THE HEIGHT STRATIFICATION ON THE MODULI OF FORMAL GROUPS

November 13th, 2025

THESE ARE THE NOTES FOR A TALK ON THE ESAGA RESEARCH SEMINAR ON THE MODULI OF FORMAL GROUPS AND APPLICATIONS TO STABLE HOMOTOPY THEORY. THIS TALK IS ESSENTIALLY JUST A COPY FROM J. LURIE'S LECTURES ON CHROMATIC HOMOTOPY THEORY, SPLICED WITH IDEAS AND PROOFS FROM L. FARGUES' COURSE ON THE GEOMETRY OF LUBIN-TATE SPACES AND SOME COMMENTS AND COMPUTATIONS BY THE SPEAKER.

1 Recollections in characteristic 0

We start by recalling the following result mentioned on Campbell's talk:

Proposition 1.1. *Let R be a ring of characteristic 0 (that is, containing $\mathbf{Q}$). Then for any formal group law $F \in R[[x,y]]$ over R there is a* unique *power-series $f = t + O(t^2)$ such that*

$$f F(x,y) = fx + fy.$$

Corollary 1.2. *Any formal group law over a characteristic 0 ring R is isomorphic to $\widehat{\mathbf{G}}_a$. Furthermore, the isomorphisms of this as a formal group is*

$$\underline{\mathrm{Isom}}(\widehat{\mathbf{G}}_a, \widehat{\mathbf{G}}_a) = \mathbf{G}_m.$$

Hence, we have that $(\mathscr{M}_{\mathrm{fg}})_{\mathbf{Q}} \cong B\mathbf{G}_m$. To put it another way, the Lie algebra functor

$$\mathrm{Lie}\colon \mathscr{M}_{\mathrm{fg}}(R) \to \mathrm{Pic}(R), \quad \mathscr{G} \mapsto \mathrm{Lie}(\mathscr{G}) = \ker\left(\mathscr{G}(R[t]/t^2) \to \mathscr{G}(R)\right)$$

is an equivalence. For non-commutative and higher dimensional formal group laws the Baker-Campbell-Hausdorff formula gives also a similar equivalence between Lie algebras over R (a finite projective R-module with a bracket) and formal group laws over R, which we shall not need, hence not explain.

Question 1.3. Suppose that R is not a $\mathbf{Q}$-algebra. Are the formal group laws $\widehat{\mathbf{G}}_a$ and $\widehat{\mathbf{G}}_m$ isomorphic?

We will find an easy critereon to distinguish the two when $p = 0$. For this, we have to start developing some theory.

2 Invariant differentials

Definition 2.1. *Let R be a ring. We define the (ad-hoc) module of (continuous) differentials on $R[[t]]$ to be $\Omega = R[[t]]\,dt$, the free module on a "variable" dt. If $F \in R[[t]]$ is a formal group-law and $\omega = h\,dt \in \Omega$ then we define*

$$F^*\omega = h(F(x,y))\left(\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy\right) \in R[[x,y]]\,dx \oplus R[[x,y]]\,dy$$

The module of invariant differentials with respect to F is defined to be the submodule Ω_F of Ω consisting on those ω such that $F^*\omega = h(x)\,dx + h(y)\,dy$.

Remark 2.2. Indeed, one easily checks that the module of *continuous* R-linear derivations

$$\partial\colon R[[t]] \to R[[t]],$$

where both sides are endowed with the t-adic topology, is one dimensional generated by $\mathrm{d}t\colon \sum_r a_r t^r \mapsto \sum_r r a_r t^{r-1}$. One can also write down a definition of F-invariant differential in this language.

Proposition 2.3. *The "taking derivative at 0" morphism $\omega = h(t)\,dt \mapsto h(0) \in R$ defines an isomorphism*

$$\Omega_F \xrightarrow{\sim} \mathrm{Lie}F = R.$$

Proof. We must solve the differential equations

$$\begin{cases} h(x) = h(x +_F y)\partial_x F \\ h(y) = h(x +_F y)\partial_y F \end{cases}$$

on the initial condition $h(0) = r \in R$. Since F is commutative it is enough to solve, say, the top one.

Now, $0 +_F x = x$ so we have that

$$r = h(0) = h(y)\frac{\partial F}{\partial x}(0,y).$$

Since $F(x,y) = x + y + \dots$ we have that $h(y) = r\left(\frac{\partial F}{\partial x}(0,y)\right)^{-1}$.

