Talk 7: Rigidity of Nori Motives

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These are notes for a talk given in the PhD-seminar on *Periods and Nori Motives* in the Summer Term 2024 at the University of Duisburg-Essen. The main references for this talk is [HM17]. I would like to thank the organiser Riccardo Tosi for discussing numerous questions with me.

Throughout let $k \subset \mathbb{C}$ be a subfield, and recall that a k-variety is a quasi-projective, reduced scheme of finite type over k. We denote the category of all k-varieties by Var.

Last time, we defined Nori motives. This was done by first considering the diagram $\mathrm{Pairs}^{\mathrm{eff}}$ of effective pairs whose vertices consist of pairs (X, Y, i) with X a k-variety, $Y \subset X$ a closed subvariety, and i an integer. We defined the category of *effective Nori motives* $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}} = \mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*)$. We then at the Tate motive $\mathbf{1}(-1) = (\mathrm{G}_m, \{1\}, 1)$ and called the resulting category $\mathcal{MM}_{\mathrm{Nori}} = \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}[\mathbf{1}(-1)^{-1}]$ the category of *Nori motives*. We write the functor $\mathrm{Pairs}^{\mathrm{eff}} \to \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ as $(X, Y, i) \mapsto H^i_{\mathrm{Nori}}(X, Y)$.

In this talk, we would like to prove rigidity. That is, we would like to prove the following Theorem:

Theorem 1 (Nori, [HM17, Thm. 9.1.5])**.** *(a) $\mathcal{MM}^{eff}_{Nori}$ has a natural structure of a commutative tensor category with unit such that H^* is a tensor functor.*

(b) $\mathcal{MM}_{Nori}$ is a rigid tensor category.

(c) $\mathcal{MM}_{Nori}$ is equivalent to the category of representations of a faithfully flat pro-algebraic group scheme $\mathrm{G}_{mot}(k, \mathbb{Z})$ over $\mathbb{Z}$.

We will spent the entire talk on proving this theorem.

Remark. It is an open question whether $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ is a full subcategory of $\mathcal{MM}_{\mathrm{Nori}}$ or equivalently, if $- \otimes \mathbf{1}(-1)$ is full on $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$. ⌟

Definition 2. The group scheme $\mathrm{G}_{\mathrm{mot}}(k, \mathbb{Z})$ is called the *motivic Galois group in the sense of Nori.* Its base change to $\mathbb{Q}$ is denoted by $\mathrm{G}_{\mathrm{mot}}(k, \mathbb{Q})$ or $\mathrm{G}_{\mathrm{mot}}(k)$ for short. ⌟

Remark. The first statement of the above theorem also holds with the coefficient ring $\mathbb{Z}$ replaced by any Noetherian ring R. The other two hold if R is a Dedeking ring or a field. ⌟

We want to do this, by applying Nori's rigidity criterion, which we proved in Talk 5.

Theorem 3 (Nori's rigidity criterion, [HM17, Prop. 8.3.4])**.** *Let $S = \{V_i \mid i \in I\}$ be a class of objects in $\mathcal{MM}^{Proj}_{Nori}$, i.e. $H^*(V_i)$ is projective for all i, with the following properties:*

(a) S generates $\mathcal{MM}_{Nori}$ as an abelian tensor category relative to H^*, i.e. its diagram category is all of $\mathcal{MM}_{Nori}$.

(b) For every V_i, there is a $W_i \in \mathcal{MM}_{Nori}^{Proj}$ and a morphism

$$q_i \colon \mathbf{1} \to V_i \otimes W_i$$

such that the dual of $H^*(q_i)$ induces a perfect pairing.

In order to be able to do this, we need to do three things:

- Define a tensor structure on $\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}}$,
- Find objects for which we can construct such a q_i, and
- Show that these objects generate.

