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PhD SS24
Talk 3

Nori's diagram category

To each diagram D and representation $T : D \to R\text{-Mod}$ (defined below) is associated a category $\mathcal{C}(D,T)$, called Nori's diagram category, which can be defined (up to equivalence of categories) in two to three ways:

Noetherian ring

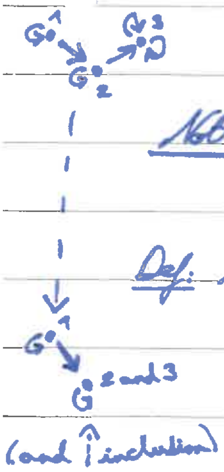
- explicitly (as a 2-colimit when D is not finite), see Ⓘ;
- by a universal property (mentioned in Ⓘ and proved in the next talk);
- as a category of comodules over a coalgebra, see Ⓘⓘ, when R is a field or a Dedekind domain (with defined as a colimit when D is not finite).

Below, we consider sets to be small (i.e. in a fixed universe) and categories (up to equivalence) to be small (or at least locally small).

Ⓘ Explicit construction of Nori's diagram category

Def: A diagram D (a.k.a. quiver, a.k.a. directed graph) is a set of "vertices" $V(D)$ together with a set of "directed edges" (a.k.a. arrows) $E(D)$, such that we have functions $pr_1^{(D)} : E(D) \to V(D)$ and $pr_2^{(D)} : E(D) \to V(D)$.
We denote $\overset{\in E(D)}{f} : \overset{\in V(D)}{v_1} \to \overset{\in V(D)}{v_2}$ to mean $pr_1(f) = v_1$ and $pr_2(f) = v_2$.

Def: A finite diagram D is a diagram such that $V(D)$ is finite.
Rk: Unlike "finite quivers", for finite diagrams D, $E(D)$ is not required to be finite.



Def: A diagram with identities $(D, \{id_v\}_{v \in V(D)})$ is a diagram D together with a chosen $id_v \in E(D)$ for each $v \in V(D)$ which verifies $id_v : v \to v$ (i.e. $pr_1(id_v) = v$ and $pr_2(id_v) = v$).

Ex: Categories: $V(D)$ = objects and $E(D)$ = morphisms. ($Id. = Id.$)

NB: $v \in D$ means $v \in V(D)$.

$D(v_1, v_2) := \{ f \in E(D) \mid pr_1(f) = v_1 \text{ and } pr_2(f) = v_2 \}$ with $v_1, v_2 \in D$.

Def: A full subdiagram D of a diagram D' is a diagram such that:
- $V(D) \subset V(D')$,
- $\forall v_1, v_2 \in D \quad D(v_1, v_2) = D'(v_1, v_2)$,
- $pr_1^D = pr_1^{D'}{}_{|E(D)}$ and $pr_2^D = pr_2^{D'}{}_{|E(D)}$.

Def: A map of diagrams $T : D \to D'$ is a map $T_{(verts)} : V(D) \to V(D')$ together with a map $T_{(edges)} : E(D) \to E(D')$ such that $T_{(verts)} \circ pr_1^D = pr_1^{D'} \circ T_{(edges)}$ and $T_{(verts)} \circ pr_2^D = pr_2^{D'} \circ T_{(edges)}$.

[Sm - - ->] [Def "with identities": $id_v \mapsto id_{Tv}$ for all $v \in D$]

Ex: Functors between categ.

Def. A representation $T : D \to C$ is a map of diagrams from a diagram to a category.
[Def. "with identities" ... "with id." ⚠ If you want to see it as a functor from the path category $P(D)$ (concatenation = composition) to C, you must use the identities id_p of D rather than the empty words as identities ($\neq$ to what you do for diag. without id.)]

Nb. R is a Noetherian commutative ring (with unit) (every ideal of R is finitely generated ($I = (i_1, \dots, i_n)$ i.e. $\forall i \in I \ \exists r_1, \dots, r_n \in R$ $i = r_1 i_1 + \dots + r_n i_n$))

R-Mod: finitely generated R-modules ($M = (m_1, \dots, m_n)$ i.e. ↑ with M)

R-Proj: fin. gen. projective R-modules: fin. gen. M s.t. $\exists$ module N s.t. $M \oplus N$ is a free module ($\simeq \bigoplus_{i \in I} R$) or equiv. s.t. $\exists$ (fin. gen.) module N s.t. $M \oplus N$ is a fin. gen. free module ($\simeq \bigoplus_{i=1}^{n} R$).

S is an (associative and) commutative R-algebra (with unit) (if S is f.g. as an R-module then S is Noetherian as a ring)

$T : D \to R\text{-Mod}$ repr. $\rightsquigarrow$ $T_S : D \to S\text{-Mod}$ repr. defined as $(- \otimes_R S) \circ T$.

Def: Let $T : D \to R\text{-Mod}$ be a representation. The ring of endomorphisms of T (which is an (associative) R-algebra (with unit) but ⚠ not commutative in general) is $\mathrm{End}(T) = \{ (e_p) \in \prod_{p \in D} \underbrace{\mathrm{End}_R(T_p)}_{R\text{-linear endom.}} \mid \forall p, q \in D \ \forall f \in D(p,q) \ e_q \circ Tf = Tf \circ e_p \}$ with addition, multiplication (given by composition) and R-linear operation (a.k.a. mult. by a scalar in R) defined component by component. Rk: $R \to \mathrm{End}(T)$ is $r \mapsto (r\, id_{T_p})_{p \in D}$.