We must now check that the above is indeed a solution (can assume $r = 1$). That is, we must check that $\partial_x F(0, F(x,y)) = \partial_x F(0,x)\partial_x F(x,y)$. We now use associativity $F(x,F(y,z)) = F(F(x,y),z)$ and take the derivative at x to get

$$\partial_x F(x,F(y,z)) = \partial_x F(F(x,y),z)\partial_x F(x,y).$$

Taking $x = 0$, $y = x$ and $z = y$ we magically get the formula we want. □

In particular, we see there is a canonical invariant differential associated to any F. For $\widehat{\mathbf{G}}_a$ it is $1\mathrm{d}t$, and for $\widehat{\mathbf{G}}_m$ it is $1/(1+t)\mathrm{d}t = (1 - t + \dots)\,\mathrm{d}t$.

Definition 2.4. *Suppose that $f \in tR[[t]]$. Then we can define*

$$f^*\colon \Omega \to \Omega, \quad \omega = h(t)\,dt \mapsto f^*\omega = g(h(t))\,dh,\ dh = h'(t)\,dt.$$

If f is a morphism $F \to G$ of formal group laws, then f carries invariant forms of G to invariant forms of F.

Proposition 2.5. *If ω_F, ω_G are the canonical invariant differentials, we have that*

$$h^*\omega_G = \lambda\Omega_F$$

for a unique $\lambda \in R$, and $h(t) = \lambda t + O(t^2)$. If $\lambda = 0$ then

- *If R is a $\mathbf{Q}$-algebra, then $h = 0$.*
- *If R is an $\mathbf{Z}_{(p)}$-algebra, then $h(t) = h'(t^p)$ for some other $h' \in tR[[t]]$.*

Proof. Most of it is straightforward from the previous proposition. We only mention that if $h^*\omega_G = 0$, then if $h = \sum_i h_i t^i$, we have that $ih_i = 0$, hence that $h_i = 0$ in characteristic 0, or that $h_i = 0$ for all i prime to p. □

3 Heights

Fix a rational prime p. From now on we work with $\mathbf{Z}_{(p)}$-algebras R and formal groups over those. We want to understand the geometry of $\mathscr{M}_{\mathrm{fg}} \times \mathbf{Z}_{(p)}$.

Definition 3.1. *Let F be a formal group law over R. We define $[n] \in R[[t]]$, the n-*series *of F, by induction:*

- $[0](t) = 0$,
- $[n](t) = F([n-1](t), t)$.

One notes that $[n](t) = nt + O(t^2)$.

Lemma 3.2. *The n-series of F is an endomorpism, that is*

$$F([n]X, [n]Y) = [n]F(X, Y).$$

Proof. Obvious for $n = 0$. By induction, associativity and commutativity of F we have

$$\begin{aligned} F(F([n-1]X, X), X), F(F([n-1]Y, Y), Y) = \\ F(F([n-1]X, [n-1]Y), F(X, Y)) = \\ F([n-1]F(X, Y), F(X, Y)) = [n]F(X, Y). \end{aligned}$$

□

Using Proposition 2.5 we see that the following definition makes sense.

Definition 3.3. *We say that F has* height $\geqq n$ *if there is $\lambda \in R$ such that*

$$[p] = \lambda t^{p^n} + O(t^{p^n} + 1).$$

We say that F has height exactly n *if λ is invertible. If F has height $\geqq n$ for all n we say that F has infinite height.*

Remark 3.4. Caution: for shorthand I will sometimes write $\mathtt{ht}\, F \geqq n$ and $\mathtt{ht}\, F = n$ for the statements that the height of F is $\geqq n$ and that the height of F is exactly n. When R is not a field however, there is no "height" $\mathtt{ht}\, F \in \mathbf{N}$; this is merely notational shorthand.

Remark 3.5. We have that $[p] = pt + O(t^2)$, hence $\mathtt{ht}\, F \geqq 0$ and $\mathtt{ht}\, F \geqq 1$ if and only if $p = 0$. Finally $\mathtt{ht}\, F = 1$ if and only if p is invertible.

Example 3.6. For $\widehat{\mathbf{G}}_a$ we have that $[p] = pt$, hence it has infinite height. For $\widehat{\mathbf{G}}_m$, we have by induction $[n] = (t+1)^n - 1$. Indeed, for $n = 0$ this is trivial, and

$$[n+1] = t + [n] + t[n] = t + (t+1)^n - 1 + t(t+1)^n - t = (t+1)^{n+1} - 1.$$

Hence if $p = 0$ in R, $[p] = t^p$, and we have that $\mathtt{ht}\, \widehat{\mathbf{G}}_m = 1$. In particular if R is not a $\mathbf{Q}$-algebra then $\widehat{\mathbf{G}}_m$ and $\widehat{\mathbf{G}}_a$ are not isomorphic!

