Babyseminar: condensed mathematics

Thiago Solovera e Nery

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The category of topological abelian groups is, despite the name of its objects, not abelian. The issue here is that, for example, the map $\mathbf{R}^\delta \to \mathbf{R}$, where $\mathbf{R}$ is the topological group of real numbers and δ means with the discrete topology, has zero kernel and cokernel while not being an isomorphism. An abelian category of locally compact abelian groups (LCA) is therefore non-existent, although $D^b(LCA)$ had had an ad hoc definition before.

In 2019, Scholze and Clausen proposed an ingenious solution based on the theory of pro-étale cohomology of rigid-spaces and schemes (which already was known to capture the right continuous cohomology groups with an intuitive definition).

The idea is to consider profinite sets as test spaces. If X is a compact (separated) space then there are enough maps $S \to X$ with S profinite as will be made precise in the seminar. As a concrete and instructive example the "binary expansion map"

$$\prod_{\omega}\{0,1\} \twoheadrightarrow [0,1], \quad (a_n) \mapsto 0,a_1a_2\cdots \in \mathbf{R}$$

presents the compact line as a quotient of the Cantor set.

Definition. Let ProfSet be the category of profinite sets. A *condensed set* is a functor

$$X : \mathrm{ProfSet}^\circ \to \mathsf{Set}$$

which satisfies a sheaf theoretic condition. A condensed abelian group, ring, module is defined by changing the codomain.

Using this definition one shows that each topological space X defines a condensed set (also denoted X) by

$$X(S) = \mathrm{Hom}_{\mathsf{Top}}(S, X)$$

and that, in fact, (compactly generated[1]) topological spaces embbed into condensed sets this way. (Note that the underlying set of X can be recovered as $X(*)$).

[1] Remember from your algebraic topology I days that this is a really mild condition to impose. Any first countable space is compactly generated!

We therefore obtain a commutative diagram (where Top is to be understood as "nice" spaces)

$$\begin{tikzcd}
\mathsf{Top} \arrow[r, hook] & \{\text{condensed sets}\} \\
\{\text{locally compact abelian groups}\} \arrow[u] \arrow[r, hook] & \{\text{condensed abelian groups}\} \arrow[u]
\end{tikzcd}$$

and, in fact, the category of condesed abelian groups has good properties. We will also see that $D^b(LCA)$ fully embeds into derived condensed groups, or which amounts to the same, that the condensed cohomology of these objects is the natural one (Talk 5). We will also show that, when G is locally profinite, the ad-hoc continuous group cohomology of G has a natural interpretation in the condensed world (Talk 8).

But our main application of the theory is the following: Recall that for each scheme X one defines a derived category $D_{\mathrm{qc}}(X)$ of quasi-coherent sheaves on X. This can be seen as a category of coefficients for the coherent cohomology of X, but, differently of what one might expect from other such formalisms, no functor $f_!$ adjoint to $f^!$ can be defined.

In fact if $X = \operatorname{Spec} A$ and $U = \operatorname{Spec} A[1/f]$ and $i\colon U \hookrightarrow X$ is the canonical open immersion then the functor $i^! = i^*$ cannot have a left adjoint as

$$M \mapsto M[1/f]$$

does not preserve infinite products. Nonetheless, we will be able to show the following theorem.

Theorem. There exists a category $D_{\blacksquare}(X) \supset D_{\mathrm{qc}}(X)$ for every scheme which carries a six functor formalism which restricts to the usual one for the five out of six defined functors in that case.

The will will require us to pursue the notions of completeness inside of the condensed formalism, and, perhaps more surprisingly, make us delve into the world of what seems to be a theory of analytic spaces over the integers (”solid geometryt’t’) which is parallel to the integral Berkovich spaces.

In particular, this will yield a new proof of coherent duality (eg. Serre duality) using condensed maths. A word of warning: we shall use a newer approach presented in the Analytic Stacks lecture course (joint Bonn/IHES 2024) in which light condensed sets are introduced. This concomitantly deals with set theoretic issues and simplifies significantly the notions of solid groups and analytic rings which are involved in the theory. The only drawback is the need for a non-orthodox YouTube$^{\mathrm{tm}}$ video reference which I hope the speakers of the first two-thirds of the talks wont mind.

Program

Here is the plan we will follow. To the prospecting speakers: feel free to massage the suggested material into the molds of a nice talk! This means in particular deciding which proofs to go into details and which to omit (even if I seem to be tacitly suggesting which proofs to omit, it is ultimately up to you) My door is always open to discuss the planning of the talks, and knocking is encouraged. The main reference (cited below as [AS]) is the Analytic Stacks lectures on YouTube$^{\mathtt{tm}}$, which, if one preferes to watching videos, I've tried to transcribed in a pdf which should be on my website. Besides that, we have the original notes from the course in condensed mathematics ("Condensed.pdf") and also some notes by Juan Esteban Rodrígez Camargo. For one of the lectures, we follow Anschütz's notes on condensed group cohomology. Have fun!

Talk 1 : Introduction (09/10)

We recall the goal of the seminar and sketch some of the basic ideas which we we'll be seing throughout the semester. I willl try and convince you to give a talk if there are still slots.

