Smooth base change; Künneth formula; the cycle class map

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We fix throughout $\Lambda = \mathbf{Z}/n$ seen as a sheaf of rings. We will usually assume that n is invertible in all schemes which we consider. All sheaves considered will be seen in the category $\mathrm{Mod}_\Lambda = \mathrm{Mod}_\Lambda(\mathrm{Ab}(X_{\text{ét}}))$ of Λ-modules. Most results remain true for $D^+(X) = D^+(X_{\text{ét}}, \Lambda)$ or even $D(X) = D(X_{\text{ét}}, \Lambda)$ when X is over a closed field, but we'll mostly stick to sheaves so as to be less scary. (Having said this, we will always work with derived functors in the level of derived categories. Leray now is written as

$$Rf_*Rg_* = R(f \circ g)_*$$

which is much cleaner.)

If M is a sheaf on X and $Y \to X$ is a morphism, we will usually denote the pullback of M to Y by the same name if no confusion arises. In particular Λ denotes the constant Λ sheaf on any scheme. We will denote geometric points $\mathrm{Spec}(\bar{k}) \to X$ as $x \to X$. In particular we wont put a bar on top of x so as not to make the notation too messy.

Main reference is [SGA4½, Arcata], but also Aaron Landesman notes for the smooth base change theorem on B. Conrad's Weil II seminar. (This is essentially an expanded version of Deligne's notes.)

1 Smooth base change

We start the talk with a variant theorem from the one seen last time (the proper base change). First we reintroduce the setup. Let X be an S-scheme and $g\colon S' \to S$ a morphism. We consider the cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

and recall there is a natural morphism $g^* Rf_* \to Rf'_* g'^*$ given by the push-pull adjunction and the natural morphism

$$g'^* (f^* Rf_* \to 1) .$$

Theorem 1 (Smooth base change)**.** *In the situation above, if S'/S is smooth and X/S qcqs then the base change morphism*

$$g^* Rf_*(M) \xrightarrow{\sim} Rf'_* g'^*(M)$$

is an isomorphism for all complexes $M \in D^+(X)$ of Λ-modules.

One important corollary is the following: if X/k is defined over a closed field and $k \subset K$ is an extension with K again closed, then

$$\mathsf{H}^q(X, \Lambda) \cong \mathsf{H}^q(X_K, M_K)$$

is an isomorphism for Λ torsion of order prime to the charactersitic of k. This corollary is not formal topos-nonsense as it may seem, and in fact, is false for k and K algebraically closed of characteristic p and $M = \mathbf{Z}/p$ by Artin-Schreier theory.

Remark. B. Zavyalov has recently provided a simple (and clear!) proof of Poincaré duality in the étale context, which implies the above base-change result almost immedetely by more or less formal reasons. This is done via the powerful machinery of 6-functors formalisms (and higher category theory). As it stands, we will be following the classical proof of Poincaré duality, which takes the smooth base change theorem as input.

Remark. We'll mention briefly that we can actually assume $X \to S$ finite type, separated and S noetherian using noetherian approximations.

Remark. In what follows, one could for simplicity do as in SGA4½ and limit ourselves to talking only about geometric points $x \to X$ with x being the closure of the residue field of its image. However, a fortiori, it follows from the smooth base change that one could then use arbitrary closed fields as these preserve cohomology groups of Λ-modules.

1.1 Universal Local Acyclicity

Our strategy for proving this result is quite interesting, as it boils down to having a precise control of the cohomological degeneration of fibers of a smooth morphism. For this we arise at the natural notion of "flatness" for étale cohomolgy.

First of all lets fix some notation. Let X be a scheme and $s \to S$ a geometric point. We will denote by $S_{(s)} = \operatorname{Spec} O^{\mathrm{sh}}_{S,s}$ the spectrum of the strict henselization at s. If X/S is an S-scheme then $X_{(s)} = X \times_S S_{(s)}$ is the pullback to this strictly henselian scheme. Similarly, X_s is the (geometric) fiber $X_{(s)} \times_{S_{(s)}} s$.