Def. Let D be a finite diagram and $T : D \to R\text{-Mod}$ be a representation. Nori's diagram category associated to D and T, denoted $\mathcal{C}(D,T)$, is the category of finitely generated left $\mathrm{End}(T)$-modules. Rk: As R-modules (via $\begin{cases} R \to \mathrm{End}(T) \\ r \mapsto (r\, id_{T_p})_{p \in D} \end{cases}$) these are also finitely generated. (see the Lemma in the margin) (*) Def and Rk

Def. Let D be a diagram and $T : D \to R\text{-Mod}$ be a representation. Nori's diagram category associated to D and T, denoted $\mathcal{C}(D,T)$, is the 2-colimit over finite full subdiagrams F of D of $\mathcal{C}(F, T_{|F})$ (see expl. on the next page), i.e.: ($\coprod_F \mathcal{C}(F, T_{|F}) / (X \sim X_{F'})_{X \in \mathcal{C}(F, T_{|F}),\, F \subset F'}$)

- the set of objects of $\mathcal{C}(D,T)$ is the colimit in the category of sets of $\mathcal{C}(F, T_{|F})$
- the set (in fact, R-module) of morphisms in $\mathcal{C}(D,T)$ from $X \in \mathcal{C}(F_X, T_{|F_X})$ to $Y \in \mathcal{C}(F_Y, T_{|F_Y})$ (with F_X, F_Y finite full subdiagrams of D) is the colimit in the category of R-modules over finite full subdiagrams F of D containing F_X and F_Y of $\mathrm{Mor}_{\mathcal{C}(F, T_{|F})}(X_F, Y_F) = \mathrm{Hom}_{\mathrm{End}(T_{|F})}(X_F, Y_F)$ where X_F (resp. Y_F) is X (resp. Y) seen as an $\mathrm{End}(T_{|F})$-module, via $\begin{cases} \mathrm{End}(T_{|F}) \to \mathrm{End}(T_{|F_X}) \\ (e_p)_{p \in F} \mapsto (e_p)_{p \in F_X} \end{cases}$ (resp. $\begin{cases} \mathrm{End}(T_{|F}) \to \mathrm{End}(T_{|F_Y}) \\ (e_p)_{p \in F} \mapsto (e_p)_{p \in F_Y} \end{cases}$). (**) Def and Rk (then (***))

(*) **Def:** If D is finite then we denote by $f_T : \mathcal{C}(D,T) \to R\text{-Mod}$ the forgetful functor and by $\tilde{T} : D \to \mathcal{C}(D,T)$ the repr. which sends $p \in D$ to T_p (✓ since $(e_q)_{q \in D} \cdot x = e_p(x)$ makes T_p into an object of $\mathcal{C}(D,T)$) and $f \in D(p,q)$ to Tf (✓ since $e_q(Tf(x)) = Tf(e_p(x))$ for all $x \in T_p$ by def. of $\mathrm{End}(T)$).

Rk: $T = f_T \circ \tilde{T}$. (**)

Def: We denote by f_T the forgetful functor (✓ to R-Mod) and by $\tilde{T} : D \to \mathcal{C}(D,T)$ the repr. which sends $p \in D$ to T_p (via $\mathcal{C}(p, T_{|p}) \to \operatorname{colim}_F \mathcal{C}(F, T_{|F})$) and $f \in D(p,q)$ to Tf (via $\mathrm{Mor}_{\mathcal{C}(\cdot_p \to \cdot_q,\, T_{|\cdot_p \to \cdot_q})}(T_p, T_q) \to \operatorname{colim}_F \mathrm{Mor}_{\mathcal{C}(F, T_{|F})}$).

Rk: $T = f_T \circ \tilde{T}$.

Lemma: If D is finite then $\mathrm{End}(T)$ is finitely generated as an R-mod.

Proof: For all $p \in D$: T_p f.g. R-mod & R Noetherian $\Rightarrow$ $\mathrm{End}_R(T_p)$ f.g. R-mod. Thus, $\prod_{p \in D} \mathrm{End}_R(T_p)$ is a f.g. R-mod since D is finite. Since R is Noetherian, every submodule of a f.g. R-module is f.g., hence $\mathrm{End}(T)$ is f.g..

(***) [Def. "with identities" coincides and $\mathcal{C}(P(D), T_{\text{induced}}) = \mathcal{C}(D,T)$] (path category)

$\operatorname{colim}_F \mathrm{Hom}_{\mathrm{End}(T_{|F})}(X_F, Y_F) \simeq \bigoplus_F \mathrm{Hom}_{\mathrm{End}(T_{|F})}(X_F, Y_F) / (\varphi \sim \varphi_{F'})$, $\varphi \in \mathrm{Hom}_{\mathrm{End}(T_{|F})}(X_F, Y_F)$, $F \subset F'$, $\varphi_{F'} \in \mathrm{Hom}_{\mathrm{End}(T_{|F'})}(X_{F'}, Y_{F'})$, $\varphi_{F'}(x) = \varphi(x)$ for all $x \in X$.

(in particular: $\forall p', q' \in D'\ \forall f' \in D'(p', q')\ e_{F(q')} \circ \overbrace{T(F(f'))}^{(T\circ F)(f')} = \overbrace{T(F(f'))}\circ e_{F(p')}$)

(Easier proof: if $(e_p) \in \prod_{p\in D} \mathrm{End}_R(T_p)$ verifies: $\forall p, q \in D\ \forall f \in D(p,q)\ e_q \circ Tf = Tf \circ e_p$ then)

(Skip for now.

