Bridgeland Stability I

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These are the notes for a talk given on the ESAGA research seminar on January 18th, 2024. We cover the basic aspects of Bridgeland stability, as defined originally by Bridgeland in [4]. I claim little to no originality in the exposition below, which is mostly based off of my understanding of Kontsevich and Bridgeland's IHES talks, the lecture notes [6] of Macrì and Schmidt, as well as many of the papers cited below.

1 Stability of vector bundles on curves

Let C be a curve over a closed field. We have defined a notion of (semi)stability for vector bundles E on this curve. (More generally even, for coherent sheaves on it.) We recall that to each such E we can assign two numbers $\deg E$ and $\operatorname{rk} E$ and finally the slope

$$\mu(E) = \deg(E)/\operatorname{rk}(E).$$

A coherent sheaf is called semistable (resp. stable) if for every (coherent) non-zero subsheaf $E' \subset E$

$$\mu(E') \leqq \mu(E) \qquad (\text{resp. } \mu(E') < \mu(E)).$$

Crucially: we can prove the following theorem.

Theorem 1. *Let E be a vector bundle on C. Then there exists a unique finite filtration by subbundles*

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_m = E,$$

whose quotients E_i/E_{i-1} are semistable of slope λ_i and $\lambda_1 > \lambda_2 > \cdots > \lambda_m$.

Remark. The same theorem also holds for coherent sheaves on C, provided that one defines the slope of rank 0 sheaves to be ∞. Alternatively, one can define the <u>phase</u> of such sheaves which work in the same way as the slope, but behaves better in the general abelian setting.

The proof of the following theorem relies on showing uniqueness first and using an induction argument to patch things up, also using finiteness of Jordan-Hölder filtrations (Noetherianness). Here is the technical heart of the proof:

Lemma 1. Let $0 \to E' \to E \to E'' \to 0$ be a short exact sequence of coherent sheaves. Then

$$\mu(E) \in [\min\{\mu(E'), \mu(E'')\}, \max\{\mu(E'), \mu(E'')\}].$$

Furthermore, if $\mu(E') \neq \mu(E'')$, then $\mu(E)$ lies in the interior of the interval.

An algebraic proof of the above lemma can be annoying to write down. Instead we can prove it geometrically by defining the following auxiliary quantities.

Definition 1. Let E be a vector bundle (or coherent sheaf) on C. We define

$$Z(E) = -\deg(E) + i\,\mathrm{rk}(E) \in \mathbf{C}$$

The <u>phase</u> $\phi(E)$ of E is defined to be the argument of $Z(E)$. Note that $\mu(E_1) < \mu(E_2)$ if and only if $\phi(E_1) < \phi(E_2)$.

Proof (of lemma 1). By additivity of rank and degree, it follows that $Z(E) = Z(E') + Z(E'')$. Now the result follows by drawing the parallelogram. □

In particular, this lemma implies the following. Let $\mathrm{Coh}_\lambda(C)$ be the full subcategory of $\mathrm{Coh}(C)$ consisting on the zero object and semistable bundles of slope λ. Then $\mathrm{Coh}_\lambda(C)$ is abelian and the inclusion $\mathrm{Coh}_\lambda(C) \subset \mathrm{Coh}(C)$ is exact.

Another important corollary of this lemma (which yields uniqueness in the theorem) is the following: If $E_i \in \mathrm{Coh}_{\lambda_i}(C)$, and $\lambda_1 < \lambda_2$, then there are no non-zero maps $E_2 \to E_1$. In short words:

> The category $\mathrm{Coh}_\lambda(C)$ is contained in the right orthogonal category to $\mathrm{Coh}_\mu(C)$ for $\lambda < \mu$.