An étale generalization of a geometric point $s \to S$ is a geometric point $t \to S_{(s)}$.

Definition 1. Let $X \to S$ be a morphism, $x \to X$ a point above $s \to S$, and let t be an étale generalization of s. We define the variety of vanishing cycles of X/S at x with respect to t.

$$\widetilde{X}^x_t = X_{(\bar{x})} \times_{S_{(\bar{s})}} \bar{t}.$$

This is not of finite type in general. Intuitively, it consists of all points which generalize x above t. Intuitively, one wants this space to be contractible, meaning that, locally at x, this family does not degenerate too badly. Turns out we can make do with a weaker notion.

Definition 2. Let $f : X \to S$ be a morphism. We say that f is locally acyclic if for all geometric points $x \to X$ mapping to $s \to S$ and generalizations $t \to \operatorname{Spec} O_{S,s} = S_{(s)}$ we have that

$$\Lambda \xrightarrow{\sim} R\Gamma(\widetilde{X}^x_t, \Lambda)$$

is an isomorphism. In other words, if the cohomology of the geometric fibers of $X_{(x)} \to S_{(s)}$ vary continuously.

The morphism $X \to S$ is said to be universally locally acyclic, or ULA, if it is so after arbitrary base change $S' \to S$. (We won't need this notion, but cf. geometric Satake).

Remark. A result of Gabber says that if S is noetherian and X/S is of finite type, then LA implies ULA. In some sense, the reason we have to use Noetherian reduction is because ULA is the definition we are really interested in. (For experts: those are the "cohomologically smooth" morphisms for the ℓ-adic 6-functors formalism).

Example. If $X \to S$ is étale, then it is (universally) locally acyclic. Indeed, the induced map on strict henselizations is an isomorphism.

Example. Note that if $S = \operatorname{Spec} k$, then any X/S is automatically locally acyclic. This is because strict henselization $X_{(x)}$ is automatically Λ-acyclic since $R\Gamma(X_{(x)}, F_{(x)}) = F_x$ is exact.

However, if X/S is of finite type, then it is a deep theorem that X/S is universally locally acyclic. In particular it is unclear that $\mathbf{A}^1_S \to S$ is locally acyclic, even for S/k of finite type.

Assuming smooth base change, one can show that smooth morphisms are (universally) locally acyclic. (Exercise, but see stacks [0GJQ].) Instead, we'll show that base change holds when $S' \to S$ is locally acyclic, and then reduce smooth base change to the above.

1.2 Locally acyclic base change

Theorem 2. *Let $g\colon S' \to S$ be locally acyclic. Then for all $X \to S$ qcqs the base change morphism is an isomorphism.*

We first start proving the above theorem in the very important case of when $X = t \to S$ is a geometric point. First some lemmas.

Lemma 1. Let $X \to S$ be finite and consider $S' \to S$ arbitrary. Let $X' = X \times_S S'$ and consider points x', x, s', s in the natrual manner. Then the natural map

$$\mathcal{O}^{\mathrm{sh}}_x \otimes_{\mathcal{O}^{\mathrm{sh}}_s} \mathcal{O}^{\mathrm{sh}}_{s'} \xrightarrow{\sim} \mathcal{O}^{\mathrm{sh}}_{x'}$$

is an isomorphism.

Lemma 2. Locally acyclic morphisms are stable under quasi-finite base change (and descend over surjective quasi-finite maps).

Proof. Take $S' \to S$ locally acyclic and $X \to S$ quasi-finite. Fix also geometric points x', x, s', s above each other and consider the diagram

$$\begin{array}{ccc} X'_{(x')} & \longrightarrow & X_{(x)} \\ \downarrow & & \downarrow \\ S'_{(s')} & \longrightarrow & S_{(s)} \end{array}$$

By properties of henselian local ring, it follows that $X_{(x)}$ is finite over $S_{(s)}$, and hence the square is cartesian by last lemma. But then we see that the vanishing cycles of X'/X agree with the vanishing cycles of S'/S, hence we're done. □

Lemma 3. Let Y be a normal, integral scheme, and let η be its generic point. Then if $f\colon X \to Y$ is an étale Y-scheme, then

$$X = \coprod_{\lambda \in f^{-1}(\eta)} X_\lambda \to Y,$$

with each X_λ (étale and) integral, normal over Y.

