3, Section 8.1]. In the previous talk we have already introduced the category $1-\mathrm{Mot}_k$ of 1-motives over $k = \overline{\mathbb{Q}}$ or $k = \mathbb{C}$, which we observed to be anti-equivalent to the subcategory of Nori motives generated by the motives of curves, roughly speaking. Our goal is to define two faithful

exact functors

$$V_{\mathrm{sing}} : 1 - \mathrm{Mot}_k \to \mathbb{Q} - \mathrm{Vect}, \quad V_{\mathrm{dR}} : 1 - \mathrm{Mot}_k \to k - \mathrm{Vect},$$

the so-called *singular* and *de Rham realization* respectively. We have already met these two functors in the context of Nori motives, and one could actually recover them in a natural way by restricting the corresponding realizations of $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}(k)$ to the category of Nori 1-motives and then apply the aforementioned anti-equivalence with $1 - \mathrm{Mot}_k$. The problem of this approach is that we have not explained how this anti-equivalence works, and the future of the seminar depends on some more explicit description of these realizations.
The singular realization can be described rather quickly, see [3, Subsection 8.1.1]. Maybe it is useful to notice that the fibered product in Definition 8.3 is taken in the category of abelian groups. For the de Rham realization, we first need to adapt the discussion of the previous talk about the universal vector extension to the case of 1-motives, which is the content of [3, Subsection 8.1.2]. We first define a vector extension of 1-motives, then we show that there is a universal vector extension M' for every object M in $1 - \mathrm{Mot}_k$. Finally the de Rham realization ([3, Subsection 8.1.3]) is obtained by taking the Lie algebra of M'. At this point, to our great surprise there is a period isomorphism between these two realizations, see [3, Subsection 8.1.4]. If after 9 talks in this seminar you are still not convinced of the existence of this period isomorphism, feel free to give the proof of Lemma 8.13.
Now comes a change of topic. The main ingredient for proving the period conjecture for 1-motives is a result due to Wüstholz, the so-called *analytic subgroup theorem*. Let me sketch the ideas behind this result.
Take a Lie subgroup B of G^{an} and define its algebraic points by $B(\overline{\mathbb{Q}}) = B \cap G(\overline{\mathbb{Q}})$. There is a Lie subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that $\exp_G(\mathfrak{b}) = B$. The analytic subgroup theorem states ([3, Theorem 6.1]) that there is an algebraic point in $B(\overline{\mathbb{Q}})$ different from 0 if and only if there is an algebraic subgroup H of G with Lie algebra $\mathfrak{h}$ such that $0 \neq \mathfrak{h} \subseteq \mathfrak{b}$. Essentially, this means that an analytic subgroup contains a non-zero algebraic point if and only if it already contains a whole algebraic subgroup of G.
We will assume this result as a black box. The proof can be found in [5] and it is a clever application of some techniques dear to transcendental number theory to this algebraic setting. Although these techniques are fairly elementary and we would be able to understand the proof with the tools developed so far, dealing with the details would lead us too far afield. In any case, a nice summary of the proof is exposed in [4], which you are invited to have a look at if you wish.
Assuming the analytic subgroup theorem, we turn to [3, Theorem 6.2], which is a refinement of the analytic subgroup theorem. Given a point $u \in \mathfrak{g}_{\mathbb{C}}$ such

that $\exp_G(u) \in B(\overline{\mathbb{Q}})$, there exists an algebraic subgroup H of G which is uniquely determined by the following properties:

1. $\exp_G(u) \in B(\overline{\mathbb{Q}})$;
2. Let $\pi : G \to G/H$ be the canonical projection. If we denote by $\mathfrak{g}_{\mathbb{C}}^{\vee}$ the dual of $\mathfrak{g}_{\mathbb{C}}$, then $(\mathfrak{g}/\mathfrak{h})^{\vee}$ embeds into $\mathfrak{g}^{\vee}$. Then $(\mathfrak{g}/\mathfrak{h})^{\vee}$ coincides with $\mathrm{Ann}(u)$. Notice that $\mathrm{Ann}(u)$ is the intersection of $(\mathfrak{g}/\langle u\rangle)^{\vee}$ with $\mathfrak{g}$ inside $\mathfrak{g}_{\mathbb{C}}$, so this is indeed a description of the algebraic points of $(\mathfrak{g}/\langle u\rangle)^{\vee}$.

This will be the main result that we are going to exploit in the proof of the period conjecture for 1-motives, and the next step will be to reinterpret it in the language of 1-motives.

Talk 11: The period conjecture for 1-motives.
We start by introducing periods in the context of 1-motives as in [3, Chapter 7]. Actually, the definitions given here are a simplified version of the ones that we have seen in Talk 8, so we should be able to just recall them without really explaining every detail. When the time for you to prepare this talk comes, we will sit down and draw the analogy between these definitions, so as to waste as little time as possible during the talk.
Our goal is to prove [3, Theorem 13.3]; the road is quite complicated, so fasten your seat belts and follow me. During the last talk, we have introduced this spectacular super powerful result, the analytic subgroup theorem. In order to put it into action, we first have to translate it into the language of 1-motives, the first ingredient for this being [3, Theorem 8.3]. This is essentially a technical result concerning the functor which associates to a 1-motive its universal vector extension.
Now turn to [3, Chapter 9]. After some bla bla concerning periods, which we should already be very familiar with at this point, we finally arrive at [3, Theorem 9.7]. This is the actual reinterpretation of the analytic subgroup theorem in terms of 1-motives, and the proof should be a not too unreachable combination of [3, Theorem 8.3] and the classical analytic subgroup theorem. Next, the period conjecture for 1-motives, namely [3, Theorem 9.10], follows rather straightforwardly. It would be nice to quickly quote Corollary 9.12, which requires no effort to be stated thanks to all the work we have already done with Nori motives.
As a very elementary application in [3, Section 10.1] a five-line proof of the transcendence of π is given. Considering how long and mysterious transcendence proofs have been at their very origin, I am pretty sure Lindemann would be proud of us if we managed to summarize his result in so few lines.

Maybe you will be angry at me for the amount of material in this talk and you will not include this in the talk, but I promise it is worth at least to have a quick glance at the argument.
Dulcis in fundo, after the Odissey that we have been through so far, we can prove [3, Theorem 13.3], which we could call the period conjecture for Nori 1-motives. The result is nothing surprising: thanks to what we have learnt so far, this boils down to restate the period conjecture in the language of Nori motives. The advantage is to have a much more solid and complete description of the relations among periods, but it is truly nothing more than exploiting the anti-equivalence with 1-motives. With this result in our hands, we can live happily ever after.

Talk 12: t.b.a.
Either the proof of the analytic subgroup theorem or the anti-equivalence between 1-motives and Nori 1-motives.

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