§8. Equivalence of the definitions of periods

1. NC-periods and cohomological periods

$k \subseteq \mathbb{C}$ subfield

Def. Let (X, D, ω, Γ) be a quadruple consisting of:

- . X smooth, dim d, variety $/k$
- . $D \subseteq X$ nc divisor (def $/k$)
- . $\omega \in \Gamma(X, \Omega^d_{X/k})$ (algebraic differential form of top degree)
- . Γ relative differentiable singular d-chain on X^{an} s.t. $\partial\Gamma \subseteq D^{an}$. [This means:

$\Gamma = \sum_{i=1}^{n} \alpha_i \gamma_i$, $\alpha_i \in \mathbb{Q}$, $\gamma_i : \Delta_d \longrightarrow X^{an}$ (that can be extended to a $\mathcal{C}^\infty$-map of a nghd of $\Delta_d \subseteq \mathbb{R}^{d+1}$), $\partial\Gamma = \sum_{i=1}^{n} \alpha_i (\partial\gamma_i)$, $\partial\gamma_i = \sum_{i=0}^{d} (-1)^i \gamma_i|_{t_i=0}$]

(1) The period of the quadruple (X, D, ω, Γ) is

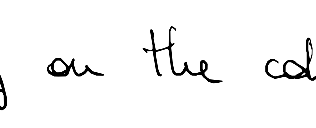
$$\int_\Gamma \omega = \sum_{i=1}^{n} \alpha_i \int_{\Delta_d} \gamma_i^* \omega .$$

(2) The algebra of effective periods is the set $\mathbb{P}^{eff}_{nc}$ of all periods, for all (X, D, Γ, ω).

(3) The periods algebra is $\mathbb{P}_{nc} = \left\{ (2\pi i)^n \alpha \;\middle|\; \begin{array}{l} \alpha \in \mathbb{P}^{eff}_{nc} \\ n \in \mathbb{Z} \end{array} \right\}$

Ex. (1) $\left(\mathbb{G}_m, \phi, \frac{1}{t}dt, \gamma : \Delta_1 \xrightarrow{\text{unit cycle}} \mathbb{C}^\times \right) \rightsquigarrow \int_{S^1} t^{-1} dt = 2\pi i$

(2) $\left(\mathbb{A}^1, V(t^3 - 2t), dt, \gamma : \Delta_1 \longrightarrow \mathbb{C}\right) \rightsquigarrow \int_\gamma \omega = \sqrt{2}$



(3) $\left(\mathbb{G}_m, V((t-2)(t-1)), t^{-1}dt, \gamma : \underset{1}{\bullet}\!\!\longrightarrow\!\!\underset{2}{\bullet} \right) \rightsquigarrow \int_\gamma \omega = \int_1^2 t^{-1} dt = \log(2)$

Lemma. For (X, D, ω, Γ) as above, the period $\int_\Gamma \omega$ depends only on the cohomology classes of ω and Γ (in the respective cohomology groups).

Proof.

. If $\Gamma' - \Gamma'' \sim \partial(\overset{\in\, C^\infty_{d+1}(X^{an}, D^{an}; \mathbb{Q})}{\Gamma_{d+1}}) \Rightarrow \int_{\Gamma'} \omega - \int_{\Gamma''} \omega = \int_{\partial(\Gamma_{d+1})} \omega = \int_{\Gamma_{d+1}} d\omega = 0$

. If $\omega' - \omega = d(\overset{\in\, \Omega^{d-1}_{X/k}}{\eta}) \Rightarrow \int_\Gamma \omega' - \int_\Gamma \omega = \int_\Gamma d(\eta) = \int_{\partial\Gamma} \eta$

Prop. $\mathbb{P}^{eff}_{nc}$ and $\mathbb{P}_{nc}$ are k-algebras.

Proof.

