

Problem sheet 3

Due date: Nov. 11, 2025.

Problem 8

Let $\varphi: A \rightarrow B$ be a ring homomorphism, and let $f: \operatorname{Spec} B \rightarrow \operatorname{Spec} A$ denote the map attached to φ .

- (a) Let $\mathfrak{b} \subseteq B$ be an ideal. Prove that

$$\overline{f(V(\mathfrak{b}))} = V(\varphi^{-1}(\mathfrak{b}))$$

(where $\bar{}$ denotes closure).

- (b) Assume that φ is surjective. Prove that f induces a homeomorphism from $\operatorname{Spec} B$ onto $V(\ker(\varphi))$.
- (c) Prove that the image of f is dense in $\operatorname{Spec} A$ if and only if every element of $\ker(\varphi)$ is nilpotent.

Problem 9 Let A be a ring, $f \in A$ and $X := D(f) \subseteq \operatorname{Spec} A$ with the subspace topology. Prove that the topological space X is quasi-compact (i.e., for every cover $X = \bigcup_{i \in I} U_i$ by open subsets U_i , there exists a finite subset $I' \subseteq I$ with $X = \bigcup_{i \in I'} U_i$).

Problem 10 Let A be ring and $f \in A$. Show that $D(f) = \emptyset$ if and only if f is nilpotent.

Problem 11 Let A be a ring. The *Jacobson radical* is defined as

$$J(A) := \bigcap_{\mathfrak{m} \subset A \text{ maximal ideal}} \mathfrak{m}.$$

- (1) Show that $J(A)$ contains the nilradical of A .
- (2) Show that $J(A)$ equals the nilradical of A if and only if every non-empty open subset of $\operatorname{Spec}(A)$ contains a closed point.
Hint: Use Problem 10.