Now, we note also that the notion of stability and slope do not depend only on the isomorphism class of the vector bundle, but only on its image in $K_0(\mathrm{Coh}(C)) = K_0(C)$. We recall the definition:

Definition 2. Let $\mathcal{A}$ be an abelian category. We denote by $K_0(\mathcal{A})$ the free abelian group on isomorphism classes of objects in $\mathcal{A}$ modulo the relations

$$[E] = [E'] + [E''], \quad (0 \to E' \to E \to E'' \to 0).$$

Then we can package the rank and degree of a coherent sheaf on C succintly via the complex number

$$Z(E) = -\deg(E) + i\,\mathrm{rk}(E) \in \mathbf{C},$$

whose slope corresponds precisely to the slope as defined before. We then obtain a homomorphism of abelian groups

$$Z\colon K_0(C) \to \mathbf{C}$$

Definition 3. Let $\mathcal{A}$ be an abelian category. Then a <u>stability function</u> is a group homomorphism

$$Z\colon K_0(\mathcal{A}) \to \mathbf{C}$$

such that the imaginary part is $\geqq 0$ and the real part is < 0 if it lies on the real line for all objects in $\mathcal{A}$.

Given a stability function we define he <u>slope</u> $\mu(A)$ to be the argument of the complex number $Z(A)$. We define the notions of stability, semi-stability as before. With the same proof, we obtain that

$$\mathcal{A}_\mu \subset \mathcal{A}$$

is an exact inclusion of abelian categories.

Example. Let Q be a quiver, meaning a finite collection of vertices V and arrows A between two vertices. (We do not impose relations on these in this talk.) A concrete example to have in mind are the quivers

$$Q = A_n : \bullet \to \bullet \to \dots \to \bullet$$
$$Q = K : \bullet \rightrightarrows \bullet$$

Let $\mathcal{A}$ be the abelian category of finite dimensional k-representations of Q. This is the category whose objects consists on finite dimensional vector spaces V_i for each vertex in V and morphisms $V_i \to V_j$ for every arrow.

Given an embedding $Q \subset \mathcal{H}$, the upper half plane, (or really just a function $z\colon V(Q) \to \mathcal{H}$) we can define a central charge

$$Z\colon K(Q) \to \mathbf{C}, \quad V \mapsto \sum_i z_i \dim V_i$$

and one shows that every central charge arises as such.

Consider the quiver $Q = \bullet \to \bullet$. What are the semistable objects of its representations for a given central charge as above? There are three cases: Let $E_{10} = k \to 0$, $E_{01} = 0 \to k$ and $E_{11} = k \xrightarrow{\sim} k$. We note the existence of a short exact sequence

$$0 \to E_{01} \to E_{11} \to E_{10} \to 0$$

All three objects are idecomposable, but only the first two are simple. Let $z_1 = Z(E_{01})$ and $z_2 = W(E_{10})$, with slopes μ_1 and μ_2 respectively.

1. If $\mu = \mu_1 = \mu_2$ then all objects are semistable of slope μ.
2. If $\mu_1 < \mu_2$, then the above sequence tell us that E_{11} is not semistable. Then only $\mathcal{A}_{\mu_1}$ and $\mathcal{A}_{\mu_2}$ are non trivial, all equivalent to k-mod.
3. If $\mu_1 > \mu_2$ then E_{11} is also semistable and we have a third non-trivial category $\mathcal{A}_{\mu(E_{11})}$, again equivalent to k-mod.

Now let $Q = K = \bullet \rightrightarrows \bullet$. Again if $\mu_2 \geqq \mu_1$ the situation is more or less trivial, so lets assume $\mu_1 < \mu_2$. Representations of K are now much more interesting. Indecomposable representations come in three types:

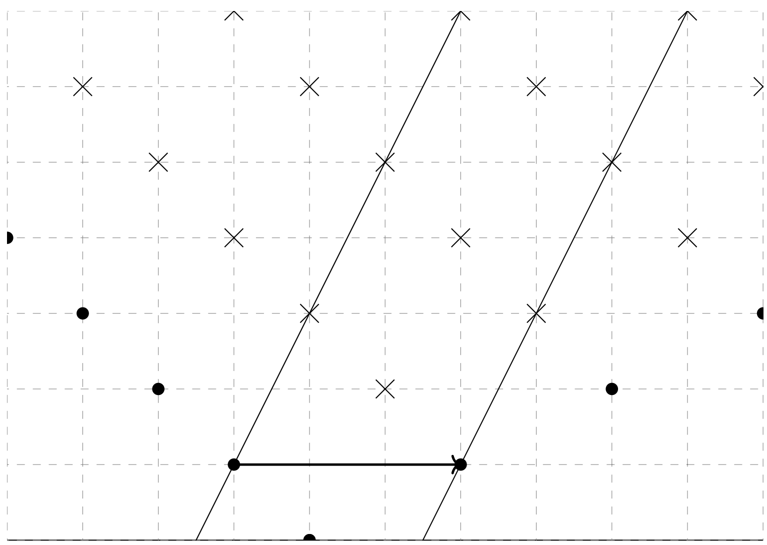
1. We have $k^n \to k^{n+1}$ and maps are inclusions in two different subspaces;
2. Dual of the above;
3. We have two maps $k^n \to k^n$ and the first one is an isomorphism; this therefore is equivalent to classifying idemcomposable modules over $k[T]$, which can be done using rational form or Jordan if k is closed. We obtain therefore a bijection between the idecomposable modules of third type and irreducible polynomials.

One curious example to have in mind for the third type is the representation

$$x_0, x_1 \colon k \rightrightarrows k$$

which will be idecomposable as soon as either x_0 or x_1 is non-zero. Up to isomorphism, such representation corresponds to a point in $\mathbf{P}^1(k)$.

Therefore, we have plenty of categories of semistable objects in the case $\mu_1 > \mu_2$, each corresponding to the slope of some idecomposable of the form above.



2 Going derived

Let $\mathcal{A}$ be an abelian category and consider $D = D^b(\mathcal{A})$ its bounded derived category. One can define the Grothendieck group of D by setting

$$[K] = [K'] + [K'']$$

whenever there is a fiber sequence $K' \to K \to K'' \to$. Noticiably, one has $[K[1]] = -[K]$ in the Grothendieck group, and in particular, it allows us to give an interpretation to vitual representations via the following proposition:

Proposition 1. Let $\mathcal{A}$ be an abelian category. Then the canonical inclusion $\mathcal{A} \subset D^b(\mathcal{A})$ induces an isomorphism of abelian groups

$$K_0(\mathcal{A}) \xrightarrow{\sim} K_0(D^b(\mathcal{A})).$$

Proof. To define the inverse, just use canonical truncations. □

In general, there is no way to recover $\mathcal{A}$ from $D(\mathcal{A})$. There is, however, purely derived structures that allows us to recover it. This is what is called a bounded t-structure.

Definition 4. Let D be a triangulated category. A <u>bounded t-structure</u> on D is the data of a full subcategory $D^\heartsuit \subset D$ such that $\mathrm{Hom}(D^\heartsuit, D^\heartsuit[-1]) = 0$. Furthermore, for every $X \in D$ there is a exhaustive and separated "filtration"

$$0 = X_{a-1} \to X_a \to \ldots \to X_b \to X_{b+1} = X, \qquad a \leqq b \in \mathbf{Z}$$

where X_r and $\mathrm{Cone}(X_{r-1} \to X_r) \in D^\heartsuit[-r]$.

We note that the maps in the filtration are not required to be monomorphisms as this notion is not well behaved in triangulated categories.For $D = D^b(\mathcal{A})$ this filtration is the canonical filtration whose graded pieces are the cohomology objects of any complex representing D.

Theorem 2. *This filtration is automatically unique up to isomorphism and $D^\heartsuit$ automatically abelian.*

Proof. We give a rough sketch. For $I = \{[m,n], \leqq m, < m, \geqq n, > n\}$ let D^I be the full subcategory of D consisting on those objects such that there is some filtration as above with $a < b$ in this range. Then given a filtration of any X as above one sees that $X_a \in D^{\leqq a}$. Using the octahedral axiom one also shows that $\mathrm{Cone}(X_a \to X) \in D^{>a}$. One also sees that $\mathrm{hom}(D^{\leqq 0}, D^{>0}) = 0$. This reduces our definition to the classical definition of a t-structure [3].