. $(X, D, \omega, \Gamma) \cdot (X', D', \omega', \Gamma') :=$

$= (X \times X', D \times X' \cup X \times D', \omega \wedge \omega', \Gamma \times \Gamma')$

. $a \in k$: $a \cdot (X, D, \omega, \Gamma) = (X, D, a \cdot \omega, \Gamma)$

Note: period of $(\mathbb{A}^1, \{0, 1\}, dt, [0, 1])$ is $\int_0^1 dt = 1$, so multiplying (X, D, ω, Γ) by does not change the period.

Hence WLOG (X, D, ω, Γ), $(X', D', \omega', \Gamma')$ s.t. $\dim X = \dim X'$:

. $(X, D, \omega, \Gamma) + (X', D', \omega', \Gamma') :=$

$= (X \sqcup X', D \sqcup D', \omega + \omega', \Gamma + \Gamma')$

$\square$

Def. Let $(X, Y, j) \in \mathrm{Pair}^{eff}$.

(1) The set of periods $\mathbb{P}(X, Y, j)$ is the image of the period pairing

$$\mathrm{per} : H^j_{dR}(X, Y) \times H^{sing}_j(X^{an}, Y^{an}; \mathbb{Q}) \longrightarrow \mathbb{C}$$

(2) The space of period is $\mathbb{P}\langle X, Y, j \rangle := \langle \mathbb{P}(X, Y, j) \rangle_{\mathbb{Q}\text{-v.s.}}$

(3) $\mathbb{P}^{eff} = \bigcup_{(X,Y,j) \in \mathrm{Pairs}^{eff}} \mathbb{P}(X, Y, j)$ (effective period algebra)

$\mathbb{P} = \left\{ (2\pi i)^n \alpha \;\middle|\; \begin{array}{l} \alpha \in \mathbb{P}^{eff} \\ n \in \mathbb{Z} \end{array} \right\}$ (period algebra)

Lemma. $\mathbb{P}^{eff}$ and $\mathbb{P}$ are k-subalgebras of $\mathbb{C}$.

Proof.

Maybe

Lemma. There are natural inclusions $\mathbb{P}^{eff}_{nc} \subseteq \mathbb{P}^{eff}$ and $\mathbb{P}_{nc} \subseteq \mathbb{P}$

Proof.

Enough to consider the effective case

. nc: $(X, D, \omega, \Gamma) \rightsquigarrow \int_\Gamma \omega$

. coh: $(X, Y, j) \rightsquigarrow \mathrm{per}\left(H^j_{dR}(X, Y) \times H^{sing}_j(X^{an}, Y^{an}; \mathbb{Q})\right)$

Assume: X sm. affine dim d, D snc divisor, $\omega \in \Omega^d_X(X) \rightsquigarrow [\omega] \in H^d_{dR}(X, D)$ (get everything by 3.3.19), $\Gamma \in H^{sing}_d(X^{an}, D^{an}; \mathbb{Q})$ (differentiable)

Then $\mathrm{per}([\omega], \Gamma) = \int_\Gamma \omega$, so any nc-period can be written as a cohomological period

In general?

$\square$

2. Formal periods

$$\begin{tikzcd} D \arrow[r, bend left, "T_1"] \arrow[r, bend right, "T_2"'] & \mathrm{Proj}_R \end{tikzcd}$$ representations.

$\rightsquigarrow \mathcal{C}(D,T_i) \simeq \mathcal{A}(D,T_i)\text{-comod}$ $(i=1,2)$,

$A_i = \mathcal{A}(D,T_i) = \varinjlim_{F\subseteq D} \mathrm{End}(T_{i|F})^\vee$ (coalgebra)

. Assume $(D,|\cdot|,\times)$ graded w/ unital commutative product structure and T_1,T_2 unital mult. reps.

$\rightsquigarrow \mathcal{C}(D,T_i)$ symm. $\otimes$-cat. w/ unit

$\rightsquigarrow A_i$ is a bialgebra and $\mathrm{Spec}\, A_i$ is a fflat unital monoid scheme $/R$

We want to compare A_1 and A_2.