EGAIV-18.10.7. Follows from a study of "geometrically unibranch" schemes. □

Proposition 1. Let $s \to S$ be a geometric point of S and consider $T \to S$ locally acyclic. Consider the cartesian diagram

$$\begin{array}{ccc} T_s & \xrightarrow{g'} & s \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

Then $g^* f_* \Lambda = g^* R f_* \Lambda \xrightarrow{\sim} R f'_* \Lambda$.

Proof. We begin by considering $X \subset S$ to be the adherence of s. We then consider the normalization Y of X in the geometric point $k(t)$. By proper base change, since $X \hookrightarrow S$ is proper, and $Y \to X$ integral, we can replace S by Y.

Now, Y is an non-noetherian scheme with the weird property that all its local rings are already strictly henselian (since they are normal and their fraction field is algebraically closed [Stacks, 0BSQ]). It is also an integral scheme and s is now identified with the generic point η of Y. Since Y is also normal we get that $g^* R f_* \Lambda = (R\eta_* \Lambda)_T \cong \Lambda$, where $R\eta_*\Lambda = \Lambda$ by the last lemma.

Finally, we consider the map $\Lambda = g^* R f_* \Lambda \to R f'_* \Lambda$ which we want to show is an isomorphism. We do this at stalks, so fix a geometric point s of T. By the formula of stalk of pushforward, we must compute the cohomology of T_η base changed to $s \to T$. But then we have a diagram

$$\begin{array}{ccccc} \widetilde{T}^s_\eta & \longrightarrow & T_\eta & \longrightarrow & \eta \\ \downarrow & & \downarrow & & \downarrow \\ s & \longrightarrow & T & \longrightarrow & Y \end{array}$$

and therefore we see that

$$\Lambda \xrightarrow{\sim} R\Gamma(\widetilde{T}^s_\eta, \Lambda) = (Rf'_*\Lambda)_x$$

is an isomorphism from local acyclicity. □

Remark. The above theorem tell us that the condition of local acyclicity is very natural from the point of view of base change. Namely if a class of morphisms is stable under quasi-finite base change and the

Proof of ULA-base change (sketch). By the proper base change and noetherian approximation we can assume that $X \to S$ is an open immersion. Then we use a strong dévissage to reduce to the case of sheaves in X which are pushfoward from geometric points. The crux of the proof lies in the fact that $U' = S' \times_S U \to U$ is also locally acyclic by quasi-finite base change.

Staring at the diagram

$$\begin{array}{ccc} X'_t & \longrightarrow & t \\ \downarrow & & \downarrow \\ U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

for a geometric point t of X, we obtain the base change for all $t_*\Lambda$ from the cases above. □

1.3 Smooth morphisms are locally acyclic

The final step of smooth base change now, of course, is to prove that smooth morphisms are (universally) locally acyclic.

Proposition 2. Composition of qcqs locally acyclic morphisms is locally acyclic.

Indeed, this is hard. We contend ourselves to proving the locally noetherian case. Our proof hinges on the following very beautiful lemma.

Lemma 4. Let $X \to S$ be locally acyclic with S noetherian. Suppose furthremore that the geometric fibers are acyclic, that is, the canonical morphism

$$\Lambda \xrightarrow{\sim} R\Gamma(X_s, \Lambda)$$

is an isomorphism for every geometric point $s \to S$. Then the canonical morphism $\Lambda \xrightarrow{\sim} Rf_*\Lambda$ is an isomorphism.

Proof of lemma. Passing to stalks, assume that S is local. The result actually holds for any sheaf F that comes from Λ by base change. Again, a strong dévissage argument reduces the case to $F = t_*\Lambda$ for $t \to S$ a geometric point.

