NORI'S DIAGRAM CATEGORY
- THE UNIVERSAL PROPERTY

<u>Recall</u>: Given a small category $\mathcal{C}$ $\rightsquigarrow$ diagram $D(\mathcal{C}) =: D$ and a representation

$$T : D \longrightarrow R\text{-mod} = \text{f.g. } R\text{-modules}$$

map of directed graphs, R noeth. commutative ring (unitary)

$$\rightsquigarrow \operatorname{End}(T) = \left\{ (e_p)_{p\in D} \in \prod_{p\in D} \operatorname{End}_R(T_p) \;\middle|\; \begin{tikzcd} T_p \arrow[r, "T_m"] \arrow[d, "e_p"'] \arrow[dr, phantom, "\circ"] & T_q \arrow[d, "e_q"] \\ T_p \arrow[r, "T_m"] & T_q \end{tikzcd} \quad \forall p,q \in D,\ m \in D(p,q) \right\}$$

Then we defined the diagram category of D w.r.t. T by

$$\mathcal{C}(D,T) = \underset{F}{2\text{-colim}}\, \underbrace{\operatorname{End}(T|_F)\text{-mod}}_{\text{f.g. } \operatorname{End}(T|_F)\text{-modules}} \quad \left(= \operatorname{End}(T)\text{-mod if } D \text{ is finite} \right)$$

(7.1.13) <u>Theorem 1</u>: Let D be a diagram and $T : D \to R\text{-mod}$ a rep. of D. Then there exists an R-linear abelian category $\mathcal{C}(D,T)$ together with a representation

$$\tilde{T} : D \longrightarrow \mathcal{C}(D,T)$$

and a faithful exact R-lin. functor f_T such that:

<u>1.</u> T factors as $D \xrightarrow{\tilde{T}} \mathcal{C}(D,T) \xrightarrow{f_T} R\text{-mod}$

<u>2.</u> $\tilde{T}$ satisfies the following univ. prop.

Given another R-linear, abelian cat. $\mathcal{A}$, R-lin. faithful exact functor $T_{\mathcal{A}} : \mathcal{A} \to R\text{-mod}$ and a rep. $F : D \to \mathcal{A}$ s.t. $T = T_{\mathcal{A}} \circ F$, then there exist a faithful exact functor $L(f)$ - unique / unique iso. of add. functors - s.t.

$$\begin{tikzcd} & \mathcal{C}(D,T) \arrow[dr, "f_T"] \arrow[dd, dotted, "\exists!\, L(f)"] & \\ D \arrow[ur, "\tilde{T}"] \arrow[rr, "T"] \arrow[dr, "F"'] & & R\text{-mod} \\ & \mathcal{A} \arrow[ur, "T_{\mathcal{A}}"'] & \end{tikzcd}$$

commute.

<u>Last time</u>: Saw that <u>1.</u> holds for $\mathcal{C}(D,T)$, i.e.

$$\begin{tikzcd} & \mathcal{C}(D,T) \arrow[dr, "f_T"] & \\ D \arrow[ur, "\tilde{T}"] \arrow[rr, "T"'] \arrow[rr, phantom, bend left=20, "\circ"] & & R\text{-mod} \end{tikzcd}$$

commutes and f_T is faithful and exact (it is the forgetful functor)

The main ingredient to prove $\underline{Z}$. is the following:

(7.1.20) Theorem 2: For R noeth. and $\mathcal{A}$ abelian R-lin. category, let $T: \mathcal{A} \to R\text{-mod}$ be exact, R-linear, and faithful, and

$$\mathcal{A} \xrightarrow{\tilde{T}} \mathcal{C}(\mathcal{A}, T) \xrightarrow{f_T} R\text{-mod}$$

the factorisation via ~~its~~ diagram category. Then $\tilde{T}$ is an equivalence of categories.

§1:

Fix: – R noeth commutative unitial
– E unitial R-alg, f.g. as R-module (not nec. comm.)
– E-mod = cat. of f.g. left E-modules
– $\mathcal{A}$ R-lin. abelian category.