Def. (1) For D,T_1,T_2 as above, define

$$\mathrm{Hom}(T_1,T_2) := \left\{ (f_p)_{p\in D} \in \prod_D \mathrm{Hom}_R(T_1p,T_2p) \;\middle|\; \begin{tikzcd} T_1p \arrow[r, "f_p"] \arrow[d, "T_1m"'] & T_2p \arrow[d, "T_2m"] \\ T_1q \arrow[r, "f_q"] & T_2q \end{tikzcd} \quad \begin{tikzcd} p \arrow[d, "m"] \\ q \end{tikzcd} \right\}$$

homomorphisms of reps $T_1 \to T_2$

(2) $A_{12} := \varinjlim_F \mathrm{Hom}(T_{1|F},T_{2|F})^\vee$

. A_{12} is fflat$/R$ since it is the colimit of loc. free modules

. In Talk 5 we saw that $A_1 = \varinjlim_{F\subseteq D} \mathrm{End}(T_{1|F})^\vee$ is a commutative algebra if D,T have mult. str. The same argument works here: $F,F'\subseteq D$ finite s.t. $\{v\times w \mid v,w\in F\}\subseteq F'$. Then:

$\to \mu_F^* : \mathrm{Hom}(T_{1|F'},T_{2|F'}) \longrightarrow \mathrm{Hom}(T_{1|F},T_{2|F}) \otimes \mathrm{Hom}(T_{1|F},T_{2|F})$

$f=(f_p)_{p\in F'} \longmapsto \mu_F^*(f)$

For $(v,w)\in F\times F$ the iso $T(v\times w) \longrightarrow T(v)\otimes T(w)$ induces an isomorphism

$\mathrm{Hom}(T_1(v),T_2(v)) \otimes \mathrm{Hom}(T_1(w),T_2(w)) \simeq \mathrm{Hom}(T_1(v\times w),T_2(v\times w))$

Then $\mu_F^*(f) \in \prod_{F\times F} \cdots \simeq \prod_{F\times F} \mathrm{Hom}(T_1(v\times w),T_2(v\times w))$ has as (v,w) component the image of $f_{v\times w}$ under the above isomorphism

$\to$ let $\mu_F : A_{12}(F)\otimes A_{12}(F) \longrightarrow A_{12}(F')$ be the R-dual of μ_F^* and define the multiplication on A_{12} by

$\mu = \varinjlim_{F,F'\subseteq D} \mu_F : A_{12}\otimes A_{12} \longrightarrow A_{12}$

Prop. $X_{12} = \mathrm{Spec}\, A_{12}$, $G_i = \mathrm{Spec}\, A_i$. $\left(\begin{array}{l} + D \text{ w/ u.c. pr. str.,} \\ T_i \text{ m. reps} \end{array}\right)$

(1) X_{12} is fflat$/R$ (if $\neq\emptyset$) and have actions

$G_1\times X_{12} \longrightarrow X_{12}$, $X_{12}\times G_2 \longrightarrow X_{12}$

(2) S/R fflat R-algebra. Then

$X_{12}(S) = \mathrm{Hom}(T_1,T_2)$ (as unital mult. rep.'s)

Proof.

(1) The maps

$\mathrm{End}(T_{1|F}) \times \mathrm{Hom}(T_{1|F},T_{2|F}) \longrightarrow \mathrm{Hom}(T_{1|F},T_{2|F})$

$\mathrm{Hom}(T_{1|F},T_{2|F}) \times \mathrm{End}(T_{2|F}) \longrightarrow \mathrm{Hom}(T_{1|F},T_{2|F})$

induce, by passing to the $\varinjlim$, R-alg. maps

$A_{12} \longrightarrow A_1\otimes_R A_{12}$

$A_{12} \longrightarrow A_{12}\otimes_R A_2$

(2) Suppose D finite. Then:

$$\mathrm{Hom}_{R\text{-lin}}(\underset{\substack{\| \\ \mathrm{Hom}_R(T_1,T_2)^\vee \leftarrow \text{f.l.f}}}{A_{12}}, S) = \mathrm{Hom}_R(T_1,T_2)\otimes_R S = \mathrm{Hom}_S(T_1\otimes S_{|F}, T_2\otimes S_{|F})$$