We look at

$$\begin{array}{ccc} X_t & \longrightarrow & t \\ \Big\downarrow{\scriptstyle \iota_t} & & \Big\downarrow \\ X & \xrightarrow{f} & S \end{array}$$

and the base change now says that we have an identification $f^*t_*\Lambda \xrightarrow{\sim} R\iota_{t,*}\Lambda$. We then have a diagram

$$\begin{array}{ccccc} (Rf_*f^*t_*\Lambda)_s & & \longrightarrow & & R\Gamma(X_s, \iota_s^* R\iota_{t,*}\Lambda) \\ \Big\downarrow{\scriptstyle \sim} & & & & \Big\uparrow{\scriptstyle \sim} \\ R\Gamma(S, Rf_*R\iota_{t,*}\Lambda) & & & & \\ \Big\| & & & & \\ R\Gamma(X_t, \Lambda) & & & & R\Gamma(X_s, \Lambda) \\ & \nwarrow{\scriptstyle \sim} & & \nearrow{\scriptstyle \sim} & \\ & & \Lambda & & \end{array}$$

which is commutative because the base change morphism commutes with the Λ-module structure on both sides (take cohomology if a Λ-module structure in the derive category scares you). □

Remark. In the above proof we used, crucially, that

$$\Lambda \xrightarrow{\sim} \iota_s R\iota_{t,*}\Lambda$$

for every étale generalization of the base. This is equivalent to local acyclicity (more on this on the next section).

Proof. Now suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ are locally acyclic. Then we want to check that $g \circ f$ is also. We can suppose that X, Y and Z are strictly local and we want to show that

$$R\Gamma(X_z, \Lambda) = \Lambda,$$

for all geometric fibers at $z \to Z$. Now we know that Y_z are Λ-acyclic because g is locally acyclic. Moreover $f_z : X_z \to Y_z$ is locally acyclic by quasi-finite base change, and its geometric fibers, being vanishing cycle varieties for f, are Λ-acyclic. We conclude that $Rf_{z,*}\Lambda = \Lambda$ and the theorem since

$$R\Gamma(X_z, \Lambda) = R\Gamma(Y_z, Rf_{z,*}\Lambda) = \Lambda.$$

□

Theorem 3. *Smooth morphisms are locally acyclic.*

Proof. Every smooth morphism is Zariski locally, an étale open of $\mathbf{A}^d_S$, hence we immediately reduce to affine space. But $\mathbf{A}^d_S \to S$ is just a bunch of compositions of $\mathbf{A}^1_T \to T$, so we may furthermore assume $d = 1$. Finally, we can assume that the base is Noetherian strictly henselian $S = \operatorname{Spec} A$ and $X = \operatorname{Spec} A\{T\}$, the (strict) henselization of $A[T]$ at $(T, \mathfrak{m}_A)$.

What we want: for every geometric point $s \to S$ the fiber X_s is Λ-acyclic. Since $A\{T\}$ is a colimit of étale neighborhoods, hence X_s is a limit of affine curves over s, and therefore $R\Gamma(X_s, \Lambda)$ is concentrated in degrees $0, 1$ by dimension results.

We now break this down into smaller propositions, each harder than the next. □

Proposition 3. The geometric fibers of $X \to S$ are connected.

Proof. Technical and not necessairly enlightening. We reduce to the excellent case by noetherian reduction and then use the powerful machinery of normalization. □

Proposition 4. The geometric fibers of $X \to S$ have no (Galois) étale covers of order n prime to the characteristic of the base field.

We have already mentioned this can fail without characteristic assumption, and this is the crucial step where the base-change results fail there.

Lemma 5 (Zariski-Nagata purity in dimension 2)**.** Let C be a regular local ring of dimension 2 and C' a finite and normal C-algebra which is furthermore étale outside of the closed point of C. Then C' is already étale over C.