For the tensor structure, we would like to define it as naturally as possible: That is, we would like to just use the product

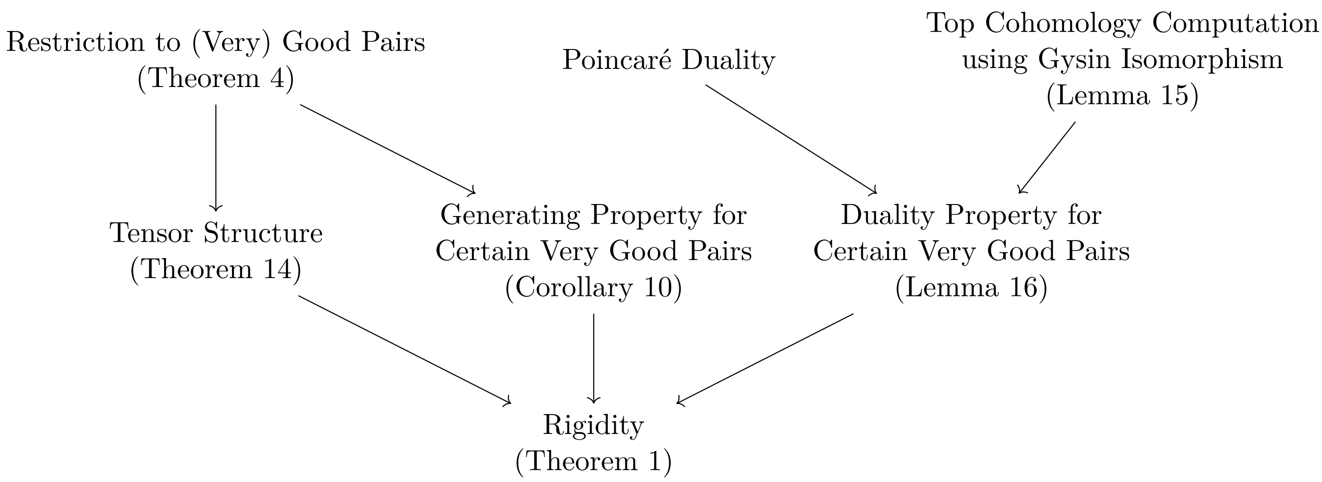
$$(X, Y, i) \times (X', Y', j) = (X \times X', Y \times X' \cup X \times Y', i + j).$$

Unfortunately, this is incompatible with taking cohomology, since the Künneth formula also involves other degrees when computing singular cohomology. Last time, we defined the notion of *good* and *very good* pairs. These were the following:

- A pair (X, Y, i) is *good* if the relative singular cohomology of (X, Y) is concentrated in degree i, and in degree i it is free.
- A good pair (X, Y, i) is *very good*, if X is affine, $X \setminus Y$ is smooth, and either X has dimension i and Y dimension $i - 1$, or $X = Y$ of dimension less than i.

The full sub-diagram of $\mathrm{Pairs}^{\mathrm{eff}}$ consisting of all (very) good pairs is denoted by $\mathrm{Good}^{\mathrm{eff}}$ ($\mathrm{VGood}^{\mathrm{eff}}$). We can easily define the tensor structure on the diagram category of good pairs. Thus, we would like to show that $\mathcal{C}(\mathrm{Good}^{\mathrm{eff}}, H^*) \simeq \mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*) = \mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}}$. (Actually for proving rigidity it is better to prove equivalence with very good pairs.)

With all of this in mind, we can talk about the strategy that we are going to employ to prove Theorem 1:



1 Every Pair is Essentially Good

We would like to prove the following theorem.

Theorem 4 ([HM17, Thm. 9.2.22]). *The diagram categories $\mathcal{C}(Pairs^{\text{eff}}, H^*), \mathcal{C}(Good^{\text{eff}}, H^*)$ and $\mathcal{C}(VGood^{\text{eff}}, H^*)$ are equivalent.*

In Talk 3, we have seen that we only need to cook up a representation $T\colon \text{Pairs}^{\text{eff}} \to \mathcal{C}(\text{VGood}^{\text{eff}}, H^*)$ such that the restriction to $\text{VGood}^{\text{eff}}$ is H^*.