(□) Lemma: Let D', D be diagrams, $F: D' \to D$ be a map of diagrams and $T: D \to R\text{-Mod}$ be a repr. The morphism of R-algebras $F^*: \begin{cases} \mathrm{End}(T) \to \mathrm{End}(T\circ F) \\ (e_p)_{p\in D} \mapsto (e_{F(p')})_{p'\in D'} \end{cases}$ is well-defined.

Proof: (too long) F induces $F^*: \begin{cases} \prod_{p\in D} \mathrm{End}_R(T_p) \to \prod_{p'\in D'} \mathrm{End}_R((T\circ F)p') & \text{(m. of } R\text{-algebras)} \\ (e_p)_{p\in D} \mapsto (e_{F(p')})_{p'\in D'} \end{cases}$

and $\tilde{F}^*: \begin{cases} \prod_{p,q\in D} \prod_{f\in D(p,q)} \mathrm{Hom}_R(T_p, T_q) \to \prod_{p',q'\in D'} \prod_{f'\in D'(p',q')} \mathrm{Hom}_R((T\circ F)p', (T\circ F)q') \\ (\varphi_{p,q,f})_{\substack{p,q\in D \\ f\in D(p,q)}} \mapsto (\varphi_{F(p'),F(q'),F(f')})_{\substack{p',q'\in D' \\ f'\in D'(p',q')}} \end{cases}$ (m. of R-modules) (a.k.a. R-linear map)

which verify: $\forall p,q \in D\ \forall f \in D(p,q)\ \forall \underset{\mathrm{End}_R(T_x)}{(e_x)_{x\in D}}\quad \tilde{F}^*(e_q \circ Tf - Tf \circ e_p)$

$$= e_{F(q')} \circ (T\circ F)f' - (T\circ F)f' \circ e_{F(p')}$$

i.e. $\tilde{F}^* \circ \Phi_T = \Phi_{T\circ F} \circ F^*$ with $\Phi_T: (e_x)_{x\in D} \mapsto (e_q \circ Tf - Tf \circ e_p)_{\substack{p,q\in D \\ f\in D(p,q)}}$ R-linear map verifying $\mathrm{End}(T) = \mathrm{Ker}(\Phi_T)$ and $\Phi_{T\circ F}: (e_{x'})_{x'\in D'} \mapsto (e_{q'} \circ (T\circ F)f' - (T\circ F)f' \circ e_{p'})_{\substack{p',q'\in D' \\ f'\in D'(p',q')}}$ R-linear map verifying $\mathrm{End}(T\circ F) = \mathrm{Ker}(\Phi_{T\circ F})$, so that $F^*(\mathrm{End}(T)) \subset \mathrm{End}(T\circ F)$.

Thus, F^* induces a morphism of R-algebras from $\mathrm{End}(T)$ to $\mathrm{End}(T\circ F)$.

Consider the category of R-linear abelian categories (i.e. abelian categories enriched in R-modules). For each diagram D and representation $T: D \to R\text{-Mod}$, each $\mathcal{C}(F, T_{|F})$ with F finite full subdiagram of D is an R-linear abelian category and if $F \subset F'$ then $\begin{cases} \mathrm{End}(T_{|F'}) \to \mathrm{End}(T_{|F}) \\ (e_p)_{p\in F'} \mapsto (e_p)_{p\in F} \end{cases}$ induces an R-linear faithful exact functor $\mathcal{C}(F, T_{|F}) \to \mathcal{C}(F', T_{|F'})$ and the 2-colimit of these categories and functors is $\mathcal{C}(D,T)$.

$\mathcal{E}(D,T)$ R-linear, well-defined

(co)products and (co)kernels in $\mathcal{E}(D,T)$

every kernel is a monom. and every cokernel is an epi

Prop: $\mathcal{E}(D,T)$ is an R-linear abelian category and $f_T: \mathcal{E}(D,T) \to R\text{-Mod}$ is an R-linear faithful exact functor.

Lemma (□) ↑ then:

Lemma: Let D', D be diagrams, $F: D' \to D$ be a map of diagrams and $T: D \to R\text{-Mod}$ be a representation. There exists an R-linear faithful exact functor $\mathcal{F}: \mathcal{E}(D', T\circ F) \to \mathcal{E}(D,T)$ such that the following diagram commutes:

$$\begin{tikzcd} D' \arrow[r, "F"] \arrow[d, "\widetilde{T\circ F}"'] & D \arrow[d, "\tilde{T}"] \\ \mathcal{E}(D', T\circ F) \arrow[r, "\mathcal{F}"] \arrow[dr, "f_{T\circ F}"'] & \mathcal{E}(D,T) \arrow[d, "f_T"] \\ & R\text{-Mod} \end{tikzcd}$$

Proof: If D' and D are finite then (□) yields our functor by restriction of scalars. Generalise to D' finite, D arbitrary, then to D' and D arbitrary.

take any $E \subset D$ finite full subdiagram and compose $\mathcal{E}(D', T\circ F) \to \mathcal{E}(E, T_{|E})$ with the canonical $\mathcal{E}(E, T_{|E}) \to \mathcal{E}(D,T) = $ 2-colim

to conclude, use the univ. prop. of the 2-colim.