Talk 2 : Profinite sets and (light) condensed mathematics (16/10)

Define profinite sets, their size and weight with examples. Talk about Stone duality and its light variant [AS, Lecture 2]. Define light condensed sets, characterize them using Cantor sets [AS, Prop. 2.6] sketching the relations between qs (light) condensed sets and (sequential, metrizable) compactly generated topological spaces [AS, Prop. 3.2]. Mention Proposition 2.5 and 2.7 in [AS] with proof if there is time. You can also mention the existence of extramelly disconnected topological spaces and compare with original definition with its set theoretic issues.

Sections 2.1 and 2.2 in Juan's notes are also of great help.

Talk 3 : Light condensed abelian groups (23/10)

Define the categories of light condensed abelian groups and give many examples. Prove that light condensed abelian groups form a Grothendieck abelian category where countable products are exact and has an internally projective generator ([AS, Thm. 3.3] and 2.3.3 in Juan's notes). Explain the tensor structure and derived variants.

Talk 4 : Condensed cohomology (30/10)

Recall how to define cohomology as a right derived functor and the theories of singular, sheaf and Čech cohomologies and their relations (briefly).

Define cohomology of light condensed sets with coefficients in a condensed abelian group show that it computes the right cohomology groups for CW complexes ([AS, Thm 4.2] or 3.3 in Condensed.pdf, but *cf.* also 2.3.6 in Juan's notes).

Talk 5 : Locally compact abelian groups (06/11)

Recall the structure theorem of LCA groups and Pontrjagin duality (Theorem 4.1 in Condensed.pdf). Talk about Breen-Deligne resolutions ([AS, Thm. 4.4] and 4.10 in Condensed.pdf). Talk about the cohomology in LCA and how it agrees with condensed cohomology by the Ext group computations (second part of [AS, Lecture 4]). Show the relation between the derived category of LCA and the derived category of condensed abelian groups (Theorem 4.9 in Condensed.pdf).

Talk 6 : Solid abelian groups (13/11)

Introduce the notion of (light) solid abelian groups [AS, Def. 5.2]. Prove the essential closure properties of solid groups [AS, Prop. 5.3 and Cor. 5.4]. Then compute the solidificatio of $\mathbf{R}$ [AS, Lemma 5.5] and talk about the internally projective generator ([AS, Thm. 5.11] and its necessary lemmas. (Check also Theorem 3.2.3 on Juan's notes.)

Mention also the derived category of solid modules [AS, Def. 6.1] and its closure properties [AS, Prop. 6.2]. You can mention that $\prod_{\omega} \mathbf{Z}$ is flat, but we won't need this.

Talk 7 : Analytic rings (20/11)

Introduce the solid affine line $\mathbf{Z}[T]$ talk about [AS, Thm. 7.2]. Mention the definition of power-bounded and topologically nilpotent elements in this context [AS, Def. 7.4] and discuss lemma [AS, Lemma 7.5]. Define analytic rings [AS, Def. 8.8] focusing on the examples arising from discrete rings. Define Huber pairs and explain their relation to analytic rings [AS, Def. 8.1, Thm. 8.14].

Talk 8 : Solid group cohomology (27/11)

In this talk we follow Anschütz's notes on solid group cohomology. Recall the definition of continuous cohomology of a profinite group G (with coefficients in a continuous G-module M). Introduce the different analytic rings associated to G (pg. 3 and 4 in *loc.cit.*) and introduce the category of solid G-modules. Prove the comparison between continuous cohomology and solid group cohomology (Sec. 2). If time permits talk about finiteness (Prop. 3.9) and/or duality (Sec. 4).

Talk 9 : The solid exceptional pushforward (4/12)

Recall once again the difficulty in defining $f_!$ for morphisms of schemes. Now we prove the existence of *solid* compactly supported cohomology for finite type $\mathbf{Z}$-algebras

(Theorems 8.1 and 8.2 in Condensed.pdf). Mention also the formulation of the relative version (Theorem 8.13).

Talk 10 : Discrete adic spaces (11/12)

Recall the definition of a Huber pair when the underlying Huber ring is discrete and define "discrete" adic spaces associated to such pairs. Talk about the two fully faithfull embeddings of schemes into discrete adic spaces and their relations. Finally, talk about globalization of the derived category of solid modules and why we need infinity categories (Condensed.pdf pg. 65). The reference is Condensed.pdf Lecture 9, but [AS, Lectures 7 and 9] is also helpful. Perhaps section 5.4.1 in Juan's notes can serve as a more advanced overview.

Talk 11 : Globalization (18/12)

Give a rough idea of what is an infinite category and the existence of a stable derived infinity category of a (condensed) ring. [2] Sketch the proof of descent for the category of solid modules on a scheme/discrete adic space (Condensed.pdf Theorem 9.8, but whose proof is actually Lecture 10).

Talk 12 : Coherent duality (08/01)

The reference is Lecture 11 in Condensed.pdf. Recall the idea of six functor formalisms (either naively or using the category of corespondences) and explain how to construct the solid quasi-coherent formalism for schemes. Give the long awaited proof of Theorem 11.1 which reproves Grothendieck duality.

Talk 13?

There are further topics which we could talk about depending on interest.

References and links

I will leave them on my webpage:

(`https://www.esaga.uni-due.de/thiago.solovera-e-nery/condensed/`).

[2]This talk is perhaps better suited for those that have seen these things once, but I can also help in preparing this overview.