Proof. The ring C' is normal and 2-dimensional at all maximal ideals. It is CM by Serre's critereon for normality, and since the base is regular we can use miracle flatness to conclude that C' is flat, and hence free over C. Now the points where C' is ramified over C is defined by the zero locus of the discriminant, which contains no point of height one, hence this locus is empty. □

Lemma 6 (Case of Abhyankar)**.** Let $S = \operatorname{Spec} V$ be a trait[1], π a uniformizer, η the generic point, X/S smooth, irreducible, and of relative dimension 1. Let $\widetilde{X}_\eta$ be a Galois covering of X_η, of degree n invertible in S, and $S_1 = \operatorname{Spec} V[\pi^{1/n}]$. Then $\widetilde{X}_{\eta,1} = \widetilde{X}_\eta \times_S S_1$ extends to a Galois covering of $X_1 = X \times_S S_1$.

Proof. Let $\widetilde{X}_1$ be the normalization of X_1 in $\widetilde{X}_{\eta,1}$. Then this is étale over X_1 generically (by construction) and in the generic point of the special fiber (by tame ramification assumption and structure of inertia). Hence we can use Zariski-Nagata purity. □

Remark. Another related theorem by Deligne: if $X \to S$ is a morphism of finite type between algebraic k-schemes, then there is an open dense subset $U \subset S$ for which $X_U \to U$ is universally locally acyclic. Again, this implies the results of this section when working over fields, and in fact Deligne's proof depends upon them.

1.4 The la-proper base change

If $f: X \to S$ is a locally acyclic morphism, then one can define cospecialization maps between the fibers of f. Given an étale generalization $t \to \operatorname{Spec} O^{sh}_{S,s}$, these are maps

$$R\Gamma(X_t, \Lambda) \to R\Gamma(X_s, \Lambda).$$

[1]means its a DVR

To understand its definition we consider the maps

$$X_t \xrightarrow{h} X_{(s)} \xleftarrow{i} X_s.$$

We define the nearby cycles of X/S to be

$$R\Psi = i^* Rh_* \Lambda \in D^+(X_s).$$

We now have an important observation:

Lemma 7. A morphism $X \to S$ is locally acyclic if and only if $K \xrightarrow{\sim} R\Psi$ is an isomorphism for all étale specializations of the base.

Proof. Since stalks commute with pullbacks, we obtain, by base change of $X_t \to X$ along $X_{(x)} \to X$, that for every geometric point x of X_s,

$$R\Psi_x \xrightarrow{\sim} R\Gamma(X_{(x)}, Rh_{(x),*}\Lambda) = R\Gamma(\widetilde{X}_t^x, \Lambda).$$

Now the canonical morphism $\Lambda \to R\Gamma(\widetilde{X}_t^x, \Lambda)$ is identified with the morphism $\Lambda \to R\Psi_x$ and we're done. □

Definition 3. Let X/S be locally acyclic and consider an étale specialization t of $s \to S$. The cospecialization morphism is defined to be the composite

$$R\Gamma(X_t, \Lambda) = R\Gamma(X_{(s)}, Rh_*\Lambda) \to R\Gamma(X_s, R\Psi) = R\Gamma(X_s, \Lambda).$$

Theorem 4. *Let $X \to S$ be a proper, locally acyclic morphism. Then all cospecialization morphisms*

$$R\Gamma(X_t, \Lambda) \to R\Gamma(X_s, \Lambda)$$

are isomorphisms. In particular, the pushfoward $Rf_*\Lambda$ is locally constant constructible. The same holds for Λ replaced by an object in $D^+(X)$.