We construct this, by essentially slowly extending the restricted representation $H^*\colon \text{VGood}^{\text{eff}} \to \mathcal{C}(\text{VGood}^{\text{eff}}, H^*)$. So, we already know what we have to do for affine very good pairs. The next step is to get to affine pairs. Fortunately, using the Basic Lemma from last time [HM17, Thm. 2.5.2] and an induction, one can prove the following

Proposition 5 ([HM17, Prop. 9.2.3]). *Every affine variety X has a filtration*

$$\emptyset = F_{-1}X \subset F_0X \subset \cdots \subset F_{n-1}X \subset F_nX = X$$

such that every pair $(F_jX, F_{j-1}X, j)$ is very good. We call such a filtration a very good filtration.

We already know what we should do with such a pair $(F_jX, F_{j-1}X, j)$, so it would be nice if we could replace X by a complex comprised of something related to the F_jX. In order to do this properly, we need to have a category of chain complexes and in order to have that we need an additive category constructed out of the category of Varieties. We are going to do this in one of the most naïve ways possible.

Definition 6 ([HM17, Def. 1.1.1]). Let $\mathbb{Z}[\text{Var}]$ be the category with the objects of Var as objects. The morphisms for two connected varieties X and Y are defined as $\mathbb{Z} \cdot \text{Hom}_{\text{Var}}(X, Y)$, and we extend this in the unique way preserving finite coproducts from Var. That is, we define for schemes $X = \bigcup_i X_i$ and $Y = \bigcup_j Y_j$, where the X_i's and Y_j's are the connective components of X and Y, respectively, the morphisms between X and Y as follows:

$$\text{Hom}_{\mathbb{Z}[\text{Var}]}(X, Y) = \bigoplus_{i,j} \text{Hom}_{\mathbb{Z}[\text{Var}]}(X_i, Y_j) = \bigoplus_{i,j} \left\{ \sum_k a_k f_k \mid a_k \in \mathbb{Z}, f_k \in \text{Hom}_{\text{Var}}(X_i, Y_j) \right\}.$$

Composition is defined by extending the composition on Var in a canonical way. ⌟

Filtrations look a bit like chain complexes, and this is exactly where we're headed. We would like to send an affine X to the complex

$$\cdots \to H^j_{\text{Nori}}(F_jX, F_{j-1}X) \to H^{j+1}_{\text{Nori}}(F_{j+1}X, F_jX) \to \cdots$$

in $D^b(\mathcal{C}(\text{VGood}^{\text{eff}}, H^*))$. Let's denote this complex by $\tilde{R}(F_\bullet X)$. This is functorial in very good filtrations. Furthermore if the pair $(X, \emptyset, i)$ were a very good pair, taking the trivial filtration F^*X, we get $H^i_{\text{Nori}}(X) = H^i(\tilde{R}(F_\bullet X))$. So this looks like a step in the right direction.

How do we extend this definition to a non-affine variety? We use, as one often does when involving cohomology, Čech covers. So for a not-necessarily affine variety X, we an finite open affine cover $\tilde{U}_X = \{U_i\}_{i \in I}$, and we replace X by its Čech complex $C_*(\tilde{U}_X)$ in $C_b(\mathbb{Z}[\text{Var}])$. Now each degree $C_n(\tilde{U}_X)$ is affine and thus, we can find a very good filtration for each of these. One can show that one can choose these filtrations compatible with the differential, such that we obtain a bigraded object $F_\bullet C_\bullet(\tilde{U}_X)$. For each n, we can construct $\tilde{R}(F_\bullet C_n(\tilde{U}_X)) \in D^b(\mathcal{C}(\text{VGood}^{\text{eff}}, H^*))$. By varying n, we obtain a bigraded complex $\tilde{R}(F_\bullet C_\bullet(\tilde{U}_X))$ of which, we can take the total complex $\text{Tot}(\tilde{R}(F_\bullet C_\bullet(\tilde{U}_X)))$. If we started with a very good pair (X, Y, i), we would have successfully extended H^*_{Nori} from very good effective pairs of the above form to all pairs of the above form – except that we didn't because assigning a Čech complex is not functorial. This is because a map of schemes does not necessarily induce a *unique* map of coverings. If it exists, there can be many such maps.