. In general, since $\otimes$ commutes w/ $\varinjlim$:

$\mathrm{Hom}_{R\text{-lin}}(\varinjlim_{F\subseteq D} \mathrm{Hom}_R(T_{1|F},T_{2|F})^\vee, S) =$

$= \varprojlim_{F\subseteq D} \mathrm{Hom}_{R\text{-lin}}(\mathrm{Hom}_R(T_{1|F},T_{2|F})^\vee, S) = \varprojlim_{F\subseteq D} \mathrm{Hom}_S(T_1\otimes S_{|F}, T_2\otimes S_{|F})$

$= \mathrm{Hom}_S(T_1\otimes S, T_2\otimes S)$

. The condition for $\phi\in \mathrm{Hom}_{R\text{-lin}}(A_{12},S)$ to respect the product translates into being a morphism of unital mult. rep's

$$\left\{ (f_p)_D \in \prod_D \mathrm{Hom}_R(T_1p,T_2p) \;\middle|\; \begin{tikzcd} T_1p \arrow[r, "f_p"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & T_2p \arrow[d] \\ T_1q \arrow[r, "f_q"] & T_2q \end{tikzcd} ; \begin{tikzcd} T_1(p\times q) \arrow[r, "f_{p\times q}"] \arrow[d] & T_2(p\times q) \arrow[d] \\ T_1p\otimes T_1q \arrow[r] & T_2p\otimes T_2q \end{tikzcd} \right\}$$

□

The algebra A_{12} is closely related with formal periods

Def. $T_1,T_2 : D \longrightarrow \mathrm{Proj}_R$ reps. The space of formal periods P_{12} is the R-module generated by symbols (p,ω,γ), with $p\in V(D)$, $\omega\in T_1p$, $\gamma\in (T_2p)^\vee$ subject to the following relations:

(1) linearity in ω,γ

(2) functoriality: $\forall\, p \xrightarrow{f} p' \in E(D)$, $\gamma\in T_2p'^\vee$, $\omega\in T_1p$

$(p', T_1f(\omega),\gamma) = (p,\omega, T_2f^\vee(\gamma))$ $\quad \begin{tikzcd} T_2p^\vee \arrow[r, "T_2f^\vee"] & T_2p'^\vee \end{tikzcd}$ $\quad \begin{tikzcd} T_1p \arrow[r, "T_1f"] & T_1p' \end{tikzcd}$

Ex. The space of effective formal period numbers $\tilde{\mathbb{P}}$ is the k-v.s. generated by symbols (X,Y,j,ω,γ), where:

. $(X,Y,j)\in \mathrm{Pair}^{\mathrm{eff}}$

. $\omega\in H^j_{dR}(X,Y)$

. $\gamma\in H_j^{sing}(X,Y;k) = H^j_{sing}(X,Y;k)^\vee$

subject to the following relations:

(1) linearity in ω,γ

(2) $\forall\ (X',Y') \xrightarrow{f} (X,Y)$, $\omega\in H^j_{dR}(X,Y)$, $\gamma\in H_j^{sing}(X',Y')$

$(X',Y',f^*\omega,\gamma) = (X,Y,\omega,f_*\gamma)$ $[(X',Y',j) \longrightarrow (X,Y,j)]$

(3) $\forall\ Z\subseteq Y\subseteq X$, $\omega\in H^j_{dR}(Y,Z)$, $\gamma\in H^{sing}_{j+1}(X,Y)$