Example 3.7. Let $E/\mathbf{F}_p$ be an elliptic curve, and $\widehat{E}$ be the formal completion at the identity. Then

- E is ordinary if and only the height of $\widehat{E}$ is 1.
- E is supersingular if and only if the height of $\widehat{E}$ is 2.

Example 3.8. Let $E/\mathbf{Q}_p$ be an elliptic curve, and $\mathcal{N}/\mathbf{Z}_p$ be its Néron model: this is in a precise sense the "best approximation" of E by a smooth group scheme, integrally. Let F be the associated group law of $\mathcal{N} \otimes \mathbf{F}_p$ by completing at identity.

- The curve E has additive reduction if and only if F has infinite height.
- The curve has good supersingular reduction if and only if F has height 2.
- If the curve has either multiplicative reduction or good ordinary reduction if and only if the height of F is 1.

The following lemma is obvious, and allows us to construct substacks associated to heights.

Lemma 3.9. *If $h : F \xrightarrow{\sim} G$ is an isomorphism then $[p]_G = h[p]_F h^{-1}$. In particular, $\mathtt{ht}\, F \geqq n$ (resp. $\mathtt{ht}\, F = n$) if and only if $\mathtt{ht}\, G \geqq n$ (resp. $\mathtt{ht}\, G = n$).*

If F is Zariski-locally on R of height $\geqq n$ (resp. exactly n) then so is F.

Definition 3.10. *We define the moduli stack of formal groups of height $\geqq n$ (resp. of height exactly n) to be the substacks*

$$\mathscr{M}_{\mathrm{fg}} \times \mathbf{Z}_{(p)} \supset \mathscr{M}_{\mathrm{fg}}^{\geqq n} \supset \mathscr{M}_{\mathrm{fg}}^{n}$$

given by those formal groups which are locally given by formal group laws of height $\geqq n$ (resp height exactly n). We also define $\mathscr{M}_{\mathrm{fg}}^{\infty} = \mathscr{M}_{\mathrm{fg}}^{\geqq \infty}$ to be the moduli of formal group laws of infinite height.

To be seen: $\mathscr{M}_{\mathrm{fg}}^{\geqq n}$ is a closed substack of $\mathscr{M}_{\mathrm{fg}} \times \mathbf{Z}_{(p)}$ and $\mathscr{M}_{\mathrm{fg}}^{n} = \mathscr{M}_{\mathrm{fg}}^{\geqq n} - \mathscr{M}_{\mathrm{fg}}^{n-1}$.

3.1 Bonus: extra curiosities about heights

In this small subsection, we assume R is of characteristic p.

Definition 3.11. *Let $F = \sum a_{ij} X^i Y^j$ be a formal group law over R. We define $F^{(p)}$ to be the formal group law $\sum a_{ij}^p X^i Y^j$.*

Clearly we have a Frobenius morphism $F \to F^{(p)}$ given by $T^p \in TR[[T]]$.

Proposition 3.12. *The height h of F is the least integer such that $[p]$ factors as*

$$[p]\colon F \to F^{(p^h)} \to F$$

Proof. Indeed, the morphism $h'\colon F^{(p^h)} \to F$ is given by Proposition 2.5. □

Remark 3.13. From this, one deduces Examples 3.7 and 3.8.

Lemma 3.14. *Let $\mathcal{G}$ be a formal group over R. If $\mathcal{G}$ has height exactly n, then $\mathcal{G}[p] = \ker([p]\colon \mathcal{G} \to \mathcal{G})$ is representable by a finite-flat group scheme of order p^n. If the height of $\mathcal{G}$ is infinite then $\mathcal{G}[p]$ is not finite-flat.*

Proof. The result is local so we can assume that $\mathcal{G}$ is a formal group law. Then we can write $\mathcal{G}[p] = \mathrm{Spec} R[[t]]/\langle [p](t)\rangle$, and then it is generated as an R-module by $1, t, \ldots, t^{p-1}$. (If the height is infinite, then the result is clear.) □

Hence we have another characterization of the height (both of which make sense in higher dimensions). It should be mentioned that this last lemma entails that $\mathcal{G}[p]$ is finite-flat if and only if the height is finite. This condition is equivalent to $\mathcal{G}$ being *p-divisible*.

A Theorem of Tate says that p-divisiible formal groups over $\mathbf{Z}_p$ (or any complete DVR) are equivalent to connected p-divisible groups! The reader is invited to take an educated guess as to which p-divisible group $\widehat{E}$ corresponds to.