Ex: Natural transformations ← **Def:** A morphism $\varphi : T_1 \to T_2$ of representations $T_1, T_2 : D \to \mathcal{C}$ (cat.) is given by
$p \in D \mapsto \varphi_p : T_1 p \to T_2 p$ morphism in $\mathcal{C}$ with:
$\forall p,q \in D\ \forall f \in D(p,q)\quad T_2 f \circ \varphi_p = \varphi_q \circ T_1 f.$

Lemma: Let $\varphi : T_1 \xrightarrow{:D\to R\text{-Mod}} T_2 \, {:}D \to R\text{-Mod}$ be an isomorphism of representations: φ induces an equivalence of categories $\Phi : \mathcal{C}(D,T_1) \to \mathcal{C}(D,T_2)$ and an isomorphism of representations $\tilde{\varphi} : \Phi \circ \tilde{T}_1 \to \tilde{T}_2$ which verifies $f_{T_2} \circ \tilde{\varphi} = \varphi$.

Proof: φ induces the iso. of R-algebras $\begin{cases} \mathrm{End}(T_1) \to \mathrm{End}(T_2) \\ (\psi_p)_{p\in D} \mapsto (\varphi_p \circ \psi_p \circ \varphi_p^{-1})_{p \in D} \end{cases}$ since:

$$\begin{aligned}\forall p,q \in D\ \forall f \in D(p,q)\quad \varphi_q \circ \psi_q \circ \varphi_q^{-1} \circ T_2 f &= \varphi_q \circ \psi_q \circ T_1 f \circ \varphi_p^{-1} \quad (\varphi^{-1} \text{ is a m. of repr.})\\ &= \varphi_q \circ T_1 f \circ \psi_p \circ \varphi_p^{-1} \quad ((\psi_p)_{p\in D} \in \mathrm{End}(T_1))\\ &= T_2 f \circ \varphi_p \circ \psi_p \circ \varphi_p^{-1} \quad (\varphi \text{ is a m. of repr.})\end{aligned}$$

which by extension of scalars induces Φ when D is finite (then take 2-colim on the right and finally 2-colim on the left). (⚠ Equivalence not iso. of categories since our categories are considered up to equivalence to manipulate "small" categories (in a given universe).) $\tilde{\varphi} : p \mapsto \varphi_p$ is well-defined.

Lemma: Let D be a finite diagram, $T : D \to R\text{-Mod}$ be a representation and S be a flat (a.k.a. c.) (alte. commu.) R-algebra (with unit) [and assume S is finitely gen. as an R-module to ensure S is a Noetherian ring]. $\mathrm{End}_{(S)}(T_S) \overset{\text{can.}}{\cong} \mathrm{End}_{(R)}(T) \otimes_R S$, where $T_S = (- \otimes_R S) \circ T$.

Proof: As S is flat over R, $- \otimes_R S$ is exact hence the following sequence is exact:

$$0 \to \mathrm{End}_R(T) \otimes_R S \xrightarrow{\subseteq \otimes \mathrm{Id}_S} \prod_{p \in D} \mathrm{End}_R(T_p) \otimes_R S \xrightarrow{((e_p) \mapsto (e_q \circ Tf - Tf \circ e_p)) \otimes_R \mathrm{Id}_S} \prod_{p,q \in D} \prod_{f \in D(p,q)} \mathrm{Hom}_R(T_p, T_q) \otimes_R S$$

↑ ⚠ not finite in gen.

↑ Because finite $\prod$ and $\otimes$ (tensor products) commute (as functors)

Note that for each $g \in (Tf)_{f \in D(p,q)} =: M_{p,q} \subseteq \mathrm{Hom}_R(T_p, T_q)$ (submodule) and $(e_p)_{p \in D} \in \mathrm{End}_R(T)$, $e_q \circ g - g \circ e_p = 0$ (since $g = \lambda_1 T f_1 + \dots + \lambda_n T f_n$, $e_q \in \mathrm{End}_R(T_q)$ and $e_p \in \mathrm{End}_R(T_p)$) and similarly if $G_{p,q} \subseteq M_{p,q}$ (subset) is a set of generators of the R-module $M_{p,q}$ for each $p,q \in D$ then it suffices that for each $p,q \in D$ and $\forall g \in G_{p,q}$ $e_q \circ g - g \circ e_p = 0$ to ensure $(e_p)_{p \in D} \in \mathrm{End}_R(T)$.

Further note that since R is Noetherian and T_p, T_q are finitely gen. over R, the R-mod $\mathrm{Hom}_R(T_p, T_q)$ and all of its submodules (in part. $M_{p,q}$) are fin. gen., so that we can take a finite set of generators $G_{p,q}$ of $M_{p,q}$. We have the exact seq.:

$$0 \to \mathrm{End}_R(T) \otimes_R S \xrightarrow{\subseteq \otimes \mathrm{Id}_S} \prod_{p \in D} (\mathrm{End}_R(T_p) \otimes_R S) \xrightarrow{((e_p) \mapsto (e_q \circ g - g \circ e_p)) \otimes_R \mathrm{Id}_S} \prod_{p,q \in D} \prod_{g \in G_{p,q}} (\mathrm{Hom}_R(T_p, T_q) \otimes_R S)$$

Furthermore, as T_p is a fin. presented R-module for each $p\in D$ (since $\mathrm{Ker}(R^n \twoheadrightarrow T_p)$ is a submodule of R^n and R is a Noetherian ring ($\Leftrightarrow$ every submodule of a f.g. R-module is f.g.)) and S is flat:

$\forall p,q\in D$ $\mathrm{End}_R(T_p)\otimes_R S \underset{\text{can.}}{\simeq} \mathrm{End}_S(\overbrace{T_p\otimes_R S}^{T_{Sp}})$ and $\mathrm{Hom}_R(T_p,T_q)\otimes_R S \underset{\text{can.}}{\simeq} \mathrm{Hom}_S(\overbrace{T_p\otimes_R S}^{T_{Sp}}, \overbrace{T_q\otimes_R S}^{T_{Sq}})$

so that we have the exact sequence:

$$0\to \mathrm{End}_R(T)\otimes_R S \xrightarrow{\hookrightarrow\otimes_R \mathrm{id}_S} \prod_{p\in D}\mathrm{End}_S(T_{Sp}) \xrightarrow{\Psi:=((\varphi_p)\mapsto(\varphi_q\circ g-g\circ\varphi_p))\otimes_R \mathrm{id}_S} \prod_{p,q\in D}\prod_{g\in G_{pq}} \mathrm{Hom}_S(T_{Sp},T_{Sq})$$

i.e. $\mathrm{End}_R(T)\otimes_R S \underset{\text{can.}}{=} \mathrm{Ker}(\Psi) \underset{\text{can.}}{\simeq} \mathrm{End}_S(T_S)$ by flatness of S over R ($M_{p,q}\otimes_R S$ is gen. over S by $G_{p,q}$ and is the S-submodule of $\mathrm{Hom}_S(T_{Sp},T_{Sq})$ generated by the $T_S f$ with $f\in D(p,q)$).

<u>Cor:</u> Let D be a diagram, $T: D\to R\text{-Mod}$ be a repr. such that each T_p is fin. pres. (e.g. in R-Proj) and S be a flat (ass. & comm.) R-algebra (with unit) [and assume S is fin. gen. as an R-module to ensure S is a Noetherian ring]. There is a canonical R-linear faithful functor $-\otimes_R S: \mathcal{E}(D,T)\to\mathcal{E}(D,T_S)$ which is compatible with $-\otimes_R S: R\text{-Mod}\to S\text{-Mod}$.

<u>Proof:</u> When D is finite, $-\otimes_R S: \mathcal{E}(D,T)\to\mathcal{E}(D,T_S)$ comes from the extension of scalars w.r.t. $-\otimes_R S: \mathrm{End}(T)\longrightarrow \mathrm{End}(T)\otimes_R S\simeq \mathrm{End}(T_S)$; then take the 2-colimit on the right and finally take the 2-colimit on the left).

<u>Thm:</u> If $\mathcal{A}$ is an R-linear abelian cat. and $T:\mathcal{A}\to R\text{-Mod}$ is an R-linear faithful exact functor then $\tilde{T}:\mathcal{A}\to\mathcal{E}(\mathcal{A},T)$ is an equiv. of cats.

<u>Proof:</u> Next talk!

<u>Rk:</u> This is the main ingredient in the proof of the following univ. property.

<u>Thm</u> (universal property): If $\mathcal{A}$ is an R-linear abelian category, $F: D\to\mathcal{A}$ is a representation and $f:\mathcal{A}\to R\text{-Mod}$ is an R-linear faithful exact functor with $T=f\circ F$ then there exists an R-linear faithful exact functor $L(F):\mathcal{E}(D,T)\to\mathcal{A}$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \mathcal{E}(D,T) \arrow[dr, "f_T"] \arrow[dd, dashed, "L(F)"] & \\
D \arrow[ur, "\tilde{T}"] \arrow[dr, "F"'] & & R\text{-Mod} \\
 & \mathcal{A} \arrow[ur, "f"'] &
\end{tikzcd}$$

$L(F)$ is unique up to unique iso. of additive exact functors.

<u>Rk:</u> $\mathcal{E}(D,T)$ (resp. $\tilde{T}$, f_T) is determined by this universal property up to unique equivalence of categories (resp. up to unique iso. of repr., up to unique natural isomorphism).

<u>Proof:</u> Next talk!

<u>Cor:</u> Let $T,T': D\to R\text{-Mod}$ be repr., S be a faithfully flat (ass. & comm.) R-algebra (with unit) [and assume S is f.g. as an R-module to ensure S is a Noetherian ring] and $\varphi: T_S\to T'_S$ be an iso. of repr. in S-Mod: φ induces an equivalence of categories $\Phi:\mathcal{E}(D,T)\to\mathcal{E}(D,T')$. $\rightsquigarrow$ generalises the second to last Lemma.

<u>Proof:</u> Next talk!

Ex: Let D_1 be a full subdiagram of D_2 and $T_2 : D_2 \to R\text{-Mod}$ be a representation.
We denote $T_1 := T_2|_{D_1} : D_1 \to R\text{-Mod}$ and by $i : \mathcal{C}(D_1,T_1) \to \mathcal{C}(D_2,T_2)$ the functor induced by the inclusion $D_1 \subset D_2$ (so that $f_{T_2} \circ i = f_{T_1}$).
"If there is a repr. $F : D_2 \to \mathcal{C}(D_1,T_1)$ and an iso. of repr. $\varphi : T_2 \xrightarrow{\sim} f_{T_1} \circ F$ then i is an equivalence of categories of "inverse" $\pi : \mathcal{C}(D_2,T_2) \to \mathcal{C}(D_1,T_1)$ given by $L(F) \circ \Phi$ with $L(F) : \mathcal{C}(D_2, f_{T_1}\circ F) \to \mathcal{C}(D_1,T_1)$ the functor of the universal property of $\mathcal{C}(D_2, f_{T_1} \circ F)$ and $\Phi : \mathcal{C}(D_2,T_2) \to \mathcal{C}(D_2, f_{T_1} \circ F)$ the equivalence of categories induced by φ (see the second to last Lemma or the last Corollary)."