We can solve this issue by attaching more data to our covers and requiring morphisms of covers to respect this data.

Definition 7. Let X be a variety. A *rigidified affine cover* is a finite open affine covering $\{U_i\}_{i\in I}$ together with the following choice: for every point $x \in X$ an index $i_x \in I$ with $x \in U_{i_x}$. We also assume that every index occurs as i_x for some $x \in X$. ⌟

Now, we can repeat the above construction with rigidified affine covers (refining to get compatibility with morphisms as needed; this does not change the total complex obtained) to actually get a functorial assignment $\mathrm{Tot}(\tilde{R}(F_\bullet C_\bullet(\tilde{U}_X)))$.

With all of this in place, we can construct the general functor $R\colon C_b(\mathbb{Z}[\mathrm{Var}]) \to D^b(\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*))$ that we would like to have.

Proposition 8 ([HM17, Prop. 9.2.18])**.** *Consider the representation $H^*_{Nori}\colon VGood^{eff} \to \mathcal{C}(VGood^{eff}, H^*)$. Then, there is a natural contravariant triangulated functor*

$$R\colon C_b(\mathbb{Z}[Var]) \to D^b(\mathcal{C}(VGood^{eff}, H^*))$$

on the category of bounded homological complexes in $\mathbb{Z}[Var]$ such that for every good pair (X, Y, i), we have

$$H^j(R(\mathrm{Cone}(Y \to X))) = \begin{cases} 0 & j \neq i, \\ H^i_{Nori}(X, Y) & j = i. \end{cases}$$

Moreover, the image of $R(X)$ in $D^b(R - Mod)$ computes the singular cohomology of X.

Proof. Okay, we need to construct R. By the discussion before this proposition, we already know what we want to do for a complex concentrated in a single degree. We will now extend this, to the case of having a complex $X_\bullet \in C_b(\mathbb{Z}[\mathrm{Var}])$. For this, we chose compatible rigidified affine covers and then we make the same construction as above just with the very good filtrations of the total complex of the Čech (double) complex that we get from these covers.

Now it is an exercise in homological algebra to show that everything is well-defined, functorial and has the desired properties. ■

Remark. The above proposition also holds if we replace $\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$ by any Abelian category $\mathcal{A}$ together with a faithful forgetful functor f to $R-\mathrm{Mod}$ with R a Noetherian ring flat over $\mathbb{Z}$, and if we replace H^*_{Nori} by any representation $T\colon \mathrm{VGood}^{\mathrm{eff}} \to \mathcal{A}$ such that $f \circ T$ is singular cohomology with R-coefficients. ⌟

Definition 9. Let $Y \subset X$ be a closed subvariety with open complement U. For $i \in \mathbb{Z}$, we put

$$R(X,Y) = R(\mathrm{Cone}(Y \to X)), \quad R_Y(X) = R(\mathrm{Cone}(U \to X)) \in D^b(\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*))$$
$$H^i_{\mathrm{Nori}}(X,Y) = H^i(R(X,Y)), \quad H^i_{\mathrm{Nori}}(R_Y(X)) \in \mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*).$$

$H^*(X,Y)$ is called *relative cohomology* and $H_Y(X,i)$ is called *cohomology with support.* Note that this notion of relative cohomology is compatible with earlier usages. ⌟