Now, one uses Yoneda to define truncation functors $\tau^{\leqq a}$ and $\tau^{\geqq b}$ as right (resp. left) adjoints of the inclusions. This implies that the filtration is unique. Finally kernels and cokernels can be computed as cones and cocones followed by truncations. □

Remark. Just as the case of abelian categories it follows (with same proof) that $K_0(D^\heartsuit) = K_0(D)$.

We can now define stability conditions[1] on a triangulated category. This definition is equivalent to the original condition appearing in [4], but nowadays its also called a pre-stability condition.

Definition 5. An stability structure on D is the collection of two data:

1. An additive morhpism (called the central charge)

$$Z\colon K_0(D) \to \mathbf{C};$$

2. a bounded t-structure on D.

This data is asked to satisfy the condition that Z becomes a stability function on $D^\heartsuit$ such that the Harder-Narasimhan filtration exists (uniquely).

The following definition can be translated into another better definition. For this we define

$$D(\phi) = D^\heartsuit_{\mu=\pi\phi}$$

to be the collection of μ-semistable objects for $\phi \in \ ]0,1]$. The advantage of going derived is that we can now exend this definition to all real numbers via

$$D(\phi + n) = D(\phi)[n].$$

The following proposition is immediate from the definition of stability structure.

Proposition 2. Let $K \in D$ be an object. Then there is a (unique) exhausted separated "filtration"

$$0 = K_0 \to K_1 \to K_2 \to \dots \to K_n = K$$

such that the cones $\mathrm{Cone}(K_{i-1} \to K_i) \in D(\phi_i)$ and $\phi_1 > \phi_2 > \dots > \phi_n$.

Proposition 3. The data of a stability condition on D is equivalent to the data of

1. A central charge $Z\colon K_0(D) \to \mathbf{C}$;

[1] But since its not a condition, I will follow Kontsevich and call these structures.

2. for each $\phi \in \mathbf{R}$ a full subcategory $D(\phi) \subset D$.

such that the argument of $Z(E) \neq 0$ is $\pi\phi$ if $0 \neq E \in D(\phi)$, for all ϕ we have $D(\phi+1) = D(\phi)[1]$, if $\phi_1 > \phi_2$ then $\mathrm{Hom}(D(\phi_1), D(\phi_2)) = 0$, and finally each object has a (unique) filtration as in the theorem above.

Proof. We sketched one direction. The other direction follows from considering

$$D^{\heartsuit} = D(]0,1]),$$

the category of those objects which have graded pieces with phase between 0 and 1. This is automatically the heart of a t-structure by our very convenient definition. $\square$

It also follows that $D(]\phi, \phi+1])$ is the heart of a t-structure on D (hence abelian) by the same reason.

Remark. For clarity's sake, when considering multiple stability conditions on the same category, we call the choice of such subcategories $D(\phi)$ by <u>slicing</u>. We will denote such by a letter such as $\mathcal{P}$ and write $\mathcal{P}(\phi)$ for $D(\phi)$.

Example. By Beilinson's theorem, we have that

$$D(\mathbf{P}^1) \cong D(\bullet \rightrightarrows \bullet)$$

In particular there are at least two different stability structures on this category. We will see later that you can deform one into the other.

3 Numerical invariants and the construction of central charges

We now want to approach the problem of constructing central charges $K_0(X) \to \mathbb{C}$ for smooth, projective k-varieties X. The main problem being the computation of $K_0(X)$.

Proposition 4. Let C be a curve. Then we have an isomorphism

$$K_0(C) \cong \mathbf{Z} \times \mathrm{Pic}(C) = \mathbf{Z} \times \mathbf{Z} \times \mathrm{Pic}^0(C),$$

where the first copy of $\mathbf{Z}$ is the rank and the second the degree. In particular all stability functions are deformations of the rank-degree stability function.