4 The geometry of $\mathcal{M}_{\mathrm{fg}}$

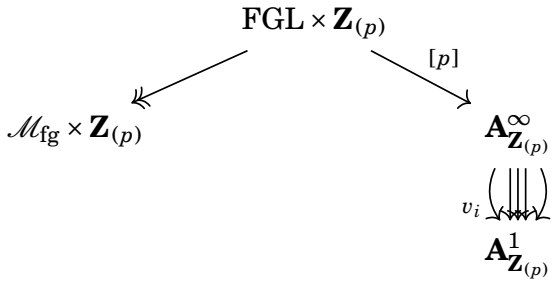
Putting together the first section of this lecture and Proposition 2.5, we have that

Proposition 4.1. *The moduli $\mathcal{M}_{\mathrm{fg}}^0$ of formal groups of height exactly zero is isomorphic to the open substack*

$$\mathcal{M}_{\mathrm{fg}}^0 \cong B\mathbf{G}_m \otimes \mathbf{Q} \cong \mathcal{M}_{\mathrm{fg}} \otimes \mathbf{Q} \hookrightarrow \mathcal{M}_{\mathrm{fg}} \times \mathbf{Z}_{(p)}.$$

The moduli $\mathcal{M}_{\mathrm{fg}}^{\geqq 1}$ of formal groups with height $\geqq 1$ is isomorphic to the reduction $\mathcal{M}_{\mathrm{fg}} \otimes \mathbf{F}_p$.

Since we are working over $\mathbf{Z}_{(p)}$ we can write $[p] = \sum v_n t^{p^n}$. Now let $\mathbf{A}^\infty_{\mathbf{Z}_{(p)}}$ be the functor over $\mathbf{Z}_{(p)}$ sending $R \mapsto \{\sum_{n=1} r_n t^{p^n}\}$. Then we have in diagram form



All this to say that v_i determine canonical elements in the Lazard ring. That is if F_u is the universal formal group law over L, then

$$[p]_{F_u} = \sum v_n t^{p^n}$$

for canonical elements $v_n \in L \otimes \mathbf{Z}_{(p)}$.

Here is the last technical proposition of today:

Proposition 4.2. *The elemnents v_n have degree $k = p^n - 1$. There is a set of generators of L over $\mathbf{Z}_{(p)}$*

$$L \otimes \mathbf{Z}_{(p)} = \mathbf{Z}_{(p)}[t_1, t_2, \dots]$$

such that $t_k = v_n$. More precisely, before localizing, one has $v_n = (p^k - 1)\bar{t}_k \in \mathbf{Z}\bar{t}_k \cong (I/I^2)_k$, for $k = p^n - 1$ where I is the ideal of positively graded elements in L.

Proof. We need to recall some details from the construction of the Lazard ring. We constructed a formal group law in the square-zero extension $\mathbf{Z} \oplus (I/I^2)_k$ coming from the key-lemma, essentially:

$$F = X + Y + \bar{t}_k C_{p^n}(X, Y).$$

for $\bar{t}_k$ the generator of $(I/I^2)_k$.

Recalling the definition of C_{p^n} we have

$$F = X + Y + \frac{\bar{t}_k}{p}\left((X+Y)^{p^n} - X^{p^n} - Y^{p^n}\right)$$

Claim: the a-series of F is $[a] = at + \frac{\bar{t}_k}{p}((at)^{p^n} - at^{p^n})$. Indeed, it is obvious for $a = 0$ and if it is true for a then $[a+1] =$

$$t + at + \frac{\bar{t}_k}{p}((at)^{p^n} - at^{p^n}) + \frac{\bar{t}_k}{p}\left((t + at + \frac{\bar{t}_k}{p}\left((at)^{p^n} - at^{p^n}\right)\right)^{p^n} - t^{p^n} - \left(at + \frac{\bar{t}_k}{p}((at)^{p^n} - at^{p^n})\right)^{p^n}\right).$$

If this obscene equation seems hopeless, remember that $\bar{t}_k$ is in fact a square-zero element, which means we can clear out the terms to get

$$(a+1)t + \frac{\bar{t}_k}{p}((at)^{p^n} - at^{p^n}) + \frac{\bar{t}_k}{p}\left(((a+1)t)^{p^n} - t^{p^n} - (at)^{p^n}\right) =$$
$$(a+1)t + \frac{\bar{t}_k}{p}(((a+1)t)^{p^n} - (a+1)t^{p^n}).$$

