

The main ingredient to prove 2. is the following:

(7.1.20) Theorem 2: For R noeth. and \mathcal{A} abelian R -lin. category, let $T: \mathcal{A} \rightarrow R\text{-mod}$ be exact, R -linear, and faithful, and

$$\mathcal{A} \xrightarrow{\tilde{T}} \mathcal{C}(\mathcal{A}, T) \xrightarrow{f_T} R\text{-mod}$$

the factorisation via its diagram category. Then \tilde{T} is an equivalence of categories.

§ 1:

- Fix:
- R noeth commutative unital
 - E unital R -alg, f.g. as R -module (not nec. comm.)
 - E -mod = cat. of f.g. left E -modules
 - \mathcal{A} R -lin. abelian category.

Def: Let $p \in \mathcal{A}$, $f: E^{op} \rightarrow \text{End}_{\mathcal{A}}(p)$ morph. of R -alg. We call p a right E -module in \mathcal{A} .

* Assume Hom -modules in \mathcal{A} are f.g.

Consider the functor $\text{Hom}_{\mathcal{A}}(p, -) : \mathcal{A} \rightarrow R\text{-mod}$

Then for $p \in \mathcal{A}$ a right E -module, can view this as a functor

$$\text{Hom}_{\mathcal{A}}(p, -) : \mathcal{A} \rightarrow E\text{-mod}$$

$$q \mapsto \underset{\varphi}{\text{Hom}}(p, q) \quad e \cdot \varphi = \varphi \circ f(e) : p \xrightarrow{f(e)} p \xrightarrow{\varphi} q$$

7.3.5) Proposition 1: For $p \in \mathcal{A}$ a right E -module, $\text{Hom}_{\mathcal{A}}(p, -)$ has an R -linear left adjoint

$$p \otimes_E - : E\text{-mod} \rightarrow \mathcal{A}$$

which is right exact s.t.

- (i) $p \otimes_E E = p$
- (ii) $p \otimes_E - : \text{End}_E(E) \rightarrow \text{End}_{\mathcal{A}}(p) \quad (\text{End}_E(E) \cong E^{op})$
 $(a: E \rightarrow E) \mapsto (f(a): p \rightarrow p)$

Proof: Describe $p \otimes_E M$ for more and more general $M \in E\text{-mod}$.
 For $M = E$ have $\text{Hom}_{\mathcal{A}}(p, q) = \text{Hom}_E(E, \text{Hom}_{\mathcal{A}}(p, q))$
 Then right exactness follows from univ. prop of adjoint func

So $E \mapsto p$.
 For $M = E^n$, $E^n \mapsto \bigoplus_{i=1}^n p$ and
 $(E^n \xrightarrow{(\alpha_{ij})} E^m) \mapsto (\bigoplus_{i=1}^n p \xrightarrow{f(\alpha_{ij})} \bigoplus_{i=1}^m p)$

For general M , take a finite presentation

$$E^m \rightarrow E^n \rightarrow M \rightarrow 0$$

and let $p \otimes_E M := \text{coker}(\bigoplus_{i=1}^n p \xrightarrow{f(\alpha_{ij})} \bigoplus_{i=1}^m p)$

- check that this satisfies adjointness
- check independence of resolution

↳ follows from univ. prop. of adjoint functors.

(R noeth. and M f.g. $E\text{-mod} \Rightarrow M$ fin. pres)

Corollary 1: $T: A \rightarrow E\text{-mod}$ exact, faithful, $p \in A$ right E -module

then $E\text{-op} \xrightarrow{f} \text{End}_E(p) \xrightarrow{Tp} \text{End}_E(Tp)$
 induces a right action on Tp and

$$\begin{array}{ccc} E\text{-mod} & \longrightarrow & A \longrightarrow E\text{-mod} \\ M & \longmapsto & p \otimes_E M \longmapsto T(p \otimes_E M) \end{array}$$

is the usual tensor functor of E -modules.

Proof: True on free modules, then use right exactness of $T(p \otimes_E -)$ and exactness of T for general $M \in E\text{-mod}$.