Def: Let $p \in \mathcal{A}$, $f: E^{op} \to \mathrm{End}_{\mathcal{A}}(p)$ morph. of R-alg. We call p a right E-module in $\mathcal{A}$.

* Assume Hom-modules in $\mathcal{A}$ are f.g.

Consider the functor $\mathrm{Hom}_{\mathcal{A}}(p,-): \mathcal{A} \to R\text{-mod}$

~~Then~~ Then for $p \in \mathcal{A}$ a right E-module, can view this as a functor

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(p,-) : \mathcal{A} &\longrightarrow E\text{-mod} \\ q &\longmapsto \mathrm{Hom}(p,q) \\ & \qquad\quad \overset{\psi}{\varphi} \qquad e\cdot\varphi = \varphi \circ f(e): \end{aligned}$$

$$p \xrightarrow{f(e)} p \xrightarrow{\varphi} q$$

7.3.5) Proposition 1: For $p \in \mathcal{A}$ a right E-module, $\mathrm{Hom}_{\mathcal{A}}(p,-)$ has an R-linear left adjoint

$$p \otimes_E - : E\text{-mod} \longrightarrow \mathcal{A}$$

which is right exact s.t.

(i) $p \otimes_E E = p$

(ii) $p \otimes_E - : \mathrm{End}_E(E) \longrightarrow \mathrm{End}_{\mathcal{A}}(p)$ $\qquad (\mathrm{End}_E(E) \cong E^{op})$

$$(a: E \to E) \longmapsto (f(a): p \to p)$$

Proof: Describe $p \otimes_E M$ for more and more general $M \in E$-mod.

Then right exactness follows from univ. prop of adjoint func

For $M = E$ have

$$\mathrm{Hom}_{\mathcal{A}}(p,q) = \mathrm{Hom}_E(E, \mathrm{Hom}_{\mathcal{A}}(p,q))$$

So $E \longmapsto p$.

For $M = E^n$, $E^n \longmapsto \bigoplus_{i=1}^n p$ and

$$(E^n \xrightarrow{(a_{ij})} E^m) \longmapsto (\bigoplus_{i=1}^n p \xrightarrow{f(a_{ij})} \bigoplus_{i=1}^m p)$$

For general M, take a finite presentation

(R noeth. and M f.g. E-mod $\Rightarrow$ M fin. pres

$$E^m \longrightarrow E^n \longrightarrow M \longrightarrow 0$$

and let

$$p \otimes_E M := \mathrm{coker}(\bigoplus_{i=1}^n p \xrightarrow{f(a_{ij})} \bigoplus_{i=1}^m p)$$

$\rightsquigarrow$ check that this satisfies adjointness

$\rightsquigarrow$ check independence of resolution

$\hookrightarrow$ follows from univ. prop. of adjoint functors.

Corollary 1: $T : \mathcal{A} \to E\text{-mod}$ exact, faithful, $p \in \mathcal{A}$ right E-module

then
$$E^{op} \xrightarrow{f} \mathrm{End}_{\mathcal{A}}(p) \xrightarrow{T_p} \mathrm{End}_E(Tp)$$
induces a right action on Tp and

$$\begin{array}{ccccc} E\text{-mod} & \longrightarrow & \mathcal{A} & \longrightarrow & E\text{-mod} \\ M & \longmapsto & p \otimes_E M & \longmapsto & T(p \otimes_E M) \end{array}$$

is the usual tensor functor of E-modules.

Proof: True on free modules, then use right exactness of $T(p \otimes_E -)$ and exactness of T for general $M \in E\text{-mod}$.