Proof of Theorem 4. Note that the inclusion of diagrams induce faithful functors

$$\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*) \to \mathcal{C}(\mathrm{Good}^{\mathrm{eff}}, H^*) \to \mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*).$$

Therefore, in order to show that these inclusions are isomorphisms, it is enough to construct a representation T of $\mathrm{Pairs}^{\mathrm{eff}}$ in the category $\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$ restricting to H^*_{Nori} for very good effective pairs. By Proposition 8, we have a functor

$$R\colon C_b(\mathbb{Z}[\mathrm{Var}]) \to D^b(\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)).$$

Consider an effective pair (X, Y, i) in $\mathrm{Pairs}^{\mathrm{eff}}$. We represent it by

$$T(X, Y, i) = H^i(R(X, Y)) = H^i(R(\mathrm{Cone}(Y \to X)) \in \mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)).$$

By construction, this restricts to H^*_{Nori} on $\mathrm{VGood}^{\mathrm{eff}}$ and we only need to specify what happens on edges. This construction is functorial. Therefore, for edges induced by morphisms of schemes, we can just use the induced morphism between the cones. For the connecting homomorphisms, we use the machinery of triangulated categories to cook up the morphisms we need. ■

This allows us to obtain that certain good pairs generate the diagram category.

Corollary 10 ([HM17, Cor. 9.2.23])**.** *Every object in $\mathcal{MM}^{eff}_{Nori}$ is a subquotient of a direct sum of object of the form $H^i_{Nori}(X, Y)$ for a good pair (X, Y, i) where $X = W \setminus W_\infty$ and $Y = W_0 \setminus (W_0 \cap W_\infty)$ with W smooth and projective, and $W_\infty \cup W_0$ a divisor with normal crossings.*

Proof. By Theorem 4, $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ is generated by very good pairs. Thus, we can assume $X \setminus Y$ to be smooth. Now, we can use resolution of singularities to cook up W, W_0, and W_∞ and the desired isomorphism of a very good pair with such a motive follows by excision. ■

2 Tensor Structure

Now that we can reduce to effective (very) good pairs, we can construct the tensor structure.

Proposition 11. *The graded diagrams $Good^{eff}$ and $VGood^{eff}$ carry a weak commutative product structure in the sense of Talk 5 [HM17, Rmk. 8.1.6] defined as follows: For all vertices $(X, Y, i), (X', Y', i')$*

$$(X, Y, i) \times (X', Y', i') = (X \times X', X \times Y' \cup Y \times X', i + i'),$$

with the obvious definition on edges. There is a unit given by $(\mathrm{Spec}\, k, \emptyset, 0)$.

Moreover, H^* is a weak graded multiplicative representation in the sense of Talk 5 [HM17, Def. 8.1.3 and Rmk. 8.1.6] with

$$\tau\colon H^{i+1}(X \times X', X \times Y' \cup Y \times X'; \mathbb{Z}) \to H^i(X, Y; \mathbb{Z}) \otimes H^{i'}(X', Y'; \mathbb{Z})$$

the Künneth isomorphism.

Proof. Note that everything is well-defined, and the claim follows by doing some tedious but straightforward compatibility checks. ■

Definition 12. Let Good and VGood be the localisations of $\mathrm{Good}^{\mathrm{eff}}$ and $\mathrm{VGood}^{\mathrm{eff}}$, respectively, with respect to the vertex $\mathbf{1}(-1) = (\mathrm{G}_m, \{1\}, 1)$. ⌟

Proposition 13 ([HM17, Prop. 9.3.3])**.** *Good and VGood are graded diagrams with a weak commutative product structure. Moreover, H^* is a graded multiplicative representation of Good and VGood.*

Proof. This follows from the effective case. ■

Putting this proposition and the results of the previous section together, we obtain the following:

Theorem 14 ([HM17, Thm. 9.3.4])**.** *(a)* $\mathcal{MM}^{eff}_{Nori} \subset \mathcal{MM}_{Nori}$ *are commutative tensor categories with a faithful fibre functor* H^*.