Remark. A homomorphism $K_0(X) \to \mathbb{C}$ always factor through the rationalization $K_0(X)_{\mathbf{Q}} \to \mathbb{C}$. By the Grothendieck-Riemann-Roch theorem, this is nothing but the rationalization of the Chow ring

$$\mathrm{ch}\colon K_0(X)_{\mathbf{Q}} \xrightarrow{\sim} \mathrm{CH}(X)_{\mathbf{Q}}$$

with isomorphism given by the Chern character. Unfortunately, the right hand side is also hard to compute.

Definition 6. Let X be a smooth projective variety over a field. For $v, w \in K_0(X)$, we define their Euler-Poincaré pairing to be

$$\chi(v,w) = \sum (-1)^i \dim \mathrm{Ext}^i(M,N) \in \mathbf{Z}$$

Let T be the set of v such that $\chi(v,w) = 0$ for all w. Then $K_{\mathrm{num}}(D) = K_0(D)/T$ is called the numerical Grothendieck group of D.

Example. If X is a curve, then $K_{\mathrm{num}}(X) = \mathbf{Z} \oplus \mathbf{Z}$.

Example. If X is a surface, then $K_{\mathrm{num}}(X)$ is the image of the Chern character map

$$\mathrm{ch}\colon K_0(X) \to \mathrm{H}^\bullet(X,\mathbf{Q}).$$

For X K3 or abelian surface then this is isomorphic to $\mathrm{H}^0(X,\mathbf{Z}) \oplus NS(X) \oplus \mathrm{H}^4(X,\mathbf{Z})$.

Theorem 3. *Assume k is of characteristic zero. If X is smooth, projective variety over k, then $K_{\mathrm{num}}(X)$ is a finitely generated (hence free!) abelian group.*

Proof. For example, see [7, Thm. 1.2]. □

Still out of all numerical stability structures, not all of them are sufficiently well behaved. The "suport property" below was found by Kontsevich-Soibelman in [5], and now is included in the definition of stability structures by most authors.

Definition 7. We say that a stability structure on X is numerical if the central charge factors through K_{num}. (There are analogous definitions for an arbitrary choice of lattice Λ and surjections $K_0(X) \twoheadrightarrow \Lambda$.)

Fix a norm $v \mapsto \|v\|$ on $K_{\mathrm{num}}(X) \otimes \mathbf{R}$. We say that a numerical stability structure satisfies the support property if

$$C_\sigma = \inf\left\{ \frac{|Z(E)|}{\|E\|} \mid 0 \neq E \in \mathcal{P}(\phi),\ \ \phi \in \mathbf{R} \right\} > 0.$$

This notion is indepenedent of the choice of norm on K_{num}.

The support property says that the image of $\{Z(E) \mid E \in \mathcal{P}\}$ is discrete in $\mathbf{C}$ and that it is not "too dense" in some sense. Its origin is physical (cf. [5, Remark 1]). It also implies that the categories $\mathcal{P}(\phi)$ are of finite length, meaning noetherian and artinian.

Proposition 5. A numerical stability structure on X satistfies the support property if and only if there exists a symmetric bilinear form Q on $K_{\mathrm{num}}(X)_{\mathbf{R}}$ satisfying the properties:

- if E is semistable then $Q(E,E) \geqq 0$;
- if $v \in K_{\mathrm{num}}$ is such that $Z(v) = 0$ then $Q(v,v) < 0$.

