Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2025/26 Dr. Andreas Pieper

Problem sheet 9

Due date: Jan. 7, 2025.

Problem 29 Let k be an algebraically closed field and $X = \operatorname{Spec} k[t]$ and $Y = \operatorname{Spec} k[x, y]/(y^2 - x^3)$. Consider the map $\varphi : X \to Y$ induced from the ring map

$$k[x, y]/(y^2 - x^3) \to k[t], \quad x \mapsto t^2,\ y \mapsto t^3.$$

(1) Show that φ is a homeomorphism on the underlying topological spaces.
Hint: You could begin by constructing an inverse to the restriction $\varphi_{|D(t)} : D(t) \to D(x)$.

(2) Show that φ is not an isomorphism of schemes.

Problem 30 Consider $X = \operatorname{Spec} k[x, y]$ and $U = X \smallsetminus \{(0, 0)\}$. Show that the scheme U is not affine.
Hint: Use Problem 21.

Problem 31 (Gluing sheaf morphisms) Let X be a topological space and $\mathcal{F}, \mathcal{G}$ sheaves on X. Suppose we are given:

- an open covering $X = \bigcup_{i \in I} U_i$,
- for every $i \in I$ a morphism $\varphi_i : \mathcal{F}_{|U_i} \to \mathcal{G}_{|U_i}$ of sheaves on U_i,

such that for every $i, j \in I$ one has $\varphi_{i|U_i \cap U_j} = \varphi_{j|U_i \cap U_j}$ (equality as maps of sheaves on $U_i \cap U_j$).

(1) Show that there exists a unique morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ such that $\varphi_{|U_i} = \varphi_i$ for all $i \in I$.
Hint: While this can be shown "directly" without too much difficulty, a simpler way may be to use the equivalence of categories between sheaves on X and sheaves on a (suitable) basis of the topology on X.

(2) Convince yourself that the statement of (1) may be rephrased by saying that the presheaf $U \mapsto \mathrm{Hom}_U(\mathscr{F}_{|U}, \mathscr{G}_{|U})$ which attaches to each open $U \subseteq X$ the set of all sheaf morphisms between the restrictions of $\mathscr{F}$ and $\mathscr{G}$ to U is a sheaf on X.