(b) $\mathcal{MM}_{Nori}$ *is equivalent ot the two diagram categories* $\mathcal{C}(Good, H^*)$ *and* $\mathcal{C}(VGood, H^*)$.

3 Rigidity of Nori Motives

Now that we have constructed a tensor structure on Nori motives, we want to prove rigidity. We have already found a rather specific generating set in Corollary 10. Thus, it remains to prove some version of Poincaré duality for these objects.

Lemma 15 ([HM17, Lemma 9.3.8])**.** *(a)* $H^{2n}_{Nori}(\mathbb{P}^N) = \mathbf{1}(-n)$ *for* $N \geq n \geq 0$.

(b) Let Z *be a projective variety of dimension* n*. Then* $H^{2n}_{Nori}(Z) \cong \mathbf{1}(-n)$.

(c) let X *be a smooth variety and* $Z \subset X$ *a smooth, irreducible, closed subvariety of pure codimension* n*. Then the motive with support satisfies*

$$H^{2n}_Z(X) \cong \mathbf{1}(-n).$$

Proof. Recall that singular cohomology is a conservative functor on Nori motives. Thus, it is enough to construct a morphism in $\mathcal{MM}_{\text{Nori}}$ and check that it induces an isomorphism in singular cohomology.

We have seen the proof of (a) in Talk 2 for Hodge structures. The proof for Nori motives is virtually the same.

For (b), choose an embedding $Z \to \mathbb{P}^N$ with $N \geq n$. Then $H^{2n}_{\text{Nori}}(Z) \leftarrow H^{2n}(\mathbb{P}^N) \cong \mathbf{1}(-n)$ is an isomorphism in $\mathcal{MM}_{\text{Nori}}$ because it is in singular cohomology.

For (c), note that the Gysin isomorphism for singular cohomology implies this assertion there. Our strategy now is to construct the Gysin isomorphism motivically, and for this we are more or less repeating its construction.

For the embedding $Z \to X$ one has the deformation to the normal cone, that is a smooth scheme $D(X, Z)$ together with a morphism to $\mathbb{A}^1$ such that the fibre of 0 is given by the normal bundle $N_Z X$ of Z in X and the other fibres by X. The product $Z \times \mathbb{A}^1$ can be embedded into $D(X, Z)$ as a closed subvariety of codimension n; this induces the zero section on $N_Z X$ on the fibre of zero and the embedding of Z in X on all other fibres.

This yields the diagram

$$\begin{array}{ccccc}
Z & \longrightarrow & Z \times \mathbb{A}^1 & \longleftarrow & Z \\
\downarrow{\scriptstyle 0} & & \downarrow & & \downarrow \\
N_Z(X) & \longrightarrow & D(X, Z) & \longleftarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
\{0\} & \longrightarrow & \mathbb{A}^1 & \longleftarrow & \{1\}.
\end{array}$$

The Gysin isomorphisms in singular cohomology for the three closed embeddings together with homotopy invariance of singular cohomology imply that the maps

$$H^{2n}_Z(X) \leftarrow H^{2n}_{Z \times \mathbb{A}^1}(D(X, Z)) \to H^{2n}_Z(N_Z X)$$

are isomorphisms of motives. Thus, we have reduced the problem to the embedding of the zero-section $Z \hookrightarrow N_Z X$. This trivialises on a dense open subset $U \subset Z$, which in turn by construction yields the isomorphism

$$H^{2n}_Z(N_Z X) \to H^{2n}_U(N_Z X|_U)$$

induced by the inclusion. Thus, we may assume that the normal bundle is trivial, but in this case, we get

$$N_Z(X) \cong N_{Z\times\{0\}}(Z \times \mathbb{A}^n) \cong Z \times N_{\{0\}}(\mathbb{A}^n).$$