$(Y,Z,\omega,\partial\gamma) = (X,Y,\delta\omega,\gamma)$ $[(Y,Z,j) \longrightarrow (X,Y,j+1)]$

where $\delta : H^j_{dR}(Y,Z) \longrightarrow H^{j+1}_{dR}(X,Y)$

$\partial : H^{sing}_{j+1}(X,Y) \longrightarrow H^{sing}_j(Y,Z)$

Prop. In the setting of the above definition, if D has unital comm. prod. str. and T_1,T_2 are unital mult. reps, then P_{12} is an R-algebra w/ product given by

$(p,\omega,\gamma)(p',\omega',\gamma') := (p\times p', \omega\otimes\omega', \gamma\otimes\gamma')$

Rmk. $\mathrm{Pairs}^{\mathrm{eff}}$ has not a mult. str., but $\tilde{\mathbb{P}}$ has the structure of a k-algebra!

$[X,D,\omega,\gamma][X',D',\omega',\gamma'] := [X\times X', D\times X'\cup X\times D', \omega\wedge\omega', \gamma\times\gamma']$

We will see how $\tilde{\mathbb{P}} = P_{12}(\mathrm{Pairs}^{\mathrm{eff}}) \overset{!}{=} A_{12}(\mathrm{Pairs}^{\mathrm{eff}}) =$
$= A_{12}(\mathrm{Good}^{\mathrm{eff}}) = A_{12}(\mathrm{VGood}^{\mathrm{eff}})$

this has a product str by above Prop, and the two coincide.

Thm Given $T_1,T_2 : D \longrightarrow \mathrm{Proj}_R$, there is a canonical R-linear isomorphism

$\psi : P_{12} \longrightarrow A_{12}$,

which is a map of R-algebras if D,T_1,T_2 "are multiplicative".

Proof.

$$\begin{tikzcd} P_{12} = \varinjlim_{F\subseteq D} P_{12}(F) \arrow[d] & P_{12}(F) \arrow[l] \arrow[d] \arrow[r, phantom, "\overset{(*)}{\subseteq}"] & \prod_{p\in F} T_1p\otimes T_2p^\vee \arrow[d, "\cong (\bigstar)"] \\ A_{12} = \varinjlim_{F\subseteq D} A_{12}(F) & A_{12}(F) \arrow[l] \arrow[r, phantom, "\overset{(*)}{\subseteq}"] & \prod_{p\in F} \mathrm{Hom}(T_1p,T_2p)^\vee \end{tikzcd}$$

$\varinjlim$ $\quad$ restriction

$(*)$ $P_{12}(F) = \left\{ (\omega_p\otimes\gamma_p)_{p\in F} \in \prod_F T_1p\otimes T_2p^\vee \;\middle|\; \forall f \begin{array}{c} p \\ \downarrow \\ p' \end{array} \begin{array}{l} T_1f(\omega_p) = \omega_{p'} \\ T_2f^\vee(\gamma_{p'}) = \gamma_p \end{array} \right\}$ and

the inclusion for $A_{12}(F)$ is still "determined" by relations induced by edges.

$(\bigstar)$ is the canonical isomorphism for finite projective R-modules

$N\otimes P^\vee \longrightarrow \mathrm{Hom}(N,P)^\vee$

$n\otimes t \longmapsto [N \xrightarrow{g} P \mapsto t(g(n))]$

and it restricts to an iso $P_{12}(F) \longrightarrow A_{12}(F)$

□

So formal periods are connected with the comparison algebra of T_1 and T_2.