Proposition 2: Let $p \in \mathcal{A}$. Then
$$\mathrm{Hom}_{\mathcal{A}}(-,p) : \mathcal{A}^{\circ} \longrightarrow R\text{-mod}$$
has a left adjoint

- $$\mathrm{Hom}_R(-,p) : R\text{-mod} \longrightarrow \mathcal{A}^{\circ}$$ (i.e. $\mathrm{Hom}_{\mathcal{A}}(q, \mathrm{Hom}_R(M,p)) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathcal{A}}(q,p))$ $\forall M \in E\text{-mod},\ q \in \mathcal{A}$.

which is left exact and $\mathrm{Hom}_R(R,p) = p$

- If $T : \mathcal{A} \to R\text{-mod}$ is exact, R-lin. then

$$\begin{array}{ccccc} R\text{-mod} & \xrightarrow{\mathrm{Hom}(-,p)} & \mathcal{A} & \xrightarrow{T} & R\text{-mod} \\ M & \longmapsto & \mathrm{Hom}_R(M,p) & \longmapsto & \mathrm{Hom}_R(M,T_p) \end{array}$$

is the usual $\mathrm{Hom}(-,Tp)$ functor in R-mod

Proof: Same as Prop. 1.

§ 2 :

Setting:

$\mathcal{A}$ R-lin. abelian category. $T : \mathcal{A} \to R\text{-mod}$ faithful and exact. (existence of T implies Hom-modules in $\mathcal{A}$ are f.g.)

Def: $S \subset \mathcal{A}$ set of objects

$\langle S \rangle$ = smallest full ~~sub~~ abelian subcat of $\mathcal{A}$ cont. S whic is closed under kernels and cokernels. then

$\langle S \rangle^{psab}$ = smallest full pseudo-abelian subcategory of $\mathcal{A}$ gen. by S, i.e. it contains S and is closed under direct sums and direct summands

~~Ex~~: if $p \in \mathcal{A}$ then $\langle p \rangle^{psab} = \{ q \in \mathcal{A} \mid p^n = q' \oplus q$ for some $n \in \mathbb{Z} \}$

Lemma 1: Let $E(p) = \mathrm{End}(T|_{\{p\}})$. Then

1. $E(p) = \mathrm{End}(T|_{\langle p \rangle^{psab}})$

(2. If p is projective and every $q \in \langle p \rangle$ is a quotient of p^n for some n, then $E(p) = \mathrm{End}(T|_{\langle p \rangle})$.)

Proof: Let $(\alpha_q)_q \in \mathrm{End}(T|_{\langle p \rangle^{psab}})$, consider the comp $\alpha_{p^n} : Tp^n \to Tp^n$

$$(Tp)^n \to (Tp)^n$$

Then

$$\begin{tikzcd} (Tp)^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & (Tp)^n \arrow[d] \\ Tp \arrow[r] & Tp \end{tikzcd}$$ project to i'th comp factor

$$\begin{tikzcd} (Tp)^n \arrow[r] & (Tp)^n \\ Tp \arrow[r] \arrow[u] & Tp \arrow[u] \end{tikzcd}$$ inclusion to j'th factor

$\Rightarrow$ α_{p^n} is given by the diagonal map α_p.

[so α_p determines all α_{p^n}]

If $p^n = ~~q~~ \oplus q'$ then ~~q~~ $\alpha_{p^n}|_q = \alpha_q$ by compatibility w. projection and inclusion

Conversely, $\alpha_p \in E(p)$ extends to $(\alpha_p)^n : Tp^n \to Tp^n$ diagonally

$\rightsquigarrow$ compatible w. all maps $p^n \to p^m$

$$\begin{tikzcd} Tp^n \arrow[r, "(\alpha_p)^n"] \arrow[d] & Tp^n \arrow[d] \\ Tp^m \arrow[r, "(\alpha_p)^m"] & Tp^m \end{tikzcd}$$

Restricting p^n to a direct summand q respects q because

$$\begin{tikzcd} p^n \arrow[r] \arrow[rr, bend right] & q \arrow[r] & p^n \end{tikzcd}$$ is an endomorphism. ■

Proof of Theorem 2 (Sketch):

Lemma 1 implies that $\mathcal{C}(\langle p\rangle^{psab}, T) = 2\text{-}\operatorname{colim}_{F\subset\langle p\rangle^{psab}} \mathrm{End}(T|_F)\text{-mod}$