Proposition 2: Let $p \in A$. Then

$$\text{Hom}_A(-, p) : A^0 \rightarrow R\text{-mod}$$

has a left adjoint

$$\text{Hom}_R(-, p) : R\text{-mod} \rightarrow A^0 \quad (\text{i.e.})$$

which is left exact and $\text{Hom}_R(R, p) = p$

$$\begin{array}{c} \text{Hom}_A(q, \text{Hom}_R(M, p)) \\ \cong \\ \text{Hom}_R(M, \text{Hom}_A(q, p)) \\ \forall M \in E\text{-mod}, q \in A. \end{array}$$

If $T: A \rightarrow R\text{-mod}$ is exact, R -lin. then

$$\begin{array}{ccc} R\text{-mod} & \xrightarrow{\text{Hom}(-, p)} & A \xrightarrow{T} R\text{-mod} \\ M & \longmapsto & \text{Hom}_R(M, p) \longmapsto \text{Hom}_R(M, Tp) \end{array}$$

is the usual $\text{Hom}(-, Tp)$ functor in $R\text{-mod}$

Proof: Same as Prop. 1.

§ 2 :

Setting :

\mathcal{A} R -lin. abelian category, $T : \mathcal{A} \rightarrow R\text{-mod}$ faithful and exact. (existence of T implies Hom-modules in \mathcal{A} are f.g.)

Def: $S \subset \mathcal{A}$ set of objects

$\langle S \rangle$ = smallest full abelian subcat of \mathcal{A} cont. S which is closed under kernels and cokernels. *free*

$\langle S \rangle^{\text{psab}}$ = smallest full pseudo-abelian subcategory of \mathcal{A} , ^{gen. by S} i.e. it contains S and is closed under direct sums and direct summands

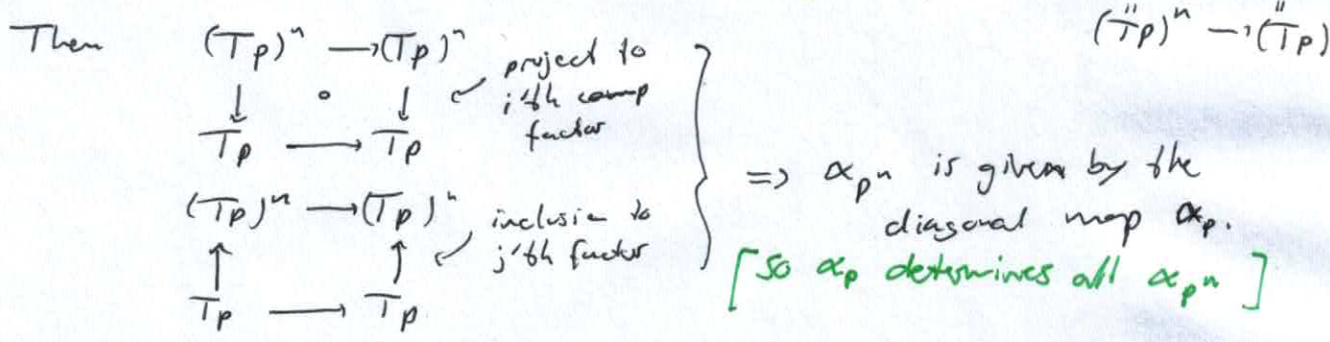
Ex 2: if $p \in \mathcal{A}$ then $\langle p \rangle^{\text{psab}} = \{ q \in \mathcal{A} \mid p^n = q' \oplus q \text{ for some } n \in \mathbb{Z} \}$

Lemma 1: Let $E(p) = \text{End}(T|_{\langle p \rangle^{\text{psab}}})$. Then

1. $E(p) = \text{End}(T|_{\langle p \rangle^{\text{psab}}})$

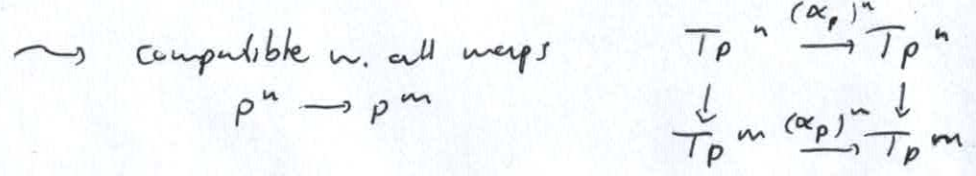
(2. If p is projective and every $q \in \langle p \rangle$ is a quotient of p^n for some n , then $E(p) = \text{End}(T|_{\langle p \rangle})$.)