Thm (!) Let $R\longrightarrow S$ be f.flat and let $\varphi : T_1\otimes S \xrightarrow{\simeq} T_2\otimes S$ be an iso. of reps. Then:

(1) $\exists\ \phi\in X_{12}(S)$ s.t. the maps

$G_{1,S} \longrightarrow X_{12,S}$, $g\longmapsto \mu_1(g,\phi)$

$G_{2,S} \longrightarrow X_{12,S}$, $g\longmapsto \mu_2(\phi,g)$

are iso. $(\mu_1 : G_1\times X_{12} \longrightarrow X_{12},\ \mu_2 \ldots)$

(2) ϕ induces an equivalence of unital tensor categories

$\Phi : \mathcal{C}(D,T_1) \xrightarrow{\simeq} \mathcal{C}(D,T_2)$

(3) A_{12} is canonically isomorphic to the comparison algebra for $\curvearrowright$ (i.e. $\mathcal{A}(\mathcal{C}; f_{T_1}, f_{T_2}\circ\Phi)$).

$$\begin{tikzcd} \mathcal{C} := \mathcal{C}(D,T_1) \arrow[r, bend left, "f_{T_1}"] \arrow[r, bend right, "f_{T_2}\circ\Phi"'] & \mathrm{Proj}_R \end{tikzcd}$$

(4) Suppose that $\mathcal{C}(D,T_1)$ is rigid. Then X_{12} is a left G_1-torsor and a right G_2-torsor in the fpqc-topology.

Proof.

(1) $G_{1,S}$, $X_{12,S}$ are fflat S-schemes. For T/S,

$G_{1,S}(T) \longrightarrow X_{12,S}(T)$ $\quad (\mu_{1,T} : G_1(T)\times X_{12}(T) \longrightarrow X_{12}(T))$

$g \longmapsto \mu_1(g,\phi_T)$

$\vdots$

(2)
$$\begin{tikzcd} & \mathcal{C}(D,T_1\otimes S) \arrow[dd, dashed, "\Phi"] \arrow[dr, "f_{T_1}"] & \\ D \arrow[ur, "\tilde{T}_1"] \arrow[dr, "\tilde{T}_2"'] \arrow[rr, bend left=10, "T_1\otimes S"] \arrow[rr, bend right=10, "T_2\otimes S"'] & & \mathrm{Proj}_{R/S} \\ & \mathcal{C}(D,T_2\otimes S) \arrow[uu, dashed, bend right, "\Phi^{-1}"'] \arrow[ur, "f_{T_2}"'] & \end{tikzcd}$$
(generalized form of universal property for diagram cats)

(3) $A_{12} \cong \mathcal{A}(D; f_{T_1}\circ\tilde{T}_1, f_{T_2}\circ\Phi\circ\tilde{T}_1)$. The map of diagrams $\tilde{T}_1 : D \longrightarrow \mathcal{C}(D,T_1) =: \mathcal{C}$ gives an alg. map

$A_{12} \longrightarrow \mathcal{A}(\mathcal{C}; f_{T_1}, f_{T_2}\circ\Phi)$ (or just R-mod.).

To check it's an iso, we can base change to S and use ϕ to replace $T_2\otimes S$ with $T_1\otimes S$. Then the above map becomes the iso $\mathcal{A}(D,T_1\otimes S) \xrightarrow{\simeq} \mathcal{A}(\mathcal{C}(D,T_1), f_{T_1})$.

(4) $\mathcal{C}(D,T_1)$ rigid $\Rightarrow$ $\mathcal{C}(D,T_2)$ rigid (they are equivalent!).

Hence $G_i = \mathrm{Spec}\, A_i$ is a group scheme and (1) $\Rightarrow$ X_{12} is a torsor under them.

3. Equivalence of the definitions

Recall: $G_{mot}(k,\mathbb{Z}) = \mathrm{Spec}(A(Good, H^*))$, affine, f flat gp. sch. $/\mathbb{Z}$

$$A(Good, H^*) = \varinjlim_{F \subseteq Good} \mathrm{End}(H^*_{|F})^\vee$$

$$G_{mot}(k) = G_{mot}(k,\mathbb{Z}) \otimes \mathbb{Q}$$

Thm The scheme $X = \mathrm{Spec}\,\tilde{\mathbb{P}}$ is (isomorphic to) the torsor of isomorphisms between singular cohomology and deRham cohomology (x_{12}).
It is a torsor under $G_{mot}(k)_k$

Proof.