$= \mathrm{End}(p)\text{-mod}$

(All $(\alpha_q)_q \in \mathrm{End}(T|_{\langle p\rangle^{psab}})$ is determined by some $\alpha_p \in E(p)$, in particular all $(\alpha_q)_q \in \mathrm{End}(T|_F)$, F finite)

Moreover, $\tilde{T}$ faithful (since $T = f_T \circ \tilde{T}$, T, f_T faithful)

Thus

$$\mathcal{C}(\mathcal{A},T) = 2\text{-}\operatorname{colim}_{F\subset Ob(\mathcal{A})\text{ fin.}} \mathrm{End}(T|_F)\text{-mod} \;\Big(= 2\text{-}\operatorname{colim}_{p\in\mathcal{A}} \underbrace{2\text{-}\operatorname{colim}_{F\subseteq\langle p\rangle^{psab}\text{ fin.}} \mathcal{C}(F, T|_F)}\Big)$$

$$= 2\text{-}\operatorname{colim}_{p\in\mathcal{A}} \mathcal{C}(\langle p\rangle^{psab}, T) \quad \leftarrow \text{cofinality}$$

because $\langle F\rangle^{psab} = \langle \bigoplus_{p\in F} p\rangle^{psab}$

$$\underset{\text{Lemma 1}}{=} 2\text{-}\operatorname{colim}_{p\in\mathcal{A}} E(p)\text{-mod}$$

We now wish to define a functor $E(p)\text{-mod} \to \mathcal{A}$ which will be given by $X(p) \otimes_{E(p)} (-)$ for some $X(p) \in \mathcal{A}$ w. a structure of right $E(p)$-module

Constructing $X(p)$:

Have our functor $\mathrm{Hom}_R(-,p) : R\text{-mod} \to \mathcal{A}^\circ$

$Tp \longmapsto \mathrm{Hom}_R(Tp, p)$

composing w. T

$\longrightarrow \mathrm{Hom}_R(Tp, Tp)$

$Tp \xrightarrow{\varphi} Tp \to p$, $\varphi \in \mathrm{End}_R(T)$ ← Then this is a ri… -module in $\mathcal{A}$.

We wish to define $X(p)$ as a subobject of $\mathrm{Hom}_R(Tp, p)$, with $E(p)$ module structure induced by $\mathrm{Hom}_R(Tp,p)$ $(E(p) \subseteq \mathrm{End}_R(Tp$…

Let $\alpha_1, \dots, \alpha_n$ be a gen. family for $\mathrm{End}_{\mathcal{A}}(p) \subset \mathrm{End}_R(Tp)$ ← This is f.g. since R noeth. + T faith…

Write

$$E(p) = \ker\Big(\mathrm{Hom}(Tp,Tp) \longrightarrow \bigoplus_{i=1}^n \mathrm{Hom}(Tp,Tp)\Big)$$
$$u \longmapsto u\circ T\alpha_i - T\alpha_i \circ u$$

Then define

$$X(p) = \ker\Big(\mathrm{Hom}_R(Tp,p) \longrightarrow \bigoplus_{i=1}^n \mathrm{Hom}_R(Tp,p)\Big)$$
$$u \longmapsto u\circ T\alpha_i - \alpha_i \circ u$$

$$\begin{array}{ccc} \mathrm{Hom}(Tp,p) & \xrightarrow{T} & \mathrm{End}(Tp \\ \cup & & \cup \\ X(p) & \longmapsto & E(p) \end{array}$$

$\Rightarrow$ $E(p)$ is a preimage ~~…~~ of $E(p)$ under T in $\mathcal{A}$.

$E(p)$-mod structure on $\mathrm{Hom}_R(Tp,p)$ restricts to one on $X(p)$ and $\tilde{T}(X(p)) = E(p) \in E(p)\text{-module} \hookrightarrow \mathcal{C}(\mathcal{A},T)$

Claim: $X(p) \otimes_{E(p)} \tilde{T}p \to p$ iso. and $X(p) \otimes_{E(p)} -$ comp. w. 2-colim.

