Talk VI - (effective) Nori motives.

We start by analyzing the main diagram category of which Nori motives are lim.

§1. effective pairs:

- an effective pair is a triple (X, Y, i) w/ X k-scheme of ft, $Y \subseteq X$ subscheme and $i \in \mathbb{N}$ (for Clémentine: we consider them up to iso)

- We define a diagram $Pairs^{eff}$ of effective pairs & isom. p's and two classes of arrows:

  - functorial type: $(X', Y', i) \to (X, Y, i)$ for all $(X, Y) \to (X', Y')$

  - boundary type: $(X, Z, i) \to (X, Y, i+1)$ ∀ $X \supset Y \supset Z$

a similar „homological" def. works. We also define $|(X,Y,i)| = i$

this diagram is tailored to encode representations for each good coh. theory

Ex. a) $H^*: \mathrm{Pairs}^{\mathrm{eff}} \longrightarrow \mathbb{Z}\text{-mod}$

$$(X,Y,i) \longmapsto H^i(X,Y;\mathbb{Z})$$

b) $H^*_{dR}: \mathrm{Pairs}^{\mathrm{eff}} \longrightarrow k\text{-Vect}$

$$(X,Y,i) \longmapsto H^i_{dR}((X,Y)/k)$$

c) $H': \mathrm{Pairs}^{\mathrm{eff}} \longrightarrow \mathcal{MHS}$

$$(X,Y,i) \longmapsto \mathrm{MHS}^i(X,Y)$$

d) $H^*_{et}: \mathrm{Pairs}^{\mathrm{eff}} \longrightarrow \mathrm{Mod}_{\mathbb{Z}_\ell}^{\Gamma_k}$

$$(X,Y,i) \longmapsto H^i_{et}(X_{\bar{k}}, Y_{\bar{k}}; \mathbb{Z}_\ell).$$

Example a) is the most important of them:

DEFINITION. The category of effective mixed Nori motives is defined to be

$$MM^{\mathrm{eff}}_{\mathrm{Nori}} = \mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*)$$

endowed w/ the tautological rep $H^i_{\mathrm{Nori}}(X,Y) \leftarrow\!\shortmid (X,Y,i)$ and the forgetful functor to $\mathbb{Z}$-mod.

· the Lefschetz motive is defined to be $H^1_{\mathrm{Nori}}(\mathbb{G}_m, \{1\}) \doteq \mathbb{1}(-1)$.

* A naive $\otimes$ structure: we define

$$(X,Y,i)\otimes(X',Y',i') = (X\times X', Y\times X' \cup Y'\times X, i+i')$$

and similarly on maps. This is a $\otimes$ structure, but H^* not monoidal:

$$H^{i+i'}(X\times X', Y\times X' \cup Y'\times X) \neq H^{i}(X,Y)\otimes H^{i'}(X',Y') \ldots$$

($\cong$ if field coeff...)

Theorem. (next time) $\exists$ VGood$^{eff} \subseteq$ Pairs^{eff} full $\otimes$ ~~subcategory~~ diagram such that H^*: Good$^{eff} \to \mathbb{Z}$-mod is $\otimes$-functor, +

$$\mathcal{C}(\text{VGood}^{eff}, H^*) \xrightarrow{\sim} \mathcal{C}(\text{Pairs}^{eff}, H^*) \cong MM_{nori}^{eff}.$$

In particular MM_{eff}^{nori} admits a $\otimes$-structure.

Even so, it'll not be the case that MM_{nori}^{eff} is rigid: the object $\mathbb{L}$ will not be dualizable.

Definition. We define the category of Nori motives to be

$$MM_{nori} \doteq MM_{nori}^{eff}\left[\mathbb{1}(-1)^{-1}\right],$$

the localization @ $\mathbb{L}$. turns out that this is also the diagram category of a digram Pairs, and hence is abelian.

§2. the localisation of $\otimes$-diagrams.

· Let D^{eff} be a graded symm. monoidal diagram w/ unit $\mathbb{1}$, $v_0 \in D^{eff}$. we define a diagram $D = D^{eff}[v_0^{-1}]$ ~~as~~ as follows:

- vertices: $v(n)$ for $v \in D^{eff}$, $n \in \mathbb{Z}$
- arrows: $\begin{cases} \alpha(n): v(n) \to w(n) & \forall \alpha: v \to w \\ + \; v \otimes v_0(n) \to v(n+1) \end{cases}$

this has a grading $|v(n)| = |v|$ and a $\otimes$-structure:

$$v(n) \otimes w(m) = (v \otimes w)(n+m)$$

and maps $(v_1(n) \to v_2(n)) \otimes w(m) = (v_1 \otimes w \to v_2 \otimes w)(m+n)$ and $(v \otimes v_0(n) \to v(n+1)) \otimes w(m) =$

$$(v \otimes v_0) \otimes w\,(n+m) \cong v \otimes (v_0 \otimes w)(n+m) \cong (v \otimes w) v_0(n+m)$$
$$\to v \otimes w\,(n+m+1).$$

· universal property: every ⊗ + commutative representation

$$T: D^{\text{eff}} \longrightarrow R\text{-proj}$$

R dedekind w/ $T(v_0)$ ~~invertible~~ dualizable extends uniquely to D (via $v \mapsto v(0)$).

