Babyseminar SS 2024 - Talk 1

Algebraic and holomorphic de Rham cohomology

Notation: · k field of char. 0 (sometimes $k \subseteq \mathbb{C}$)

· Sch/k category of schemes over k which are separated and of finite type

· Var/k category of varietés over k (quasiprojective reduced $X \in$ Sch/k)

· Sm/k category of smooth varieties over k

· An category of complex analytic spaces (locally $\cong$ vanishing locus of some holomorphic functions)

§0. Hypercohomology

X top. space, $Sh(X)$ category of sheaves of abelian groups on X

$\mathcal{F}^\bullet : \dots \to 0 \to 0 \to \mathcal{F}^N \to \mathcal{F}^{N+1} \to \dots$ bounded below complex of sheaves of abelian groups on X

$R\Gamma(X, \mathcal{F}^\cdot) := \Gamma(X, \mathcal{I}^\cdot)$, $H^i(X, \mathcal{F}^\cdot) = H^i(\Gamma(X, \mathcal{I}^\cdot))$, $\mathcal{F}^\cdot \overset{e}{\hookrightarrow} \mathcal{I}^\cdot$ injective resolution.

Here we mean that $\mathcal{I}^\cdot$ is a bounded below complex of injective sheaves and e quasi-iso

Equivalently: $H^i(X, \mathcal{F}) = \mathrm{Hom}_{D^+(Sh(X))}(\mathbb{Z}, \mathcal{F}^\cdot[i])$

Note: $\mathcal{F}^\cdot$ admits also a so-called Cartan-Eilenberg resolution $\mathcal{F}^\cdot \to \mathcal{J}^{\cdot,\cdot}$

In particular $R\Gamma(X, \mathcal{F}^\cdot) \cong \mathrm{Tot}(\Gamma(\mathcal{J}^{\cdot,\cdot})) \rightsquigarrow$ one obtains two spectral sequences:

$$E_1^{ij} = H^j(X, \mathcal{F}^i) \Rightarrow H^{i+j}(X, \mathcal{F}^\cdot) \quad \text{(affording the Hodge filtration!)}$$

$$E_2^{ij} = H^i(X, \mathcal{F}^j) \Rightarrow H^{i+j}(X, \mathcal{F}^\cdot)$$

§1. Algebraic de Rham cohomology - smooth case

$X \in$ Sm/k, $\Omega^1_{X/k} = \Omega^1_X$ sheaf of k-linear algebraic differentials on X

(Zariski sheafification of $\mathrm{Spec}(A) \rightsquigarrow \Omega^1_{A/k}$ $\quad \mathrm{Der}_k(A, M) \cong \mathrm{Hom}_A(\Omega^1_{A/k}, M)$)

Recall: · X smooth of dimension $n \Rightarrow \Omega^1_X$ locally free of rank n

· $d : \mathcal{O}_X \to \Omega^1_X$ universal derivation (⚠ only k-linear)

Set $\Omega^p_X := \Lambda^p \Omega^1_X$ for $p \geq 0$ $\quad (\Omega^0_X = \mathcal{O}_X)$.

Then d induces $d^p : \Omega^p_X \to \Omega^{p+1}_X$ uniquely characterised by:

- $d^0 = d$ $\quad$ • $d^{p+1} \circ d^p = 0$
- $d^{p+q}(\omega \wedge \eta) = d^p\omega \wedge \eta + (-1)^p \omega \wedge d^q \eta$ for local sections ω (resp η) of Ω^p_X (resp. Ω^q_X)

Explicit formula: $x \in X$, $t_1 \dots t_n$ system of local parameters at $x \in X$, then locally near x a section ω of Ω^p_X can be described as

$$\omega = \sum_{1 \leq i_1 < \dots < i_p \leq n} f_{i_1 \dots i_p} \, dt_{i_1} \wedge \dots \wedge dt_{i_p}$$

$$\Rightarrow d\omega = \sum_{1 \leq i_1 < \dots < i_p \leq n} df_{i_1 \dots i_p} \wedge dt_{i_1} \wedge \quad \wedge dt_{i_p}$$

Def: $X \in Sm/k$. The algebraic de Rham complex of X is $(\Omega^\bullet_X, d^\bullet)$ as above.
The algebraic de Rham cohomology of X is given by:

$R\Gamma_{dR}(X) := R\Gamma(X, \Omega^\bullet_X)$ $\leadsto$ hypercohomology

Rmk: (i) $X \in Sm/k$ affine $\Rightarrow$ $H^i_{dR}(X) = H^i(\Omega^\bullet_X(X))$ (use your favourite spectral seq.)