Proof: By the uniqueness in the universal property of $\mathcal{C}(D_2, f_{T_1} \circ F)$,

$$i \circ L(F) \underset{nat.}{\simeq} L(i \circ F) \underset{nat.}{\simeq} \Phi^{-1} \text{ hence } i \circ \pi \underset{nat.}{\simeq} Id_{\mathcal{C}(D_2,T_2)}$$

in next talk (or work it out from the second to last Lemma)

By the uniqueness in the univ. prop. of $\mathcal{C}(D_1, f_{T_1} \circ F_{|D_1})$,

$$L(F) \circ i' \underset{nat.}{\simeq} L(F_{|D_1}) \underset{nat.}{\simeq} \Phi_{D_1}^{-1} \text{ hence } \pi \circ i \underset{nat.}{\simeq} Id_{\mathcal{C}(D_1,T_1)} \text{ since } \pi \circ i = L(F) \circ \underbrace{i' \circ \Phi_{D_1}}_{\Phi \circ i}$$

in next talk (or ...)

$\Phi \circ i$ (since $\varphi_{|D_1} \rightsquigarrow \Phi_{D_1}$ and $\varphi \rightsquigarrow \Phi$; see the second to last Lemma)

where $i' : \mathcal{C}(D_1, f_{T_1} \circ F_{|D_1}) \to \mathcal{C}(D_2, f_{T_1} \circ F)$ is the functor induced by the inclusion $D_1 \subset D_2$ and $\Phi_{D_1} : \mathcal{C}(D_1,T_1) \to \mathcal{C}(D_1, f_{T_1} \circ F_{|D_1})$ is the equiv. of cats induced by $\varphi_{|D_1}$.

III) Nori's diagram category as comodules over a coalgebra

Lemma (Prop 5.2 in Quantum groups (2007) by Street): Let R be a Noetherian (commut.) ring (with unit) and E be an R-algebra which is f.g. as an R-module.
E is projective iff $\forall M$ R-module $\rho : \overset{Hom_R(E,R)}{E^\vee} \otimes_R M \to Hom_R(E,M)$ is an isomorphism
$\sum_i \varphi_i \otimes m_i \mapsto (x \mapsto \sum_i \varphi_i(x).m_i)$

Rk: In general, even $\rho : E^\vee \otimes_R E^\vee \to Hom_R(E,E^\vee) \simeq (E \otimes_R E)^\vee$ is not an isomorphism so that when E is an algebra, $E^\vee$ is not necessarily a coalgebra.

Def: A coalgebra C over R is an R-module together with a "comultiplication" (a.k.a. "coproduct") $\Delta : C \to C \otimes_R C$ and a "counit" $\varepsilon : C \to R$ which are R-linear maps such that:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowleft"] & C \otimes_R C \arrow[d, "Id_C \otimes \Delta"] \\ C \otimes_R C \arrow[r, "\Delta \otimes Id_C"'] & C \otimes_R C \otimes_R C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "Id_C"] & C \otimes_R C \arrow[d, "Id_C \otimes \varepsilon"] \\ C \otimes_R C \arrow[r, "\varepsilon \otimes Id_C"'] & R \otimes_R C \simeq C \simeq C \otimes_R R \end{tikzcd}$$

(coassociativity) (counit property)

then

Def: A morphism $f : C_1 \to C_2$ of coalgebras is an R-linear map s.t.:

$$\begin{tikzcd} C_1 \arrow[r, "\Delta_1"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & C_1 \otimes_R C_1 \arrow[d, "f \otimes f"] \\ C_2 \arrow[r, "\Delta_2"'] & C_2 \otimes_R C_2 \end{tikzcd}$$

and

$$\begin{tikzcd} C_1 \arrow[r, "f"] \arrow[dr, "\varepsilon_1"'] & C_2 \arrow[d, "\varepsilon_2"] \\ & R \end{tikzcd}$$

Def: Let C be a coalgebra over R. A right comodule over C is an R-module M together with a "co-scalar multiplication" $\psi: M \to M \otimes_R C$ which is an R-linear map s.t.:

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\circlearrowleft"] & M \otimes_R C \arrow[d, "\psi \otimes \mathrm{Id}_C"] \\ M \otimes_R C \arrow[r, "\mathrm{Id}_M \otimes \Delta"'] & M \otimes_R C \otimes_R C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M \arrow[r, "\psi"] \arrow[dr, "\cong"'] & M \otimes_R C \arrow[d, "\mathrm{Id}_M \otimes \varepsilon"] \\ & M \otimes_R R \end{tikzcd}.$$

(compatibility with Δ) (comp. with ε)

Def: A morphism $f: M_1 \to M_2$ of right C-comodules is an R-linear map s.t.:

$$\begin{tikzcd} M_1 \arrow[r, "\psi_1"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & M_1 \otimes_R C \arrow[d, "f \otimes \mathrm{Id}_C"] \\ M_2 \arrow[r, "\psi_2"'] & M_2 \otimes_R C \end{tikzcd}.$$

Lemma: Let $E \in R\text{-Proj}$ be an algebra.

