HERMITIAN SPACES OVER p-ADIC FIELDS

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Abstract

These notes contain a concise proof of the classification of hermitian spaces over p-adic fields for odd p, avoiding the classification of quadratic spaces and Hasse invariants.

We fix F_0 a field of characteristic not 2. Recall that quadratic extensions are, up to isomorphism, in bijection with the set of $a \in F^*/(F^*)^2 - \{1\}$ via

$$a \mapsto F_0(\sqrt{a})/F_0.$$

We fix an extension F/F_0 of degree 2 (equivalently, an a as above). The unique automorphism of F/F_0 is denoted by σ.

Definition 1. Let V be a d-dimensional vector space over F. A *hermitian form* over V is a map

$$H : V \times V \to F$$

which is linear on the first coordinate, σ-linear on the second and $H(x,y) = H(y,x)^\sigma$. A *hermitian space* is an F-vector space endowed with a hermitian form.

If you fix a basis of V, then we can write any hermitian form can be as

$$H(x,y) = x^t H y^\sigma$$

where H is a matrix with $H^t = H^\sigma$. Two Hermitian forms H, H' are equivalent when the are identified after some linear isomorphism $g : V \xrightarrow{\sim} V'$. In matrix equations, this says that

$$H' = g^t H g^\sigma.$$

The *discriminant* of a Hermitian form is the determinant of the matrix defining it in some basis. By the eqution above that is only defined up to a norm element in F. We therefore define representatives

$$S^0 \cong F_0^*/N(F^*)$$

and we have that the discriminant of H is an element of S^0.

We now fix F_0 a non-archimedean local field with ring of integral elements $\mathcal{O}$, uniformizer π and residue k of prime characteristic not equal to 2. The units of F_0 are written as

$$F_0^* = \pi^{\mathbf{Z}} \oplus k^* \oplus U_1$$

where U_1 is a pro-p group and k^* is a cycic group of order $q-1$. Hence

$$F_0^*/(F_0^*)^2 \cong \{1, \delta, \pi, \delta\pi\}$$

is the Klein 4 group with $\delta \in \mathcal{O}$ any element which is not a square mod π.

Proposition 1 (Local class field theory for F)**.** One has an isomorphism $S^0 = F_0^*/N(F^*) \cong \mathrm{Gal}(F/F_0) = \{1, \sigma\}$.

Proof. Indeed, the norm subgroup contains all squares and hence one has a short exact sequence

$$1 \to N(F^*)/(F_0^*)^2 \to F_0^*/(F_0^*)^2 \to F_0^*/N(F^*) \to 1$$

Now one checks that unramified extensions will kill the $\langle\delta\rangle$ subgroup whereas ramified extensions will kill the $\langle\pi\rangle$ or $\langle\delta\pi\rangle$ subgroups (depending on whether (-1) is a square modulo p or not). □

This already tells us that there exists exactly two Hermitian forms of dimension 1, which are described precisely by the discriminant of H, as an element of S^0. Choose now representatives of S^0 in F_0^*.

Proposition 2. Any hermitian form can be diagonalized. That is, there is some basis for which H is expressed as

$$H(x, y) = \sum a_i x_i y_i^\sigma$$

for certain $a_i \in K^*$. Furthermore, the a_i can be chosen to be in $S^0 \cup \{0\}$.

Proof. We proceed by induction. If $H = 0$ there is nothing to do. Else, we claim that $Q(v) = H(v,v) \neq 0$. Indeed H can be reconstructed as

$$H(x,y) = \frac{1}{2}(Q(x+y) + Q(x+\omega y) - 2Q(x) - Q(y) - Q(\omega y)).$$

Let v be a vector for which $Q(v) \neq 0$. Then the map $w \mapsto H(w,v)$ is surjective on F and we get a F-hyperplane V' of elements H-orthogonal to v. Then clearly one has $V = Fw \oplus V'$ and this direct sum preserves H. □

A hermitian form is called non-degenerate if all a_i appearing in the decomposition above are non-zero, or, equivalently, if its discriminant is non-zero.

Theorem 1. *Any non-degenerate hermitian form over F is completely determined, up to equivalence, by the rank and discriminant. In particular, there are precisely two hermitian forms in each positive rank.*

Proof. By the diagonalization algorithm given above, it suffices to show that if H is the two dimensional form given by

$$H(x,y) = \delta(x_1 y_1^\sigma + x_2 y_2^\sigma),$$

with δ some element which is not a square modulo π, then H is equivalent to $x_1 y_1^\sigma + x_2 y_2^\sigma$. In matrix equations, this is equivalent to finding a matrix in $T \in \mathrm{GL}_2$ such that

$$T^t T^\sigma = \begin{bmatrix} \delta & \\ & \delta \end{bmatrix}.$$

We are therefore looking for elements $a,b,c,d \in F$ such that

$$\begin{cases} N(a) + N(b) = \delta \\ N(c) + N(d) = \delta \\ a^\sigma c + b^\sigma d = 0 \end{cases}$$

Now, the equation $N(a) + N(b)$ is a non-degenerate bilinear form, and hence $N(a) + N(b) = \delta$ defines a smooth scheme over $\mathcal{O}$. Since $\mathcal{O}$ is henselian, to find a solution in $\mathcal{O}$ it suffices to find one modulo π.

Now, by local class field theory, there are $(p+1)/2$ elements of the form $N(a)$ in k. Similarly, there are $(p+1)/2$ elements of the form $\delta - N(b)$ in k, and hence, by the pidgeonhole principle, we find a solution to $N(a) + N(b) = \delta \bmod \pi$.

Now, neither a nor b are zero, otherwise δ would be a norm element, and we can write, for example,

$$c = -\frac{b^\sigma}{a^\sigma} d$$

and therefore $N(c) = (N(\delta) - 1)N(d)$. Hence we need to find d with $N(d) = 1$, but this is clearly possible. □

Remark. What changes from the local field $\mathbf{R}$ to p-adic local fields? Local class field theory still holds true for $\mathbf{R}$ since $S^0 = \mathbf{R}/\mathbf{R}_{>0} = \{\pm 1\}$ but note that -1, the analogue of delta, cannot be written as a sum of norms, as that would be positive! For $\mathbf{R}$, the Hermitian form is determined by its signature, as is determined by the law of inertia.