Proof. It follows by proper base change that $Rf_*\Lambda$ is constructible, and the stalks are the $R\Gamma(X_s, \Lambda)$ for $s \to S$, so the first part implies the second. Now using again proper base change for the proper morphism $X_{(s)} \to S_{(s)}$ applied to the torsion (derived) sheaf $Rh_*\Lambda$ we get

$$R\Gamma(X_t, \Lambda) = R\Gamma(X_{(s)}, Rh_*\Lambda) = (Rh_*\Lambda)_s \xrightarrow{\sim} R\Gamma(X_s, i^*Rh_*\Lambda) \cong R\Gamma(X_s, \Lambda)$$

which is what we want. □

Corollary 1. *Let S be a DVR, say $S = \operatorname{Spec} \mathbf{Z}_p$, with closed point s and generic point η. Then if X/S is a proper and smooth S-scheme then, after suitable choice of geometric points, there is a canonical isomorphism*

$$R\Gamma_{\text{ét}}(X_s, \mathbf{Z}_p) \xrightarrow{\sim} R\Gamma_{\text{ét}}(X_\eta, \mathbf{Z}_p).$$

Remark. In the proof above, one has to be a bit careful. Namely, what we really use is that the cospecialization maps are, when X/S is proper, inverses of the specialization maps:

$$R\Gamma(X_s, K) \xleftarrow{\sim} (Rf_*K)_s = R\Gamma(X_{(s)}, K) \to R\Gamma(X_t, K).$$

Granted this, using noetherian approximation, one can then use that a lcc sheaf are the constructible sheaves whose specialization morphisms are equivalences.

Define $R\Phi(K)$ to be the cone of $K \to R\Psi(K)$. Then there is an exact triangle

$$R\Gamma(X_s, K) \to R\Gamma(X_t, K) \to R\Gamma(X_s, R\Phi(K)) \to$$

and hence a proper morphism is locally acyclic precisely when the specialization maps are isomorphisms.

2 The cup product and the Künneth formula

As a corollary, we obtain a Künneth formula for étale cohomology in great generality. First we recall the projection formula: Let K be an object in $D(X, \mathcal{O}_X)$ and E in $D(Y, \mathcal{O}_Y)$. There a map

$$\pi : Rf_*K \otimes^L E \to Rf_*(K \otimes^L f^*E)$$

coming by adjunction from the counit $f^*(Rf_*K \otimes^L E) = f^*Rf_*K \otimes^L f^*E \to K \otimes^L f^*E$. We say that the projection formula holds if the above map π is an isomorphism.

Proposition 5. Let X/k be a variety over a field with finite Λ-cohomological dimension. Then the projection formula holds for every pair of objects $K, L \in D(X, \Lambda)$.

For the talk, we will assume this theorem. Since I didn't find a good reference for this, I will sketch the proof using Bhatt-Scholze's article.

Definition 4. Let (X, O_X) be a ringed topos. An object K in $D(X, O_X)$ is said to be perfect if, locally on X, it is quasi-isomorphic to a complex of the form

$$K = [M_{n_1} \to \ldots \to M_{n_r}],$$

for finite-locally free O_X-modules M_i.

Lemma 8. Let $f : X \to Y$ be a morphism of ringed topoi. Let $K \in D(X, O_X)$ and $E \in D(Y, O_Y)$ and suppose furthermore E perfect. Then, the canonical morphism

$$Rf_*K \otimes^L E \xrightarrow{\sim} Rf_*(K \otimes^L f^*E)$$

is an isomorphism.

Proof. Essentially, a dévissage to the case where $E = \Lambda$, where it is obvious. □

Theorem 5. *Let X/k be a variety over a field with finite cohomological dimension. Then every object in $D(X, \Lambda)$ can be written as a filtered colimit of constructible*[2] *objects.*

Proof. This is Proposition 6.4.8 on Bhatt-Scholze "The pro-étale topology for schemes". □

Proof, of proposition. Writing K as a colimit of constructibles we can assume that K is constructible. But then, by the previous lemma, and compatibility with base change, the map π is an isomorphism on a stratification, and hence is an isomorphism. □

Theorem 6 (Geometric Künneth)**.** *Let X, Y be varieties over a field with finite cohomological dimension. Let $E \in D(X, \Lambda)$ and $K \in (Y, \Lambda)$. (The order of Λ is again assumed not to be divisible by the characteristic of k.) Then there is a canonical isomorphism*

$$R\Gamma(X \times Y, \mathrm{pr}_X^* E \otimes^L \mathrm{pr}_Y^* K) \cong R\Gamma(X, E) \otimes^L R\Gamma(Y, K).$$

[2] The right notion of constructibility here is that there is a constructible decomposition $X = \sqcup X_\lambda$ with the derived sheaf $M \in D(X)$ locally coming from a locally perfect object, that is, $M_\lambda \in D(X_\lambda)$ is the pullback of some perfect object in $D(\lambda)$, at least after passing to an étale cover of X_λ.