1. $E^\vee := \mathrm{Hom}_R(E, R)$ is naturally a coalgebra.
2. Any left E-module which is f.g. over R is naturally a right $E^\vee$-comodule.
3. The cat. of left E-modules which are f.g. over R is equivalent to the cat. of right $E^\vee$-comodules which are f.g. over R.

Proof: Should be straightforward thanks to the preceding Lemma.

Not: Let C be a coalgebra over R. C-Comod is the cat. of right C-comodules which are f.g. as R-modules.

Ex: Let R be a field or a Dedekind domain (an integral domain which is a Noetherian ring and whose localization at each maximal ideal is a discrete valuation ring) and $T: D \to R\text{-Proj}$ be a representation. The colimit in the cat. of R-modules over finite full subdiagrams F of D of $\mathrm{End}(T_{|F})^\vee$, denoted $A(D,T)$, has the structure of a coalgebra such that $A(D,T)$-Comod is equivalent to $\mathcal{C}(D,T)$.

Nori's diagram cat.

Proof: Let F be a finite full subdiagram of D.

$\forall p \in F$ $T_p \in R\text{-Proj}$ (f.g. proj. R-module) $\Rightarrow \mathrm{End}_R(T_p) \in R\text{-Proj}$ and finite products of f.g. projective R-modules are f.g. proj. R-modules so $\prod_{p \in F} \mathrm{End}_R(T_p) \in R\text{-Proj}$. As a submodule of this, $\mathrm{End}(T_{|F})$ is f.g. as R is Noetherian. $\triangle$ R Noetherian $\not\Rightarrow$ submodules of R-Proj. are R-Proj.

Since R is a field or a Dedekind domain, f.g. proj. R-modules are exactly f.g. torsion-free R-modules (f.g. M s.t. $\forall r \in R \, \forall m \in M$ $r.m = 0 \Rightarrow (r = 0$ or $m = 0))$.

or r zero-divisor in gen. (but here R is an int. dom.)

As a submodule of a f.g. proj. hence f.g. torsion-free module, $\mathrm{End}(T_{|F})$ is f.g. torsion-free hence f.g. projective. By the previous Lemma, $\mathrm{End}(T_{|F})^\vee := \mathrm{Hom}_R(\mathrm{End}(T_{|F}), R)$ is naturally a coalgebra over R and $\mathcal{C}(F, T_{|F})$ is equivalent to $\underbrace{\mathrm{End}(T_{|F})^\vee}_{A(F,T_{|F})}$-Comod.

Since the underlying R-module of a colimit of a direct system of R-coalgebras (here a morphism for each $F' \subset F$) is the colimit of this direct system in the category of R-modules, $A(D,T)$ has a natural structure of coalgebra over R.

[Direct(ed)/filtered system: a m. for each $F' \leq F$ with $\leq$ refl. and trans. (preorder) s.t. $\forall a, b \in$ set $\exists c \geq a$ and $\geq b$]

We now want to show that $A(D,T)$-Comod is equivalent to $\mathcal{E}(D,T) = 2\text{-}\underset{F}{\mathrm{colim}}\, \mathcal{E}(F,T_{|F})$.
Since for each finite full subdiagram $F \subset D$ $\mathcal{E}(F,T_{|F})$ is equivalent to $A(F,T_{|F})$-Comod, $\mathcal{E}(D,T)$ is equivalent to $2\text{-}\underset{F}{\mathrm{colim}}\,(A(F,T_{|F})\text{-Comod})$.
By the universal property of the 2-colimit, since for each F we have a functor $\Phi_{F,D} : A(F,T_{|F})\text{-Comod} \to A(D,T)\text{-Comod}$ which is induced by the canonical morphism of R-coalgebras $A(F,T_{|F}) \to A(D,T)$ and we have for each $F \subset F'$ finite full subdiagrams of D: $\Phi_{F',D} \circ \Phi_{F,F'} = \Phi_{F,D}$ (with $\Phi_{F,F'}$ the functor induced by the canonical m. of R-coalgebras $A(F,T_{|F}) \to A(F',T_{|F'})$), there exists a unique functor (up to unique natural iso.) $u : \mathcal{E}(D,T) \to A(D,T)\text{-Comod}$ such that the following diagram commutes for every finite full subd. $F \subset D$:

$$\begin{tikzcd}
A(F,T_{|F})\text{-Comod} \arrow[r, "\text{can}"] \arrow[dr, "\Phi_{F,D}"'] & \mathcal{E}(D,T) \simeq 2\text{-}\underset{F'}{\mathrm{colim}}\,(A(F',T_{|F'})\text{-Comod}) \arrow[d, "u"] \\
 & A(D,T)\text{-Comod}
\end{tikzcd}$$