. We apply the discussion of section 2 to

$$\begin{tikzcd} Good^{eff} \subseteq Pairs^{eff} \arrow[r, bend left, "T_1 := H^*_{dR}(-,-)"] \arrow[r, bend right, "T_2 := H^*_{sing}(-,-;k)"'] & Proj_k = k\text{-v.s.} \end{tikzcd}$$

. $\tilde{\mathbb{P}}^{eff} = P_{12}(Pairs^{eff}) = A_{12}(Pairs^{eff})$ as k-v.s., and $A_{12}(Pairs^{eff}) \overset{!}{=} A_{12}(Good^{eff})$ (as k-v.s.). Indeed, $A_{12}(Pairs^{eff}) = \mathcal{A}(\mathcal{C}(Pairs^{eff}, H^*_{dR}); f_{H^*_{dR}}, f_{H^*_{sing}} \circ \Phi)$ by Thm(!)(3). But we know $\Phi: \mathcal{C}(Pairs^{eff}, H^*_{dR}) \xrightarrow{\simeq} \mathcal{C}(Pairs^{eff}, H^*_{sing})$ that $\mathcal{C}(Pairs^{eff}, H^*_{dR}) \simeq \mathcal{C}(Good^{eff}, H^*_{dR})$ and using still Thm(!)(3) we get that $A_{12}(Pairs^{eff}) \cong A_{12}(Good^{eff})$.

. <u>Claim</u>: The isomorphism of k-v.s.

$$A_{12}(Good^{eff}) \cong \tilde{\mathbb{P}} \text{ is of } k\text{-algebras.}$$

. Same reasoning for Pairs and Good. Since localization of diagram $\leftrightsquigarrow$ localization of corresponding algebras, we get that

$$A_{12}(Good) \cong P_{12}(Good) = \tilde{\mathbb{P}},$$

so $X = \mathrm{Spec}(A_{12}(Good))$.

. By Thm(!)(4), X is a torsor under the group $G_2 = \mathrm{Spec}(A_2(Good))$ which is the base change to k of the motivic Galois group.

□

Rmk Implicit in the use of Thm(!) in the proof above, is the fact that the period isomorphism $per_{(X,Y)}: H^*_{dR}(X,Y) \otimes_k \mathbb{C} \xrightarrow{\cong} H^*_{sing}(X,Y;k)$, which gives the isomorphism of rep's $per: H^*_{dR} \longrightarrow H^*_{sing}$ needed to apply Thm(!).
By the discussion in Section 2, per corresponds to a $\mathbb{C}$-valued point of X

Def. Let $ev: \tilde{\mathbb{P}} \longrightarrow \mathbb{C}$ be the k-algebra homomorphism induced by per

Clearly, $ev(\tilde{\mathbb{P}}) = \mathbb{P}$ is the space of cohomological periods.
Now we prove that $\mathbb{P} = \mathbb{P}_{nc}$.

Cor The algebra $\mathbb{P}^{eff}$ is generated, as a $\mathbb{Q}$-v.s., by periods of (X, D, ω, γ) with X smooth affine, D a nc-divisor and $\omega \in \Omega^d_X(X)$

Proof.

. We know(?) that MM^{eff}_{Nori} is generated by motives of the form $H^d_{Nori}(X,Y)$, with X smooth affine and $Y \subseteq X$ nc-divisor.

. Now, we have $\tilde{\mathbb{P}}^{eff} = A_{12}(Pairs^{eff}) = A_{12}(MM^{eff}_{Nori}) = A_{12}(D)$, where $D \subseteq Pairs^{eff}$ is the subdiagram consisting of pairs (X, Y, j) with X sm. aff and Y nc-divisor.

. For such pairs, any cohomology class in $H^d_{dR}(X,Y)$ comes from a global section $\omega \in \Omega^d_X(X)$ and this concludes the proof.

□