(ii) $X \in Sm/k$ of dimension n $\Rightarrow$ $H^i_{dR}(X) = 0$ if $i > 2n$ $\quad (i > n$ if X affine$)$

$\hookrightarrow$ pf: the affine case follows from (i). In general we know that:

$H^i(X, \Omega^j_X) = 0$ if $j > n$ or if $i > n$ $\quad (\Omega^j_X$ coherent sheaf$)$

the assertion follows again by the spectral seq. $\blacksquare$

Lemma 1 (Properties of dR cohomology)

(i) (pull-back) given $f : X \to Y$ morphism in Sm/k, there is an induced functorial morphism $f^* : R\Gamma_{dR}(Y) \to R\Gamma_{dR}(X)$

(ii) (cup product) given $X, Y \in Sm/k$, the wedge product induces a cup product

$$R\Gamma_{dR}(X) \otimes_k R\Gamma_{dR}(X) \xrightarrow{\cup} R\Gamma_{dR}(X)$$

(iii) (Künneth formula) Given $X, Y \in Sm/k$, there is a natural quasi-isomorphism:

$$\begin{array}{ccc} R\Gamma_{dR}(X) \otimes_k R\Gamma_{dR}(Y) & \longrightarrow & R\Gamma_{dR}(X \times_k Y) \\ \text{"} (\omega, \eta) & \longmapsto & p^*\omega \cup q^*\eta \text{"} \end{array}$$

$$\begin{tikzcd} & X \times_k Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

Hence for every $n \geq 0$ we have induced isomorphisms:

$$H^n_{dR}(X \times_k Y) \cong \bigoplus_{i=0}^{n} H^i_{dR}(X) \otimes_k H^{n-i}_{dR}(Y)$$

(iv) (homotopy invariance) $X \in Sm_k$ then $R\Gamma_{dR}(X) \cong R\Gamma_{dR}(X \times_k \mathbb{A}^1)$

(v) (scalar extension) K/k field extension, $X \in Sm/k$ then there is a natural quasi-iso $R\Gamma_{dR}(X) \otimes_k K \to R\Gamma_{dR}(X_K)$

(vi) (scalar restriction) K/k <u>finite</u> field extension, $Y \in Sm/K$, then $\Omega^\bullet_{Y/k} \xrightarrow{\sim} \Omega^\bullet_{Y/K}$

so in particular $R\Gamma_{dR}(Y/k) \xrightarrow{\sim} R\Gamma_{dR}(Y/K)$

<u>Lemma 2</u>: $H^i_{dR}(X)$ is a finite dimensional k-vs for every $X \in Sm/k$ and every $i \in \mathbb{Z}$

<u>Pf</u> (sketch): embed $X \overset{j}{\hookrightarrow} \bar{X}$ with
- $\bar{X}$ projective
- $D := \bar{X} \setminus X$ is a simple normal crossing divisor

one can prove: $\Omega^\bullet_{\bar{X}}\langle D\rangle \hookrightarrow j_* \Omega^\bullet_X$ quasi-iso

↳ complex of differentials with log poles along D ("allow $\frac{dt}{t}$" along the divisor)

$\Omega^j_{\bar{X}}\langle D\rangle$ coherent sheaves on $\bar{X}$ $\Rightarrow$ $H^i(\bar{X}, \Omega^j_{\bar{X}}\langle D\rangle)$ fin. dim. k-vs

$\Rightarrow$ (spectral seq. argument) $H^i_{dR}(X) \cong H^i(\bar{X}, \Omega^\bullet_{\bar{X}}\langle D\rangle)$ are fin. dim k-vs $\blacksquare$

§ 2. Algebraic de Rham cohomology – general case

We want to define $H^*_{dR}(X)$ for $X \in Sch/k$ (not nec. smooth).

Ideally, if $k \subseteq \mathbb{C}$, we expect/would like $\dim_k H^i_{dR}(X) = \dim_{\mathbb{Q}} H^i_{sing}(X, \mathbb{Q})$.

Sadly, the de Rham complex $\Omega^\bullet_{X/k}$ (which would make sense for $X \in Sch/k$) does not always provide the correct answer.

Exercise (for those who will NOT give a talk in the BS): compute $H^1(X, \Omega^\bullet_{X/k})$ and $H^1_{sing}(X, \mathbb{Q})$ for

$$X : \quad s^5 + t^5 + s^2 t^2 = 0$$

The "cleanest" workaround to this problem is given by introducing the so-called h-topology (Voevodsky, 1996) and h-sheafify the constructions.

Def: (i) A morphism of schemes $p : X \to Y$ is a topological epimorphism if the topology on Y is the quotient topology wrt p. The morphism p is a universal top. epi if any base change of p is a top. epi.