Proof: Let $(\alpha_q)_q \in \text{End}(T|_{\langle p \rangle^{\text{psab}}})$, consider the comp $\alpha_{p^n} : T p^n \rightarrow T p^n$



If $p^n = q' \oplus q$ then $\alpha_{p^n}|_q = \alpha_q$ by compatibility w. projection and inclusion

Conversely, $\alpha_p \in E(p)$ extends to $(\alpha_p)^n : T p^n \rightarrow T p^n$ diagonally



Restricting p^n to a direct summand q respects q because

$p^n \rightarrow q \rightarrow p^n$ is an endomorphism. ■

Proof of Theorem 2 (Sketch):

Lemma 1 implies that $\mathcal{C}(\langle p \rangle^{psab}, T) = 2\text{-colim}_{F \in \langle p \rangle^{psab}} \text{End}(T|_F)\text{-mod} = \text{End}(p)\text{-mod}$

(All $(\alpha_q)_q \in \text{End}(T|_{\langle p \rangle^{psab}})$ is determined by some $\alpha_p \in E(p)$, in particular all $(\alpha_q)_q \in \text{End}(T|_F)$, F finite)

Moreover, \tilde{T} faithful (since $T = f_T \circ \tilde{T}$, T, f_T faithful)

Thus

$$\begin{aligned} \mathcal{C}(A, T) &= 2\text{-colim}_{F \in \text{Ob}(A) \text{ fin.}} \text{End}(T|_F)\text{-mod} (= 2\text{-colim}_{p \in A} 2\text{-colim}_{F \in \langle p \rangle^{psab} \text{ fin.}} \mathcal{C}(F, T|_F)) \\ &= 2\text{-colim}_{p \in A} \mathcal{C}(\langle p \rangle^{psab}, T) \\ &\stackrel{\text{Lemma 1}}{=} 2\text{-colim}_{p \in A} E(p)\text{-mod} \end{aligned}$$

because $\langle F \rangle^{psab} = \left(\bigoplus_{p \in F} p \right)^{psab}$

cofinality

We now wish to define a functor $E(p)\text{-mod} \rightarrow A$ which will be given by $X(p) \otimes_{E(p)} (-)$ for some $X(p) \in A$ w. a structure of right $E(p)$ -module

Constructing $X(p)$:

Have our functor $\text{Hom}_R(-, p) : R\text{-mod} \rightarrow A$
 $T_p \longmapsto \text{Hom}_R(T_p, p)$

composing w. T

$$\text{Hom}_R(T_p, T_p)$$

$T_p \otimes T_p \rightarrow p, p \in \text{End}_R(T)$ -module in A .
 Then this is a right $E(p)$ -module

We wish to define $X(p)$ as a subobject of $\text{Hom}_R(T_p, p)$, with $E(p)$ -module structure induced by $\text{Hom}_R(T_p, p)$ ($E(p) \subseteq \text{End}_R(T_p)$)

Let $\alpha_1, \dots, \alpha_n$ be a gen. family for $\text{End}_R(p) \subseteq \text{End}_R(T_p)$

write

$$E(p) = \ker(\text{Hom}(T_p, T_p) \rightarrow \bigoplus_{i=1}^n \text{Hom}(T_p, T_p))$$

$$u \longmapsto u \circ \alpha_i; -\alpha_i \circ u$$

This is fig. since R noeth. + T faithful

Then define

$$X(p) = \ker(\text{Hom}_R(T_p, p) \rightarrow \bigoplus_{i=1}^n \text{Hom}_R(T_p, p))$$

$$u \longmapsto u \circ \alpha_i; -\alpha_i \circ u$$

$$\begin{array}{ccc} \text{Hom}(T_p, p) & \xrightarrow{T} & \text{End}(T_p) \\ \cup & & \cup \\ X(p) & \xrightarrow{} & E(p) \end{array}$$

$\Rightarrow E(p)$ is a preimage of $E(p)$ under T in A .

$E(p)$ -mod structure on $\text{Hom}_R(T_p, p)$ restricts to one on $X(p)$ and

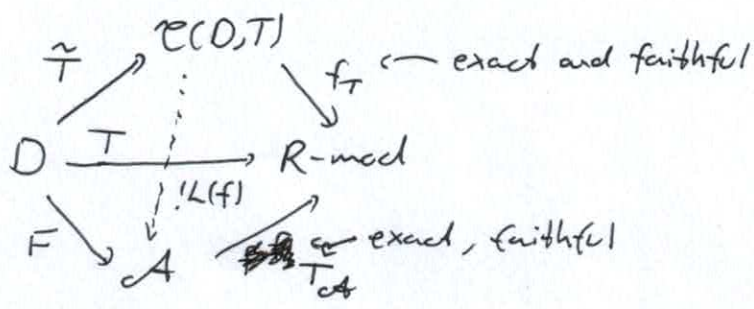
$$\tilde{T}(X(p)) = E(p) \in E(p)\text{-module} \iff \mathcal{C}(A, T)$$