Taking $a = p$ yields the desired result. □

Corollary 4.3. *The stack $\mathcal{M}_{\mathrm{fg}}^{\geqq n} \subset \mathcal{M}_{\mathrm{fg}}$ is a closed substack, and we have $\mathcal{M}_{\mathrm{fg}}^{n} = \mathcal{M}_{\mathrm{fg}}^{\geqq n} - \mathcal{M}_{\mathrm{fg}}^{\geqq n+1}$.*

Proof. Indeed, it is enough to check this after pulling back to the cover $\mathrm{FGL} \twoheadrightarrow \mathcal{M}_{\mathrm{fg}}$. Now the first stack is just the vanishing of $v_0, \dots, v_{n-1}$ and the second is given by this and inverting v_n: Indeed, for height $\geqq 1$ this is just the fact that $v_0 = p$, and for greater heights we can assume R is an $\mathbf{F}_p$-algebra, and $F \in \mathrm{FGL}(R)$ corresponds to a morphism

$$L \to R, \quad F = F_u \otimes_L R,$$

where F_u is the universal group law, as before. Hence the p-series of F is

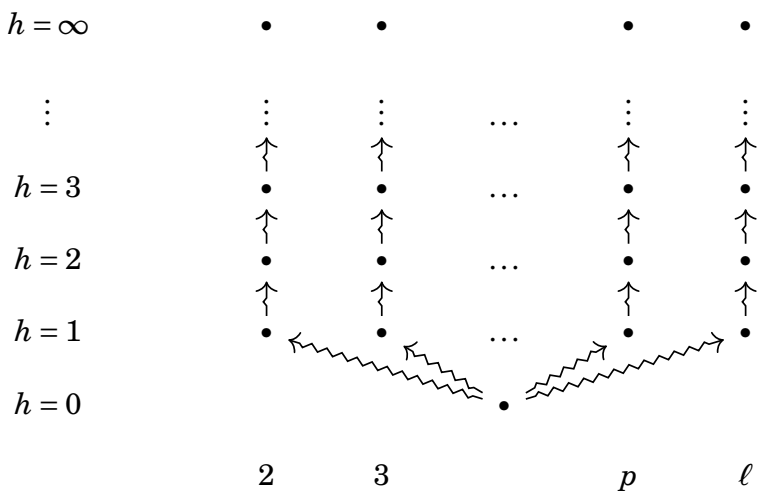
$$[p]_F = \sum_n f(v_n) t^{p^n}$$

where the $v_n \in L$ as above, hence the claim. □

Finally, we can tease a bit the results of next talk: from the proposition above it follows that the strata $\mathscr{M}_{\mathrm{fg}}^n$ ($n = 1,2,\ldots,\infty$) are non-empty. What is surprising is that, in fact, they are isomorphic to the classifying stack of a group scheme $\mathbf{G}_n$, the *Morava stabilizer group*.

Recall that any algebraic stack $\mathscr{X}$ has an underlying topological space $|\mathscr{X}|$ whose points are given by equivalence classes of field morphisms $\mathrm{Spec}K \to \mathscr{X}$: that is $|\mathscr{X}|$ is the union of $\mathscr{X}(K)$ modulo isomorphism over some extension of k (eg. $|B\mathrm{G}|$ always has one point). The topology is induced as the quotient topology of a presentation.

Our stack $\mathscr{M}_{\mathrm{fg}}$ is not algebraic, but it still has a presentation, so we can still define $|\mathscr{M}_{\mathrm{fg}}| = |\mathrm{FGL}|/|G^+|$ (cf. section [04XE] on the Stacks Project). Hence, we can draw the underlying topological space of $\mathscr{M}_{\mathrm{fg}}$ thus:



THE TOPOLOGICAL SPACE $|\mathscr{M}_{\mathrm{fg}}|$.

By no coincidence, this is in fact isomorphic to the Balmer's spectrum of the tensor-triangulated category of finite spectra! A surprising connection with algebraic topology which shows that, in some sense, $\mathbf{S} \to MU$ should be a "cover" of $\mathbf{E}_\infty$-rings. (Whatever this MU is, one knows that $\pi_*(MU)$ is the Lazard ring, but the map $\mathbf{S} \to MU$ is not flat in any reasonable way.)

Remark 4.4. The elements v_i are *not* elements in $\Gamma(\mathscr{M}_{\mathrm{fg}}, \mathscr{O}_{\mathscr{M}_{\mathrm{fg}}})$. Rather, they are sections of a tensor power of the Lie algebra line bundle $\omega\colon F \mapsto \mathrm{Lie}(F) \in \mathrm{Pic}(R)$, that is

$$v_i \in \Gamma(\mathscr{M}_{\mathrm{fg}}, \omega^{\otimes p^n - 1}).$$

This reminds me of modular forms.