Algebraic Geometry II Prof. Dr. Ulrich Görtz
SS 2026 Dr. Andreas Pieper

Problem sheet 9

Due date: June 23th, 2026.

Problem 25 (Characterization of $\mathbb{P}^1_k$.) Let k be an algebraically closed field and C be an integral smooth projective curve over k. Show that the following are equivalent:

i) $C \cong \mathbb{P}^1_k$,

ii) There is a point $P \in C$ such that $l(P) > 1$.

Problem 26 (Elliptic curves) Let k be an algebraically closed field and (E, O_E) be an elliptic curve over k (in the sense of Problem 23).

i) Show that $l(n[O_E]) = n$ for all $n \geqslant 1$.

ii) Pick elements $x \in \Gamma(E, \mathscr{O}_E(2[O_E])) \setminus \Gamma(E, \mathscr{O}_E([O_E]))$, $y \in \Gamma(E, \mathscr{O}_E(3[O_E])) \setminus \Gamma(E, \mathscr{O}_E(2[O_E]))$. Show that, after rescaling x, y if necessary, there exist $a_1, a_2, a_3, a_4, a_6 \in k$ such that

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Hint: Consider $y^2, xy, \ldots$ as sections of $\mathscr{O}_E(6[O_E])$.

Problem 27 Let k be a field and C be an integral smooth proper curve over k. The goal of this exercise is to show that C is projective.

i) Choose an affine open cover $C = \bigcup_{i=1}^r U_i$. Choose closed immersions $\varphi_i : U_i \hookrightarrow \mathbb{A}^{n_i}_k$, $i = 1, \ldots, r$ for some $n_i \in \mathbb{N}$.

ii) Show that the φ_i extend to morphisms $\tilde{\varphi}_i : C \to \mathbb{P}^{n_i}_k$.

iii) Consider the composition

$$\varphi : C \to \prod_{i=1}^r \mathbb{P}^{n_i}_k \hookrightarrow \mathbb{P}^N_k,$$

where $N = \prod_{i=1}^r (n_i + 1) - 1$, the first map is $(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r)$, and the second is the Segre embedding (Problem 5). Show that each $\varphi_{|U_i} : U_i \to \mathbb{P}^N_k$ $i = 1, \ldots, r$ is an immersion.

iv) Use [Görtz-Wedhorn, Algebraic Geometry I, 2nd ed.] Lemma 14.18 (or Lemma 15.19 in the 1st edition) to show that φ is a closed immersion.