Proof. By locally acyclic base change we can write the diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{\mathrm{pr}_X} & X \\ \downarrow{\scriptstyle \mathrm{pr}_Y} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & \operatorname{Spec} k \end{array}$$

and we get that $g^* R\Gamma(X, E) \xrightarrow{\sim} \mathrm{pr}_{Y,*}\mathrm{pr}_X^* E$. Using the projection formula we get

$$R\Gamma(X \times Y, \mathrm{pr}_X^* E \otimes^L \mathrm{pr}_Y^* K) \cong R\Gamma(Y, \mathrm{pr}_{Y,*}(\mathrm{pr}_X^* E \otimes^L \mathrm{pr}_Y^* K)) \cong R\Gamma(Y, K \otimes^L g^* R\Gamma(X, E)).$$

But this means that

$$R\Gamma(Y, K \otimes^L g^* R\Gamma(X, E)) = R\Gamma(Y, K) \otimes^L R\Gamma(X, E)$$

and we're done. □

We observe that the Künneth formula is just base change and the projection formula in disguise. Outside of the geometric setting we must then use proper base change and the proper projection formula to get the result.

Theorem 7 (Künneth variation)**.** *Let $f: X \to S, g: Y \to S$ be S-schemes with X/S separated finite type and S qcqs. Then there is a canonical isomorphism*

$$R(f \times_S g)_! \left(R\mathrm{pr}_X^* K \otimes^L R\mathrm{pr}_Y^* L \right) \cong Rf_! K \otimes^L Rg_* L$$

for K, L in D^+.

Proof. By proper base change, and the proof above, it is enough to check the projection formula (with Rf_* replaced by $Rf_!$). If f is quasi compact open immersion, this is easy by looking at stalks; the étale case follows with a bit more care. Then the proper case can be checked by reducing to the case where L is of the form $g_!\Lambda$ for $g: U \to X$ étale, using smooth base change and the cases above. □

2.1 The cup product

We observe in this subsection that the above theorem has a very concrete interpretation in terms of cohomology sheaves, at least when $\Lambda = \mathbf{Z}/\ell$ or in the limit $\Lambda = \mathbf{Q}_\ell$. For some $K \in D(\mathbf{Z})$, we define the graded cohomology group to be

$$\mathrm{H}^*(K) = \bigoplus_p \mathrm{H}^p(K).$$

Now we have the classical Künneth theorem.

Proposition 6. Let K, K' be complexes of abelian groups. Then there is a short exact sequence

$$0 \to \bigoplus_{p+q=k} \mathrm{H}^p(K) \otimes \mathrm{H}^q(K') \to \mathrm{H}^k(K \otimes^L K') \to \bigoplus_{i+j=k-1} \mathrm{Tor}_1(\mathrm{H}^i(K), \mathrm{H}^j(K')) \to 0.$$

In general, we have a canonical map (called the cup product)

$$\cup : \mathrm{H}^*(K) \otimes \mathrm{H}^*(K') \to \mathrm{H}^*(K \otimes K'),$$

where the left hand side is the graded tensor product. Its construction is pretty straightfoward: interpreting $\mathrm{H}^j(K')$ as $\mathrm{Hom}(\mathbf{Z}, K'[j])$, one simply composes

$$\mathrm{Hom}(\mathbf{Z}, K'[j]) \to \mathrm{Hom}(K, K \otimes^L K'[j]) \to \mathrm{Hom}(K[i], K \otimes^L K'[i+j]),$$

and then one uses the composition map (which is additive and bilinear)

$$\mathrm{H}^i(K) = \mathrm{Hom}(\mathbf{Z}, K[i]) \otimes \mathrm{Hom}(K[i], K \otimes^L K'[i+j]) \to \mathrm{H}^{i+j}(K \otimes^L K').$$