Claim: $X(p) \otimes_{E(p)} \tilde{T}p \rightarrow p$ iso. and $X(p) \otimes_{E(p)} -$ comp. w. 2-colim.

$$\Rightarrow A \xrightarrow{\tilde{T}} \mathcal{C}(A, T) \text{ comp. to id} \stackrel{\tilde{T} \text{ faithful}}{\Rightarrow} \text{eq. of cat.}$$

§ 3:

Setting:



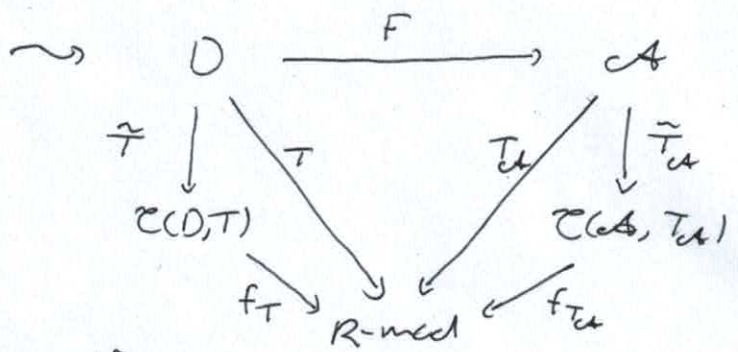
Existence:

Regard A as a diagram and get a rep.

$$T_{ct}: A \rightarrow R\text{-mod.}$$

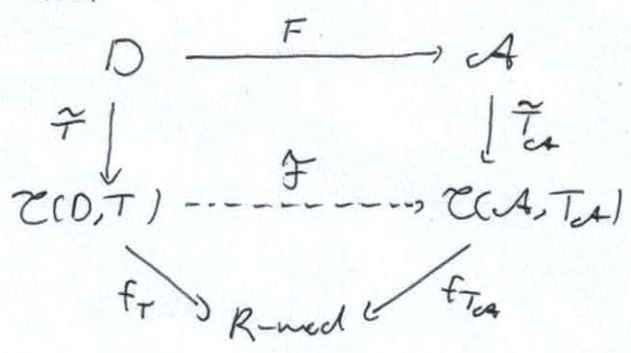
and factor it as

$$A \xrightarrow{\tilde{T}_{ct}} C(A, T_{ct}) \xrightarrow{f_{T_{ct}}} R\text{-mod}$$

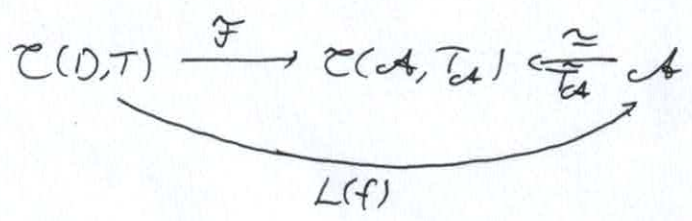


Lemma (7.2.6)

From last time



F is R -lin., faithful, exact.



$L(f)$ is R -lin., faithful and exact because F and \tilde{T}_{ct}^{-1} are.

Uniqueness:

Assume L' is another such functor. Let

$\mathcal{C}' \subset \mathcal{C}(D, T)$ be the subcat. on which $L' = L(f)$.

Claim: $\mathcal{C} \hookrightarrow \mathcal{C}(D, T)$ is an eq. of categories (the inclusion is an eq. of cat.)

- wlog assume D finite.
- $T_A : A \rightarrow R\text{-mod}$ faithful $\Rightarrow \mathcal{C}$ full.
- $\tilde{T}_p \in \mathcal{C}'$ for all $p \in D$.
- $L', L(f)$ additive \Rightarrow they agree on fin. direct sums of objects
- $L', L(f)$ exact \Rightarrow agree on kernels and cokernels

$\Rightarrow \mathcal{C}'$ is the full abelian subcat of $\mathcal{C}(D, T)$ gen. by $\tilde{T}(D)$

Prop. 7.3.24

$\Rightarrow \mathcal{C}' \cong \mathcal{C}(D, T)$.

from last time