Proof. For one direction we define

$$Q(w,w) = \frac{1}{C_\sigma^2}|Z(w)|^2 - ||w||^2.$$

By construction $Q(w,w) \geqq 0$ on semistable objects, and clearly $Q(v,v) < 0$ if $Z(v) = 0$. Conversely, since $Q(v) \geqq 0$ then $||v|| \geqq 0$ hence v is non-zero in K_{num}. Now we see that $|Z(E)|^2$ is > 0 on the set where $-Q(v) \leqq 0$. By compacntess of the unit ball, there is some $C > 0$ such that $C|Z(v)|^2 - Q(v)$ is a positive definive quadratic form. $\square$

4 The moduli of stability structures and Bridgeland's "deformation theorem"

Definition 8. Let X be a smooth projective k-variety. We define $\mathrm{Stab}(X)$ to be the set of numerical stability structures on $D(X)$.

The group $\widetilde{\mathrm{GL}}_2^+(\mathbf{R})$ acts on $\mathrm{Stab}(X)$. Recall that this group is the universal cover of $\mathrm{GL}(2,\mathbf{R})$ and has the presentation

$$\widetilde{\mathrm{GL}}_2^+(\mathbf{R}) = \Big\{(T,f) : f\colon \mathbb{R} \to \mathbb{R} \text{ monotone}, \\ f(\phi+1) = f(\phi)+1,\ T \in \mathrm{GL}_2^+(\mathbf{R}),\ f_{\mathbf{R}/2\mathbf{Z}} = T_{\mathbf{R}^2-\{0\}/\mathbf{R}_{>0}}\Big\}$$

The action is given by $(T,f) \star (\mathcal{P},Z) = (\mathcal{P}',Z')$ where $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$ and $Z' = T^{-1} \circ Z$.

Definition 9. Let $\sigma = (\mathcal{P}, Z)$ be a stability structure on $D(X)$. Consider some object $E \in D$ and let $\phi_\sigma^-(E)$ and $\phi_\sigma^+(E)$ be the smallest (resp. largest) slope in its HN filtration. Then we endow $\mathsf{Stab}(X)$ with the coarsest topology making the functions

$$\sigma \mapsto \phi_\sigma^+(E), \phi_\sigma^-(E) \in \mathbf{R}$$

continuous for all E.

The above topology is actually locally given by a metric, and hence this is a reasonable topological space. (See [4] for details.) We finish this talk with two remarkable theorems.

Theorem 4. *The action of $\widetilde{\mathsf{GL}}_2^+(\mathbf{R})$ on $\mathsf{Stab}(X)$ is continuous and the map*

$$Z \colon \mathsf{Stab}(X) \to \mathsf{Hom}(K_{\mathrm{num}}(X), \mathbf{C})$$

sending a stability structure to its central charge has the following property: its restriction to each connected component Σ of $\mathsf{Stab}(X)$ is a local homeomorphism $\Sigma \xrightarrow{\sim} V(\Sigma)$ for a certain linear subspace $V(\Sigma)$ of the right hand side.

Proof. This is the main theorem of [4]. □

This says that any small deformation of the central charge can be uniquely lifted to a deformation of the whole stability structure. The following theorem now tell us the significance of the condition of the support property.

Theorem 5. *A numerical stability structure $\sigma \in \mathsf{Stab}(X)$ satisfies the support property if and only if it is <u>full</u>, meaning $V(\Sigma) = \mathsf{Hom}(K_{\mathrm{num}}(X), \mathbf{C})$ for the connected component Σ of σ.*

Given a full stability structure σ, and Q the quadratic form for which it satisfies the support property, then the set U of those $Z \colon K_{\mathrm{num}} \to \mathbf{C}$ such that Q is negative definite on $\ker Z$ is open. If U_0 is the connected component containing Z_σ then

$$\mathcal{U}_0 \to U$$

is a covering map, where $\mathcal{U}_0$ is the connected component of the preimage of U_0 containing σ.

Proof. The first part is [1, Prop. B.4]. The second is in [2, Prop. A.5]. □

Example. We know all stability spaces for curves.

- genus 0: $\mathrm{Stab}(\mathbf{P}^1) = \mathbf{C}^2$.
- genus g > 0: the action of $\widetilde{\mathrm{GL}}_2^+(\mathbf{R})$ on $\mathrm{Stab}(C)$ is free and transitive (and non-empty).

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