The Gysin isomorphism yields that the cohomology $H^*_{\{0\}}(N_{\{0\}}(\mathbb{A}^n))$ is concentrated in degree $2n$ and thus, the Künneth formula with support yields

$$H^{2n}_Z(N_Z X) \cong H^{2n}_{\{0\}}(N_{\{0\}}(\mathbb{A}^n)) \cong H^{2n}_{\{0\}}(\mathbb{A}^n) \cong (H^2_{\{0\}}(\mathbb{A}^1))^{\otimes n} \cong \mathbf{1}(-n),$$

which yields the formula of the statement. ∎

With this preparatory work done, we can show that the pairs of the from in Corollary 10 satisfy a duality condition.

Lemma 16 ([HM17, Lemma 9.3.9])**.** *Let W be a smooth projective variety of dimension i and $W_0, W_\infty \subset W$ divisors such that $W_0 \cup W_\infty$ is a normal crossing divisor. Let*

$$X = W \setminus W_\infty, \quad Y = W_0 \setminus (W_0 \cap W_\infty), \quad X' = W \setminus W_0, \quad \textit{and} \quad Y' = W_\infty \setminus (W_0 \cap W_\infty).$$

We assume that (X, Y) is a very good pair.

Then, there is a morphism in $\mathcal{MM}_{Nori}$

$$q\colon \mathbf{1} \to H^i_{Nori}(X, Y) \otimes H^i_{Nori}(X'Y')(i)$$

such that the dual of $H^*(q)$ is a perfect pairing.

Proof. The two pairs (X, Y) and (X', Y') are Poincaré dual, see for example [HM17, Prop. 2.4.5] for a proof. Thus, both of them are good pairs, and hence

$$H^i_{\mathrm{Nori}}(X, Y) \otimes H^i_{\mathrm{Nori}}(X', Y') \to H^{2i}_{\mathrm{Nori}}(X \times X', X \times Y' \cup Y \times X')$$

is an isomorphism. Let $\Delta = \Delta(W \setminus (W_0 \cup W_\infty))$ via the diagonal embedding Δ. Now, we have

$$X \times Y' \cup Y \times X' \subset (X \times X') \setminus \Delta.$$

Hence by functoriality and the definition of the motive with support, we have a map

$$H^{2i}_{\mathrm{Nori}}(X \times X', X \times Y' \cup Y \times X') \leftarrow H^{2i}_\Delta(X \times X').$$

Again by functoriality, there is a map

$$H^{2i}_\Delta(X \times X') \leftarrow H^{2i}_{\tilde\Delta}(W \times W)$$

with $\tilde\Delta = \Delta(W)$. By Lemma 15 (c), this is isomorphic to $\mathbf{1}(-i)$. We now get the map q by twisting this composition by i. Since the dual of this map realises Poincaré duality in singular cohomology, it is a perfect pairing. ∎

Theorem 17 (Nori, [HM17, Thm. 9.3.10])**.** *Let $k \subset \mathbb{C}$ be a field. Then $\mathcal{MM}_{Nori}(k)$ is rigid, hence a neutral Tannakian category. It is equivalent to the category of linear algebraic representations of the affine faithfully flat group scheme over $\mathbb{Z}$*

$$G_{mot}(k, \mathbb{Z}) := \mathrm{Spec}(A(\mathit{Good}, H^*)).$$

Proof. We can apply Nori's rigidity criterion: Let S be the set of objects of the form occurring in Lemma 16. By this lemma, they admit a perfect pairing. Furthermore, they generate by Corollary 10. Thus by Nori's rigidity criterion and some statements around it, we get the isomorphism to the category of representations of the above monoid scheme, which by Nori's rigidity criterion is actually a group scheme. ■

Proof of Theorem 1. Theorem 17 is just a more spelled out version of Theorem 1. ■

References

[HM17] Annette Huber and Stefan Müller-Stach. *Periods and Nori motives.* Vol. 65. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer, Cham, 2017. xxiii+372.