$\Rightarrow$ $\mathcal{A} \underset{\tilde{T}}{\rightleftarrows} \mathcal{C}(\mathcal{A},T)$ comp. to id $\underset{\tilde{T}\text{ faithf.}}{\Rightarrow}$ eq. of cat. ∎

§ 3:

Setting:

$$\begin{tikzcd}
 & \mathcal{C}(D,T) \arrow[dr, "f_T"] \arrow[dd, dashed, "!L(f)" near end] & \\
D \arrow[ur, "\tilde{T}"] \arrow[rr, "T"] \arrow[dr, "F"'] & & R\text{-mod} \\
 & \mathcal{A} \arrow[ur, "T_{\mathcal{A}}"'] &
\end{tikzcd}$$

f_T ⟵ exact and faithful

$T_{\mathcal{A}}$ ⟵ exact, faithful

Existence:

Regard $\mathcal{A}$ as a diagram and get a rep.

$$T_{\mathcal{A}} : \mathcal{A} \longrightarrow R\text{-mod}.$$

and factor it as

$$\mathcal{A} \xrightarrow{\tilde{T}_{\mathcal{A}}} \mathcal{C}(\mathcal{A}, T_{\mathcal{A}}) \xrightarrow{f_{T_{\mathcal{A}}}} R\text{-mod}$$

$\rightsquigarrow$

$$\begin{tikzcd}
D \arrow[rr, "F"] \arrow[d, "\tilde{T}"'] \arrow[dr, "T"] & & \mathcal{A} \arrow[d, "\tilde{T}_{\mathcal{A}}"] \arrow[dl, "T_{\mathcal{A}}"'] \\
\mathcal{C}(D,T) \arrow[r, "f_T"'] & R\text{-mod} & \mathcal{C}(\mathcal{A}, T_{\mathcal{A}}) \arrow[l, "f_{T_{\mathcal{A}}}"]
\end{tikzcd}$$

Lemma (7.2.6)

$\longrightarrow$

From last time

$$\begin{tikzcd}
D \arrow[rr, "F"] \arrow[d, "\tilde{T}"'] & & \mathcal{A} \arrow[d, "\tilde{T}_{\mathcal{A}}"] \\
\mathcal{C}(D,T) \arrow[rr, dashed, "\mathcal{F}"] \arrow[dr, "f_T"'] & & \mathcal{C}(\mathcal{A}, T_{\mathcal{A}}) \arrow[dl, "f_{T_{\mathcal{A}}}"] \\
 & R\text{-mod} &
\end{tikzcd}$$

$\mathcal{F}$ is R-lin., faithful, exact.

$$\begin{tikzcd}
\mathcal{C}(D,T) \arrow[r, "\mathcal{F}"] \arrow[rr, bend right, "L(f)"'] & \mathcal{C}(\mathcal{A}, T_{\mathcal{A}}) & \mathcal{A} \arrow[l, "\simeq"', "\tilde{T}_{\mathcal{A}}"]
\end{tikzcd}$$

$L(f)$ is R-lin, faithful and exact because $\mathcal{F}$ and $\tilde{T}_{\mathcal{A}}^{-1}$ are.

Uniqueness:

Assume L' is another such functor. Let

$\mathcal{C}' \subset \mathcal{C}(D,T)$ be the subcat. on which $L' = L(f)$.

Claim: $\mathcal{C}' \hookrightarrow \mathcal{C}(D,T)$ is an eq. of categories (the inclusion is an eq. of cat.)

- wlog assume D finite.
- $T_{\mathcal{A}} : \mathcal{A} \to R\text{-mod}$ faithful $\Rightarrow$ $\mathcal{C}'$ full.
- $\tilde{T}p \in \mathcal{C}'$ for all $p \in D$.
- $L', L(f)$ additive $\Rightarrow$ they agree on fin. direct sums of objects
- $L', L(f)$ exact $\Rightarrow$ agree on kernels and cokernels

$\Rightarrow \mathcal{C}'$ is the full abelian subcat of $\mathcal{C}(D,T)$ gen. by $\tilde{T}(D)$

$\overset{\text{Prop. 7.3.24}}{\Longrightarrow} \mathcal{C}' \cong \mathcal{C}(D,T)$.

from last time