The Künneth theorem tells us conditions for this map being an isomorphism, but it still allows us to define a ring structure on cohomology regardless: Namely, if we have map of sheaves/complexes $K \otimes^L K' \to K''$ then this induces a map

$$\mathrm{H}^*(K) \otimes \mathrm{H}^*(K') \xrightarrow{\cup} \mathrm{H}^*(K \otimes^L K') \to \mathrm{H}^*(K'').$$

In particular we collect the results of this section in the following.

Corollary 2. *If $K \in D^+(X, \Lambda)$ has an algebra structure for the tensor product $\otimes^L$, for example $K = \Lambda$, then Rf_*K inherits one also, whose multiplication is given by the "cup-product". If X/k is a variety over a field, then there is a canonical isomorphism*

$$\mathrm{H}^*(X, \mathbf{Q}_p) \otimes \mathrm{H}^*(Y, \mathbf{Q}_p) \xrightarrow{\sim} \mathrm{H}^*(X \times Y, \mathbf{Q}_p)$$

which is induced by the cup product.

3 The cycle class map

We finish this talk by mentioning the cycle class map and its basic properties. Let $Z \hookrightarrow X$ be a closed subsqueme of a k-variety X. Let c be the codimension of Z in X. Recall that we have a purity isomorphism

$$\mathsf{H}^r(Z,M) \xrightarrow{\sim} \mathsf{H}_Z^{r+2c}(X,M(c))$$

whenever both Z and X are smooth.

Therefore, if Z is a connected, smooth closed subvariety of X, we have a canonical class

$$\mathtt{cl}_X(Z)\colon \Lambda = \mathsf{H}^0(Z,\Lambda) \xrightarrow{\sim} \mathsf{H}_Z^{2c}(X,\Lambda(c)) \to \mathsf{H}^r(X,\Lambda(c)).$$

Theorem 8. *The above constrution extends uniquely to a natural morphpism of rings*

$$\mathtt{cl}_X\colon \mathsf{CH}^*(X) \to \bigoplus_r \mathsf{H}^r(X,\Lambda(r)).$$

The left hand side is the Chow ring of X: it is defined to be the free abelian group $C^*(X)$ on irreducible closed subvarieties of X, modulo an equivalence relation of rationality: intuitively, two k-cycles are equivalent if there is an irreducible $k+1$-subvariety W and a rational function $f \in k(\eta)$ whose zero divisor is their difference.

The Chow ring is functorial on X: if $f\colon Y \to X$ is flat, then

$$f^*\colon \mathsf{CH}^*(X) \to \mathsf{CH}^*(Y)$$

is exactly what you would expect: the map induced by the inverse image of prime divisors.

Finally the ring structure can be defined as follows. Given Z, Z' prime divisors intersecting transversely, meaning that the codimention of $Z \cap Z'$ is the sum of the codimension of Z and Z', then $[Z].[Z]' = [Z \cap Z']$. This pins down a unique graded-ring structure on $\mathsf{CH}^*(X)$.

Example. Let $X = \mathbf{P}^n$. Then in fact the class map $\mathtt{cl}\colon \mathsf{CH}^*(X) \xrightarrow{\sim} \mathsf{H}^*(X)$ is an isomorphism. This is in some sense a big coincidence, and very far away from the general case. However, this works whenever we can stratify X by affine subvarieties, such as toric varieties and flag varieties of reductive groups. (Essentially, cl is an isomorphism if it is so on Z and $U = X - Z$ for Z a closed subvariety. Hence it is enough to stratify X by things for which we know this to be true. See Fulton Ex. 19.1.11.)