(ii) The h-topology on the category Sch/X of separated schemes of finite type$/X$ is the Grothendieck topology with coverings given by finite families $p_i : U_i \to Y$ $i \in I$ $\# I < \infty$ such that $\coprod U_i \to Y$ is a univ. top. epi.

We write $(Sch/X)_h$ to denote the h-site$/X$ ($(Sch/k)_h$ when $X = \mathrm{Spec}(k)$)

Facts: (i) examples of h-covers are: (in part. étale covers)

- flat covers with finite index set
- proper surjective morphisms
- quotients by finite group actions

(ii) for $X \in Sch/k$ $X^{red} \to X$ is clearly an h-cover and for every h-sheaf $\mathcal{F}$ it holds $\mathcal{F}(X) = \mathcal{F}(X^{red})$ ($\Rightarrow$ the h-topology is NOT subcanonical)

Def: Given $X \in Sch/k$, we set $R\Gamma_{dR}(X_h) := R\Gamma_h(X, \Omega^{\bullet}_h)$, where:
$\forall p \geq 0$ Ω^p_h is the h-sheafification of the presheaf $Y \mapsto \Omega^p_Y(Y)$ on Sch/k.
$R\Gamma_{dR}(X_h)$ is called the h-de Rham cohomology of X

Thm 3 (Geissen, Huber-Jörder): If $X \in Sm_k$, then $\forall p \geq 0$ $\Omega^p_h(X) = \Omega^p_X(X)$ and
$\forall i \geq 0$ $H^i_{dR}(X_h) = H^i_{dR}(X)$.

Def: An abstract blow-up of $X \in Sch/k$ is a cover of X of the form
$(f: X' \to X, i: Z \hookrightarrow X)$, i closed immersion, f proper and an iso above $X \setminus Z$.

Ex: Any abstract blow-up is an h-cover.

Thm 4 (G, HJ) Let $(f: X' \to X, i: Z \hookrightarrow X)$ be an abstract blow-up of $X \in Sch/k$.
Then there is a natural long exact sequence:

$$\dashrightarrow H^i_{dR}(X_h) \to H^i_{dR}(X'_h) \oplus H^i_{dR}(Z_h) \to H^i_{dR}((X' \times_X Z)_h) \to H^{i+1}_{dR}(X_h) \to \cdots$$

⌈ Actually, if $e_Y: Y_h \to Y_{Zar}$ is the obvious morphism of sites, one has a dist. triangle:

$$Re_{X*}F \to Rf_* Re_{X'*}F \oplus i_* Re_{Z*}F \to i_* Rf'_* Re_{E*}F \xrightarrow{+1}$$

$E := X' \times_X Z$, $f': E \to Z$ base change of f, F any h-sheaf ⌋

Def: Given $X \in Sch/k$ and $i: Z \hookrightarrow X$ a closed immersion, we set for $p \in \mathbb{Z}_{\geq 0}$

$$\Omega^p_{h/(X,Z)} := \ker(\Omega^p_{h/X} \to i_* \Omega^p_{h/Z})$$ an abelian sheaf on $(Sch/X)_h$

The relative de Rham cohom. (of X wrt Z) is given by:

$$R\Gamma_{dR}(X,Z) := R\Gamma_h(X, \Omega^{\bullet}_{h/(X,Z)})$$

Fact: The properties (i) – (vi) of lemma 1 remain true in this more general setup and admit a relative counterpart

§ 3. Holomorphic de Rham cohomology

Def Let X be a complex manifold. The holomorphic de Rham cohom. of X is

$$R\Gamma_{dR,an}(X) = R\Gamma(X, \Omega_X^{\bullet\, hol})$$

Proposition 5 (holom. Poincaré lemma): Let X be a complex manifold, then the natural map of sheaves $\mathbb{C} \hookrightarrow \mathcal{O}_X^{hol}$ induces an isomorphism $H^i_{sing}(X, \mathbb{C}) \xrightarrow{\sim} H^i_{dR\, an}(X)$.

Pf We can compute $H^i_{sing}(X, \mathbb{C})$ as sheaf cohomology. Hence it is enough to show that $\mathbb{C} \hookrightarrow \Omega_X^{\bullet\, hol}$ is a quasi-isomorphism, i.e., that $0 \to \mathbb{C} \to \mathcal{O}_X^{hol} \to \Omega_X^{1\, hol} \to \dots$ is exact.