but: $1\!\!1(-1) = v_0^\vee$!

note: $v(n) = v \otimes v_0(n) \leftarrow v \otimes v_0^{\otimes n}$ but map not nec an iso in ...

proof: define $T(v(n)) = T(v) \otimes T(v_0)^{\otimes n}$ and $T(\alpha(n)) = T(\alpha) \otimes 1\!\!1^{\otimes}$ for $\alpha: v \to w$. if now $\beta: v(n) \otimes v_0 \to v(n+1)$ then

$$T(\beta): T(v) \otimes Tv_0^{\otimes n} \otimes Tv_0 \to T(v) \otimes Tv_0^{\otimes n+1}$$

is the obvious map.

proposition. $C(D,T)$ is the localization of $C(D^{\text{eff}},T)$ wrt $\tilde{T}v_0$. Furthermore, $A(D,T) = A(D^{\text{eff}},T)_\chi$ where $\chi \in \mathrm{End}(T(v_0)^*)^\vee$ is the dual of the identity (eg. Σx_{ii}).

Cor. MM^{nori} is an abelian ⊗-category, if/when defined.

§3. Good pairs and the BASIC LEMMA.

to finish the talk, we introduce the diagrams (V)Good$^{\text{eff}}$ ⊂ (V)Good:

DEFINITION: An effective pair (X,Y,i) is said to be "good" (very good) if $H^j(X,Y) = \delta_{ij}\mathbb{Z}^a$. (and X, Y affine, $X \setminus Y$ smooth + $\left(\begin{smallmatrix}\dim X = i \\ \operatorname{cod} Y = 1\end{smallmatrix}\right)$ or $Y = X$ dim(i))

(V)Good$^{\text{eff}}$ is a ⊗ diagram by künneth and (V)Good $:=$ (V)Good$^{\text{eff}}\left[\frac{1}{1\!\!1(-1)}\right]$.

Cor II. universal property:

$$\begin{tikzcd} & MM_{\text{nori}} \arrow[dr, "s_r"] \arrow[dd, "\exists!"] & \\ \text{Pairs}^{\text{eff}} \arrow[ur] \arrow[dr] & & S\text{-mod} \\ & A[H^*(1\!\!1(-1))^{-1}] \arrow[ur] & \end{tikzcd}$$

S/R flat + $R/\mathbb{Z}$ flat (so O_K or $F/\mathbb{Q}$)

~~the BASIC LEMMA: Let (X,Y) be a pair~~

FACT: if X affine variety of dim $n \Rightarrow X \underset{top}{\simeq}$ n-skeletal CW complex!

$\Rightarrow H^i(X;\mathbb{Z}),\ H^i(X,Y;\mathbb{Z}) = 0 \quad i > n \quad (Y \subseteq X)$.

„weak Lefschetz".

BASIC LEMMA: Let (X,Y) be a pair w/ $\dim X = n$, $Y \hookrightarrow X$ of dim $< n$. $\exists Z \hookrightarrow X$, $Y \subseteq Z \subseteq X$ with (X,Z,n) good pair.

* Reduction to field coefficients:

Bockstein: $\cdots \to H^i(X,Y;\mathbb{Z}) \xrightarrow{[p]} H^i(X,Y;\mathbb{Z}) \to H^i(X,Y;\mathbb{F}_p) \to H^{i+1} \cdots$

$\Rightarrow H^n(X,Y;\mathbb{Z})$ p torsfree + $H^i(X,Y;\mathbb{Z})$ divisible $i \neq n$

LEFSCHETZ HYPERPLANE PRINCIPLE:

H ample div on $X^{\text{smooth + proper}}$, $U = X \setminus H$. Note: $H^i(X,H) = H^i_c(U)$.

$\Rightarrow$ Poincaré Duality: $H^i_c(U) = H_{2n-i}(U) = 0 \quad i \le n$ (AV)

Suppose H' another ample divisor transverse to H. Relative Gysin sequence + weak purity:

$$R\Gamma(H, H \cap H')[-2] \to R\Gamma(X,H) \to R\Gamma(X \setminus H, H' \setminus H)$$

previous result $\Rightarrow R\Gamma(X \setminus H, H' \setminus H)$ in degree n.

In general: can suppose $W \subset X$ cont. sing locus.

$$\begin{array}{ccc} X & \hookrightarrow & \bar{X} \\ \cup\!\! | & & \cup\!\! | \\ W & \hookrightarrow & \bar{W} \end{array}$$

blowup $\tilde{\bar{X}}$ along ∂X and $\bar{W}$ $\Rightarrow$ get

$$\begin{array}{ccc} \tilde{X} = \tilde{\bar{X}} \setminus \partial\tilde{X} & \xrightarrow{\pi} & X \\ \cup\!\! | & & \cup\!\! | \\ \tilde{W} & \longrightarrow & W \end{array}$$

can increase $\tilde{W}$ to $\underbrace{\tilde{H} + m\tilde{W}}_{\tilde{D} =}$, $m \gg 0$: ample + transverse. take new $D = \pi\tilde{D}$, then $R\Gamma(X,D) = R\Gamma(\tilde{X},\tilde{D})$ concentrated in one degree by above.