We claim that $u : \mathcal{E}(D,T) \to A(D,T)\text{-Comod}$ is an equivalence of categories of "inverse" $v : A(D,T)\text{-Comod} \to \mathcal{E}(D,T)$ which to a right $A(D,T)$-comodule M f.g. over R, $M = (x_1, \dots, x_n)$ over R, with comultiplication $m : M \to M \underset{R}{\otimes} A(D,T)$ given by: $\forall i \in \{1,\dots,n\}$ $m(x_i) = \sum_{k=1}^{n} x_k \otimes a_{k,i}$, $a_{k,i} \in A(D,T)$ coming from $A(F_{k,i}, T_{|F_{k,i}})$, $F_{k,i}$ finite full $\subset D$, associates the image of the $A(\bigcup_{i=1}^{n} \bigcup_{k=1}^{n} F_{k,i}, T_{|\bigcup_i \bigcup_k F_{k,i}})$-comodule M (f.g. over R) in $\mathcal{E}(D,T) \simeq 2\text{-}\underset{F}{\mathrm{colim}}\,(A(F,T_{|F})\text{-Comod})$ via the canonical morphism, and to a morphism $f : M_1 \to M_2$ of right $A(D,T)$-comodules f.g. over R associates in a similar way a morphism $vf : vM_1 \to vM_2$ in $\mathcal{E}(D,T)$. Our claim (that $u \circ v$ is nat. iso. to $\mathrm{Id}_{A(D,T)\text{-Comod}}$ and $v \circ u$ is nat. iso. to $\mathrm{Id}_{\mathcal{E}(D,T)}$) follows from the universal property used to define u.

__Lemma:__ Let R be a field or a Dedekind domain, $T : D \to R\text{-Proj}$ be a repr. and S be a flat (ass. & comm.) R-algebra (with unit) [and assume S is f.g. as an R-module to ensure S is a Noetherian ring]. $A(D,T_S) \underset{\text{can}}{\simeq} A(D,T) \underset{R}{\otimes} S$.

__Proof:__ Since $- \underset{R}{\otimes} S$ commutes with colimits, it suffices to prove this when D is finite. If D is finite then $A(D,T) = \mathrm{End}(T)^{\vee} = \mathrm{Hom}_R(\underbrace{\mathrm{End}(T)}_{\substack{\text{fin. gen. hence} \\ \text{fin. pres. over} \\ R \text{ Noetherian}}}, R)$ and, as S is flat, $A(D,T) \underset{R}{\otimes} S \underset{\text{can}}{\simeq} \mathrm{Hom}_S(\underbrace{\mathrm{End}(T) \underset{R}{\otimes} S}_{\underset{\text{can}}{\simeq} \mathrm{End}(T_S) \text{ (see the last Lemma of (I))}}, S)$.
Hence $A(D,T) \underset{R}{\otimes} S \underset{\text{can}}{\simeq} A(D,T_S)$.

<u>Not:</u> Let k be a field. k-Vect is the category of finite-dim. k-vector spaces.

<u>Prop:</u> Let k be a field, $\mathcal{A} \subset \mathcal{B}$ be a full abelian subcategory which is closed under subquotients and $T: \mathcal{B} \to k\text{-Vect}$ be a faithful exact functor. The morphism of k-coalgebras $A(\mathcal{A}, T_{|\mathcal{A}}) \to A(\mathcal{B}, T)$ induced by $\mathcal{A} \subset \mathcal{B}$ is injective.

($\uparrow$ colimits and $\mathrm{End}(T_{|\mathcal{A}})^\vee \to \mathrm{End}(T)^\vee$ comes from $\left\{ \begin{array}{l} \mathrm{End}(T) \overset{\text{proj.}}{\to} \mathrm{End}(T_{|\mathcal{A}}) \\ (e_p)_{p \in \mathcal{B}} \mapsto (e_p)_{p \in \mathcal{A}} \end{array} \right.$ when $\mathcal{A}$ and $\mathcal{B}$ are finite)

<u>Proof:</u> By the Theorem mentioned just before the univ. prop. (in I),
$\tilde{T}: \mathcal{B} \to \mathcal{E}(\mathcal{B}, T)$ and $\widetilde{T_{|\mathcal{A}}}: \mathcal{A} \to \mathcal{E}(\mathcal{A}, T)$ are equiv. of cat.
and by the last Corollary $\mathcal{E}(\mathcal{B}, T) \simeq A(\mathcal{B}, T)\text{-Comod}$ and $\mathcal{E}(\mathcal{A}, T_{|\mathcal{A}}) \simeq A(\mathcal{A}, T_{|\mathcal{A}})\text{-Comod}$
so $\mathcal{B} \simeq \underbrace{A(\mathcal{B}, T)}_{A:=}\text{-Comod}$ and $\mathcal{A} \simeq \underbrace{A(\mathcal{A}, T_{|\mathcal{A}})}_{A':=}\text{-Comod}$.

The morphism of k-coalgebras $A' \to A$ turns A' into an A-comodule.
We denote by B the image of A' in A. B is an A-comodule (since the category of A-comodules is abelian) and even an A'-comodule (since $\mathcal{A}$ is closed under subquotients in $\mathcal{B}$).

We have the following commutative diagram (with $\varepsilon: A \to k$ the counit of A):

$$\begin{tikzcd}
B \arrow[r, "A'\text{-comod.}"] & B \otimes_k A' \arrow[r, "\varepsilon_{|B} \otimes \mathrm{Id}"] & A' \\
A' \arrow[u, two heads, "\text{induced by } A' \to A"] \arrow[r, "A\text{-comod.}"] \arrow[urr, bend right, "\mathrm{Id}_{A'}"'] & A' \otimes_k A \arrow[ur, "\mathrm{Id} \otimes \varepsilon"'] &
\end{tikzcd}$$

so that $A' \twoheadrightarrow B$ is injective hence $A' \to A$ is injective.

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