Since the question is local, we may assume $X = \Delta^n$, Δ open unit disc in $\mathbb{C}$

and since $\Omega^\bullet_{\Delta^n} \cong (\Omega^\bullet_\Delta)^{\boxtimes n} \left(= p_1^*\Omega^\bullet_\Delta \otimes \dots \otimes p_n^*\Omega^\bullet_\Delta\right)$ $\quad p_i : \Delta^n \to \Delta$ i-th projection

we can assume $n=1$, in which case we have:

$$0 \to \mathbb{C} \to \mathcal{O}^{hol}(\Delta) \xrightarrow{d} \mathcal{O}^{hol}(\Delta)\cdot dt \to 0$$

$$\sum_{j\geq 0} a_j t^j \longmapsto \sum_{j\geq 0} (j+1)\, a_{j+1} t^j\, dt$$

Clearly $\ker(d) = \mathbb{C}$, d is surjective because taking a primitive doesn't change the radius of convergence. ∎

Given $X \in Sch/\mathbb{C}$ one can attach to it a complex analytic space X^{an} (functorially ...) and there is a "universal" map of locally ringed spaces $\alpha = \alpha_X : (X^{an}, \mathcal{O}^{hol}_{X^{an}}) \to (X, \mathcal{O}_X)$.

Let us assume that X is smooth, so that X^{an} is a complex manifold.

α induces a morphism of complexes $\alpha^{-1}\Omega^\bullet_X \to \Omega^\bullet_{X^{an}}$

$\rightsquigarrow \alpha^* : H^i_{dR}(X) \to H^i_{dR,an}(X^{an})$

Proposition 6 (de Rham GAGA) For $X \in \mathrm{Sm}/\mathbb{C}$, $\alpha^*: H^i_{dR}(X) \to H^i_{dR,an}(X^{an})$ is an iso.

Pf (sketch) For X projective, this follows from Hodge-to-de Rham spectral sequence and the standard GAGA result by Serre for coherent sheaves.

In general one embeds $X \hookrightarrow \overline{X}$ with $\overline{X}$ smooth & projective and s.t. $D := \overline{X} \setminus X$ is a SNC divisor and works with differentials with log poles along D (algebraic and holomorphic version) and mimics the same proof. ∎

As in the algebraic case, one can define $H^*_{dR,an}(X)$ for $X \in \mathrm{An}$ (not nec. a complex manifold) introducing a finer Grothendieck topology on An/X.

Def: Given $X \in \mathrm{An}$, the h'-topology on An/X is the smallest Grothendieck top. such that proper surjective morphisms & open covers are coverings.
We define the h'-de Rham cohomology as $R\Gamma_{dR,an}(X_{h'}) = R\Gamma(\mathrm{An}/X, \Omega^\bullet_{h'})$

h'-sheafification ...

Thm 7 If X is a complex manifold, then $\Omega^p_X(X) = \Omega^p_{h'}(X)$ $\forall p \geq 0$
and $H^i_{dR,an}(X_{h'}) \cong H^i_{dR,an}(X)$ $\forall i \geq 0$.

One can prove an h'-version of the Poincaré lemma, so that

Thm 8: There are natural isomorphisms $H^i_{sing}(X, \mathbb{C}) \xrightarrow{\sim} H^i_{dR,an}(X)$ $\forall X \in \mathrm{An}$ $\forall i \geq 0$.

One can also upgrade prop. 6 to the case of a general $X \in \mathrm{Sch}/\mathbb{C}$.

Idea: we have morphisms of sites

obvious morphism of sites: it induces an equivalence on the categories of abelian sheaves

$$(\mathrm{An}/X^{an})_{h'} \xrightarrow{\alpha_{h'}} (\mathrm{Sch}/X)_{h'} \longleftarrow (\mathrm{Sch}/X)_h$$

h'-top. on schemes: smallest Groth. top. such that proper surj morphisms & open covers are coverings

$\rightsquigarrow$ get natural maps $\alpha^*: H^i_{dR}(X_h) \longrightarrow H^i_{dR,an}(X^{an}_{h'})$ $\forall i \geq 0$

Thm 9 : α^* is an isomorphism $\forall i \geq 0$.

Rmk: One can use the h'-topology on analytic spaces to define relative h'-de Rham cohomology (for $Z \hookrightarrow X$ closed subspace).
Theorems 7-8-9 admit a relative version and the relative holomorphic h'-de Rham cohom. satisfies the same properties of its algebraic counterpart (Künneth, ...).

Rmk: There exists a p-adic version of the whole story. H. Guo (ex-student of B. Bhatt) has developed éh-topology for (proper) rigid spaces / K $K/\mathbb{Q}_p$ finite ext. and obtained similar comparison results with p-adic étale cohom. of rigid spaces (= étale cohom. of the corr. alg. variety in the "algebraizable" case)